Chapter 2

Frequency Approach for Measuring Process Capability

2.1. The Index C_p

2.1.1. Process Quality and the Index C_p

Several authors have promoted the use of various process capability indices and examined with differing degrees of completeness their associated properties. The first process capability index appearing in the literature was the precision index C_p and defined as (see Juran (1974), Sullivan (1984, 1985) and Kane (1986)):

$$C_p = \frac{USL - LSL}{6\sigma},$$

where USL is the upper specification limit, LSL is the lower specification limit, and σ is the process standard deviation. The numerator of C_p gives the range over which the process measurements are acceptable. The denominator gives the width of the range over which the process is actually varying. The index C_p was designed to measure the magnitude of the overall process variation relative to the manufacturing tolerance, which is to be used for processes with data that are normal, independent, and in statistical control. Obviously, it is desirable to have a C_p as large as possible, small values of C_p (particularly less than 1.00) would not be acceptable because this indicates that the natural range of variation of the process does not fit within the tolerance band. Finley (1992) refers to this index as CPI, which he says stands for Capability Potential or Capability Potential Index; Montgomery (1996) refers to C_p as PCR, for Process Capability Ratio. Clearly, the index only measures the potential of a process to reproduce acceptable product and does not take into account whether the process is centered.

For processes with two-sided specification limits, the percentage of non-conforming items ($\%NC$) can be calculated as $1 - F(USL) + F(LSL)$, where $F(\cdot)$ is the cumulative distribution function of the process characteristic X. On the assumption of normality, $\%NC$ can be expressed as:

$$\%NC = 1 - \Phi\left(\frac{USL - \mu}{\sigma}\right) + \Phi\left(\frac{LSL - \mu}{\sigma}\right).$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. If the process is perfectly centered at the specification range ($\mu = M$), then the $\%NC$ can be expressed as $2\Phi(-3C_p)$. For example, C_p = 1.00 corresponds to $\%NC$ = 2700 PPM, and C_p = 1.33 corresponds to $\%NC$ = 63 PPM. However, C_p does not refer to the mean of the process, it will not give an exact measure of percentage NC in the general case, i.e. $\mu \neq M$. Therefore, it provides a lower bound on $\%NC$ with $2\Phi(-3C_p)$.

2.1.2. Estimation of C_p

The index C_p contains only one parameter, σ, to be estimated. If one single sample is given as $\{x_1, x_2, \cdots, x_n\}$, we may consider the following estimator $\hat{C}_p$ of C_p defined as:

$$\hat{C}_p = \frac{USL - LSL}{6s},$$

where $s = [\sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)]^{1/2}$ is the sample standard deviation, which can be obtained from a stable process.

2.1.3. Distributional and Inferential Properties of the Estimated C_p

Under the assumption of normality, Chou and Owen (1989) obtained the probability density function (PDF) of the natural estimator $\hat{C}_p$, which can be expressed as:

$$f_{\hat{C}_p}(y) = \frac{2\left[\sqrt{(n-1)/2}\,C_p\right]^{n-1}}{\Gamma[(n-1)/2]} y^{-n} \exp\left[\frac{-(n-1)(C_p)^2}{2y^2}\right], \ y > 0.$$

The r-th moment of $\hat{C}_p$, therefore can be calculated as the following:

$$E(\hat{C}_p^r) = \frac{\Gamma\left(\frac{n-r-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\left[\frac{(n-1)}{2}\right]^{\frac{r}{2}} C_p^r,$$

where $\Gamma(k) = \int_0^\infty t^{k-1} e^{-t}\,dt$ is a gamma function. The first two moments as well as the variance may be obtained (see also Chou and Owen (1989), Pearn *et al.* (1992), and Kotz and Johnson (1993)). It can be shown that the coefficient of $E(\hat{C}_p)$, is larger than 1 for all n. For $n \ge 15$, this coefficient can be accurately approximated by $(4n-7)/(4n-4)$. Therefore, the natural estimator $\hat{C}_p$ is biased, which overestimates the actual value of C_p. For the percentage bias to be less than one percent, i.e. $|E(\hat{C}_p) - C_p|/\hat{C}_p \le 0.01$, it requires the sample size $n > 80$. Pearn *et al.* (1998) obtained an unbiased estimator by adding a correction factor to $\hat{C}_p$ as $\tilde{C}_p = b_{n-1}\hat{C}_p$, $b_{n-1} = \sqrt{\frac{2}{n-1}}\Gamma\left(\frac{n-1}{2}\right)/\Gamma\left(\frac{n-2}{2}\right)$. Pearn *et al.* (1998) also showed that the unbiased estimator $\tilde{C}_p$ is indeed the uniformly minimum variance unbiased estimator (UMVUE) of C_p, which is consistent, asymptotically efficient, and that $n^{1/2}(\tilde{C}_p - C_p)$ converges to $N(0, C_p^2/2)$ in distribution.

Due to sampling variation maybe introduced by estimation, it is critical to construct a confidence interval to provide a range which includes the true C_p with high probability. Thus, the $100(1-\alpha)\%$ (equal tails) confidence interval of C_p may be derived as (see, e.g., Chou *et al.* (1990) and Pearn *et al.* (1998)),

$$\left[\frac{\sqrt{\chi^2_{n-1,1-\alpha/2}}}{\sqrt{n-1}}\hat{C}_p,\ \frac{\sqrt{\chi^2_{n-1,\alpha/2}}}{\sqrt{n-1}}\hat{C}_p\right] \text{ or } \left[\frac{\sqrt{\chi^2_{n-1,1-\alpha/2}}}{\sqrt{n-1}b_{n-1}}\tilde{C}_p,\ \frac{\sqrt{\chi^2_{n-1,\alpha/2}}}{\sqrt{n-1}b_{n-1}}\tilde{C}_p\right],$$

where $\chi^2_{n-1,\alpha/2}$ and $\chi^2_{n-1,1-\alpha/2}$ are the upper $\alpha/2$ and $1-\alpha/2$ quantiles of chi-squared distribution with $n-1$ degrees of freedom. And the $100(1-\alpha)\%$ lower confidence limit (C_p^L) of C_p can be obtained by using only the lower limit.

2.1.4. Hypothesis Testing with C_P

In the study of process capability testing, to judge if the process satisfies the preset capability requirement (capable), we can consider the following testing hypothesis, with null hypothesis $H_0: C_p \le C$ (the process is not capable), versus the alternative $H_1: C_p > C$ (the process is capable), where C is the predetermine capability requirement. For cases with one single sample, Pearn *et al.* (1998) considered the test $\phi(x)=1$ if $\tilde{C}_p > c_0$, and $\phi(x)=0$, otherwise. The test ϕ rejects the null hypothesis if $\tilde{C}_p > c_0$, with type I error $\alpha(c_0)=\alpha$, the chance of incorrectly judging an incapable process as capable, and the critical value c_0 can be obtained below. Pearn *et al.* (1998) showed that the test ϕ is the uniformly most powerful (UMP) test of α level, which has minimal type II error among all unbiased tests.

$$c_0 = \frac{\sqrt{n-1}\,b_{n-1}}{\sqrt{\chi^2_{n-1,1-\alpha}}}C.$$

Therefore, if $\tilde{C}_p > c_0$ then $\phi(x)=1$ and we reject the null hypothesis H_0 and conclude that the process meets the capability requirement ($C_p > C$).

2.2. The Index C_{pk}

2.2.1. Process Quality and the Index C_{pk}

Most companies are no longer solely relying on C_p to quantify process capability because of perceived weakness in the index. The major weakness of this index lies in the fact that it measures potential capability as defined by the actual process spread and does not consider the mean of the process. Therefore, C_p gives no indication of actual process performance. It does not reflect the impact that shifting the process mean has on a process's ability to produce product within specification (Kane (1986) and Chan *et al.* (1988)). For this reason, the C_{pk} was developed by taking the magnitude of process variation as well as process location into consideration. It is defined as (Kane (1986)):

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma},\ \frac{\mu-LSL}{3\sigma}\right\}, \tag{2.1}$$

where USL is the upper specification limit, LSL is the lower specification limit, μ

is the process mean and σ is the process standard deviation. The index C_{pk} has been regarded as a yield-based index since it provides bounds on the process yield for a normally distributed process given a fixed value of C_{pk}. That is, $2\Phi(3C_{pk})-1 \le p \le \Phi(3C_{pk})$ (see Boyles (1991)). The upper and lower bounds of nonconforming units in parts per million (NCPPM) are plotted in Figure 2.1 as a function of C_{pk}. Table 2.1 displays some index values with the upper and lower bounds of NCPPM for a normally distributed process.

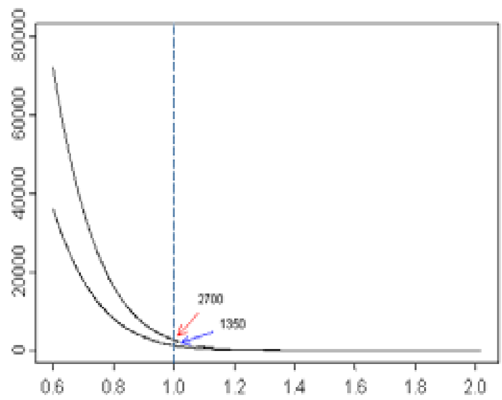

Figure 2.1 The bounds on NCPPM versus C_{pk}

Table 2.1 Index values and the corresponding bounds on NCPPM for a normally distributed process

Index	Lower bound	Upper bound	Index	Lower bound	Upper bound
0.60	35930	71861	1.33	33	66
0.70	17864	35729	1.40	13	27
0.80	8198	16395	1.45	6.807	13.614
0.90	3467	6934	1.50	3.398	6.795
1.00	1350	2700	1.60	0.793	1.587
1.10	483	967	1.67	0.272	0.544
1.20	159	318	1.70	0.170	0.340
1.24	100	200	1.80	0.033	0.067
1.25	88	177	1.90	0.006	0.012
1.30	48	96	2.00	0.001	0.002

In a purchasing contract, a minimum C_{pk} value is usually specified. If the prescribed minimum C_{pk} fails to be met, the process is determined to be incapable. Otherwise, the process is considered capable. For a C_{pk} level of 1, statistically, one would expect that the product's fractions of defectives, is no more than 2700 parts per million (PPM) fall outside the specification limits. At a C_{pk} level of 1.33, the defect rate drops to 66 PPM. To achieve less than 0.544 PPM defect rate, a C_{pk} level of 1.67 is needed. At a C_{pk} level of 2.0, the likelihood of a defective part drops to 2 parts per billion (PPB).

This bound may be established by noting that for a process with fixed C_{pk} the number of nonconformities (product items fallout the specification interval $[LSL, USL]$) but the exact number of nonconformities will vary depending upon the location of the process mean and the magnitude of the process variation. Thus, for $C_{pk} > 0$, the exact expected fraction nonconforming measure formula can be expressed in terms of C_p and C_{pk} together, as follows (see, e.g., Kotz and Lovelace (1998) and Kotz and Johnson (2002)):

$$\%NC = \Phi\left[-3C_{pk}\right] + \Phi\left[-3(2C_p - C_{pk})\right].$$

Montgomery (2001) recommended some minimum capability requirements for processes runs under some designated quality conditions (see Table 2.2). In particular, $C_{pk} \geq 1.33$ for existing processes, and $C_{pk} \geq 1.50$ for new processes; $C_{pk} \geq 1.50$ also for existing processes on safety, strength, or critical parameter, and $C_{pk} \geq 1.67$ for new processes on safety, strength, or critical parameter. Finley (1992) also found that required C_{pk} values on all critical supplier processes are 1.33 or higher and C_{pk} values of 1.67 or higher are preferred. Many companies have recently adopted criteria for evaluating their processes that include process capability objectives more stringent than before. Motorola's "Six Sigma" program essentially requires the process capability at least 2.0 to accommodate the possible 1.5σ process shift (see Harry (1988)), and no more than 3.4 PPM of nonconformities.

Table 2.2 Some minimum capability requirements of C_{pk} for existing, new, special processes.

Production Process Types	C_{pk} Value
Existing Processes	1.33
New Processes, or Existing Processes on Safety, Strength, or Critical Parameters	1.50
New Processes on Safety, Strength, or Critical Parameters	1.67

On the other hand, in current practice, a process is called "Inadequate" if $C_{pk} < 1.00$; it indicates that the process is not adequate with respect to the production tolerances (specifications), either process variation (σ^2) needs to be reduced or process mean (μ) needs to be shifted closer to the target value. A process is called "Capable" if $1.00 \leq C_{pk} < 1.33$; it indicates that caution needs to be taken regarding to process distribution, some process control is required. A process is called "Satisfactory" if $1.33 \leq C_{pk} < 1.50$; it indicates that process quality is satisfactory, material substitution may be allowed, and no stringent quality control is required. A process is called "Excellent" if $1.50 \leq C_{pk} < 2.00$; it indicates that process quality exceeds "Satisfactory". Finally, a process is called "Super" if $C_{pk} \geq 2.00$. Table 2.3 summarizes the five process conditions and the corresponding capability requirements with the fraction of nonconformities (in PPM).

Table 2.3 Five commonly used capability requirements with nonconformities and the corresponding process conditions.

C_{pk} value	Nonconformities	Process conditions
$C_{pk} < 1.00$	> 2700 PPM	Inadequate
$1.00 \le C_{pk} < 1.33$	< 2700 PPM	Capable
$1.33 \le C_{pk} < 1.50$	< 66 PPM	Satisfactory
$1.50 \le C_{pk} < 2.00$	< 6.795 PPM	Excellent
$2.00 \le C_{pk}$	< 0.002 PPM	Super

2.2.2. Estimation of C_{pk}

Utilizing the identity $\min\{a,b\} = (a+b)/2 - |a-b|/2$, the definition of index C_{pk} in (2.1) can be alternatively written as:

$$C_{pk} = \frac{d - |\mu - M|}{3\sigma},$$

where $d = (USL - LSL)/2$ is half length of the specification interval, $M = (USL + LSL)/2$ is the mid-point between the lower and the upper specification limits. The natural estimator $\hat{C}_{pk}$ defined below can be obtained by replacing the process mean μ and the process standard deviation σ by their sample estimators $\bar{x} = \sum_{i=1}^{n} x_i / n$ and $s = [\sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)]^{1/2}$. We note that the process must be demonstrably stable (under statistical control) in order to produce a reliable estimate of process capability.

$$\hat{C}_{pk} = \frac{d - |\bar{x} - M|}{3s} = \left\{1 - \frac{|\bar{x} - M|}{d}\right\}\left(\frac{d}{3s}\right) = \left\{1 - \frac{|\bar{x} - M|}{d}\right\}\hat{C}_p. \qquad (2.2)$$

2.2.3. Distributional and Inferential Properties of the Estimated C_{pk}

Under normality assumption, Kotz *et al.* (1993) obtained the *r*-th moment of $\hat{C}_{pk}$. Numerous methods for constructing approximate confidence intervals of C_{pk} have been proposed in the literature. Examples include Chou *et al.* (1990), Zhang *et al.* (1990), Franklin and Wasserman (1992a, 1992b), Kushler and Hurley (1992), Nagata and Nagahata (1994), Tang *et al.* (1997), Hoffman (2001), Pearn and Shu (2003a) and many others. Further, from the estimated $\hat{C}_{pk}$ expressed in (2.2), since $\hat{C}_p$ is distributed as $(n-1)^{1/2} C_p(\chi_{n-1}^{-1})$, and $n^{1/2}|\bar{x} - M|/\sigma$ is distributed as the folded normal distribution with parameter $n^{1/2}|\mu - M|/\sigma$ (see Leone *et al.* (1961) for details about this distribution). Thus, $\hat{C}_{pk}$ is a mixture of χ_{n-1}^{-1} and the folded normal distribution (Pearn *et al.* (1992)). The probability density function (PDF) of $\hat{C}_{pk}$ can be obtained as (Pearn *et al.* (1999)), where $D' = (n-1)^{1/2} d/\sigma$, $a' = [(n-1)/n]^{1/2}$, $\lambda = n(\mu - M)^2/\sigma^2$.

$$
f_{\hat{C}_{pk}}(y) = \begin{cases} 4A_n \sum_{\ell=0}^{\infty} P_\ell(\lambda)\, B_\ell \times \dfrac{D'^{n+2\ell}}{a'^{2\ell+1}} \int_0^{\infty} (1-yz)^{2\ell} z^{n-1} \\ \quad \times \exp\left\{ -\dfrac{D'^2}{18a'^2}\left(a'^2 z^2 + 9(1-yz)^2 \right) \right\} dz, & y \le 0, \\ 4A_n \sum_{\ell=0}^{\infty} P_\ell(\lambda)\, B_\ell \times \dfrac{D'^{n+2\ell}}{a'^{2\ell+1}} \int_0^{1/y} (1-yz)^{2\ell} z^{n-1} \\ \quad \times \exp\left\{ -\dfrac{D'^2}{18a'^2}\left(a'^2 z^2 + 9(1-yz)^2 \right) \right\} dz, & y > 0, \end{cases}
$$

$$
P_\ell(\lambda) = \frac{e^{-(\lambda/2)}(\lambda/2)^\ell}{\ell!}, \quad A_n = \frac{1}{3^{n-1}\, 2^{n/2}\, \Gamma((n-1)/2)}, \quad B_\ell = \frac{1}{2^\ell\, \Gamma((2\ell+1)/2)}.
$$

Using the integration technique similar to that presented in Vännman (1997), an exact and explicit form of the cumulative distribution function (CDF) of the natural estimator $\hat{C}_{pk}$ can be obtained, under the assumption of normality. The CDF of $\hat{C}_{pk}$ is expressed in terms of a mixture of the chi-square distribution and the normal distribution (see Pearn and Lin (2003)):

$$
F_{\hat{C}_{pk}}(y) = 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n}-t)^2}{9ny^2} \right) \left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n}) \right] dt, \tag{2.3}
$$

for $y > 0$, where $b = d/\sigma$, $\xi = (\mu - M)/\sigma$, $G(\cdot)$ is the CDF of the chi-square distribution with degree of freedom $n-1$, χ^2_{n-1}, and $\phi(\cdot)$ is the PDF of the standard normal distribution $N(0,1)$. It is noted that we would obtain an identical equation if we substitute ξ by $-\xi$ into equation (2.3) for fixed values of y and n.

2.2.4. Hypothesis Testing with C_{pk}

Statistical hypothesis testing used for examining whether the process capability meet the customers' demands, can be stated as follows:

H_0: $C_{pk} \le C$ (process is not capable),
H_1: $C_{pk} > C$ (process is capable).

We define the test $\phi(x)$, the decision making rule, as the following: $\phi(x) = 1$ if $\hat{C}_{pk} > c_0$, and $\phi(x) = 0$ otherwise. Thus, the test ϕ rejects the null hypothesis H_0 ($C_{pk} \le C$) if $\hat{C}_{pk} > c_0$, with type I error $\alpha(c_0) = \alpha$, the chance of incorrectly concluding an incapable process ($C_{pk} \le C$) as capable ($C_{pk} > C$). Based on the CDF of $\hat{C}_{pk}$ expressed in (2.3), given values of capability requirement C, parameter ξ, sample size n, and risk α, the critical value c_0 can be obtained by solving the equation $P(\hat{C}_{pk} > c_0 \mid C_{pk} = C) = \alpha$ using available numerical methods.

$$
\int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n}-t)^2}{9nc_0^2} \right) \left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n}) \right] dt = \alpha. \tag{2.4}
$$

For processes with symmetric tolerances ($T = M$), the index may be rewritten as $C_{pk} = (d/\sigma - |\xi|)/3$. Given $C_{pk} = C$, $b = d/\sigma$ can be expressed as $b = 3C + |\xi|$, the *p*-value corresponding to c^*, a specific value of $\hat{C}_{pk}$ calculated from the sample data, is:

$$p-value = \Pr(\hat{C}_{pk} \geq c^* \mid C_{pk} = C)$$

$$= \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n}-t)^2}{9n(c^*)^2}\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right]dt . \quad (2.5)$$

Furthermore, based on the CDF of $\hat{C}_{pk}$ expressed in (2.3), the lower confidence limit which conveying critical information regarding the true process capability is also developed as below. In fact, given the sample of size n, the confidence level γ, the estimated value $\hat{C}_{pk}$, and ξ, the lower confidence bounds C_{pk}^L can be obtained using numerical integration technique with iterations, to solve the following equation (2.6) with $b_L = 3C_{pk}^L + |\xi|$ (see Pearn and Shu (2003a) for more details).

$$\int_0^{b_L\sqrt{n}} G\left(\frac{(n-1)(b_L\sqrt{n}-t)^2}{9n\hat{C}_{pk}^2}\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right]dt = 1-\gamma . \quad (2.6)$$

2.3. The Index C_{pm}

2.3.1. Process Quality and the Index C_{pm}

The C_p and C_{pk} indices are appropriate measures of progress for quality improvement paradigms in which reduction of variability is the guiding principle and process yield is the primary measure of success. However, they are not related to the cost of failing to meet customers' requirement. Taguchi, on the other hand, emphasizes the loss in a product's worth when one of its characteristics departs from the customers' ideal value T. To help account for this, Hsiang and Taguchi (1985) introduced the index C_{pm}, which was also proposed independently by Chan *et al.* (1988). The index is related to the idea of squared error loss and this loss-based process capability index C_{pm}, sometimes called the Taguchi index. The index emphasizes on measuring the ability of the process to cluster around the target, which therefore reflects the degrees of process targeting (centering). The index C_{pm} incorporates with the variation of production items with respect to the target value and the specification limits preset in the factory. The index C_{pm} is defined as:

$$C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}} = \frac{USL-LSL}{6\tau} = \frac{d}{3\tau} \quad (2.7)$$

where $USL-LSL$ is the allowable tolerance range of the process,

$d = (USL - LSL)/2$ is the half length of the specification interval, and τ is the measure of the average product deviation from the target value T. The term $\tau^2 = E[(X-T)^2] = \sigma^2 + (\mu - T)^2$ incorporates two variation components: (i) variation to the process mean and (ii) deviation of the process mean from the target.

By observing the definition of C_{pm}, it is easy to see that if the process variance increases (decreases) then the denominator will increase (decrease) and C_{pm} will decrease (increase). Also, if the process mean moves away from (closer to) the target value, then the denominator will increase (decrease) and C_{pm} will decrease (increase). Obviously, C_{pm} adds an additional penalty for being off-target. It is most often assumed that the target lies at the mid-point of the tolerance range (symmetric tolerances, $T = M$). When this is not the case, there are serious disadvantages in the casual use of C_{pm} (the situation, $T \neq M$, sometimes called as asymmetric tolerances).

Note that C_{pm} differs from C_p, the first-generation index, only in the measure of the process variation. It follows that

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{C_p}{\sqrt{1 + \left(\frac{\mu - T}{\sigma}\right)^2}}$$

$$= \frac{C_{pk}}{\left(1 + \frac{|\mu - M|}{\sigma}\right)\sqrt{1 + \left(\frac{\mu - T}{\sigma}\right)^2}}.$$

For the case with $T = M$, Parlar and Wesolowsky (1999) have noted that C_p, C_{pk} and C_{pm} are related by the formula

$$C_{pk} = C_p - \frac{1}{3}\sqrt{\left(\frac{C_p}{C_{pm}}\right)^2 - 1} = C_p - \frac{1}{3}\left|\frac{\mu - M}{\sigma}\right|,$$

or, equivalently,

$$C_{pm} = \frac{C_p}{\sqrt{1 + 9(C_p - C_{pk})^2}}.$$

And note that $C_p \geq \max(C_{pk}, C_{pm})$ will hold obviously from the definition.

Boyles (1991) has provided a definitive analysis of C_{pm} and its usefulness in measuring process centering. He notes that both C_{pk} and C_{pm} coincide with C_p when $\mu = T$ and decrease as μ moves away from T. However, $C_{pk} < 0$ for $\mu > USL$ or $\mu < LSL$, whereas C_{pm} of process with $|\mu - T| > 0$ is strictly bounded above by the C_p value of a process with $\sigma = |\mu - T|$. That is,

$$C_{pm} < \frac{USL - LSL}{6|\mu - T|}. \tag{2.8}$$

The index C_{pm} approaches to zero asymptotically as $|\mu - T|$ tends to infinity. On the other hand, while $C_{pk} = (d - |\mu - M|)/(3\sigma)$ increases without bound for fixed μ as σ tends to zero, C_{pm} is bounded above $C_{pm} < d/(3|\mu - T|)$. The right-hand side of above equation is the limiting value of C_{pm} as σ tends to zero, and is equal to C_p value of a process with $\sigma = |\mu - T|$. It follows from (2.8) that a necessary condition for $C_{pm} \geq 1$ is $|\mu - T| < d/3$.

As mentioned earlier, for a normally distributed process, the C_{pk} index provides a lower bound on the process yield, $Yield \geq 2\Phi(3C_{pk}) - 1$, or $\%NC \leq 2\Phi(-3C_{pk})$ for $LSL \leq \mu \leq USL$. Furthermore, based on the C_{pm} index, Ruczinski (1996) obtained a lower bound on the process yield as $Yield \geq 2\Phi(3C_{pm}) - 1$, or equivalently $\%NC \leq 2\Phi(-3C_{pm})$ for $C_{pm} > \sqrt{3}/3$.

2.3.2. Estimation of C_{pm}

Due to the index C_{pm} involves the unknown parameters μ and σ, which must be estimated from sample. Chan *et al*. (1988) and Boyles (1991) proposed the following two estimators of C_{pm}, respectively.

$$\tilde{C}_{pm} = \hat{C}_{pm(CCS)} = \frac{d}{3\hat{\tau}_{CCS}} = \frac{d}{3\sqrt{\sum_{i=1}^{n}(x_i - T)^2/(n-1)}} = \frac{d}{3\sqrt{s^2 + \frac{n}{n-1}(\bar{x} - T)^2}},$$

$$\hat{C}_{pm} = \hat{C}_{pm(B)} = \frac{d}{3\hat{\tau}_B} = \frac{d}{3\sqrt{\sum_{i=1}^{n}(x_i - T)^2/n}} = \frac{d}{3\sqrt{s_n^2 + (\bar{x} - T)^2}},$$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$, $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)$ and $s_n^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/n = \frac{n-1}{n}s^2$.

2.3.3. Distributional and Inferential Properties of the Estimated C_{pm}

Assuming that the process measurement are normally distributed and $T = M$, Chan *et al*. (1988) derived the PDF of $Y = \tilde{C}_{pm}$

$$f_Y(y) = \frac{a}{2^{n/2-1}y^3}\exp\left[-\frac{1}{2}\left(\frac{a}{y^2} + \lambda\right)\right]\sum_{j=0}^{\infty}\frac{\lambda^j(\frac{a}{y^2})^{\frac{n}{2}+j-1}}{j!\Gamma(\frac{n}{2}+j)2^{2j}}, \quad y > 0.$$

where $a = C_{pm}^2(1 + \lambda/n)(n-1)$ and $\lambda = n(\mu - T)^2/\sigma^2$. An alternative equivalent formula was provided by Pearn *et al*. (1992). When the case of $T = M$, $\tilde{C}_{pm}$ is a biased estimator of C_{pm}, but is asymptotically unbiased. Detailed descriptions and proofs of the properties of $\tilde{C}_{pm}$ are given in Chan *et al*. (1988). On the other hand, Boyles (1991) considered that it would be more appropriate to replace the factor $n-1$ by n in the denominator since the term $\hat{\tau}_B^2 = s_n^2 + (\bar{x} - T)^2 = \sum_{i=1}^{n}(x_i - T)^2/n$

and $E[\hat{\tau}_B^2] = \sigma^2 + (\mu - T)^2$ in the denominator of $\hat{C}_{pm}$ is the uniformly minimum variance unbiased estimator (UMVUE) of the term $\sigma^2 + (\mu - T)^2$. In fact, the two estimators, $\tilde{C}_{pm}$ and $\hat{C}_{pm}$, are asymptotical equivalent. We note that $\bar{x}$ and s_n^2 are the maximum likelihood estimators (MLEs) of μ and σ^2, respectively. Hence, the estimated $\hat{C}_{pm}$ is also the MLE of C_{pm}. Therefore, it is reasonable, for reliability purpose, that we use the estimator $\hat{C}_{pm} = \hat{C}_{pm(B)}$ to evaluate process performance.

Under the assumption of normality, Kotz and Johnson (1993) derived formulas for the *r*-th moment of $\hat{C}_{pm}$, note that if $r \geq n$, $E(\hat{C}_{pm}^r)$ is infinite. Note that the quantity $n\hat{\tau}_B^2 / \sigma^2$ has a non-central chi-square $\chi_{n,\lambda}^2$ distribution with n degrees of freedom and non-centrality parameter $\lambda = n\xi^2$, $\xi = (\mu - T)/\sigma$. Boyles (1991) and Pearn *et al*. (1992) showed that $\hat{C}_{pm}$ is distributed as $C_p\sqrt{n/\chi_{n,\lambda}^2}$, which can be alternatively expressed as

$$\hat{C}_{pm} \sim C_{pm}\sqrt{1 + \frac{\lambda}{n}}\sqrt{\frac{n}{\chi_{n,\lambda}^2}}.$$

Vännman and Kotz (1995) derived the CDF of a generalized process capability index $C_p^*(u, v)$. The special case with $u = 0$ and $v = 1$ reduces their estimator to Boyles' (1991) estimator $\hat{C}_{pm}$. Furthermore, by rewriting $\hat{C}_{pm(B)} = D/(3\sqrt{K + H})$, where $D = n^{1/2}d/\sigma$, $K = ns_n^2/\sigma^2 \sim \chi_{n-1}^2$, $H = n(\bar{x} - T)^2/\sigma^2$, an exactly explicit form of the CDF of $\hat{C}_{pm}$ can be derived as:

$$F_{\hat{C}_{pm}}(y) = 1 - \int_0^{b\sqrt{n}/(3y)} G\left(\frac{b^2 n}{9y^2} - t^2\right)\left[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right]dt, \quad \text{for} \quad y > 0. \quad (2.9)$$

2.3.4. Capability Testing with C_{pm}

To test whether a given process is capable based on C_{pm} index, we may consider the statistical testing hypothesis as H_0: $C_{pm} \leq C$ (process is not capable), versus H_1: $C_{pm} > C$ (process is capable). Based on a given $\alpha(c_0) = \alpha$, the decision rule is to reject H_0 if $\hat{C}_{pm} > c_0$ and fails to reject H_0 otherwise. For processes with target value setting on the middle of the specification limits, the index may be rewritten as $C_{pm} = b/[3(1+\xi^2)^{1/2}]$. Given $C_{pm} = C$, $b = d/\sigma = 3C(1+\xi^2)^{1/2}$. The *p*-value corresponding to c^*, a specific value of $\hat{C}_{pm}$ calculated from the sample data, is:

$$\begin{aligned} p - value &= P(\hat{C}_{pm} \geq c^* \mid C_{pm} = C) \\ &= \int_0^{b\sqrt{n}/(3c^*)} G\left(\frac{b^2 n}{9(c^*)^2} - t^2\right)\left[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right]dt. \end{aligned} \quad (2.10)$$

Given values of C, ξ, n, and α, the critical value c_0 can be obtained by solving the equation $\Pr(\hat{C}_{pm} \geq c_0 \mid C_{pm} = C) = \alpha$ as

$$\int_0^{b\sqrt{n}/(3c_0)} G\left(\frac{b^2 n}{9c_0^2} - t^2\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right]dt = \alpha . \qquad (2.11)$$

Moreover, given the sample of size n, the confidence level γ, the estimated value $\hat{C}_{pm}$, and the parameter ξ, the lower confidence bounds (denoted as C_{pm}^L) can be obtained by solving the following equation (2.12) with $b_L = 3C_{pm}^L(1+\xi^2)^{1/2}$ (Pearn and Shu (2003b)).

$$\int_0^{b_L\sqrt{n}/\left(3\hat{C}_{pm}\right)} G\left(\frac{b^2 n}{9\hat{C}_{pm}^2} - t^2\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right]dt = 1-\gamma . \qquad (2.12)$$

2.4. The Index C_{pmk}

2.4.1. Process Quality and the Index C_{pmk}

Pearn *et al*. (1992) proposed the process capability index C_{pmk}, which combines the merits of the three earlier indices C_p, C_{pk} and C_{pm}. The index C_{pmk} alerts the user if the process variance increases and/or the process mean deviates from its target value. The index C_{pmk}, referred to as the third-generation capability index, has been defined as the following.

$$C_{pmk} = \min\left\{\frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-LSL}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\} = \frac{d-|\mu-M|}{3\sqrt{\sigma^2+(\mu-T)^2}} . \qquad (2.13)$$

By comparing the pair of indices (C_{pmk}, C_{pm}), similar to (C_{pk}, C_p), we have the relation $C_{pmk} = C_{pm} \times (1-|\mu-M|/d) = C_{pm} \times C_a$ and hence $C_{pmk} = (C_{pm} \times C_{pk})/C_p$. The index C_{pmk} is constructed by combining the yield-based index C_{pk} and the loss-based index C_{pm}, taking into account the process yield (meeting the manufacturing specifications) as well as the process loss (variation from the target). Note that a process meeting the capability requirement "$C_{pk} \ge C$" may not be meeting the capability requirement "$C_{pm} \ge C$". On the other hand, a process meets the capability requirement "$C_{pm} \ge C$" may not be meeting the capability requirement "$C_{pk} \ge C$" either. The discrepancy between the two indices may be contributed to the fact that the C_{pk} index primarily measures the process yield, but the index C_{pm} focuses mainly on the process loss. But, if the process meets the capability requirement "$C_{pmk} \ge C$", then the process must meet both capability requirements "$C_{pk} \ge C$" and "$C_{pm} \ge C$" since $C_{pmk} \le C_{pk}$ and $C_{pmk} \le C_{pm}$. According to today's modern quality improvement theory, reduction of the process loss is as important as increasing the process yield. While the C_{pk} remains the more popular and widely used index, the index C_{pmk} is considered to be an advanced and useful index for processes with two-sided specification limits. The four indices C_p, C_{pk}, C_{pm}, and C_{pmk}, ranked in terms of sensitivity to differences between the process mean and the target, are (1) C_{pmk}, (2) C_{pm}, (3) C_{pk} and (4) C_p (Pearn and Kotz (1994)).

2.4.2. Estimation of C_{pmk}

For a normally distributed process that is demonstrably stable (under statistical control), Pearn *et al*. (1992) suggested using the following estimator, which defined as:

$$\hat{C}_{pmk} = \min\left\{\frac{USL-\bar{x}}{3\sqrt{s_n^2+(\bar{x}-T)^2}}, \frac{\bar{x}-LSL}{3\sqrt{s_n^2+(\bar{x}-T)^2}}\right\} = \frac{d-|\bar{x}-T|}{3\sqrt{s_n^2+(\bar{x}-T)^2}},$$

where $\bar{x}=\sum_{i=1}^{n}x_i/n$ and $s_n^2=\sum_{i=1}^{n}(x_i-\bar{x})^2/n$ are the MLEs of μ and σ^2 respectively. We note again that $s_n^2+(\bar{x}-T)^2=\sum_{i=1}^{n}(x_i-T)^2/n$ which is in the denominator of $\hat{C}_{pmk}$ is the UMVUE of $\sigma^2+(\mu-T)^2=E[(X-T)^2]$ in the denominator of C_{pmk}.

2.4.3. Distributional and Inferential Properties of the Estimated C_{pmk}

In fact, the estimator of C_{pmk} can be expressed as $\hat{C}_{pmk}=(D-\sqrt{H})/(3\sqrt{K+H})$, where $D=n^{1/2}d/\sigma$, $K=ns_n^2/\sigma^2$, $H=n(\bar{x}-T)^2/\sigma^2$, and $\eta=n^{1/2}|\mu-T|/\sigma$. Under the assumption of normality, K is distributed as χ_{n-1}^2, a chi-square distribution with $n-1$ degrees of freedom, H is distributed as $\chi_{1,\lambda}'^2$ a non-central chi-square distribution with one degree of freedom and non-centrality parameter $\lambda=n(\mu-T)^2/\sigma^2$, and $\sqrt{H}$ is distributed as the normal distribution $N(\eta,1)$ with mean η and variance 1. That is, the estimator $\hat{C}_{pmk}$ is a mixture of the chi-square distribution and the non-central chi-square distribution, as expressed in the following (Pearn *et al.* (1992)):

$$\hat{C}_{pmk} \sim \frac{\frac{d\sqrt{n}}{\sigma}-\chi_{1,\lambda}'}{3\sqrt{\chi_{n-1}^2+\chi_{1,\lambda}'^2}}.$$

Chen and Hsu (1995) investigated the asymptotic sampling distribution of the estimated C_{pmk} and showed that the estimator $\hat{C}_{pmk}$ is consistent, asymptotically unbiased estimator of C_{pmk}, and if the fourth moment of the distribution of X is finite, then $\hat{C}_{pmk}$ is asymptotically normal. Vännman and Kotz (1995) obtained the distribution of the estimated $C_p(u,v)$ for cases with $T=M$. By taking $u=1$ and $v=1$, the distribution of $C_p(1,1)=C_{pmk}$ can be obtained. Wright (1998) derived an explicit but rather complicated expression for the PDF of the estimated C_{pmk}. Using variable transformation and the integration technique similar to that presented in Vännman (1997), the CDF and the PDF of the estimated index $\hat{C}_{pmk}$ may be expressed alternatively in terms of a mixture of the chi-square distribution and the normal distribution. The explicit form of the CDF considerably simplify the complexity for analyzing the statistical properties of the estimated index, which can be expressed below (see Pearn and Lin (2002)):

$$F_{\hat{C}_{pmk}}(y) = 1-\int_0^{b\sqrt{n}/(1+3y)} G\left(\frac{(b\sqrt{n}-t)^2}{9y^2}-t^2\right)\left[\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n})\right]dt, \quad (2.14)$$

for $y > 0$, where $b = d/\sigma$, $\xi = (\mu - T)/\sigma$, $G(\cdot)$ is the CDF of the χ^2_{n-1} distribution, $\phi(\cdot)$ is the PDF of $N(0,1)$, and it is noted that for $\mu > USL$ or $\mu < LSL$, the capability $C_{pmk} < 0$, and for $\mu = USL$ or $\mu = LSL$, the capability C_{pmk} = 0.0. The requirement with $LSL < \mu < USL$ has been a minimum capability requirement applies to most start-up engineering applications or new processes.

2.4.4. Capability Testing with C_{pmk}

Using the index C_{pmk}, the engineers can access the process performance and monitor the manufacturing processes on a routine basis. To test whether a given process is capable, we can consider the following statistical testing hypothesis:

H_0: $C_{pmk} \le C$ (process is not capable),
H_1: $C_{pmk} > C$ (process is capable).

For cases with $T = M$, we let $C_{pmk} = C$ then the $b = d/\sigma$ can be rewritten as $b = 3C\sqrt{1+\xi^2} + |\xi|$. Hence, given values of capability requirement C, parameter ξ, sample size n, and risk α, the critical value c_0 can be obtained by solving the equation $P(\hat{C}_{pmk} \ge c_0 \mid C_{pmk} = C) = \alpha$ using available numerical methods. That is,

$$\int_0^{b\sqrt{n}/(1+3c_0)} G\left(\frac{(b\sqrt{n}-t)^2}{9c_0^2} - t^2\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right]dt = \alpha . \qquad (2.15)$$

In addition, given a value of C, the p-value corresponding to c^*, a specific value of $\hat{C}_{pmk}$ calculated from the sample data, is:

$$p - value = P(\hat{C}_{pmk} \ge c^* \mid C_{pmk} = C)$$
$$= \int_0^{b\sqrt{n}/(1+3c^*)} G\left(\frac{(b\sqrt{n}-t)^2}{9(c^*)^2} - t^2\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right]dt. \qquad (2.16)$$

If the estimated value $\hat{C}_{pmk}$ is greater than the critical value c_0 ($\hat{C}_{pmk} > c_0$) or the calculated p-value is smaller than α (p-value $< \alpha$), then we conclude that the process meets the capability requirement ($C_{pmk} > C$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C_{pmk} \le C$. On the other hand, given the sample of size n, the confidence level γ, the estimated values $\hat{C}_{pmk}$ and ξ, the lower confidence bounds C^L_{pmk} can be obtained by solving the following equation with $b_L = 3C^L_{pmk}\sqrt{1+\xi^2} + |\xi|$ (see Pearn and Shu (2004)).

$$\int_0^{b_L\sqrt{n}/(1+3\hat{C}^2_{pmk})} G\left(\frac{(b\sqrt{n}-t)^2}{9\hat{C}^2_{pmk}} - t^2\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right]dt = 1-\gamma . \qquad (2.17)$$

2.5. The Indices C_{PU} and C_{PL}

2.5.1. Process Quality and the Indices C_{PU} and C_{PL}

Several process capability indices including C_p, C_{pk}, C_{pm} and C_{pmk} have been commonly used in the manufacturing industry to measure whether a process is capable of reproducing product items within the specified manufacturing tolerance. However, those indices are appropriate measures only for processes with two-sided specifications (which require both USL and LSL). For the unilateral tolerance situation where only a single specification limit is given, Kane (1986) considered the following indices as

$$C_{PU} = \frac{USL-\mu}{3\sigma}, \quad C_{PL} = \frac{\mu - LSL}{3\sigma},$$

The index C_{PU} measures the capability of a smaller-the-better process with an upper specification limit USL, whereas the index C_{PL} measures the capability of a larger-the-better process with a lower specification limit LSL.

For normally distributed processes with one-sided specification limit USL, the process yield is:

$$P(X < USL) = P\left(\frac{X-\mu}{3\sigma} < \frac{USL-\mu}{3\sigma}\right) = P(Z < 3C_{PU}) = \Phi(3C_{PU}),$$

where Z follows the standard normal distribution $N(0,1)$ with the cumulative distribution function $\Phi(\cdot)$. Similarly, for normally distributed processes with one-sided specification limit LSL, the process yield is:

$$P(X > LSL) = P\left(\frac{\mu - X}{3\sigma} < \frac{\mu - LSL}{3\sigma}\right) = P\left(-\frac{1}{3}Z < C_{PL}\right) = \Phi(3C_{PL}).$$

For convenience of presentation, we let C_I denote either C_{PU} or C_{PL}. Thus, process capability index C_I provides an exact measure on the potential process yield for processes with one-sided manufacturing specifications and the corresponding fraction of the nonconformities $\%NC$ for a well controlled normally distributed process may be calculated as $\%NC = 1-\Phi(3C_I)$.

2.5.2. Estimations of C_{PU} and C_{PL}

In practice, sample data must be collected in order to calculate those indices since the process mean μ and standard deviation σ are usually unknown. To estimate the indices C_{PU} and C_{PL}, Chou and Owen (1989) considered $\hat{C}_{PU}$ and $\hat{C}_{PL}$, the natural estimators of C_{PU} and C_{PL}, which are defined as the following:

$$\hat{C}_{PU} = \frac{USL-\bar{x}}{3s}, \qquad \hat{C}_{PL} = \frac{\bar{x}-LSL}{3s},$$

where $\bar{x}=\sum_{i=1}^{n}x_i/n$ is the sample mean, and $s^2=(n-1)^{-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$ is the sample variance, which may be obtained from a process that is demonstrably stable (under statistical control). Under the assumption of normality, the estimator $\hat{C}_{PU}$ is distributed as $(3\sqrt{n})^{-1}t_{n-1}(\delta)$, where $t_{n-1}(\delta)$ is a non-central t distribution with $n-1$ degrees of freedom and non-centrality parameter $\delta=3\sqrt{n}C_{PU}$. The estimator $\hat{C}_{PL}$ has the same sampling distribution as $\hat{C}_{PU}$ but with $\delta=3\sqrt{n}C_{PL}$. However, both estimators $\hat{C}_{PU}$ and $\hat{C}_{PL}$ are biased. Pearn and Chen (2002) showed that by adding the correction factor $b_{n-1}=[2/(n-1)]^{1/2}\,\Gamma[(n-1)/2]\,/\,\Gamma[(n-2)/2]$ to $\hat{C}_{PU}$ and $\hat{C}_{PL}$, we could obtain unbiased estimators $b_{n-1}\hat{C}_{PU}$ and $b_{n-1}\hat{C}_{PL}$ which have been denoted as $\tilde{C}_{PU}$ and $\tilde{C}_{PL}$. That is, $E(\tilde{C}_{PU})=C_{PU}$ and $E(\tilde{C}_{PL})=C_{PL}$. Since $b_{n-1}<1$, then $Var(\tilde{C}_{PU})<Var(\hat{C}_{PU})$ and $Var(\tilde{C}_{PL})<Var(\hat{C}_{PL})$. And due to the estimators $\tilde{C}_{PU}$ and $\tilde{C}_{PL}$ only based on the complete and sufficient statistics $(\bar{x},s^2)$, we can conclude that $\tilde{C}_{PU}$ and $\tilde{C}_{PL}$ are the uniformly minimum variance unbiased estimators (UMVUEs) of C_{PU} and C_{PL}, respectively.

2.5.3. Distributional and Inferential Properties of the Estimated C_{PU} and C_{PL}

The r-th moment (about zero) and the variance of $\tilde{C}_{PU}$ can be obtained as in the following, where $Z=\sqrt{n}(USL-\bar{x})/\sigma$ is distributed as $N(3\sqrt{n}C_{PU},1)$. It is easy to verify that $E(\tilde{C}_{PU})=C_{PU}$. The results of the r-th moment, the expected value, and the variance of the other estimator $\tilde{C}_{PL}$ are the same.

$$E\left(\tilde{C}_{PU}\right)^r=\frac{(\Gamma[(n-1)/2])^{r-1}\,\Gamma[(n-1-r)/2]}{\left(3\sqrt{n}\right)^r(\Gamma[(n-2)/2])^r}E(Z)^r,$$

$$Var\left(\tilde{C}_{PU}\right)=\left\{\frac{\Gamma[(n-1)/2]\,\Gamma[(n-3)/2]}{(\Gamma[(n-2)/2])^2}-1\right\}(C_{PU})^2+\frac{1}{9n}\frac{\Gamma[(n-1)/2]\,\Gamma[(n-3)/2]}{(\Gamma[(n-2)/2])^2}.$$

By changing the variables with the non-central t distribution with $n-1$ degrees of freedom and non-centrality parameter $\delta=3\sqrt{n}C_I$, we let $Y=\tilde{C}_I$ $=b_{n-1}(3\sqrt{n})^{-1}t_{n-1}(\delta)$, the CDF of $\tilde{C}_I$ can be derived directly as:

$$F(y)=\frac{1}{2^{(n-3)/2}\Gamma[(n-1)/2]}\int_0^{\infty}t^{n-2}e^{\frac{-t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_0^{3\sqrt{n}yt/b_{n-1}\sqrt{n-1}}\exp\left[-\frac{1}{2}(u-\delta)^2\right]du\,dt\,.\quad(2.18)$$

Differentiating the equation of the CDF in (2.18) with respect to y gives the probability density function (PDF) of $\tilde{C}_I$ as:

$$f(y)=\frac{3\sqrt{n/(n-1)}2^{-n/2}}{b_{n-1}\sqrt{\pi}\Gamma[(n-1)/2]}\times\int_0^{\infty}t^{(n-2)/2}\exp\left\{-\frac{1}{2}\left[t+\left(\frac{3y\sqrt{nt}}{b_{n-1}\sqrt{n-1}}-\delta\right)^2\right]\right\}dt,\quad(2.19)$$

where $b_{n-1}=[2/(n-1)]^{1/2}\Gamma[(n-1)/2]/\Gamma[(n-2)/2]$.

2.5.4. Testing Hypothesis with C_{PU} and C_{PL}

For processes with one-sided manufacturing specifications (which require only USL or LSL, but not both), the index C_{PU} can be used to measure the capability of a smaller-the-better process with upper specification limit USL, whereas the index C_{PL} can be used to measure the capability of a larger-the-better process with a lower specification limit LSL. To test whether a given process meets the capability requirement, we consider the following statistical testing hypothesis with $H_0 : C_I \le C$ (the process is incapable), versus the alternative $H_1 : C_I > C$ (the process is capable). Thus, one can consider the test $\varphi(x) = 1$ if $\tilde{C}_I > c_0$, and $\varphi(x) = 0$, otherwise. The test φ rejects the null hypothesis if $\tilde{C}_I > c_0$, with type I error $\alpha(c_0) = \alpha$, the chance of incorrectly judging an incapable process as a capable one.

Furthermore, the calculations of p-value (rejection probability) and critical value are provided in the following. Suppose the observed value of the statistic $\tilde{C}_I = c^*$, then we can calculate those values as the following, where $\delta = 3\sqrt{n}C$.

$$p-value = P\left\{\tilde{C}_I \ge c^* \ | C_I \le C\right\}$$
$$= P\left\{t_{n-1}(\delta) \ge \frac{3\sqrt{n}c^*}{b_{n-1}} \ | C_I \le C\right\}. \tag{2.20}$$

The critical value, c_0, is determined by

$$\alpha = P\left\{\tilde{C}_I \ge c_0 \ | C_I = C\right\}$$
$$= P\left\{t_{n-1}(\delta) \ge \frac{3\sqrt{n}c_0}{b_{n-1}} \ | C_I = C\right\}.$$

Hence, we have

$$\frac{3\sqrt{n}c_0}{b_{n-1}} = t_{n-1,\,\alpha}(\delta) \ \ , \text{ or } \ c_0 = b_{n-1}t_{n-1,\,\alpha}(\delta)/(3\sqrt{n}). \tag{2.21}$$

where $t_{n-1,\alpha}(\delta)$ is the upper α quantile of non-central t distribution with $n-1$ degrees of freedom satisfies $P(t_{n-1}(\delta) \ge t_{n-1,\alpha}(\delta)) = \alpha$.