Appendix II

Derivation of expression (4.10)

For the multiple samples, given a pre-specified capability level $w > 0$, the posterior probability based on index C_{pm} that a process is capable is given as

$$p = \Pr\{C_{pm} > w \mid \mathbf{X}\} = \Pr\left\{\frac{USL - LSL}{6\tau} > w \;\middle|\; \mathbf{X}\right\} = \Pr\left\{\tau < \frac{USL - LSL}{6w} \;\middle|\; \mathbf{X}\right\}$$

$$= \Pr\left\{\sigma^2 + (\mu\text{-}T)^2 < \left(\frac{USL - LSL}{6w}\right)^2 \;\middle|\; \mathbf{X}\right\} = \int_0^{\left(\frac{USL-LSL}{6w}\right)} \int_{T-\sqrt{a^2-\sigma^2}}^{T+\sqrt{a^2-\sigma^2}} f(\mu,\sigma \mid \mathbf{X})\, d\mu d\sigma\,.$$

Denote $a = (USL - LSL)/6w$ and $g(\sigma) = \sqrt{a^2 - \sigma^2}$. Then

$$p = \int_0^a \int_{T-g(\sigma)}^{T+g(\sigma)} f(\mu,\sigma \mid \mathbf{X}\,)\, d\mu d\sigma$$

$$= \int_0^a \int_{T-g(\sigma)}^{T+g(\sigma)} \frac{2\sqrt{\sum_{i=1}^m n_i}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(\sum_{i=1}^m n_i+1)} \times \exp\left(-\frac{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \mu)^2}{2\sigma^2}\right) d\mu d\sigma$$

$$= \int_0^a \frac{2\sqrt{\sum_{i=1}^m n_i}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(\sum_{i=1}^m n_i+1)} \times \exp\left(-\frac{1}{\beta\sigma^2}\right) \int_{T-g(\sigma)}^{T+g(\sigma)} \exp\left(-\frac{\sum_{i=1}^m n_i(\mu - \bar{\bar{x}})^2}{2\sigma^2}\right) d\mu\, d\sigma$$

$$= \int_0^a \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(\frac{T - \bar{\bar{x}} + g(\sigma)}{\sigma/\sqrt{\sum_{i=1}^m n_i}}\right) - \Phi\left(\frac{T - \bar{\bar{x}} - g(\sigma)}{\sigma/\sqrt{\sum_{i=1}^m n_i}}\right)\right] d\sigma\,. \quad \text{(A.5)}$$

where $\alpha = (\sum_{i=1}^m n_i - 1)/2$, $\beta = [\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \bar{\bar{x}})^2 / 2]^{-1}$, $\bar{\bar{x}} = \sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij} / \sum_{j=1}^{n_i} x_{ij}$, $s_p^2 = \sum_{i=1}^m (n_i - 1)s_i^2 / \sum_{i=1}^m (n_i - 1)$ is the pooled variance of subsamples and $\Phi(\cdot)$ is the CDF of $N(0,1)$.

By changing the variable, we let $y = \beta\sigma^2$. Then, $dy = 2\beta\sigma\, d\sigma$, and $s_p/\sigma = \sqrt{2r/\sum_{i=1}^m (n_i - 1)y}$. Thus, the posterior probability p based on the index C_{pm} for multiple samples, which given in (A.5) can be simplified to:

$$p = \Pr\{C_{pm} > w \mid \mathbf{X}\}$$

$$= \int_0^t \frac{1}{\Gamma(\alpha)y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times [\,\Phi(\,b_1(y) + b_2(y)\,) - \Phi(\,b_1(y) - b_2(y)\,)\,]\; dy\,,$$

where

$$b_1(y)=\frac{T-\overline{\overline{x}}}{\sigma/\sqrt{\sum_{i=1}^{m}n_i}}=\sqrt{\sum_{i=1}^{m}n_i}\left(\frac{T-\overline{\overline{x}}}{s_p}\right)\left(\frac{s_p}{\sigma}\right)=\sqrt{\frac{2r\sum_{i=1}^{m}n_i}{\sum_{i=1}^{m}(n_i-1)y}}\ \delta,$$

$$b_2(y)=\frac{g(\sigma)}{\sigma/\sqrt{\sum_{i=1}^{m}n_i}}=\sqrt{\sum_{i=1}^{m}n_i}\left(\frac{g(\sigma)}{\sigma}\right)=\sqrt{\sum_{i=1}^{m}n_i}\left(\frac{a^2-\sigma^2}{\sigma^2}\right)^{1/2},$$

$$=\sqrt{\sum_{i=1}^{m}n_i}\left(\frac{a^2}{\sigma^2}-1\right)^{1/2}=\sqrt{\sum_{i=1}^{m}n_i}\left(\frac{\beta a^2}{y}-1\right)^{1/2}=\sqrt{\sum_{i=1}^{m}n_i}\left(\frac{t}{y}-1\right)^{1/2}.$$

and

$$t=\beta a^2=\frac{2ra^2}{\sum_{i=1}^{m}(n_i-1)s_p^2}=\frac{2r}{\sum_{i=1}^{m}(n_i-1)s_p^2}\left(\frac{USL-LSL}{6w}\right)^2=\frac{2r}{\sum_{i=1}^{m}(n_i-1)w^2}\left(\frac{USL-LSL}{6s_p}\right)^2$$

$$=\frac{2r}{\sum_{i=1}^{m}(n_i-1)w^2}\left(\frac{USL-LSL}{6\hat{\tau}'}\right)^2\left(\frac{\hat{\tau}'}{s_p}\right)^2=\frac{2r}{\sum_{i=1}^{m}(n_i-1)}\left(\frac{\hat{C}_{pm}^*}{w}\right)^2\left(\frac{\hat{\tau}'}{s_p}\right)^2$$

$$=\frac{2}{\sum_{i=1}^{m}(n_i-1)}\left(\frac{\hat{C}_{pm}^*}{w}\right)^2\left(\frac{\sum_{i=1}^{m}(n_i-1)}{\sum_{i=1}^{m}n_i}+r\delta^2\right).$$

Therefore, the posterior probability p for multiple control samples, which given in (4.10) can be derived.