Appendix III

Derivation of expression (4.12)

For the multiple samples, given a pre-specified capability level $w > 0$, the posterior probability based on the one-sided index C_{PU} that a process is capable is given as

$$p = \Pr\{C_{PU} > w \mid \mathbf{X}\} = \Pr\left\{\frac{USL-\mu}{3\sigma} > w \;\middle|\; \mathbf{X}\right\} = \Pr\{\mu + 3\sigma w < USL \mid \mathbf{X}\}$$

$$= \int_0^\infty \int_{-\infty}^{USL-3\sigma w} f(\mu,\sigma \mid \mathbf{X})\, d\mu d\sigma$$

$$= \int_0^\infty \int_{-\infty}^{USL-3\sigma w} \frac{2\sqrt{\sum_{i=1}^m n_i}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(\sum_{i=1}^m n_i + 1)} \times \exp\left(-\frac{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij}-\mu)^2}{2\sigma^2} \right) d\mu d\sigma$$

$$= \int_0^\infty \frac{2\sqrt{\sum_{i=1}^m n_i}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(\sum_{i=1}^m n_i + 1)} \times \exp\left(-\frac{1}{\beta\sigma^2} \right) \int_{-\infty}^{USL-3\sigma w} \exp\left(-\frac{\sum_{i=1}^m n_i (\mu-\bar{\bar{x}})^2}{2\sigma^2} \right) d\mu\, d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2} \right) \times \Phi\left(\frac{USL - 3\sigma w - \bar{\bar{x}}}{\sigma / \sqrt{\sum_{i=1}^m n_i}} \right) d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2} \right) \times \Phi\left(\frac{USL - \bar{\bar{x}}}{s_p / \sqrt{\sum_{i=1}^m n_i}} \times \frac{s_p}{\sigma} - 3\sqrt{\sum_{i=1}^m n_i} w \right) d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2} \right) \times \Phi\left(3\sqrt{\sum_{i=1}^m n_i} (\hat{C}_{PU}^* \times \frac{s_p}{\sigma} - w) \right) d\sigma . \qquad \text{(A.6)}$$

where $\alpha = (\sum_{i=1}^m n_i - 1)/2$, $\beta = [\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \bar{\bar{x}})^2 / 2]^{-1} = [\sum_{i=1}^m (n_i - 1) s_p^2 / 2r]^{-1}$, $\bar{\bar{x}} = \sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij} / \sum_{j=1}^{n_i} x_{ij}$, $s_p^2 = \sum_{i=1}^m (n_i - 1) s_i^2 / \sum_{i=1}^m (n_i - 1)$ is the pooled variance of subsamples and $\Phi(\cdot)$ is the CDF of $N(0,1)$.

By changing the variable, we let $y = \beta\sigma^2$. Then, $dy = 2\beta\sigma\, d\sigma$, and $s_p / \sigma = \sqrt{2r / \sum_{i=1}^m (n_i - 1) y}$. Thus, the posterior probability p based on the index C_{PU} for multiple samples, which given in (A.6) can be rewritten as:

$$p = \Pr\{C_{PU} > w \mid \mathbf{X}\}$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y} \right) \times \Phi\left[3\sqrt{\sum_{i=1}^m n_i} \left(\hat{C}_{PU}^* \times \sqrt{\frac{2r}{\sum_{i=1}^m (n_i - 1) y}} - w \right) \right] dy$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \Phi\left[3\sqrt{\sum_{i=1}^m n_i}\left(\frac{\tilde{C}_{PU}^*}{b_{\sum_{i=1}^m (n_i-1)}} \times \sqrt{\frac{2r}{\sum_{i=1}^m (n_i-1) y}} - w\right)\right] dy. \quad \text{(A.7)}$$

On the other hand, the posterior probability based on index C_{PL} that a process is capable is given as

$$p = \Pr\{C_{PL} > w \mid \mathbf{X}\} = \Pr\left\{\frac{\mu - LSL}{3\sigma} > w \;\middle|\; \mathbf{X}\right\} = \Pr\{\mu - 3\sigma w > LSL \mid \mathbf{X}\}$$

$$= \int_0^\infty \int_{LSL+3\sigma w}^\infty \frac{2\sqrt{\sum_{i=1}^m n_i}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(\sum_{i=1}^m n_i + 1)} \times \exp\left(-\frac{\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \mu)^2}{2\sigma^2}\right) d\mu d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[1 - \Phi\left(\frac{LSL + 3\sigma w - \bar{\bar{x}}}{\sigma / \sqrt{\sum_{i=1}^m n_i}}\right)\right] d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \Phi\left(3\sqrt{\sum_{i=1}^m n_i}(\hat{C}_{PL}^* \times \frac{s_p}{\sigma} - w)\right) d\sigma$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \Phi\left[3\sqrt{\sum_{i=1}^m n_i}\left(\frac{\tilde{C}_{PL}^*}{b_{\sum_{i=1}^m (n_i-1)}} \times \sqrt{\frac{2r}{\sum_{i=1}^m (n_i-1) y}} - w\right)\right] dy. \quad \text{(A.8)}$$

For convenience of presentation, we let $\tilde{C}_I^*$ denote as either $\tilde{C}_{PU}^*$ or $\tilde{C}_{PL}^*$, then from the equations (A.7) and (A.8), the posterior probability based on the one-sided indices C_{PU} and C_{PL} can be simplified as

$$p = \Pr\{\text{the process is capable} \mid \mathbf{X}\} = \Pr\{C_I > w \mid \mathbf{X}\}$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \Phi\left[3\sqrt{N}\left(\frac{\tilde{C}_I^*}{b_{N-m}} \times \sqrt{\frac{2r}{(N-m)y}} - w\right)\right] dy.$$

where $b_g = (2/g)^{1/2}\Gamma(g/2)/\Gamma[(g-1)/2]$, $g = N - m$, $\alpha = (N-1)/2$, $N = \sum_{i=1}^m n_i$ Therefore, the posterior probability p based on the one-sided indices C_{PU} and C_{PL} for multiple samples, which given in (4.12) can be derived.