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博 士 論 文

決策球模式之建構與應用

Decision Ball Models: Methods and Applications

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中 華 民 國 九 十 五 年 六 月

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摘　要

決策者偏好常受到背景資訊的影響，本研究發展一套決策球系統，提供決策者視覺化資訊及相似性分析，將決策資訊視覺化，以輔助決策。 此一決策球系統分為法蘭克運算、對等交換、成對比較、群集分析四模式。 法蘭克運算模式是用於單一方案取捨的決策問題， 對等交換及成對比較模式主要是解決多個替選方案的排序問題，而群集分析模式是應用於替選案的分群問題。 本研究成果可廣泛應用於經營管理決策及財務投資決策等。

關鍵字 ： 決策球，視覺化，決策，偏好，不一致性

Decision Ball Models: Methods and Applications

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ABSTRACT

Decision makers' preferences are often influenced by background information. This study develops a Decision Ball system to provide visual context and similarity analysis to help decision makers to reach a better decision. The proposed Decision Ball system includes four types of Decision Ball models: Franklin's Moral Algebra models, Even Swap models, Pairwise Comparison models, and Classification models. Franklin's Moral Algebra Decision Ball models solve "Yes" or "No" decision problem. Even Swap and Pairwise Comparison Decision Ball models are for ranking problems with multiple alternatives. Classification Decision Ball models treat group problem. The proposed approach can be applied in a variety of decision problems. For instance, a Decision Ball system can assist decision makers in personal decision-making problem, operational and managerial decision problems, and financial decision problems, etc.

Keywords: Decision Balls, Visualization, Decision-Making, Preference, Inconsistency

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Chapter 1 Introduction

Ranking and grouping alternatives are two of major challenges in decision-making. The more decision alternatives and criteria are being considered, the more difficulties the decision maker (DM) has to face. Therefore, how to assist the decision maker make a more reliable and knowledgeable decision is a very important issue.

1.1 Research Motivation and Purposes

Consumer choice theories show that consumer choice is often affected by context (Seiford and Zhu, 2003). For instance, a circle appears large when surrounded by small circles and small when surrounded by larger ones, as shown in Figure 1.1(a). Similarly, a product may appear attractive against a background of less attractive alternatives and unattractive when compared to more attractive alternatives (Simonson and Tversky, 1992). Tversky and Simonson (1993) showed the relative attractiveness of *x* compared to *y* often depends on the presence or absence of a third option *z*. In addition, Keeney (2002) identified 12 important mistakes frequently made that limit one's ability to determine useful value trade-offs, in which "not understanding the Decision Context" is the first critical mistake.

Even animals' choice is heavily affected by what visual background they have seen. In a famous experiment (Waite, 2001), a biologist set up an experiment for a gray jay as shown

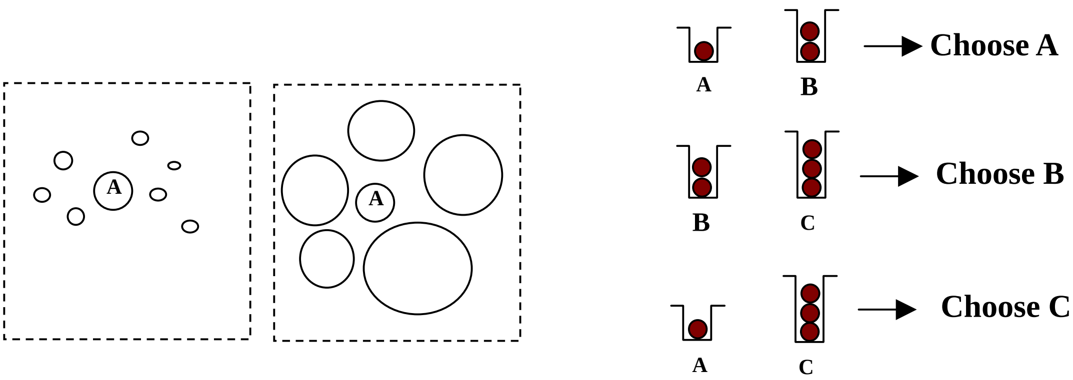


Figure 1.1 Visual background in decision environment (a) Influence of visual background (b) Gray jay's choice

in Figure 1.1(b). There are three options: A, B and C. A is for one raisin in a short tube. B is for two raisins in a medium length tube, and C is for three raisins in a long tube. When displaying A and B to a jay, it will choose A. When displaying B and C to a jay, it prefers B. However, by displaying A and C to a jay, it prefers C. If the choices, A, B and C could be displayed to the gray jay simultaneously, it might make a better decision. Therefore, how to assist decision makers visualize the background information is an important issue in decision-making.

Ranking alternatives is one of the most important challenges in decision-making, especially when involving inconsistencies. If a decision maker's judgment is highly inconsistent, different ranking methods may produce wildly different priorities. That is, the decision maker may not make a reliable decision. Hence, how to assist the decision makers detect and improve these inconsistencies is another important issue in decision-making.

This study proposes Decision Ball models to provide visual representation of ranks of and similarities among alternatives, thus to help the decision makers make a more knowledgeable decision. Four types of Decision Ball models are constructed to meet the decision makers with different decision preferences and requirements. In addition, this study tries to help a decision maker improve the quality of his/her decision-making by reducing serious inconsistencies in judgment.

The major advantages of the proposed approach for a decision maker are summarized as below:

(i) Make a more knowledgeable decision through visualizing background information and decision processes.

(ii) Make a more reliable decision by improving inconsistencies iteratively.

(iii) Select a type of Decision Ball models based on his/her decision preferences and requirements.

(iv) Observe the ranks of and similarities among alternatives on Decision Balls directly.

(v) See the grouping relationships among alternatives layer-by-layer on Decision Balls, and perceive the benchmark alternatives if the DM would like to upgrade the performance of an alternative from one group to another.

1.2 Advantages of Decision Balls

Decision Ball models display alternatives on the surface of a ball. The arc length between two alternatives is used to represent the dissimilarity between them: the larger the

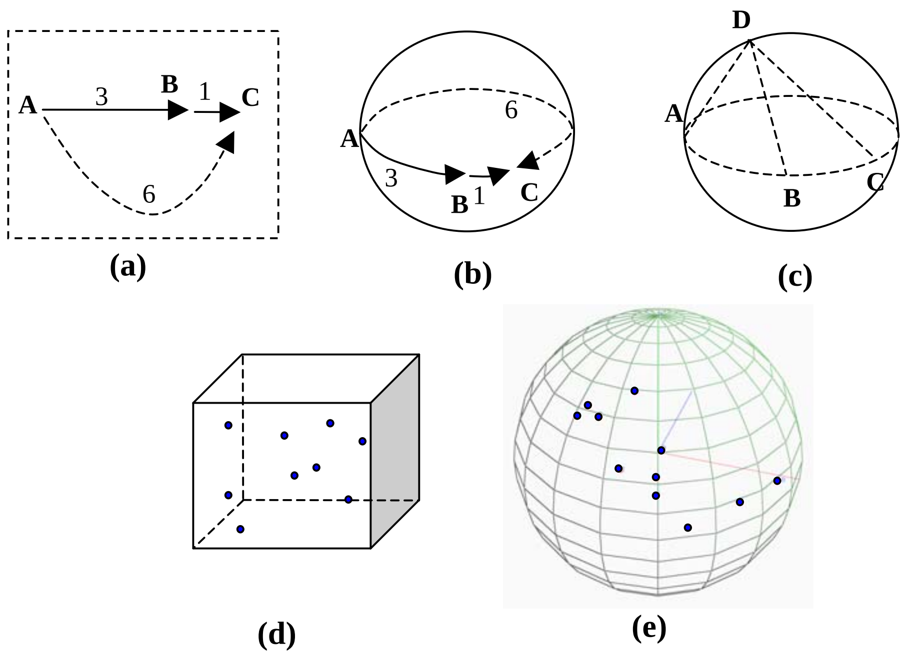


Figure 1.2 Advantages of Decision Balls (a) Display line segments on a 2-D plane (b) Display curves on a ball (c) Display four points that are not on the same plane (d) Display points in a 3-D cube (e) Display points on the surface of a ball

difference, the longer the arc length. In addition, the alternative with a higher score is designed to be closer to the North Pole so that alternatives will be located on the concentric circles in the order of rank from top view.

The advantages of Decision Balls are illustrated as follows:

(i) Comparing with 2-Dimensional plane models, Decision Balls can depict three points that do not obey the triangular inequality (i.e. the total length of any two edges must be larger than the length of the third edge). For instance, given three options A, B, and C. Suppose the distance between A and B is 3; the distance between B and C is 1; the distance between A and C is 6. We cannot draw three lines to connect A, B and C (Figure

1.2(a)). However, it is convenient to draw three arcs on the surface of a ball to illustrate their relationships (Figure 1.2(b)).

(ii) Decision Ball models are better than 2-Dimensional plane models because the former can show four points which are not on the same plane, as shown in Figure 1.2(c).

(iii) Comparing with 3-Dimensional cube models (Figure 1.2(d)), Decision Ball models are easier for a decision maker to observe the relationship among alternatives than 3-Dimensional cube models because the former can exhibit points on the surface of a ball, as shown in Figure1.2 (d) and (e).

(iv) Decision Ball models can depict inconsistencies in the decision makers' judgments. (Discussed in Chapter 5 and 6).

(v) A Decision Ball can display both ranks of and similarities among alternatives.

(vi) A Decision Ball involves no edges.

1.3 Framework of the Proposed Decision Ball System

Different decision makers may have various decision preferences and requirements because of personality of a decision maker, complexity of a decision problem, availability of decision data, …etc. This study summarizes four popular types of decision patterns and proposes corresponding Decision Ball models as follows: (Figure 1.3)

(i) Type I Pattern

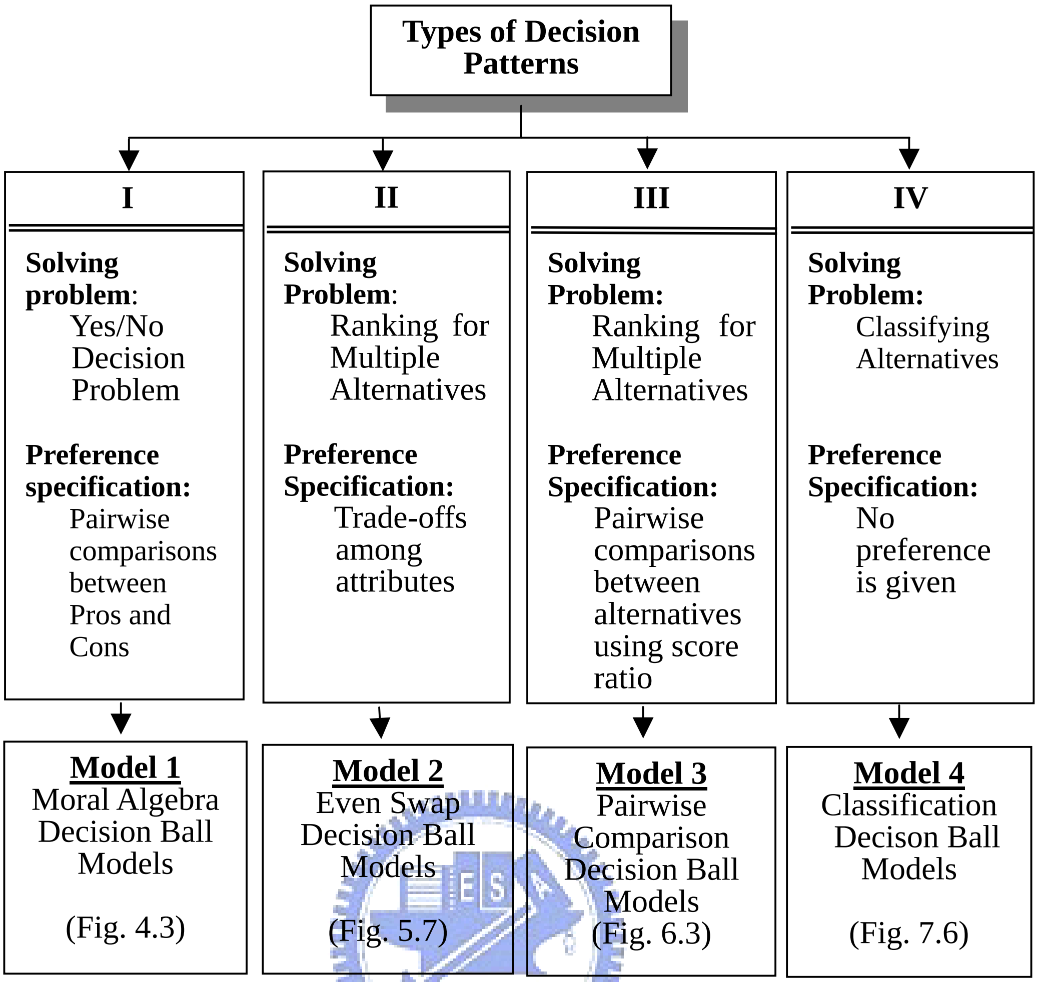


Figure 1.3 Framework of the proposed Decision Ball System

The decision makers are assumed to make a binary choice, or a "Yes or No" decision problem. This is the simplest decision pattern because the decision makers have not to estimate the value of each criterion for each alternative in advance.

Franklin (1956) proposed a process to help a decision maker make a rational choice under this decision pattern, called Franklin's Moral or Prudential Algebra. Franklin's Moral Algebra for making choices was first to divide a sheet of paper into two columns; one for pro, and another for con. Then, write down the various motives, for or against the choice. If a reason pro equaled a reason con, then both would be crossed out. If a reason pro equaled two reasons con, the three were crossed out. After a day or two of consideration, if nothing new came to mind for either side, the decision maker could then come to a determination.

Franklin's Moral Algebra is an intelligent way of simplifying the complexity of a decision. However, it is not easy for a decision maker to tell explicitly which pro(s) and con(s) can be eliminated simultaneously.

This study proposes Moral Algebra Decision Ball models to improve the insufficiencies of Franklin's Moral Algebra. Decision makers are assumed to be able to make pairwise comparisons between pro and con reasons with words such as "equally important", "slightly more important", "more important" and "significantly more important". By visualizing the relationships between pros and cons on Decision Balls, the decision makers can make a more knowledgeable decision.

(ii) Type II Pattern

Ranking for multiple alternatives is the major type of decision problem considered here. This pattern is sophisticated because the decision makers must be capable of making clear trade-offs among a range of criteria across a range of alternatives.

Hammond et al. (1998) developed a mechanism of Even Swaps to provide a useful way of making trade-offs. "Even" implies equivalence and "Swap" represents exchange. An even swap increases the value of one criterion while decreasing the value by an equivalent amount in terms of another criterion. By iteratively crossing out equally rated criteria to reduce the number of criteria, the most preferred alternative could be found.

Even swap approach is a rational and practically useful way in finding the best alternative. However, the ranks of rest of alternatives are not known, and there may exist large inconsistencies among even swaps that the DM could not know.

This study presents Even Swap Decision Ball models to assist the DM observe the ranks of and similarities among alternatives on the Decision Ball. The superiority relationship between alternatives can be observed by checking the longitude of alternatives. The inconsistencies between even swaps can also be known by checking the latitude of alternatives.

(iii) Type III Pattern

Ranking for multiple alternatives is the type of decision problem solved in this pattern too. However, instead of making trade-offs explicitly among values of criteria in Type II pattern, the decision makers of this decision pattern make pairwise comparisons between alternatives using score ratios.

The analytic hierarchy process (AHP)(Saaty, 1977, 1980; Saaty and Vargas, 1984, 1994) has been used widely to determine relative ranking of the decision alternatives through the pairwise comparison of alternatives at each level of the hierarchy. However, if perturbations from consistency are large, the information available cannot be used to derive a reliable answer (Saaty, 1977). That is, different ranking methods may produce wildly different priorities if a preference matrix is highly inconsistent. Hence, how to help the decision makers detect and adjust these inconsistencies becomes an important issue in this decision pattern.

This study illustrates Pairwise Comparison Decision Ball models to help the DM make a more reliable decision by detecting and improving inconsistencies in judgments. In addition to the ranks of and similarities among alternatives, the DM can observe the suggestions for effectively reducing inconsistencies on Decision Balls.

(iv) Type IV pattern

In this decision pattern, the decision makers do not have personal preferences about alternatives. They are interested in classifying alternatives more than ranking alternatives.

Discriminant Analysis (DA) is a statistical technique and popular method for predicting group membership. The GP (Goal Programming)-based DA, first proposed by Freed and Glover (1981), can estimate weights of criteria by minimizing sum of deviations (MSD, Freed and Glover, 1986) or minimizing misclassified alternatives (MMO, Banks and Abad, 1991). Those weights yield an evaluation score, which is compared with a threshold value for classifying alternatives. Sueyoshi (1999) first proposed a DEA-DA analysis incorporating the non-parametric feature of Data Envelopment Analysis (DEA, Charnes et al., 1978) into the DA. DEA-DA approach can effectively improve hit rates. However, it includes too many binary variables, and the decision makers cannot "see" the grouping relationships via graphical representation.

This study presents Classification Decision Ball models to aid the decision makers observe the grouping relationships on Decision Balls layer by layer. In addition to the ranks of and similarities among alternatives, the DMs can perceive the benchmark alternatives if the DMs would like to upgrade the performance of an alternative from one group to another. The number of binary variables can also be reduced significantly.

The framework of the proposed Decision Ball system is shown in Figure 1.3. Each type of decision patterns is illustrated as solving problem and preference specification parts. The corresponding Decision Ball models are depicted in the lower part of Figure 1.3.

1.4 Structure of the dissertation

The structure of this dissertation is depicted in Figure 1.4 and briefly introduced as follows:

Chapter 2 reviews two popular visualization tools: Multidimensional Scaling (Cox and Cox, 2000) and Gower Plots (Gower, 1977; Genest and Zhang, 1996). Their advantages and insufficiencies are also discussed.

Chapter 3 introduces Decision Ball techniques. The properties of additive score functions and multiplicative score functions are discussed first. Then, the Decision Ball techniques, based on the concept of Multidimensional Scaling, are presented. How to display alternatives on Decision Balls is demonstrated as an illustrative example.

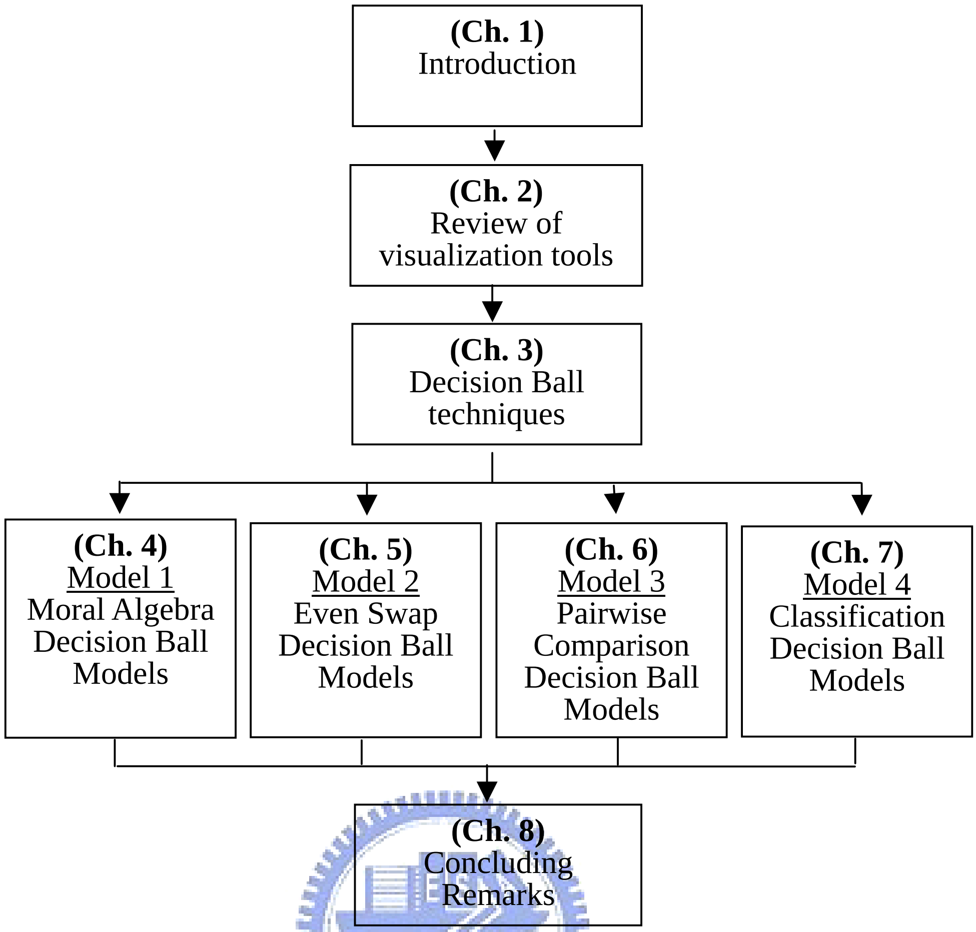


Figure 1.4 Structure of the dissertation

Chapter 4 presents Model 1 – Moral Algebra Decision Ball models for Type I decision pattern. The process of Franklin's Moral Algebra is described first. Moral Algebra Decision Ball models are then constructed. An example of a CEO's dilemma is illustrated to demonstrate the decision processes.

Chapter 5 discusses Model 2 – Even Swap Decision Ball models for Type II decision pattern. The method of Even Swaps is introduced. Then corresponding Even Swap Decision Ball models are built. An office-renting problem is used as an illustrative example. Chapter 6 addresses Model 3 – Pairwise Comparison Decision Ball models for Type III decision pattern.

This chapter first describes the basic concept of pairwise comparison, and then creates Pairwise Comparison Decision Ball models. Gower Plots are adopted to detect alternatives causing major inconsistencies. Optimization models are proposed to help the DM improve these inconsistencies conveniently. Two examples, investment in mutual funds and selection of universities, are demonstrated in this chapter.

Chapter 7 presents Model 4 – Classification Decision Ball models for Type IV decision pattern. DEA-DA analysis is introduced, and the Classification Decision Ball models are formed. Then, a corporate bankruptcy example and an example of Japanese banks are demonstrated. Chapter 8 presents concluding remarks and suggests directions for future research.

Chapter 2 Review of Visualization Tools

Several graphical techniques have been developed to aid the DM visualize background information. For instance, Li (1999) used deduction graphs to treat decision problems associated with expanding competence sets. Gower (1977), Genest and Zhang (1996) proposed a powerful graphical tool, the so-called Gower Plot, to detect the cardinal and ordinal inconsistencies in decision maker's preferences. Multidimensional Scaling (Borg and Groenen, 1997; Cox and Cox, 2000) is a classical technique used to provide a visual representation of similarities among a set of alternatives.

This chapter briefly reviews two popular visualization techniques, Multidimensional Scaling techniques and Gower Plots, which are adopted and compared in this study.

The structure of this chapter is organized as follows. Section 2.1 illustrates the Multidimensional Scaling technique. Section 2.2 briefly reviews Gower Plots method. Summary of this chapter is made in Section 2.3.

2.1 Review of Multidimensional Scaling (MDS) Techniques

Multidimensional Scaling (Borg and Groenen, 1997; Cox and Cox, 2000) is a classical technique to provide a visual representation of similarities among a set of alternatives, which allows one to map similarities between points in a high dimensional space into a lower dimensional space (usually Euclidean).

There are two major forms of MDS: metric and non-metric MDS. In metric scaling, the dissimilarities between all objects are known numbers, which can be approximated by distances directly. In non-metric MDS, only the rank order of the dissimilarities is approximated: the larger the dissimilarity, the longer the distance. Several MDS models (Cox and Cox, 2000) have been developed. One of commonly used model is proposed by Kruskal (1964a, 1964b). He developed a numerical measure of the closeness between the dissimilarities in the lower dimensional and the original spaces, called Stress. Denote $d_{i,j}$ as distance and $\delta_{i,j}$ as dissimilarity between alternative A_i and A_j. Stress can be formulated as

$$\text{Stress} = \sqrt{\frac{\sum_i \sum_{j>i} (d_{i,j} - f(\delta_{i,j}))^2}{\sum_i \sum_{j>i} d_{i,j}^2}}, \qquad (2.1)$$

where $f(\delta_{i,j})$ is the transformation of the $\delta_{i,j}$. In metric scaling, $f(\delta_{i,j})$ is a linear transformation of $\delta_{i,j}$. In non-metric scaling, $f(\delta_{i,j})$ is a weakly monotonic transformation of $\delta_{i,j}$. That is, if $\delta_{i,j} < \delta_{p,q}$, $f(\delta_{i,j}) \le f(\delta_{p,q})$**.** The Stress has a value between 0 and 1, with 0 indicating perfect fit and 1 implying worst possible fit**.** The rule of thumb for the value of Stress is that anything under 0.1 is excellent and over 0.15 is unacceptable. Based on Kruskal's approach, an initial configuration is randomly specified. Then an iterative procedure based on the steepest descent method is applied to move toward a local optimum by minimizing (2.1).

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 2 & 1 & 2 & 5 \\ 4 & 2 & 1 & 3 \\ 5 & 5 & 3 & 1 \end{pmatrix}$$

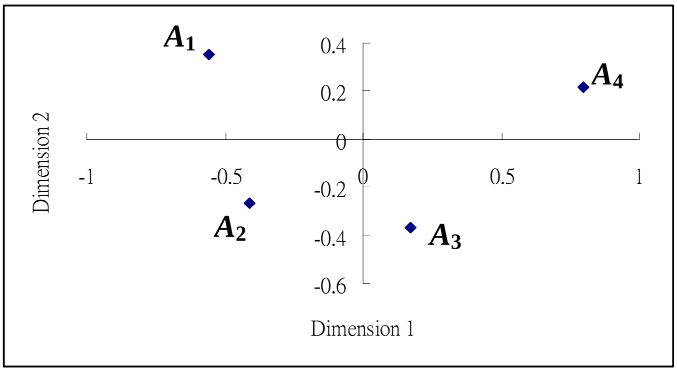


Figure 2.1 Displaying a distance matrix $\mathbf{R}_1$ by non-metric MDS techniques

For instance, a distance matrix $\mathbf{R}_1$, with four alternative A_1, A_2, A_3, and A_4, can be visualized by non-metric MDS techniques as shown in Figure 2.1. The Stress of this visual presentation is 0.57%.

Conventional MDS models, including Kruskal's approach, can effectively provide a visual representation of dissimilarities among objects. However, the conventional multidimensional scaling technique cannot show the ranks of alternatives and is incapable of detecting and adjusting inconsistencies in the decision makers' preferences.

2.2 Review of Gower Plots

Genest and Zhang (1996) proposed a graphical method, which is close in spirit to MDS, to graph a skew-symmetric matrix on a 2-Dimensional plane. Their method, based on the work of Gower (Gower, 1977), can display both the inconsistencies of data matrix and the ranks of alternatives. This section briefly introduces the mathematical properties of Gower Plots. The detail explanations of Gower Plots can refer to Genest and Zhang (1996).

The singular values of a matrix **M** of rank *n* are the positive square roots of the eigenvalues of the symmetric matrix $\mathbf{M}^{\mathbf{T}}\mathbf{M}$, where $\mathbf{M}^{\mathbf{T}}$ stands for transposition of **M**. If **M** is skew-symmetric, i.e. $\mathbf{M}^{\mathbf{T}} = -\mathbf{M}$, the singular values of the matrix **M** are equal to the norm of its purely imaginary eigenvalues.

Let $\lambda_1 \geq \ldots \geq \lambda_m \geq 0$ (and $\lambda_{m+1} = 0$ if *n* is an odd number) represent those singular values, with *m* indicating the integer part of *n/2*. Using singular value decomposition (Horn and Johnson, 1985), a skew-symmetric matrix **M** can be decomposed into the form

$$\mathbf{M} = \sum_{j=1}^{m} \lambda_j (\mathbf{U}_{2j-1}\mathbf{U}_{2j}^T - \mathbf{U}_{2j}\mathbf{U}_{2j-1}^T), \tag{2.2}$$

where $\mathbf{U}_{2j-1}$ and $\mathbf{U}_{2j}$ are orthonormal eigenvectors of $\mathbf{M}^{\mathbf{T}}\mathbf{M}$ corresponding to λ_j^2.

The matrix $\mathbf{M}^* = \lambda_1(\mathbf{U}\mathbf{V}^{\mathbf{T}} - \mathbf{V}\mathbf{U}^{\mathbf{T}})$ with $\mathbf{U} = \mathbf{U_1}$ and $\mathbf{V} = \mathbf{U_2}$ provides the best approximation of a skew-symmetric matrix **M** of rank two, because the first term of **M** gives the best least-squares fit of rank two to **M** (Eckart and Young, 1936). Let $\mathbf{U} = (u_1, \ldots, u_n)^T$ and $\mathbf{V} = (v_1, \ldots, v_n)^T$ as *n* points $P_j = (u_j, v_j)$ in the plane. A Gower Plot of a skew-symmetric matrix **M** is a two-dimensional graph composed of all P_j, $1 \leq j \leq n$, on the graph.

The measure of the faithfulness of the graphical representation of **M** is provided by

$$\text{variability} = \frac{\|\mathbf{M}^*\|}{\|\mathbf{M}\|} = \frac{\lambda_1^2}{\sum_{j=1}^{m} \lambda_j^2} . \tag{2.3}$$

Consider a set of n alternatives $A_1, A_2, \ldots, A_n$. Denote $r_{i,j}$ as the ratio of the weights of A_i to that of A_j, specified as,

$$r_{i,j} = \frac{w_i}{w_j} e_{i,j} , \tag{2.4}$$

where w_i is the weight of A_i, $w_i > 0$, for all i. $e_{i,j}$ is a multiplicative term accounting for inconsistencies. It is assumed that $r_{i,j} = \frac{1}{r_{j,i}}$, as illustrated in AHP (Saaty, 1977). Let $\mathbf{R} = (r_{i,j})$, for all i, j, be a $n \times n$ preference matrix. Following Genest and Zhang (1996), a tournament matrix $\mathbf{T} = (t_{i,j})$ corresponding to $\mathbf{R}$, is defined as $t_{i,j} = 1$ if $r_{i,j} > 1$; $t_{i,j} = 0$ if $r_{i,j} = 1$; $t_{i,j} = -1$ if $r_{i,j} < 1$.

Since $\mathbf{T}$ is a skew-symmetric matrix, a Gower Plot based on $\mathbf{T}$ can be depicted, called the ordinal Gower Plot of $\mathbf{R}$. From the work of Genest and Zhang (1996), we summarize the following rules to detect the ordinal consistency of **R.** Examining the ordinal Gower Plot of $\mathbf{R}$, $\mathbf{R}$ is close to be ordinal consistent, if (i) the location of alternatives (points $P_1, \ldots, P_n$) are equidistant from origin within a 180 degree arc; (ii) the angles between consecutive points are equal to $180/n$ degrees; (iii) the faithfulness of the graphical representation is demonstrated by variability factor, expressed in (2.3), being approximately 1. The points are arranged counter-clock-wise in the order of preference.

Let $\mathbf{S} = (s_{i,j})$, for all i,j, where $s_{i,j} = ln(r_{i,j})$. $\mathbf{S}$ is then a skew symmetric matrix. A Gower

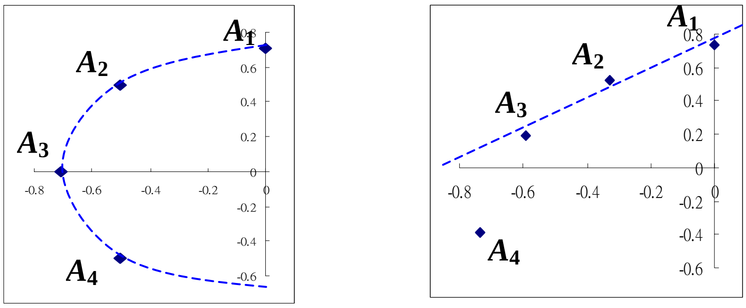
Plot based on **S** can be depicted, called the cardinal Gower Plot of **R**. Examining the cardinal Gower Plot of **R**, **R** is close to be cardinal consistent, if (i) $P_1, \ldots, P_n$ are collinear, and (ii) variability factor is approximately 1. The first condition means that $s^*_{i,k} + s^*_{k,j} = s^*_{i,j}$, for all $1 \le i, k, j \le n$.

For instance, suppose a DM specifies a preference matrix as $\mathbf{R_2}$. $\mathbf{T_2}$ is the tournament matrix corresponding to $\mathbf{R_2}$. The ordinal Gower Plot is depicted in Figure 2.2(a). Examining the ordinal Gower Plot, the matrix $\mathbf{R_2}$ is ordinal consistent because (i) all its points

$$\mathbf{R}_2 = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1/2 & 1 & 2 & 5 \\ 1/4 & 1/2 & 1 & 3 \\ 1/5 & 1/5 & 1/3 & 1 \end{pmatrix} \qquad \mathbf{T}_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

are located on a half-circle; (ii) the angles between every two consecutive points are equal to 180/*n* degrees; (iii) variability factor = 97.1%. Let $\mathbf{S_2} = ln(\mathbf{R_2})$, the cardinal Gower Plot of $\mathbf{R_2}$ is depicted in Fig. 2.2(b) representing 99.9% variability. The matrix $\mathbf{R_2}$ is not cardinal consistent because A_4 is away from the collinear line. The ranking of alternatives is $A_1 \succ A_2 \succ A_3 \succ A_4$ ("$\succ$" means superior to).

Gower Plots are powerful tools for detecting inconsistencies in data matrix, and can also display ranks of alternatives. However, it can neither show the similarities among alternatives nor provide any suggestions about how to adjust inconsistencies. In addition, a Gower Plot can be drawn only if the preference matrix is complete (discussed in Chapter 6).



(a) **(b)**

Figure 2.2 Gower Plots of R_2 (a) ordinal Gower Plot of R_2 (b) cardinal Gower Plot of R_2

2.3 Summary

A decision maker's choice is often affected by background information. This chapter briefly reviews two commonly used visualization techniques, Multidimensional Scaling and Gower Plots, and illustrates their advantages and insufficiencies.

Chapter 3 Decision Ball Techniques

This chapter illustrates the Decision Ball techniques with additive and multiplicative score functions respectively, based on the concept of Multidimensional Scaling techniques.

An additive score function is the most commonly used form in practice (Belton and Stewart, 2002) since it is more understandable for the decision maker. However, the linear additive score function is restricted to a fixed rate of substitution between criteria. A multiplicative score function is good at reflecting reasonable marginal rate of substitution, but is more complicated than the additive one. Both score functions are provided here to allow a decision maker to choose a proper one.

The structure of this chapter is organized as follows. Section 3.1 introduces the properties of additive score functions. Section 3.2 illustrates the properties of multiplicative score functions. Section 3.3 proposes the Decision Ball techniques with additive and multiplicative score functions respectively. Section 3.4 uses an example to demonstrate how to display alternatives on Decision Balls. Summary of this chapter is made in Section 3.5.

3.1 Properties of Additive Score Functions

Let $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$ be a set of n alternatives for solving a decision problem, where each alternative contains m criteria. An alternative A_i is expressed as $A_i\,(c_{i,1}, \ldots, c_{i,m})$. Denote

w_k as the weight of criterion k. In order to make sure all weights of criteria are positive, a criterion $c_{i,k}$ with cost feature (i.e., a DM likes to keep it as small as possible) is transformed from $c_{i,k}$ to $(\overline{c_k} - c_{i,k})$ in advance, where $\overline{c_k}$ is the largest value of criterion k.

Notation 3.1 The score function of A_i is assumed in an additive form, expressed below

$$S_i(\mathbf{w}) = \sum_{k=1}^{m} w_k \frac{c_{i,k} - \underline{c_k}}{\overline{c_k} - \underline{c_k}}, \tag{3.1}$$

where (i) $w_k \geq 0,\ \forall k$ and $\sum_{k=1}^{m} w_k = 1$. $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ is a weight vector obtained by other decision methods in advance, (ii) $\overline{c_k}$ and $\underline{c_k}$ are respectively the largest and smallest values of a criterion k. (iii) $0 \leq S_i(\mathbf{w}) \leq 1$.

Notation 3.2 The dissimilarity between A_i and A_j is defined as

$$\delta_{i,j}(\mathbf{w}) = \sum_{k=1}^{m} w_k \frac{|c_{i,k} - c_{j,k}|}{\overline{c_k} - \underline{c_k}}, \tag{3.2}$$

where $0 \leq \delta_{i,j}(\mathbf{w}) \leq 1$ and $\delta_{i,j}(\mathbf{w}) = \delta_{j,i}(\mathbf{w})$.

For the purpose of comparison, we define an ideal alternative A_*, where $A_* = A_*(\overline{c_1}, \overline{c_2}, \ldots, \overline{c_m})$ and $S_* = 1$. A_* is designed to be located at the north pole of a ball (radius = 1) with coordinate (x_*, y_*, z_*) = (0, 1, 0). Denote $\delta_{i,*}$, $d_{i,*}$ as the dissimilarity, distance between A_i and A_* respectively. We then have following propositions:

Proposition 3.1 $\delta_{i,*}(\mathbf{w}) = 1 - S_i(\mathbf{w})$ (3.3)

<Proof> $\delta_{i,*}(\mathbf{w}) = \sum_{k=1}^{m} w_k \frac{|c_{i,k} - \overline{c_k}|}{\overline{c_k} - \underline{c_k}} = \sum_{k=1}^{m} w_k \frac{(\overline{c_k} - \underline{c_k}) - (c_{i,k} - \underline{c_k})}{\overline{c_k} - \underline{c_k}}$

$$= (\sum_{k=1}^{m} w_k \frac{(\overline{c_k} - \underline{c_k})}{\overline{c_k} - \underline{c_k}} - \sum_{k=1}^{m} w_k \frac{(c_{i,k} - \underline{c_k})}{\overline{c_k} - \underline{c_k}}) = 1 - S_i(\mathbf{w})$$

<u>Notation 3.3</u> Denote the Euclidean distance between A_i and A_j as

$$d_{i,j} = \sqrt{2}\delta_{i,j}, \quad (3.4)$$

such that if $\delta_{i,j} = 0$ then $d_{i,j} = 0$ and if $\delta_{i,j} = 1$ then $d_{i,j} = \sqrt{2}$, where $\sqrt{2}$ is used because the distance between the north pole and equator is $\sqrt{2}$ when radius = 1. The relationship between y_i and S_i is expressed as

<u>Proposition 3.2</u> $y_i = 2S_i - S_i^2$. (3.5)

<Proof> Following Proposition 3.1 and Notation 3.3,

$$d_{i,*}^2 = (x_i - 0)^2 + (y_i - 1)^2 + (z_i - 0)^2 = 2\delta_{i,*}^2 = 2(1 - S_i)^2.$$

Therefore, we can obtain $y_i = 2S_i - S_i^2$.

Assume the weights of criteria are obtained from other decision methods in advance. The scores of and dissimilarities among alternatives can be calculated based on Notation 3.1 and 3.2. From Proposition 3.2, if $S_i = 0$, then $y_i = 0$; if $S_i = 1$, then $y_i = 1$. That is, the alternative with a higher score is located to be closer to the North Pole.

3.2 Properties of Multiplicative Score Functions

Before applying multiplicative score functions, all criterion values have to be normalized into interval [1, 10] with $\underline{c_k} = 1$, and $\overline{c_k} = 10$.

<u>Notation 3.4</u> The multiplicative score function of A_i is assumed in a non-linear Cobb-Douglas

(1928) form with constant return to scale, expressed below

$$S_i(\mathbf{w}) = w_0 c_{i,1}^{w_1} c_{i,2}^{w_2} \ldots c_{i,m}^{w_m}, \qquad (3.6)$$

where $w_0, w_1, \ldots, w_m \geq 0$ and $\sum_{k=1}^{m} w_k = 1$.

Let $1 \leq S_i \leq 10$, then $w_0 = 1$.

Notation 3.5 The dissimilarity between A_i and A_j is expressed as

$$\delta_{i,j}(\mathbf{w}) = [\frac{Max\{c_{i,1}, c_{j,1}\}}{Min\{c_{i,1}, c_{j,1}\}}]^{w_1} \times \cdots \times [\frac{Max\{c_{i,m}, c_{j,m}\}}{Min\{c_{i,m}, c_{j,m}\}}]^{w_m}, \qquad (3.7)$$

where $\delta_{i,j}(\mathbf{w}) = \delta_{j,i}(\mathbf{w})$ and $1 \leq \delta_{i,j}(\mathbf{w}) \leq 10$.

Notation 3.6 Let the Euclidean distance between A_i and A_j be

$$d_{i,j} = \frac{\sqrt{2}\ln(\delta_{i,j})}{\ln(10)}, \qquad (3.8)$$

such that if $\delta_{i,j} = 1$ then $d_{i,j} = 0$ and if $\delta_{i,j} = 10$ then $d_{i,j} = \sqrt{2}$.

Because $d_{i,j} = \frac{\sqrt{2}(\sum_{k=1}^{m} w_k(\ln(Max\{c_{i,k}, c_{j,k}\}) - \ln(Min\{c_{i,k}, c_{j,k}\}))}{\ln(10)}$, the relationship between $d_{i,*}$ and S_i can be expressed as

$$d_{i,*} = \frac{\sqrt{2}(\sum_{k=1}^{m} w_k(\ln(\overline{c_k}) - \ln(c_{i,k})))}{\ln(10)} = \frac{\sqrt{2}(\ln(10) - \ln(S_i))}{\ln(10)} = \sqrt{2}(1 - \frac{\ln(S_i)}{\ln(10)}). \qquad (3.9)$$

We then have following proposition:

Proposition 3.3 $y_i = \frac{2\ln(S_i)}{\ln(10)} - (\frac{\ln(S_i)}{\ln(10)})^2$ (3.10)

<Proof> Since $x_i^2 + (y_i - 1)^2 + z_i^2 = d_{i,*}^2 = 2(1 - \frac{\ln(S_i)}{\ln(10)})^2$,

then $y_i = \frac{2\ln(S_i)}{\ln(10)} - (\frac{\ln(S_i)}{\ln(10)})^2$.

From Proposition 3.3, if $S_i = 1$, then $y_i = 0$; if $S_i = 10$, then $y_i = 1$.

3.3 Display Techniques

From the basis of Multidimensional Scaling techniques, this section proposes Decision Ball techniques to provide spatial relationships among alternatives. The arc length between two alternatives is used to represent the dissimilarity between them: the larger the difference, the longer the arc length. However, because the arc length is monotonically related to the Euclidean distance between two points and both approximation methods make little difference to the resulting configuration (Cox and Cox, 1991), the Euclidean distance is used here for simplification.

In addition, the alternative with a higher score is designed to be closer to the North Pole so that alternatives will be located on the concentric circles in the order of rank from the top view.

Let $\hat{d}_{i,j} = f(\delta_{i,j})$, where $f(\delta_{i,j})$ is a monotonic transformation of $\delta_{i,j}$ (i.e. if $\delta_{i,j} < \delta_{p,q}$, then $\hat{d}_{i,j} < \hat{d}_{p,q}$). A Decision Ball technique with additive score functions is developed as follows.

Model 3.1 (A Decision Ball model – An additive score function)

Min $\quad \mathbb{Z} = \sum_{i=1}^{n}\sum_{j>i}^{n}(d_{i,j} - \hat{d}_{i,j})^2$

s.t. $\quad y_i = 2S_i - S_i^2, \ \forall i,$ (3.11)

$\hat{d}_{i,j} \le \hat{d}_{p,q} - \varepsilon, \ \forall \delta_{i,j} < \delta_{p,q},$ (3.12)

$$d_{i,j}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2, \quad \forall i, j, \tag{3.13}$$

$$x_i^2 + y_i^2 + z_i^2 = 1, \quad \forall i, \tag{3.14}$$

$$-1 \le x_i, z_i \le 1, \ 0 \le y_i \le 1, \ \forall i, \ \varepsilon \text{ is a tolerable error.} \tag{3.15}$$

The objective function of Model 3.1 is to minimize the sum of difference between $d_{i,j}$ and $\hat{d}_{i,j}$. (3.11) is from Proposition 3.2. (3.12) is the monotonic transformation from $\delta_{i,j}$ to $\hat{d}_{i,j}$. All alternatives are graphed on the surface of a semi-sphere (3.14)(3.15).

The stress value can be measured by

$$\text{Stress} = \sqrt{\frac{\mathbb{Z}}{\sum_{i=1}^{n}\sum_{j>i}^{n} d_{i,j}^2}} \tag{3.16}$$

If a decision maker chooses to use a multiplicative score function, Model 3.1 can be reformulated as follows.

<u>Model 3.2 (A Decision Ball model – A multiplicative score function)</u>

Min $\quad \mathbb{Z} = \sum_{i=1}^{n}\sum_{j>i}^{n} (d_{i,j} - \hat{d}_{i,j})^2$

s.t. $\quad y_i = \frac{2\ln(S_i)}{\ln(10)} - (\frac{\ln(S_i)}{\ln(10)})^2, \qquad (3.17)$

(3.12) ~ (3.15).

Expression (3.17) is from Proposition 3.3.

3.4 An Illustrative Example – Visualization on Decision Balls

This section uses a numerical example to demonstrate how to display alternatives on Decision Balls with additive and multiplicative score functions respectively.

<Example 3.1> Visualization on Decision Balls

Suppose a decision maker has three criteria (c_1, c_2, and c_3) to fulfill. He hopes all criteria values to be as large as possible. Assume the weights of criteria are known as follows: (w_1, w_2, w_3) = (0.2, 0.5, 0.3). Four alternatives are under considerations as listed in Table 3.1.

Assume the decision maker chooses to use an additive score function. Following Notation 3.1, the scores of alternatives can be obtained as (S_1, S_2, S_3, S_4) = (0.3, 0.66, 0.45, 0.8). The dissimilarities among alternatives are calculated based on Notation 3.2, as listed in Table 3.2 (a). Applying Model 3.1 to this example yields the coordinate of each alternative, as

Table 3.1 Data matrix of Example 3.1

$c_{i,k}$	c_1	c_2	c_3
A1	20	100	1.2
A2	35	165	0.8
A3	40	140	0.6
A4	30	180	1

Table 3.2 Results of Example 3.1 with an additive score function

(a) dissimilarity (b) coordinates of alternatives

(a)

$\delta_{i,j}$	**A1**	**A2**	**A3**	**A4**
A1		0.76	0.75	0.70
A2			0.31	0.24
A3				0.55
A4				

(b)

	x	***y***	***z***
A1	-0.78	0.52	-0.34
A2	-0.40	0.89	0.21
A3	-0.60	0.71	0.37
A4	-0.28	0.96	-0.02

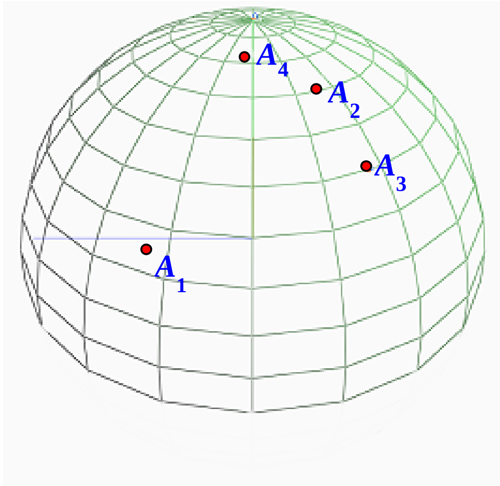


Figure 3.1 The Decision Ball of Example 3.1 with an additive score function

listed in Table 3.2(b). The corresponding Decision Ball is shown in Figure 3.1.

Assume the decision maker selects a multiplicative score function in Example 3.1. From Notation 3.4, the scores of alternatives are (S_1, S_2, S_3, S_4) = (1.06, 4.06, 3.19, 4.45). Based on Notation 3.5, the dissimilarities among alternatives are calculated, as listed Table 3.3(a). Applying Model 3.2 to the example yields the coordinates of alternatives, as listed in Table 3.3(b). The Decision Ball with a multiplicative score function is depicted in Figure 3.2.

Table 3.3 Results of Example 3.1 with a multiplicative score function (a) dissimilarity (b) coordinates of alternatives

(a)

$\delta_{i,j}$	**A1**	**A2**	**A3**	**A4**
A1	0.00	1.62	1.67	1.54
A2	0.00	0.00	1.22	1.15
A3	0.00	0.00	0.00	1.40
A4	0.00	0.00	0.00	0.00

(b)

	x	*y*	*z*
A1	-0.92	0.06	-0.39
A2	-0.52	0.86	-0.01
A3	-0.61	0.74	0.27
A4	-0.41	0.87	-0.29

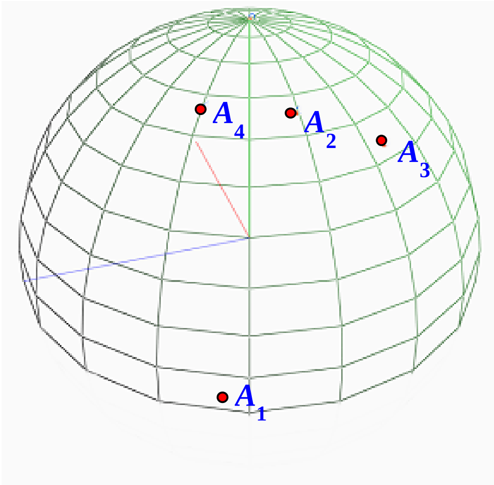


Figure 3.2 The Decision Ball of Example 3.1 with a multiplicative score function

3.5 Summary

This section proposes Decision Ball techniques with additive and multiplicative score functions respectively to provide a useful visual representation of ranks and similarities among alternatives. An illustrative example is also demonstrated about how to display alternatives on Decision Balls.

Chapter 4 Model 1: Moral Algebra Decision Ball Models

This chapter presents Model 1 – Moral Algebra Decision Ball models for Type I decision pattern. The decision problems solved in this pattern are Yes/No decision problems. This is the simplest decision pattern because the decision makers have not to estimate the value of each criterion for each alternative in advance. Decision makers are assumed to be capable of making pairwise comparisons between pro and con reasons. Based on Franklin's Moral Algebra, this study develops a mechanism to visualize the decision alternatives and processes on Decision Balls.

The structure of this chapter is organized as follows. Section 4.1 introduces the concept of Franklin's Moral Algebra. Section 4.2 constructs Moral Algebra Decision Ball models. Section 4.3 uses an example to demonstrate how to apply Moral Algebra on Decision Balls. Summary of this chapter is made in Section 4.4.

4.1 Introduction to Franklin's Moral Algebra

More than 230 years ago, Joseph Priestly, a noted scientist, asked for advice from Benjamin Franklin about what option to choose when making a decision. Franklin replied to his friend that he could not advise what to determine, but would like to tell how. Franklin called his method of choices a Moral or Prudential Algebra, which had brought him great success in making rational decisions. (The letter from Benjamin Franklin to Joseph Priestly is listed in the Appendix)

Franklin thought, the difficulty of making decision was because the reasons pro and con were not present in the mind at the same time; sometimes one set present themselves, and at other times another, while the first was out of sight.

Franklin's Moral Algebra for making choices was first to divide a sheet of paper into two columns; one for pro, and another for con. Then, write down the various motives, for or against the choice. Franklin then attempted to estimate the respective weights of these reasons at one time. If a reason pro equaled a reason con, then both would be crossed out. If a reason pro equaled two reasons con, the three were crossed out. After a day or two of consideration, if nothing new came to mind for either side, Franklin would then come to a determination.

Franklin thought that since all the reasons lay before him, and since each reason was considered separately and comparatively; he could judge better, and was less liable to make a rash choice. In fact, Franklin benefited a lot from this kind of choice method.

Franklin's Moral Algebra is an intelligent way of simplifying the complexity of a decision. By eliminating reasons pro and con step-by-step, the original list of pros and cons can be replaced with an equivalent but compact list. Then, a clear choice can then be reached. However, this algebra is not used widely today because of the following facts.

First, Franklin's Moral Algebra requires a decision maker to list equivalent pros and cons.

However, it is not easy for a decision maker to tell explicitly which pro(s) and con(s) can be eliminated simultaneously. Second, the key point in Franklin's Moral Algebra is to present all the pros and cons to the mind at the same time, the decision maker therefore can make whole comparisons about these pros and cons. However, the table listing may not be a proper way to display complete information to a decision maker. Since a table can only list the items of pros and cons but can not tell the similarities or differences between them.

This study therefore proposes Moral Algebra Decision Ball models to visualize and enrich Franklin's Moral Algebra. The merits of this approach in making choices are listed below:

(i) The decision maker is not required to directly list equivalent pros and cons. But to roughly express the comparisons between pros and cons with words such as "equally important", "slightly more important", "more important" and "significantly more important".

(ii) After making the comparisons, the differences of importance between pros and cons are displayed on the surface of a ball. By examining the ball, the decision maker can detect the closest sets of pros and cons, and then eliminate them simultaneously.

(iii) The whole decision process can now be visualized. By "seeing and choosing", the decision maker is more confident when making comparisons, updating preferences, eliminating pros and cons, simplifying complexity, and finally reaching a decision.

4.2 Construction of Moral Algebra Decision Ball Models

To illustrate the relationship between pros and cons, we can compare the differences between them. Suppose an option represents a pro or a con. If two options are equally important, then the difference of importance between them should be small. If one option is slightly more important than the other, then their difference becomes larger. If one option is much more important than the other, then their difference is significantly larger. To visualize the difference of importance means to convert them into physical distances.

Two rules of allocating all options on the surface of a ball are as follows:

Rule 1 : The more the difference of importance between two options, the longer the physical distance between them.

Rule 2 : The more important an option is, the closer it is to the north pole.

The decision maker's preferences between two options A and B are classified and expressed in Table 4.1.

Table 4.1 The relationship between two options

Preference between A and B	**Expression**
A is equally important as B	$A \approx B$
A is slightly more important than B	$A \succ B$
A is more important than B	$A \succ\succ B$
A is significantly more important than B	$A \succ\succ\succ B$

The essence of Franklin's Moral Algebra is to simultaneously display the complete information of pros and cons to the decision maker. This study intends to utilize computer graphic technologies to develop a decision support system to visualize a decision maker's preferences on a ball.

On the surface of a Decision Ball, the distance between two reasons is designed to be the relationship between them: the more the difference of importance, the longer the distance. The relationship between relationship type and distance is defined as listed in Table 4.2.

Table 4.2 The mapping table of relationship type and distance

	Relationship Type ($r_{i,j}$)	**Minimum Distance ($\underline{d'_{i,j}}$)**	**Target Distance ($d'_{i,j}$)**	**Maximum Distance ($\overline{d'_{i,j}}$)**
1	$\approx$	0	$0\times q$	$0.2\times q$
2	$\succ$	$0.2\times q$	$1\times q$	$2\times q$
3	$\succ\succ$	$1\times q$	$2\times q$	$3\times q$
4	$\succ\succ\succ$	$2\times q$	$3\times q$	$4\times q$

In Table 4.2, q is a scaling constant, and $r_{i,j}$ is the relationship type between two options i and j. There are four relationship types, including "$\approx$", "$\succ$", "$\succ\succ$"and "$\succ\succ\succ$". Each type of relationship is mapped to a target distance $d'_{i,j}$, with upper and lower bound $\overline{d'_{i,j}}$ and $\underline{d'_{i,j}}$ respectively. Let $d_{i,j}$ be the actual distance between reason i and reason j, and (x_i, y_i, z_i) be the mapping coordinates of reason i on the Decision Ball. For simplicity, let the radius of the Decision Ball be 1. The Decision Ball is formulated as follows:

Model 4.1 A Pro-Con Decision Ball Model

Min $\mathbb{Z}$

s.t.

$$-\mathbb{Z} \le d_{i,j} - d_{i,j}^{'} \le \mathbb{Z},\ \forall r_{i,j} \ne \phi, \tag{4.1}$$

$$\underline{d_{i,j}^{'}} \le d_{i,j} \le \overline{d_{i,j}^{'}},\ \ \forall r_{i,j} \ne \phi, \tag{4.2}$$

$$y_i \ge y_j + g,\ \text{ if } r_{i,j} \in \{"\succ","\succ\succ","\succ\succ\succ"\}, \tag{4.3}$$

$$g \ge \underline{g},\ \ q \ge \underline{q}, \tag{4.4}$$

(3.13) ~ (3.15),

where $\underline{g}$, $\underline{q}$ are lower bounds of g and q respectively.

The objective is to minimize the difference between the actual distance and target distance (4.1). Expression (4.2) is used to set the upper and lower bound of $d_{i,j}$. The latitudes of Pro or Con reasons stand for the order of importance. If a reason P_i is important than P_j, the latitude of P_i is designed to be higher than that of P_j, as listed in Expression (4.3), where g is a gap in y coordinate between two reasons with different importance. The lower bounds of g and q are set in (4.4) in order to avoid all reasons located too close to each other. The suggested values are $\underline{g} = 0.1$, $\underline{q} = 0.25$.

4.3 An illustrative Example – A CEO's Dilemma

<Example 4.1> A CEO's Dilemma

Here we use an example, called a CEO's dilemma, to illustrate the process of utilizing a Decision Ball to assist a manager in making choices.

Imagine a manager, David, who faces a difficult choice. David is the department director of SOFTCOM, a famous software company with 2000 employees. David came to the U.S.A from Shanghai, China. After obtaining his PhD from Wharton business School, David was recruited by SOFTCOM. Because of his outstanding ability in analysis, he has been promoted to a senior position in SOFTCOM. David has a lovely family, his wife Lisa and two children Ivy and Paul. Ivy is 10 and Paul is 6.

Because of the boom in the Chinese market, SOFTCOM plans to establish a subsidiary in Shanghai. One week ago, David was asked to be the CEO of the China subsidiary of SOFTCOM. The rewards of this new position are quite promising. The salary will be doubled, and David may be promoted to the Asia's director of SOFTCOM in the future. In addition, David can take care of his old parents in Shanghai. However, Lisa, Ivy and Paul do not want to leave. After staying at home for 5 years to take care of kids, Lisa cherishes her current job. Ivy and Paul love their current schools very much. In addition, Ivy and Paul cannot speak Chinese and may not make many friends in China. David is very excited about the new position; however, he does not want to be separated from his family. David needs to choose this week. How can he make this decision?

Many quantitative tools learned from school do not seem useful for David's decision, since all of these tools ask David to specify explicitly the trade-offs between "job and family" or between "money and love". David does not like it. Now we assist David to make his decision via a Decision Ball.

There are five steps of making a choice:

Step 1 Listing of Pros and Cons

Suppose David lists five pros in order of importance (roughly) for accepting the new position. First, this is a great promotion opportunity. If he accepts this new position, it is very possible he will be promoted to be the director of Asia in three years. Second, David's parents live in Shanghai. Both of them are over 75 years old. He can give his aging parents attention if he moves back to China. Third, the salary of the new position is more than twice as high as his salary now. Fourth, to be the CEO of a Chinese subsidiary, he could make more contributions to his homeland. Last, David has an aggressive personality and likes a career that offers a challenge. To be a CEO of Chinese subsidiary is an exciting challenge for him.

David also lists five cons in order of importance (roughly). First, both kids were born in the U.S.A. They cannot speak Chinese. They may have a tough time transforming to a new culture. Besides, both kids enjoy their American-style school life very much and object to leaving. Second, David's wife is an accountant. Lisa has worked hard and has recently got a promotion to section manager. She is not willing to quit her job. Third, the population density is very high in China, which results in a polluted environment. Fourth, the family just bought a new house in the U.S.A. one year ago. The house has a great view and a beautiful yard. The family likes the house very much and they are not willing to move out. Finally, David and Lisa have lived in the U.S.A. for over 16 years. Most of their friends are in the U.S.A. They cherish their friendships very much.

The summary of pros and cons are listed in Table 4.3.

Table 4.3 David's list of pros and cons for accepting the new position

Pros		Cons	
$\mathbf{P_1}$	Career promotion	$\mathbf{C_1}$	Children's education
$\mathbf{P_2}$	Parents' care	$\mathbf{C_2}$	Lisa's job
$\mathbf{P_3}$	High salary	$\mathbf{C_3}$	Polluted environment
$\mathbf{P_4}$	Working homeland	$\mathbf{C_4}$	Abandoning new house
$\mathbf{P_5}$	New challenge	$\mathbf{C_5}$	Loss of friendships

Step 2 Comparison of Pros

David selects some pros for comparison, as listed in Figure 4.1(a).

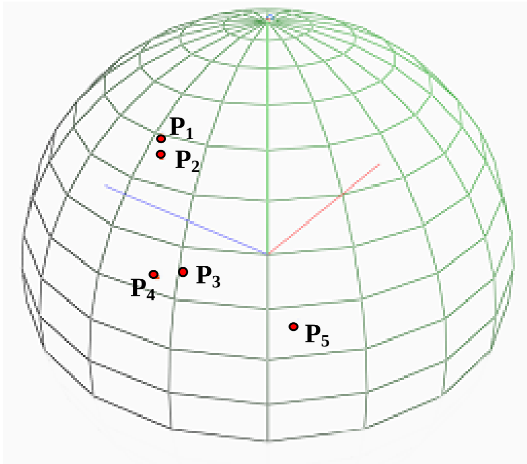
- Comparing Career promotion (P_1) with other pros, David thinks career promotion is equally important as Care for parents (P_2), more important than High salary (P_3), more important than Working for the homeland (P_4), and significantly more important than a New challenge (P_5). These preferences are expressed as $P_1 \approx P_2$, $P_1 \succ\succ P_3$, $P_1 \succ\succ P_4$, and $P_1 \succ\succ\succ P_5$.

- Comparing Parents' care (P_2) with other pros, David thinks it is slightly more important compared to a High salary (P_3) as well as Working for homeland (P_4), denoted as $P_2 \succ P_3$ and $P_2 \succ P_4$.

	P_1	P_2	P_3	P_4	P_5
P_1 : Career promotion		$\approx$	$\succ\succ$	$\succ\succ$	$\succ\succ\succ$
P_2 : Parents' care			$\succ$	$\succ$	
P_3 : High salary					$\succ$
P_4 : Working homeland					$\succ\succ$
P_5 : New Challenge					

$\approx$: equally important; $\succ$: slightly more important;
$\succ\succ$: more important; $\succ\succ\succ$: significantly more important.

(a)



(b)

Figure 4.1 Pro Ball of Example 4.1 (a) Relationships among pros (b) David's Pro Ball

- Comparing High salary (P_3) with other pros, David is unclear about the comparison of High salary (P_3) and Working for homeland (P_4). What he can sure is High salary (P_3) is slightly more important than a New challenge (P_5) ($P_3 \succ P_5$). Working for homeland (P_4) seems more important than a New challenge (P_5) ($P_4 \succ\succ P_5$).

After David finishes filling out preferences in Figure 4.1(a), the Decision Ball system then maps David's preferences into a Pro Ball in Figure 4.1(b). Figure 4.1(b) illustrates the relationships among the five pros. The arc length between two pros indicates their differences of importance: the longer the distance, the larger the difference. For instance, because the importance of Career promotion (P_1) over a New challenge (P_5) is higher than that of Career promotion (P_1) over High salary (P_3), the distance between P_1 and P_5 is much longer than that of P_1 and P_3. Moreover, the latitude of a pro stands for the order of importance. For example, because the importance of Career promotion (P_1) is higher than a New challenge (P_5), the latitude of P_1 is much higher than P_5.

Figure 4.1(b) shows that Career promotion (P_1) and Parents' care (P_2) are the closest to each other, and Career promotion (P_1) and a New challenge (P_5) are the longest distance apart; which fit the preference values in Figure 4.1(a). It is noteworthy that High salary (P_3) and Working for homeland (P_4) are close to each other, which implies P_3 and P_4 may be of similar importance. This relationship was not realized by David before; but it is visually illustrated by the ball. Moreover, David could also choose to revise the relationship between pro reasons in Figure 4.1(a) to modify his Pro-Ball iteratively.

<u>Step 3</u> Comparison of Cons

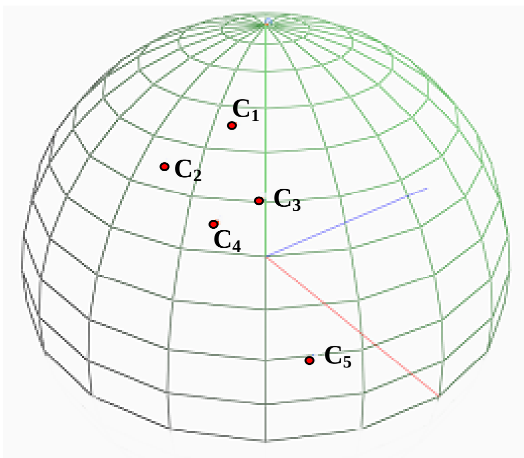
David selects some cons for comparisons, as listed in Figure 4.2(a).

- Considering his Children's education (C_1), it seems slightly more important than Lisa's job (C_2) ($C_1 \succ C_2$), because David thinks Ivy and Paul can only have a childhood once.

	C_1	C_2	C_3	C_4	C_5
C_1: Children's education		$\succ$	$\succ$		$\succ\succ\succ$
C_2 : Lisa's job				$\succ$	
C_3 : Polluted environment					$\succ\succ$
C_4 : Abandoning new house					$\succ\succ$
C_5 : Loss of friendships					

$\approx$: equally important; $\succ$: slightly more important;
$\succ\succ$: more important; $\succ\succ\succ$: significantly more important.

(a)



(b)

Figure 4.2 Con Ball of Example 4.1(a) Relationships among con reasons (b) David's Con Ball

His children's education is slightly more important than a Polluted environment (C_3), and is significantly more important than Loss of friendships (C_5) ($C_1 \succ C_3$, $C_1 \succ\succ\succ C_5$).

- Lisa's job (C_2) is slightly more important than Abandoning their new house (C_4).
- Both a Polluted environment (C_3) and Abandoning new house (C_4) are more important than the Loss of friendships (C_5).

A Con Ball associated with Figure 4.2(a) is depicted in Figure 4.2(b).

<u>Step 4</u> Comparison between Pro(s) and Con(s)

Next, David needs to specify the relationship between pro and con reasons, as listed in Figure 4.3(a).

- Since Lisa had stayed at home for 5 years to care for the kids before she got her current job, the job means a lot to her. David therefore thinks his Promotion opportunity (P_1) is equally important as Lisa's job (C_2).
- It is difficult to compare Care for parents (P_2) with any con. David therefore does not make any comparison here.
- Working for homeland (P_4) is equally important as the problems caused by a Polluted environment (C_3).
- David thinks his family's emotional reluctance to Abandon their new house (C_4) is slightly more important than the pleasure due to a Higher salary (P_3), denoted as $P_3 \prec C_4$.

After filling out Figure 4.3(a), the system generates a Pro-Con Ball (Figure 4.3(b)), which merges the Pro-Ball in Figure 4.1(b) and the Con-Ball in Figure 4.2(b). During the merging process, the system reallocates all pros and cons in order to let Career promotion (P_1) and Lisa's job (C_2), Working for homeland (P_4) and a Polluted environment (C_3) be as close as possible, to let the latitude of Abandoning the house (C_4) be higher than that of a High salary (P_3). This is because David feels $P_1 \approx C_2$, $P_4 \approx C_3$, and $P_3 \prec C_4$, as specified in Figure 4.3(a).

Step 5 Swapping Equivalent Pros and Cons

By examining the Pro-Con Ball in Figure 4.3(b), David finds that Career promotion (P_1) and Lisa's job (C_2) are very close to each other, that means P_1 and C_2 are equally important (as specified in Figure 4.3(a)); therefore, P_1 and C_2 can be eliminated (marked with a dash oval in Figure 4.3(b)). Similarly, a Polluted environment (C_3) and Working for homeland (P_4) can be eliminated. It is worthy to notice that Loss of friendships (C_5) and a New challenge (P_5) are also close to each other, which means they may be of similar importance although David did not realize it in Figure 4.3(a). This can only be visualized on a ball. Suppose David decides to eliminate a New challenge (P_5) and the Loss of friendship (C_5). The final Decision Ball is displayed in Figure 4.3(c).

	C_1 : Children's education	C_2 : Lisa's job	C_3 : Polluted environment	C_4:Abandoning new house	C_5 : Loss of friendship
P_1 : Career promotion		≈			
P_2 : Parents' care					
P_3 : High salary				≺	
P_4 : Working homeland			≈		
P_5 : New challenge					

≈ : equally important; ≻ : slightly more important;
≻≻ : more important; ≻≻≻ : significantly more important.

(a)

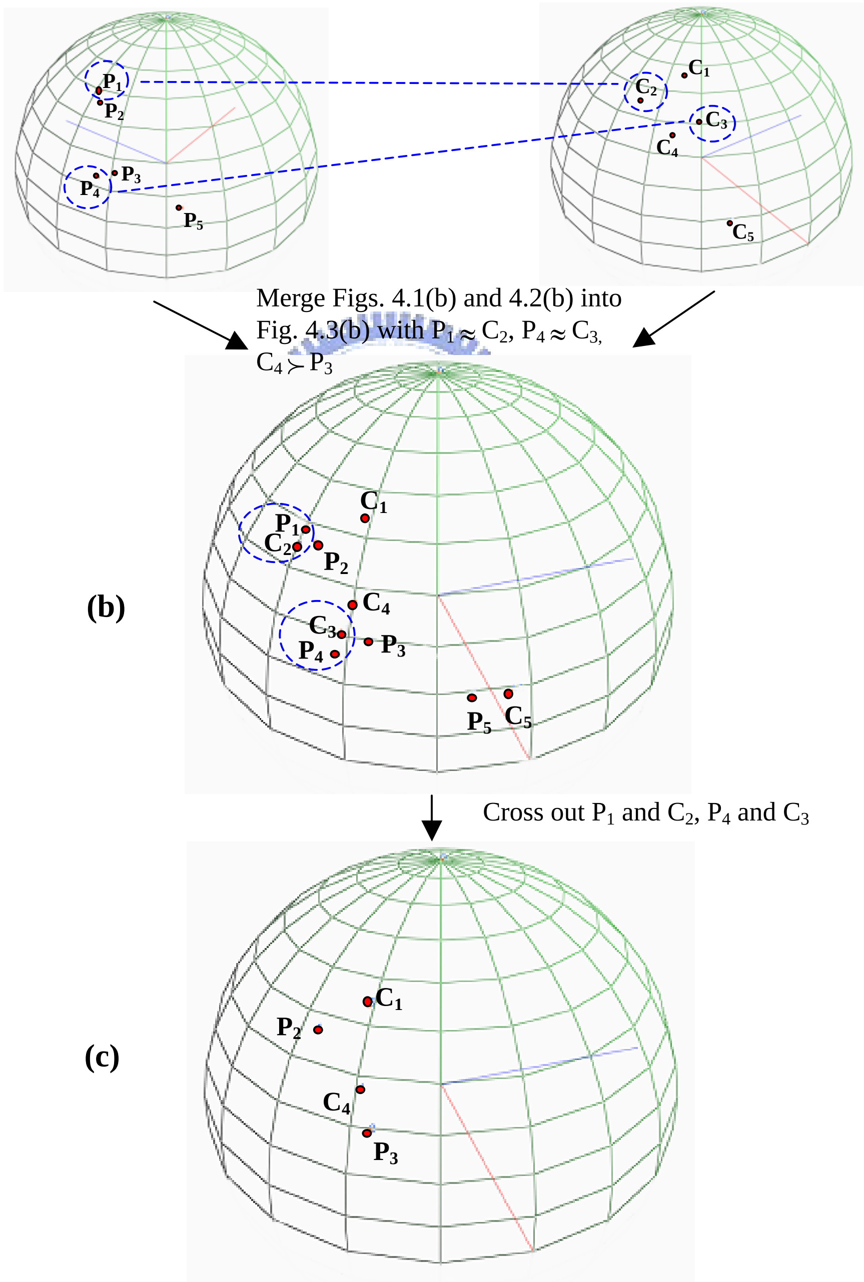


Figure 4.3 Pro-Con Ball of Example 4.1 (a) Relationships between pros and cons (b) David's Pro-Con Ball

By checking the latitude of the rest of the reasons in Figure 4.3(c), we can see that Children's education (C_1) is more important than Parents' care (P_2) and Abandoning the house (C_4) is more important than a High salary (P_3), where $C_4 \succ P_3$ is already shown in Figure 4.3(a). David now is quite clear about his mindset: Children's education (C_1) seems more urgent than Parents' care (P_2). He did not realize this before his Decision Ball showed him. Because the reasons con are more significant than the reasons pro as illustrated in Figure 4.3(c). David therefore decides not to accept the new position.

4.4 Summary

From the basis of Franklin's Moral Algebra, this study proposes Moral Algebra Decision Ball models to assist a manager make choice more confidently. By presenting all pros and cons related to a choice on a ball simultaneously, a decision maker can make a more knowledgeable decision.

The merits of this approach in making choices are listed below:

(i) The decision maker is not required to directly list equivalent pros and cons. But to roughly express the comparisons between pros and cons.

(ii) By examining the ball, the decision maker can detect the closest sets of pros and cons, and then eliminate them simultaneously.

(iii) By “seeing and choosing”, the decision maker is more confident when making comparisons, updating preferences, eliminating pros and cons, simplifying complexity, and finally reaching a decision.

(iv) Comparing with traditional 2-dimentional plane models, the proposed approach is more flexible because it can display points not on the same plane. For instance, as shown in Figure 4.3(b), there are 10 points, which are not on the same plane.

(v) Comparing with traditional 3-dimensional cube models, the decision maker can observe the difference of importance and priority of importance between pro and con reasons on Decision Balls more easily because all points are displayed on the surface of balls.

Chapter 5　Model 2 : Even Swap Decision Ball Models

This chapter introduces Model 2 – Even Swap Decision Ball models for Type II decision pattern. Ranking for multiple alternatives is the major decision problem considered here. This decision pattern is sophisticated because the decision makers are assumed to be capable of making clear trade-offs among a range of criteria across a group of alternatives. Even Swaps (Hammond et. al., 1998) processes are adopted here for making comparisons among alternatives. The whole decision processes are visualized on Decision Balls.

The structure of this chapter is organized as follows. Section 5.1 introduces the Even Swap processes. Section 5.2 builds Even Swap Decision Ball models. Section 5.3 uses an office-renting example to demonstrate the decision processes. Summary of this chapter is made in Section 5.4.

5.1　Introduction to Even Swap Processes

From the basis of Franklin's moral algebra, Hammond, Keeney and Raiffa (1998) developed a reliable mechanism for making trade-offs among a range of objectives across a group of alternatives. "Even" implies equivalence and "Swap" represents exchange. An Even Swap increases the value of one criterion while decreasing the value by an equivalent amount in terms of another criterion. By iteratively crossing out equally rated criteria to reduce the number of criteria, the most preferred alternative could be found.

Given a set of n alternatives A_1, A_2, …, A_n , where each alternative A_i contains multiple criteria. The conventional even-swap method (Hammond et al., 1998) begins with creating a consequences table, specified by the DM. Such a table contains the consequences that the alternatives have for the given criteria. The DM can find the best alternative based on the following three steps.

Step 1 Eliminating dominated alternatives.

The even-swap method intends to eliminate the alternatives as many as possible. Since the fewer the alternatives, the fewer trade-offs the DM has to make. A_i is said to dominate A_j if alternative A_i is better than A_j in some criteria and no worse than A_j in all other criteria. All dominated alternatives are eliminated first.

Step 2 Choosing a target criterion

After eliminating dominated alternatives, the even-swap method suggests the DM to choose a target criterion where the values of this criterion for all alternatives are ready to be adjusted as the same. Normally, a criterion with close values for most of alternatives is considered as a target criterion.

Step 3 Making Even Swaps

Choosing another criterion ready for compensating the changes in the target criterion. Assessing what changes in this criterion would compensate for the needed change in the target criterion. Make the Even Swaps, and cancel out the target criterion.

Steps 1 through Step 3 are applied iteratively until the best alternative is chosen.

Even Swap approach is a rational and practically useful way in finding the best alternative. However, current Even Swap method has following drawbacks remained to be improved:

(i) Only the best alternative is found. Other alternatives are not ranked. In actual decision environment, the DM may like to know the second best and/or the third best alternatives.

(ii) Various trade-off values among alternatives, which are specified by the DM, may not consist with each other. Current methods have no mechanism to check the consistency among these trade-offs.

(iii) The dissimilarities among alternatives are not taken into account. Actually, the DM hopes to know not only the ranks of alternatives but also the dissimilarities among alternatives.

In order to improve the insufficiencies of the conventional Even Swap method, this study proposes Even Swap Decision Ball models to rank and display alternatives. The DM can see the ranks of and differences among all alternatives on a Decision Ball. In addition, by examining the moving trajectories of alternatives on a Decision Ball, the DM can check the consistency among Even Swaps.

5.2 Construction of Even Swap Decision Ball Models

By mapping all A_i into the points (denoted as P_i) on a ball, the spatial relationships among these points are discussed below:

Notation 5.1 Given two alternatives $A_i\,(c_{i,1}, \ldots, c_{i,m})$ and $A_j\,(c_{j,1}, \ldots, c_{j,m})$, A_i "dominates" A_j, denoted as $A_i \succ A_j$ if (i) $c_{i,k} \geq c_{j,k}$ for all k and (ii) there is at least a k such that $c_{i,k} > c_{j,k}$.

Consider the following propositions.

Proposition 5.1 Suppose there are two alternatives A_i and A_j with $S_i > S_j$. P_*, P_i and P_j are on the same longitude if and only if $d_{i,j}(\mathbf{w}) = d_{j,*}(\mathbf{w}) - d_{i,*}(\mathbf{w})$.

<Proof> If P_*, P_i and P_j are on the same longitude with $S_i > S_j$, then $\overset{\frown}{P_iP_j} = \overset{\frown}{P_*P_j} - \overset{\frown}{P_*P_i}$. That is, the value of $\overset{\frown}{P_*P_i} + \overset{\frown}{P_iP_j} - \overset{\frown}{P_*P_j}$ is minimal for known S_i and S_j. Since the arc length is monotonically related to Euclidean distance between two points, $d_{*,i}(\mathbf{w}) + d_{i,j}(\mathbf{w}) - d_{*,j}(\mathbf{w})$ is minimal. Because $d_{*,i}(\mathbf{w}) + d_{i,j}(\mathbf{w}) - d_{*,j}(\mathbf{w}) \geq 0$, we then have $d_{i,j}(\mathbf{w}) = d_{j,*}(\mathbf{w}) - d_{i,*}(\mathbf{w})$. On the other hand, since $d_{i,*}(\mathbf{w})$ and $d_{j,*}(\mathbf{w})$ are expressed as $\overline{P_iP_*}$ and $\overline{P_jP_*}$ which are monotonically related to $\overset{\frown}{P_iP_*}$ and $\overset{\frown}{P_jP_*}$ respectively, if $d_{i,j}(\mathbf{w}) = d_{j,*}(\mathbf{w}) - d_{i,*}(\mathbf{w})$, then P_*, P_i and P_j are located on the same arc along the great circle. That is, P_*, P_i and P_j are on the same longitude.

Proposition 5.2 Consider a DB($\mathbf{w}$, I) with two alternatives A_i and A_j only, i.e., I = $\{i, j\}$. If dominance exists between A_i and A_j (i.e. $A_i \succ A_j$ or $A_j \succ A_i$), then P_i and P_j are on the same longitude.

<Proof> $A_i \succ A_j$ implies $c_{i,k} \geq c_{j,k}$, for all k.

(i) For an additive score function, from (3.2) and (3.4),

$$d_{i,j} = \sqrt{2}\delta_{i,j} = \sqrt{2}\sum_{k=1}^{m} w_k \frac{\left|c_{i,k} - c_{j,k}\right|}{\overline{c_k} - \underline{c_k}} = \sqrt{2}\sum_{k=1}^{m} w_k \frac{(\overline{c_k} - c_{j,k}) - (\overline{c_k} - c_{i,k})}{\overline{c_k} - \underline{c_k}}$$

$$= \sqrt{2}(\delta_{j,*} - \delta_{i,*}) = d_{j,*} - d_{i,*}$$

(ii) For a multiplicative score function, from (3.8) and (3.9)

$$d_{i,j} = \frac{\sqrt{2}\ln(\delta_{i,j})}{\ln(10)} = \frac{\sqrt{2}}{\ln(10)}(\sum_{k=1}^{m} w_k (\ln(Max(c_{i,k}, c_{j,k}) - \ln(Min(c_{i,k}, c_{j,k})))$$

$$= \frac{\sqrt{2}}{\ln(10)}(\ln(S_i) - \ln(S_j)) = \frac{\sqrt{2}}{\ln(10)}(1 - \frac{\ln(10)d_{i,*}}{\sqrt{2}} - 1 + \frac{\ln(10)d_{j,*}}{\sqrt{2}})$$

$$= d_{j,*} - d_{i,*}.$$

From Proposition 5.1, P_*, P_i and P_j are on the same longitude.

Proposition 5.3 For a DB(**w**, I) for I = {*i*, *j*}. If $S_i(\mathbf{w}) > S_j(\mathbf{w})$, and P_i and P_j are on the same longitude, then $A_i \succ A_j$.

<Proof> Since $S_i(\mathbf{w}) > S_j(\mathbf{w})$ and P_i, P_j are on the same longitude,

$$d_{i,j}(\mathbf{w}) = d_{j,*}(\mathbf{w}) - d_{i,*}(\mathbf{w}).$$

(i) For an additive score function, from (3.3) and (3.4),

$$d_{i,j} = \sqrt{2}\sum_{k=1}^{m} \frac{\left|c_{i,k} - c_{j,k}\right|}{\overline{c_k} - \underline{c_k}} = \sqrt{2}\sum_{k=1}^{m} w_k \frac{c_{*,k} - c_{j,k} - c_{*,k} + c_{i,k}}{\overline{c_k} - \underline{c_k}} = \sqrt{2}\sum_{k=1}^{m} w_k \frac{c_{i,k} - c_{j,k}}{\overline{c_k} - \underline{c_k}}.$$

(ii) For a multiplicative score function, from (3.7) and (3.8),

$$d_{i,j} = \frac{\sqrt{2}}{\ln(10)}(\sum_{k=1}^{m} w_k (\ln(\overline{c_k}) - \ln(c_{j,k}) - \ln(\overline{c_k}) + \ln(c_{i,k}))) = \frac{\sqrt{2}}{\ln(10)}(\sum_{k=1}^{m} w_k (\ln(c_{i,k}) - \ln(c_{j,k}))).$$

Both (i) and (ii) imply $c_{i,k} \geq c_{j,k}$, for all k. That is $A_i \succ A_j$.

We then deduce following theorem:

Theorem 5.1 Given A_i and A_j where $S_i(\mathbf{w}) > S_j(\mathbf{w})$. Let P_i and P_j be the mapping points of A_i and A_j on a DB(**w**, I), I = {i, j}. If and only if dominance exists between A_i and A_j (i.e. $A_i \succ A_j$ or $A_j \succ A_i$), then P_i and P_j are on the same longitude of the ball connecting P_*, P_i and P_j.

Notation 5.2 DS(i_1, i_2, …, i_p) is denoted as a dominant set composed of p alternatives with dominant relationships $A_{i_1} \succ A_{i_2} \succ \cdots \succ A_{i_p}$.

Proposition 5.4 Consider a dominant set DS(1, 2, …, k), let DB(**w**, I), I = {1, 2, …, k} be the corresponding Decision Ball for the alternatives A_1, A_2, …, A_k, where $A_1 \succ A_2 \succ \ldots \succ A_k$. Connecting points P_*, P_1, P_2, …, P_k forms a longitude on the surface of this Decision Ball. That implies $ax_i + cz_i = 0$ for i = 1, 2, …, k, where a and c are constants.

<Proof> Similar to Propositions 5.2 and 5.3.

Notation 5.3 Given an alternative set **A** = (A_1, A_2, …, A_n) and a weighted vector **w**, a corresponding Decision Ball of A and **w** is denoted as

DB(**w**, I) = $\{(x_i, y_i, z_i) \mid i \in \mathrm{I} = \{1,2,\ldots,n\}\}$, where ($x_i$, y_i, z_i) are obtained by solving following models.

Model 5.1 (Even Swap -- Decision Ball model with an additive score function)

$$\text{Min} \quad \mathbb{Z} = \sum_{i=1}^{n} \sum_{j>i}^{n} (d_{i,j} - \hat{d}_{i,j})^2$$

s.t. $x_i z_j = x_j z_i, \ \forall A_i \succ A_j \ , \ \forall i, j,$ (5.1)

(3.11) ~ (3.15).

The objective of Model 5.1 is to minimize the sum of difference between $d_{i,j}$ and $\hat{d}_{i,j}$.

(5.1) is from Proposition 5.4.

<u>Model 5.2 (Even Swap --Decision Ball model with a multiplicative score function)</u>

Min $\mathbb{Z} = \sum_{i=1}^{n}\sum_{j>i}^{n}(d_{i,j} - \hat{d}_{i,j})^2$

s.t. $y_i = \frac{2\ln(S_i)}{\ln(10)} - (\frac{\ln(S_i)}{\ln(10)})^2, \ \forall i,$ (5.2)

(5.1), and (3.12) ~ (3.15),

where (5.2) is from Proposition 3.3.

<u>Notation 5.4</u> Let $A_i^{'}$ be the alternative converted from A_i by the DM through making Even Swaps, A_i and $A_i^{'}$ are called concurrent alternatives. P_i and $P_i^{'}$, which are mapping points of A_i and $A_i^{'}$, are called concurrent points.

<u>Remark 5.1</u> Given two alternatives A_i and A_j, suppose the DM can stably make Even Swaps, then P_j can be converted into another concurrent point $P_j^{'}$ such that P_*, P_i and $P_j^{'}$ are on the same longitude.

We can use Figure 5.1 to interpret Remark 5.1. Here $S_i \geq S_j$ but A_i does not dominate A_j. Via Even Swap processes, we can convert A_j to $A_j^{'}$ where $A_i \succ A_j^{'}$. From Theorem 5.1, P_j therefore can be moved to a concurrent point $P_j^{'}$ where P_*, P_i and $P_j^{'}$ are on the same longitude.

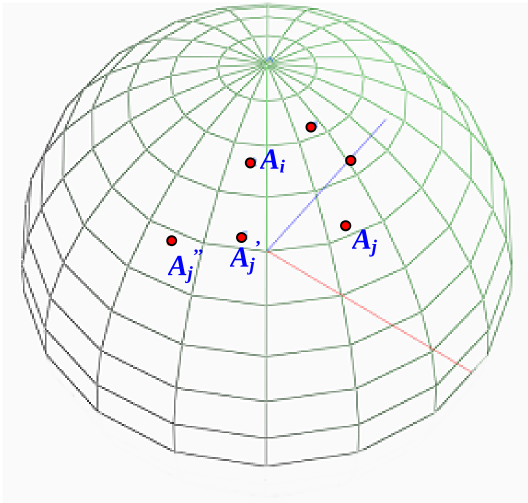


Figure 5.1 Moving trajectory of concurrent points

__Notation 5.5__ A_i is said consistently even swapped into A_i' if $\frac{|S_i - S_i'|}{S_i} \leq \varepsilon$, where ε is a tolerable error. Normally we may set $\varepsilon \leq 0.05$.

__Theorem 5.2__ Given A_i with its concurrent alternative A_i', and P_i with its concurrent point P_i', A_i is consistently even swapped into A_i' if and only if P_i and P_i' are on the same latitude.

<Proof>

(i) If A_i is consistently even swapped into A_i', then $S_i = S_i'$, it implies $y_i = y_i'$ (referred to Proposition 3.2 or 3.3). Therefore, P_i and P_i' are on the same latitude.

(ii) If P_i and P_i' are on the same latitude, then $y_i = y_i'$ which implies $S_i = S_i'$. A_i' therefore is consistently even swapped from A_i.

Theorem 5.2 is useful in checking the consistency of Even Swap processes made by the DM. Take Figure 5.1 for instance, A_j is consistently even swapped into A_j', however, A_j'' is not even swapped from A_j consistently. The more inconsistent swap the DM has made, the bigger differences in score before and after even swap. That is, the difference between coordinate y_j and $y_j^{'}$ is bigger.

Both Theorem 5.1 and Theorem 5.2 are utilized in this study to develop a mechanism to visualize the Even Swap processes via Decision Balls. By examining the moving trajectories of related points on a Decision Ball, the DM can rank the alternatives more confidently.

The solving processes are summarized as follows:

Step 1 (Initialization) The system asks the DM to input a consequence table, to select criteria with cost feature, to quantify the non-numerical criteria, to choose a type of score function, and to specify the initial weights **w**(J) for J = 0. A dominant set is initialized as DS(J) = ϕ, for J = 0.

Step 2 (Computing scores) Based on **w**(J), the system computes $S_i(\mathbf{w})$ and $d_{i,j}(\mathbf{w})$.

Step 3 (Displaying a Decision Ball) A Decision Ball DB(**w**, I) is displayed to the DM after solving Model 5.1 or 5.2. The alternative $A_i \notin$ DS(J) with the highest score is chosen as the next swap alternative by the system. The process stops if all alternatives are in DS(J) or the DM ceases to make further even swaps.

Step 4 (Making Even Swaps) The DM makes even swaps between A_i and alternatives in DS(J). A_i is changed to a concurrent alternative $A_i^{'}$.

Step 5 (Weight adjustment) For each even swap, the system computes the related weights by solving following linear programs:

<u>Model 5.3 (Weight adjustment model with an additive score function)</u>

Min α

s.t. $\left| w_p(c_{i,p} - c'_{i,p}) + w_q(c_{i,q} - c'_{i,q}) \right| \le \alpha$, for an even swap ($c_p$, c_q) in A_i, (5.3)

$$\sum_{k=1}^{m} w_k = 1, \quad (5.4)$$

$$w_k \ge 0, \quad \forall k, \quad (5.5)$$

where $c_{i,k}$ and $c'_{i,k}$ are the value of criterion k of A_i before and after the even swap respectively. The weights of unadjusted criteria are kept the same as previous step. J = J+1.

The weight adjustment model with a multiplicative score function can be formulated as follows:

<u>Model 5.4 (Weight adjustment model with a multiplicative score function)</u>

Min α

s.t. $\left| w_p(\ln(c_{i,p}) - \ln(c'_{i,p})) + w_q(\ln(c_{i,q}) - \ln(c'_{i,q})) \right| \le \alpha$, (5.6)

for an even swap (c_p, c_q) in A_i,

(5.4) ~ (5.5).

<u>Step 6</u> (Updating the dominant set) A_i is added into DS(J). Reiterate Step 2 to Step 6.

5.3 An Illustrative Example – An Office-Renting Problem

<Example 5.1> An office-renting problem

This example is slightly modified from Harvard Business Review (Hammond et al., 1998), which describes a business problem for determining where to rent an office. The DM has five major decision criteria to fulfill (Table 5.1): (c_1) a short commute time from home to office, (c_2) good access to his clients, (c_3) good office services, (c_4) sufficient space, and (c_5) low costs. The commuting time is the average time in minutes needed to travel to work during rush hour. The percentage of his clients within an hour's drive of the office is used to measure the access to clients. A simple three-letter scale is used to describe the office services provided: "A" indicates full service; "B" means partial service; and "C" implies no service available. Office size is measured in square feet, and cost is measured by monthly rent. Five alternative locations from A_1 through A_5 are under considerations.

The 1st iteration: At Step 1 of the first iteration, the DM inputs his consequences table, maximal and minimal values of each criterion (Table 5.1), where c_1, c_5 are criteria with cost feature (The DM would like it as small as possible). The DM chooses a multiplicative score function. The system asks the DM to answer some questions. Suppose the dialogue is as

Table 5.1 The consequence table of Example 1 ($A_2 \succ A_5$)

Criteria \ Alternative		A_1	A_2	A_3	A_4	A_5	Max	Min
c_1	**Commute (Mins)**	45	25	20	25	30	60	0
c_2	**Customer Access (%)**	50	80	70	85	75	100	0
c_3	**Office Services**	A	B	C	A	C	A	C
c_4	**Office Size (Square Feet)**	800	700	500	950	700	1200	500
c_5	**Monthly Cost ($)**	1850	1700	1500	1900	1750	2000	1500

follows:

<Q1> Consider criterion c_3, how do you quantify the values of service level A, B and C?

<A1> 4, 2, 1.

<Q2> Input the initial weights for c_1, c_2, c_3 , c_4 and c_5. **<A2>** 0.2, 0.2, 0.2, 0.2, 0.2.

At Step 2, based on the initial weights, the similarities among alternatives and scores of alternatives are calculated. At Step 3, a Decision Ball (Figure 5.2(a)) is displayed to the DM. The figure illustrates that because $A_2 \succ A_5$ (Table 5.1), A_2 and A_5 are on the same longitude. At Step 4, the DM are suggested to make even swaps between A_2 and A_4 first because A_2 and A_4 yield the highest score. Suppose the dialogue is as follows.

<Q3> Consider Figure 5.2(a), now A_2 needs to be moved to a new point $A_2^{'}$ which has the same longitude of A_4. Please choose a target criterion of A_2 from {c_2, c_3, c_4, c_5}, and adjust its value. **<A3>** c_5 and 1900.

<Q4> Consider A_2, to compensate the increase of c_5 from 1700 to 1900, choose one criterion from {c_2, c_3, c_4 } and specify the value being adjusted. **<A4>** c_4 and 850 (A_2 is changed to a concurrent point $A_2^{'}$, and $A_4 \succ A_2^{'}$).

The weights of criteria are adjusted as (w_1 , w_2, w_3 , w_4, w_5) =(0.2, 0.2, 0.2, 0.263, 0.137). The criteria values and scores of alternatives after even swap are listed in Figure 5.2(b).

The 2nd iteration: A Decision Ball is shown in Figure 5.3(a). A_1, which has the higher score than A_5 and A_3, is then chosen as the next swap alternative. At Step 4, the system asks the DM following sample questions.

<Q5> Consider Figure 5.3(a), here A_1 needs to be moved to $A_1^{'}$ which has the same longitude of A_4 and $A_2^{'}$. Firstly, c_3 of A_1 is adjusted to B. To compensate the increase of c_3 from A to B, choose one criterion from {c_1, c_2, c_4, c_5} and specify the value you want to adjust. **<A5>** c_2 and 75.

<Q6> Do you want to adjust another criterion pair of A_1 ? **<A6>** Yes.

<Q7> Consider A_1, choose a target criterion from {c_1, c_2, c_4, c_5}, and specify the equated value. **<A7>** c_5, and 1900.

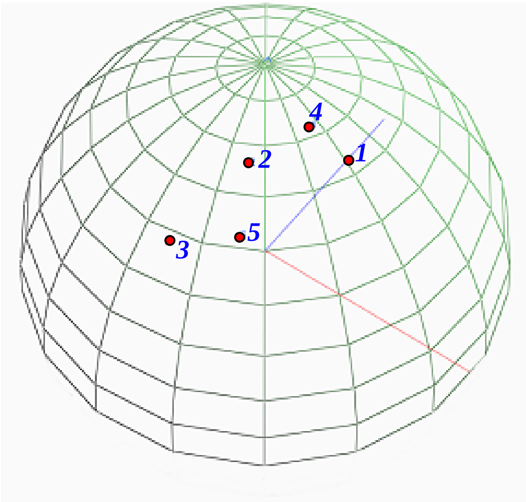
<Q8> Consider A_1, to compensate the increase of c_5 from 1850 to 1900, choose one criterion from {c_1, c_2, c_4} and specify value being adjusted. **<A8>** c_4 and 850. (A_1 is changed to a concurrent point $A_1^{'}$, and $A_4 \succ A_2^{'} \succ A_1^{'}$).

The weights of criteria become **w** = (0.2, 0.291, 0.109, 0.277, 0.123).

The 3rd iteration: A Decision Ball (Figure 5.4(a)) is displayed to the DM, where A_5 is then chosen as a swap alternative. Suppose the DM equates an increase in c_3 from C to B with a 200 increase in c_5, and equates an increase in c_1 from 30 to 45 with a 100 increase in c_4. The consequences table after even swaps is listed in Fig. 5.4(b). The weights become **w** = (0.176, 0.291, 0.101, 0.301, 0.131).

The 4th iteration: Figure 5.5(a) is displayed to the DM. A_3 is chosen as a swap alternative. Suppose the consequences table after even swaps is listed in Figure 5.5(b). The final weights on criteria become **w** = (0.178, 0.291, 0.099, 0.232, 0.2). Since all alternatives are on the same longitude, the process is terminated. The final Decision Ball and consequences table are depicted in Figure 5.6(a) and (b). The ranks for these alternatives are $A_4 \succ A_2 \succ A_1 \succ A_5 \succ A_3$.

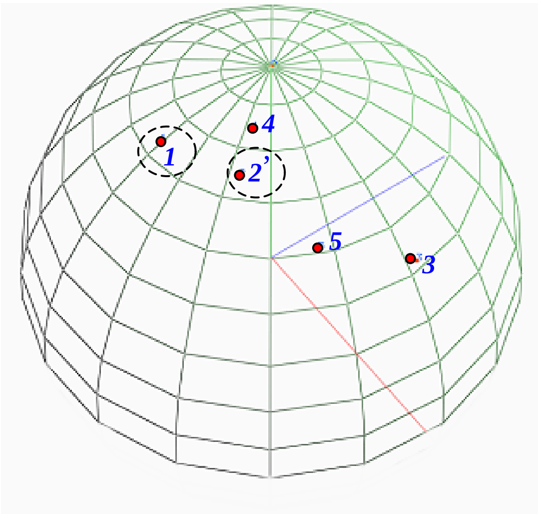
The moving trajectories of concurrent points A_3 and A_4 for the whole processes are shown in Figure 5.7, where 3^i stands for concurrent point of A_3 after the i^{st} iteration. The most inconsistent even swaps the DM has made are at Iteration 2 and 5 because 3^2 and 3^5 are furthest away from the latitude formed by all 3^i . The DM can therefore examine the moving trajectories of A_3 and A_4 to discover and update these inconsistencies. The system may also warn the DM about these inconsistencies, thus to help the DM to update his preferences at the Iteration 2 and 5.



Criteria \ Alternative		Weight	A_4	A_2	A_2'
c_1	Commute (Mins)	0.2	25	25	25
c_2	Customer Access (%)	0.2	85	80	80
c_3	Office Services	0.2	A	B	B
c_4	Office Size (Square Feet)	0.263	950	700 →	850
c_5	Monthly Cost ($)	0.137	1900	1700 →	1900
	Score		6.71	5.42	5.23

(a) (b)

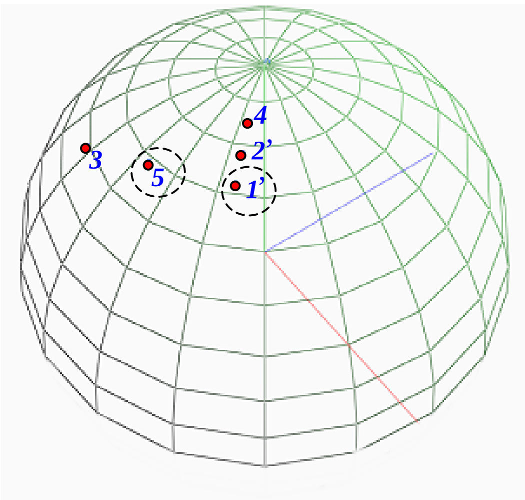
Figure 5.2 The decision ball and even swaps after Iteration 1. The shaded area is the swap inputted by the DM



Criteria \ Alternative		Weight	A_2'	A_1	A_1'
c_1	Commute (Mins)	0.200	25	45	45
c_2	Customer Access (%)	0.291	80	50 →	75
c_3	Office Services	0.109	B	A →	B
c_4	Office Size (Square Feet)	0.277	850	800 →	850
c_5	Monthly Cost ($)	0.123	1900	1850 →	1900
	Score		5.63	5.11	4.86

(a) (b)

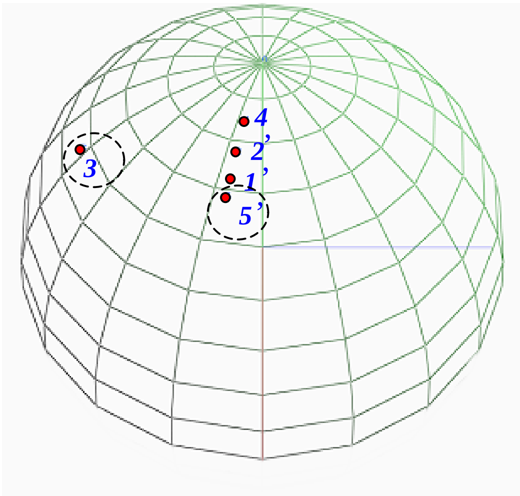
Figure 5.3 The decision ball and even swaps after Iteration 2



Criteria \ Alternative		Weight	A_1'	A_5	A_5'
c_1	Commute (Mins)	0.176	45	30 →	45
c_2	Customer Access (%)	0.291	75	75	75
c_3	Office Services	0.101	B	C →	B
c_4	Office Size (Square Feet)	0.301	850	700 →	800
c_5	Monthly Cost ($)	0.131	1900	1750 →	1950
	Score		4.91	4.48	4.49

(a) (b)

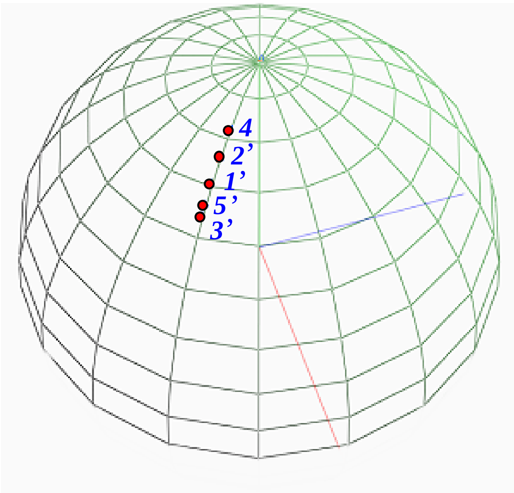
Figure 5.4 The decision ball and even swaps after Iteration 3



Criteria \ Alternative		Weight	A_5'	A_3	A_3'
c_1	Commute (Mins)	0.178	45	20	45
c_2	Customer Access (%)	0.291	75	70	70
c_3	Office Services	0.099	B	C	B
c_4	Office Size (Square Feet)	0.232	800	500	750
c_5	Monthly Cost ($)	0.200	1950	1500	1950
	Score		4.21	3.40	4.00

(a) **(b)**

Figure 5.5 The decision ball and even swaps after Iteration 4



Criteria \ Alternative		Weight	A_1'	A_2'	A_3'	A_4	A_5'
c_1	Commute (Mins)	0.178	45	25	45	25	45
c_2	Customer Access (%)	0.291	75	80	70	85	75
c_3	Office Services	0.099	B	B	B	A	B
c_4	Office Size (Square Feet)	0.232	850	850	750	950	800
c_5	Monthly Cost ($)	0.200	1900	1900	1950	1900	1950
	Score		4.68	5.35	4.00	6.24	4.21

(a) **(b)**

Figure 5.6 The final decision ball and consequences table

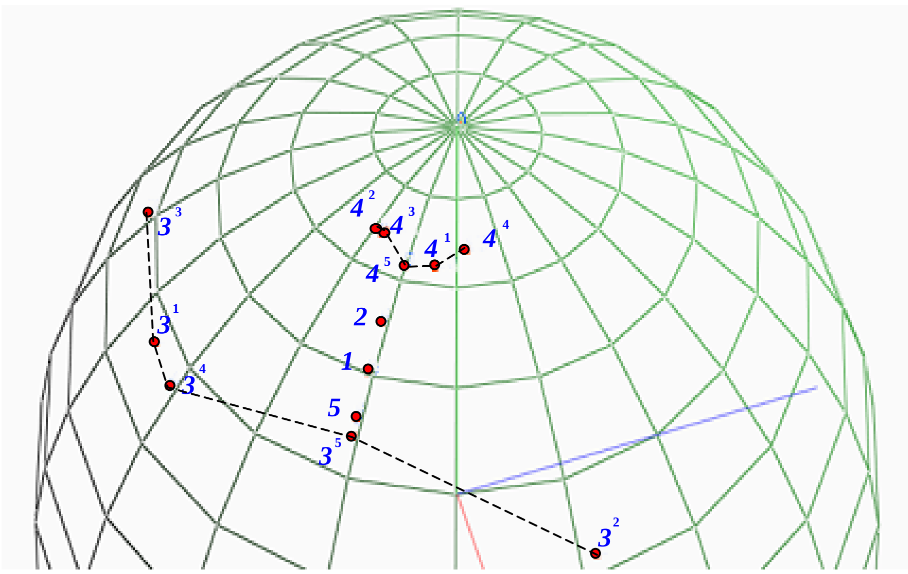


Figure 5.7 The moving trajectories of A_3 and A_4 after even swaps

5.4 Summary

Even Swap method is a straightforward process, which provides a useful way for making trades among criteria thus to assist the DM find out the best alternatives. However, only the best alternative can be found. The ranks of rest of alternatives are not known. In addition, there may exist large inconsistencies among even swaps that the DM could not know.

This study develops Even Swap Decision Ball models to visualize the Even Swap process via a Decision Ball. By mapping the alternatives into the points on the surface of this Decision Ball, the Even Swap processes are illustrated as the moving trajectories among related points. The DM can examine these trajectories to obtain intelligences below:

(i) To know the dissimilarities among alternatives. The longer the distance, the larger the dissimilarity.

(ii) To know the superiority (or dominance) relationship between alternatives by checking the longitude of alternatives. The alternatives, which are on the same longitude, exist dominance relationship.

(iii) To know the inconsistencies in decision processes by checking the latitude of alternatives. The even swap, which causes the alternative the furthest away from the latitude, is the most inconsistent one.

The proposed Decision Ball models can display alternatives on the same longitude and latitude of a sphere to indicate special relationships among alternatives, which are difficult to be plotted and examined by traditional 2-dimensional or 3-dimensional models.

Chapter 6　Model 3 : Pairwise Comparison Decision Ball Models

Ranking multiple alternatives with inconsistent preferences is one of the most important issues in decision science. This study proposes Pairwise Comparison Decision Ball models for Type III decision pattern to help the decision makers improve inconsistent preferences and rank alternatives. The decision makers are assumed to be capable of making pairwise comparisons between alternatives using score ratios.

After a decision maker specifies pairwise comparisons between alternatives, an Adjusting model will suggest options for adjusting the inconsistent judgments. These options are then illustrated on Gower Plots to aid in detecting the causes of any ordinal inconsistency. Following that, Decision Ball techniques are used to display the spatial distances among alternatives based on their dissimilarities.

By cycling through the above three steps iteratively, a decision maker can rank decision alternatives more confidently. Proposed approach can aid the decision maker detect and improve inconsistencies conveniently. In addition, incomplete preference matrix can also be treated.

The structure of this chapter is organized as follows. Section 6.1 briefly introduces the concept of pairwise comparisons. Section 6.2 forms Pairwise Comparison Decision Ball models. Section 6.3 uses two examples to demonstrate the decision processes. Summary of this chapter is made in Section 6.4.

6.1 Introduction to Pairwise Comparisons

The Analytic Hierarchy Process (Saaty, 1977) is a popular method for establishing priorities in multicriteria decision problems by evaluating the strength of individual preferences through the pairwise comparison of alternatives at each level of the hierarchy.

Let $\mathbf{A} = \{A_i \mid i = 1,\ldots,n\}$ be a set of n alternatives for solving a decision problem. Denote $r_{i,j}$ as $r_{i,j} = \frac{w_i}{w_j} \times e_{i,j}$, where $e_{i,j}$ is a multiplicative term accounting for inconsistencies. The ratio $\frac{w_i}{w_j}$ measures the relative dominance of A_i over A_j in terms of underlying priority weights $w_1 > 0$, …, $w_n > 0$, taken to sum up to one by convention. Following Saaty, it is convenient to let $\mathbf{R} = (r_{i,j})$, $i, j \in \{1,\ldots,n\}$, be an $n \times n$ preference matrix. It is assumed that $r_{i,j} = \frac{1}{r_{j,i}}$.

Several methods have been proposed (e.g., Saaty, 1977; Jesen, 1984; Genest and Rivest, 1994) to rank alternatives in AHP. The ranks they yield do not vary much when the decision makers' preferences are consistent. However, if a preference matrix is ordinally inconsistent or highly cardinally inconsistent, different ranking methods may produce wildly different priorities and rankings. Hence, how to help the decision makers detect and improve these inconsistencies becomes an important issue in pairwise comparison models.

Consider a set of decision alternatives $\boldsymbol{A} = \{A_1, A_2, \ldots, A_n\}$ for solving a problem, where each A_i contains m criteria $c_{i,1}$, …, $c_{i,m}$. Denote S_i as the score function of an alternative A_i. S_i can be either an additive function or a multiplicative function of $c_{i,k}$. Denote **C** = ($c_{i,k}$) as the criterion matrix of the decision problem. Assume a decision maker can tell which score function to choose, and can specify the ratio of the score of one alternative to another alternative in a pairwise fashion.

Denote **R** = ($r_{i,j}$) as a decision maker's preference matrix where $r_{i,j}$ is the ratio of S_i to S_j, $r_{i,j}$ = 1/$r_{j,i}$. If the decision maker is unclear about the ratio of S_i to S_j, $r_{i,j}$ is remained blank (denoted as $r_{i,j} = \phi$). **R** is ordinally inconsistent (intransitive) if for some $i, j, k \in$ {1, 2, 3, …, n} there exists $r_{i,j} > 1$, $r_{j,k} > 1$, but $r_{i,k} < 1$. **R** is cardinally inconsistent if for some $i, j, k \in$ {1, 2, 3, …, n} there exists $r_{i,k} \neq r_{i,j} \times r_{j,k}$ (Genest and Zhang, 1996). **R** is incomplete if there exists any $r_{i,j} = \phi$.

The problem in this study is as follows:

"Given a data set **C** = ($c_{i,k}$) and a preference matrix **R** = ($r_{i,j}$), how should one rank the decision alternatives A_1, A_2, …, A_n?"

If **R** is complete and ordinally consistent, all A_i can be ranked immediately; otherwise, **R** should be adjusted. This study develops three models to assist the decision maker adjust **R** = ($r_{i,j}$) and rank A_1, A_2, …, A_n, as follows: (Figure 6.1)

(i) The first model is an Adjusting model used to convert **R** = ($r_{i,j}$) into some new complete matrix $\mathbf{R}' = (r'_{i,j})$. It also provides some options as to how to adjust $r'_{i,j}$.

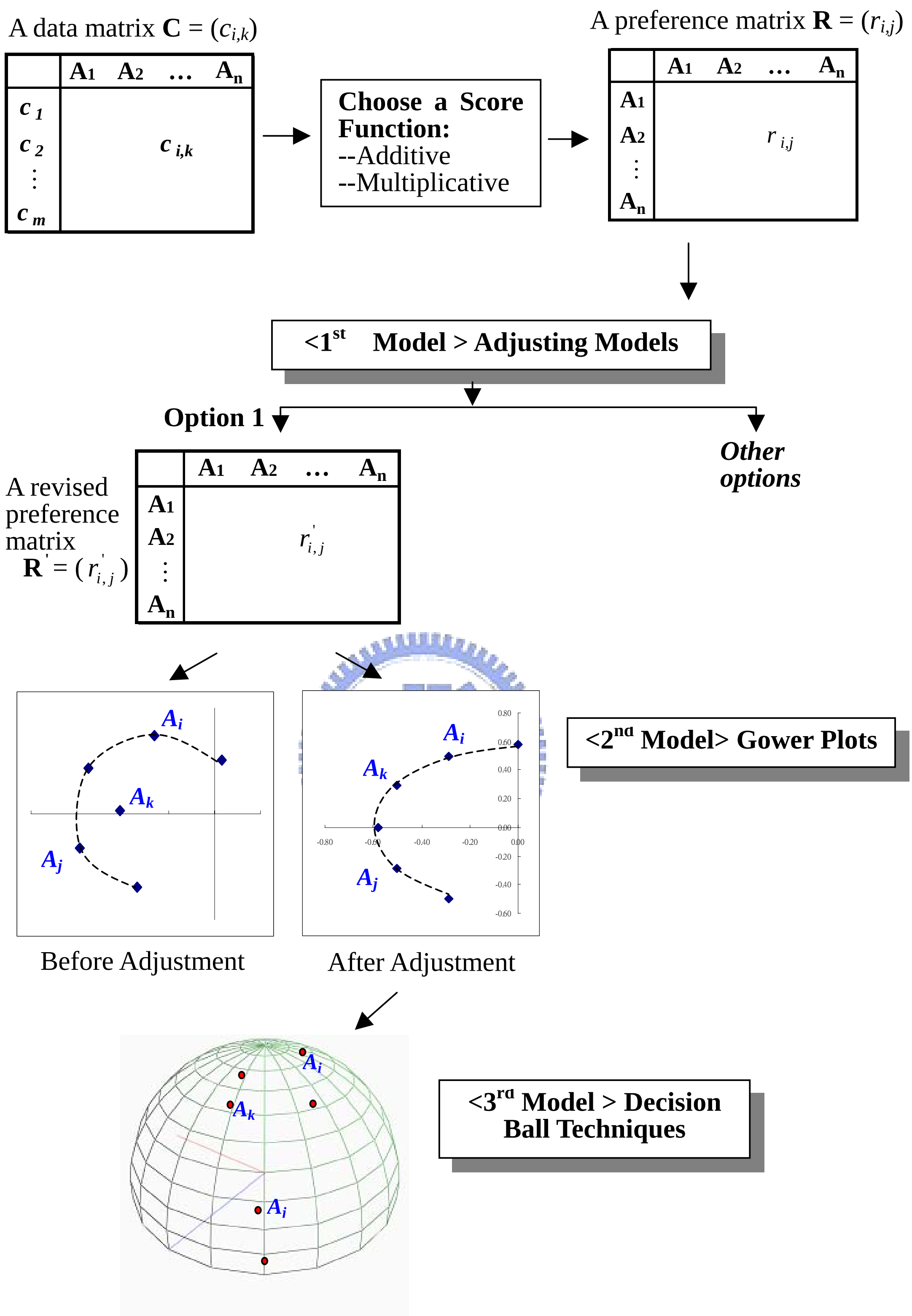


Figure 6.1 Solution procedure of Pairwise Comparison Decision Ball models

(ii) The second model is a Gower Plot model originally developed by Gower (1977) and Genest and Zhang (1996). Following an option provided by the first model (i.e., $\mathbf{R}'$), a Gower Plot model could graphically detect the decision alternative(s) which violates ordinal consistency.

(iii) The third model is a Decision Ball model. The Decision Ball not only shows the ranks of alternatives but displays the spatial distances associated with the dissimilarities between alternatives following the revised preference matrix $\mathbf{R}'$.

Through the iterative operations of (i), (ii) and (iii) the decision maker can finally rank alternatives more confidently.

6.2 Construction of Pairwise Comparison Decision Ball Models

This section develops a systematical approach for ranking and displaying alternatives. The approach includes three models: the Adjusting model, the Gower Plot model, and the Decision Ball model.

Given a $\mathbf{C} = (c_{i,k})$ and a $\mathbf{R} = (r_{i,j})$, where $\mathbf{R}$ may be incomplete or inconsistent, a model of adjusting $\mathbf{R}$ with addition score functions is formulated below:

Model 6.1 (Adjusting model – Additive score functions)

$$\underset{\{w_k\}}{\text{Min}} \quad M \times \mathbb{Z}_1 + \mathbb{Z}_2$$

$$\mathbb{Z}_1 = \sum_{i=1}^{n} \sum_{j>i}^{n} u_{i,j}$$

$$\mathbb{Z}_2 = \sum_{i=1}^{n} \sum_{j>i}^{n} \alpha_{i,j}$$

s.t. $$\left(\frac{S_i}{S_j} - 1\right) \times (r_{i,j} - 1) + M \times u_{i,j} \ge \varepsilon, \text{ for all } i, j \text{ where } r_{i,j} \ne \phi \text{ and } r_{i,j} \ne 1, \quad (6.1)$$

$$-\left|S_i - S_j\right| + M \times u_{i,j} \ge 0, \text{ for all } i,j \text{ where } r_{i,j} = 1, \quad (6.2)$$

$$\left|\frac{S_i}{S_j} - r_{i,j}\right| \le \alpha_{i,j}, \ \forall i, j, \quad (6.3)$$

$$S_i(\mathbf{w}) = \sum_{k=1}^{m} w_k \frac{c_{i,k} - \underline{c_k}}{\overline{c_k} - \underline{c_k}}, \quad (6.4)$$

$$\sum_{k=1}^{m} w_k = 1, \quad (6.5)$$

$$\underline{w_k} \le w_k \le \overline{w_k}, \ \forall k, \quad (6.6)$$

$$u_{i,j} \in \{0,1\}, \ M \text{ is a large value, } \varepsilon \text{ is a tolerable error.} \quad (6.7)$$

The variables in Model 6.1 are (w_1, w_2, …, w_m). The first objective ($\mathbb{Z}_1$) of Model 6.1 is to achieve ordinal consistency by minimizing the number of preferences (i.e., $r_{i,j}$) being revised. The elements of matrix **U**, $u_{i,j}$, are binary variables, i.e. $u_{i,j}$ = 1 if $r_{i,j}$ is reversed, and otherwise $u_{i,j}$ = 0. Constraint (6.1) means: when $r_{i,j} \ne \phi$ and $r_{i,j} \ne 1$, $u_{i,j}$ = 0, if (i) $(\frac{S_i}{S_j} > 1)$ and $(r_{i,j} > 1)$ (ii) $(\frac{S_i}{S_j} < 1)$ and $(r_{i,j} < 1)$; and otherwise $u_{i,j}$ = 1. A tolerable positive number ε is used to avoid $\frac{S_i}{S_j} = 1$. Constraint (6.2) means: when $r_{i,j}$ = 1, $u_{i,j}$ = 0 if $S_i = S_j$; and otherwise $u_{i,j}$ = 1. The second objective ($\mathbb{Z}_2$) is to achieve cardinal consistency by minimizing the $\alpha_{i,j}$ values, i.e. to minimize the difference between $\frac{S_i}{S_j}$ and $r_{i,j}$. Since ordinal consistency ($\mathbb{Z}_1$) is more important than cardinal consistency ($\mathbb{Z}_2$), $\mathbb{Z}_1$ is multiplied by a large value M in the objective function. Constraints (6.4) and (6.5) come from Notation 3.1. Constraint (6.6) sets the upper and lower bound of weights.

Model 6.1 is a nonlinear model, which can be converted into the following linear mixed 0-1 program:

$$\underset{\{w_k\}}{\text{Min}} \quad M \times \mathbb{Z}_1 + \mathbb{Z}_2$$

$$\mathbb{Z}_1 = \sum_{i=1}^{n} \sum_{j>i}^{n} u_{i,j}$$

$$\mathbb{Z}_2 = \sum_{i=1}^{n} \sum_{j>i}^{n} \alpha_{i,j}$$

s.t. $(S_i - S_j) \times (r_{i,j} - 1) + M \times u_{i,j} \geq \varepsilon$, for all i, j where $r_{i,j} \neq \phi$ and $r_{i,j} \neq 1$, (6.8)

$-M \times u_{i,j} \leq S_i - S_j \leq M \times u_{i,j}$, for all i,j where $r_{i,j} = 1$, (6.9)

$S_j \times r_{i,j} - \alpha_{i,j} \leq S_i \leq S_j \times r_{i,j} + \alpha_{i,j}$, $\forall i, j$, (6.10)

(6.4) ~ (6.7).

Where (6.8), (6.9) and (6.10) are converted from (6.1), (6.2) and (6.3) respectively.

After the weight vector, (w_1, w_2, …, w_n), is found, $S_i(\mathbf{w}) = \sum_{k=1}^{m} w_k \frac{c_{i,k} - \underline{c_k}}{\overline{c_k} - \underline{c_k}}$ can be calculated and a complete matrix can be obtained as

$$\mathbf{R}' = (r'_{i,j}), \tag{6.11}$$

where $r'_{i,j} = \frac{S_i}{S_j}$ if $r_{i,j} = \phi$ or $u_{i,j} = 1$; otherwise, $r'_{i,j} = r_{i,j}$.

The Adjusting model with multiplicative score functions is formulated as follows.

Model 6.2 (Adjusting Model –Multiplicative score functions)

$$\underset{\{w_k\}}{\text{Min}} \quad M \times \mathbb{Z}_1 + \mathbb{Z}_2$$

$$\mathbb{Z}_1 = \sum_{i=1}^{n}\sum_{j>i}^{n} u_{i,j}$$

$$\mathbb{Z}_2 = \sum_{i=1}^{n}\sum_{j>i}^{n} \alpha_{i,j}$$

s.t. $$\ln(\frac{S_i}{S_j}) \times \ln(r_{i,j}) + M \times u_{i,j} \geq \varepsilon, \text{ for all } i, j \text{ where } r_{i,j} \neq \phi \text{ and } r_{i,j} \neq 1, \quad (6.12)$$

$$-\left|\ln \frac{S_i}{S_j}\right| + M \times u_{i,j} \geq 0, \quad \text{for all } i,j \text{ where } r_{i,j} = 1, \quad (6.13)$$

$$\left|\ln(\frac{S_i}{S_j} e_{i,j}) - \ln(r_{i,j})\right| \leq \alpha_{i,j}, \quad \forall i, j, \quad (6.14)$$

and (6.5)~(6.7).

Constraints (6.12), (6.13), and (6.14) correspond to constraints (6.1), (6.2) and (6.3) respectively. Model 6.2 can be linearized as follows.

$$\underset{\{w_k\}}{\text{Min}} \quad M \times \mathbb{Z}_1 + \mathbb{Z}_2$$

$$\mathbb{Z}_1 = \sum_{i=1}^{n}\sum_{j>i}^{n} u_{i,j}$$

$$\mathbb{Z}_2 = \sum_{i=1}^{n}\sum_{j>i}^{n} \alpha_{i,j}$$

s.t. $$(\sum_{k=1}^{m} w_k \ln(c_{i,k}) - \sum_{k=1}^{m} w_k \ln(c_{j,k})) \times \ln(r_{i,j}) + M \times u_{i,j} \geq \varepsilon,$$

$$\text{for all } i, j \text{ where } r_{i,j} \neq \phi \text{ and } r_{i,j} \neq 1, \quad (6.15)$$

$$-M \times u_{i,j} \leq \sum_{k=1}^{m} w_k \ln(c_{i,k}) - \sum_{k=1}^{m} w_k \ln(c_{j,k}) \leq M \times u_{i,j}, \ \forall r_{i,j} = 1, \ \forall i, j, \quad (6.16)$$

$$\left|\sum_{k=1}^{m} w_k \ln(c_{i,k}) - \sum_{k=1}^{m} w_k \ln(c_{j,k}) - \ln(r_{i,j}) + e'_{i,j}\right| \leq \alpha_{i,j}, \ \forall i, j, \quad (6.17)$$

(6.5)~(6.7),

where $e_{i,j} = \exp(e'_{i,j})$.

After a complete matrix $\mathbf{R}'$ is obtained, the ordinal Gower Plots can be used to aid in detecting the causes of any ordinal inconsistency. The concept of Gower Plots refers to Section 2.2. The Gower Plot model with a multiplicative score function is the same as that with an additive score function.

At last, Decision Ball techniques, as proposed in Section 3.3, with additive or multiplicative score functions are adopted to display ranks and similarities among alternatives.

The solution processes are shown in Figure 6.1 and illustrated below:

Step 1 The decision maker specifies a data matrix **C** = ($c_{i,k}$), chooses a type of score function, and inputs a preference matrix **R** = ($r_{i,j}$), where **R** can be an incomplete matrix.

Step 2 Applying Model 6.1 (if an additive score function is selected) or Model 6.2 (if a multiplicative score function is chosen) to the data and preference matrix yields a set of weights **w**, and a revised preference matrix $\mathbf{R}' = (r'_{i,j})$, where $\mathbf{R}'$ is a complete matrix.

Step 3 Applying Gower Plot model to $\mathbf{R}'$, the ordinal Gower Plots before and after adjustment are displayed based on $\mathbf{R}'$.

Step 4 Based on the weights **w** obtained in Step 2, the score of alternatives $S_i(\mathbf{w})$ and dissimilarities $\delta_{i,j}(\mathbf{w})$ among alternatives are calculated.

Step 5 Applying Decision Ball techniques (Model 3.1 if an additive score function is chosen; or Model 3.2 if a multiplicative score function is chosen) to $S_i(\mathbf{w})$ and $\delta_{i,j}(\mathbf{w})$ yields the coordinates (x_i, y_i, z_i) of alternatives on the Decision Ball. The Decision Ball is then displayed to the decision maker.

Step 6 The decision maker can observe the ranks and similarities among alternatives on the ball. Other options can be obtained through Step 3 to Step 5 by setting some $u_{i,j} = 0$. The decision maker can also adjust preferences in **R** (Step 1) or $\mathbf{R}'$ (Step 2) directly based on the information provided by Gower Plots and Decision Balls, and observe the corresponding changes.

6.3 Illustrative Examples

Two examples are used to illustrate the Decision Maker's problem solving processes. For simplicity, only additive score functions are illustrated here.

<Example 6.1> Investment in Mutual Funds

The first example is about an investor who would like to invest in mutual funds. The investor has four major decision criteria to fulfill: (c_1) a high total return, (c_2) large fund size (economies of scale), (c_3) low risk (β : Beta), and (c_4) low turnover. Six alternatives (A_1, **…,** A_6) are under considerations as listed in the $\mathbf{C_1}$ in Figure 6.2, where c_3 and c_4 are cost criteria. Suppose the investor chooses to use an additive score function and specifies an incomplete preference matrix $\mathbf{R_1} = (r_{i,j})$, where $r_{1,6}$, $r_{2,3}$, and $r_{3,6}$ are left blank because it is difficult for the investor to make comparisons between these alternatives. The data set, preference matrix, and the solving process are depicted in Figure 6.2. (Here $\mathbf{R_1}$ can be checked as ordinally inconsistent since $r_{1,2} < 1$ and $r_{2,4} < 1$ and but $r_{1,4} > 1$.)

Applying Model 6.1 to $\mathbf{C_1}$ and $\mathbf{R_1}$ yields the solution as $\mathbb{Z}_1 = 1$, $\mathbb{Z}_2 = 3.64$, $u_{1,4} = 1$, $(w_1, w_2, w_3, w_4) = (0.38, 0.19, 0.05, 0.38)$, and $(S_1, S_2, S_3, S_4, S_5, S_6) = (0.56, 0.65, 0.41, 0.82, 0.58, 0.35)$. The values of unspecified preferences can be computed as $r_{1,6} = \frac{S_1}{S_6} = 1.59$, $r_{2,3} = \frac{S_2}{S_3} = 1.58$, and $r_{3,6} = \frac{S_3}{S_6} = 1.17$. The system's results suggest reversing $r_{1,4}$ from 2 to 0.68 ($\frac{S_1}{S_4}$) to minimize both ordinal and cardinal inconsistencies. This is regarded as Option 1 of adjusting the preferences. The ordinal Gower Plots, with $r_{1,4} > 1$ and $r_{1,4} < 1$, are also depicted in Figure 6.2. Examining the Gower Plots before reversal (i.e., when $r_{1,4} > 1$), the preference matrix is ordinally inconsistent because A_1 and A_4 lie off the half circle, which implies A_1 and A_4 are the alternative causing major ordinal inconsistencies. By following the suggestion of revising $r_{1,4}$ as $r_{1,4} < 1$, the Gower Plot will show the preference matrix is ordinally consistent and the alternatives will be ranked as $A_4 \succ A_2 \succ A_5 \succ A_1 \succ A_3 \succ A_6$.

Applying a Decision Ball technique (Model 3.1) based on the results of Model 6.1 ($S_i(\mathbf{w})$ and $\delta_{i,j}(\mathbf{w})$) yield a set of coordinates for $A_1, \ldots, A_6$, with Stress = 6.9%. The corresponding Decision Ball is shown on the left bottom of Figure 6.2. Examining the Decision Ball, the investor can observe that (i) if he reverses $r_{1,4}$ from larger than 1 to smaller than 1, the ranks of alternatives is $A_4 \succ A_2 \succ A_5 \succ A_1 \succ A_3 \succ A_6$ from top to down along a latitudinal line (ii) A_4, A_2 and A_5 have higher similarities because they are close to each other. For diversifying the investment, the investor may avoid investing A_4, A_2 and A_5 simultaneously.

If the investor does not like to reverse $r_{1,4}$, another option can be generated by setting $u_{1,4} = 0$ in Model 6.1. Applying Model 6.1 (with a new constraint $u_{1,4} = 0$) again yields Option 2 where $\mathbb{Z}_1 = 2$, $\mathbb{Z}_2 = 3.57$, $u_{1,2} = 1$, $u_{1,5} = 1$, $(w_1, w_2, w_3, w_4) = (0.63, 0.10, 0.05, 0.22)$, $(S_1, S_2, S_3, S_4, S_5, S_6) = (0.73, 0.70, 0.26, 0.72, 0.53, 0.40)$. The corresponding Gower Plots and Decision Ball (with Stress 5.9%) are shown on the right bottom of Figure 6.2.

If the investor does not like Option 1 and Option 2, he may modify $\mathbf{R_1}$ (or $\mathbf{R}_1^{'}$) directly (Option 3) based on the information provided on the Gower Plot about the alternative(s) causing major inconsistencies, or based on the information provided on the Decision Ball about the scores and similarities among alternatives.

C_1

$c_{k,i}$	A_1	A_2	A_3	A_4	A_5	A_6
c_{1}: Return	67.76	57.82	20.56	48.30	43.45	42.12
c_{2}: Size(million)	1,102	1,100	782	2,215	1,522	1,825
c_{3}: Beta	1.43	1.41	0.85	1.08	1.45	0.99
c_{4}: Turnover	305.26	180.97	139.2	127.94	187.48	402.42

R_1

$r_{i,j}$	A_1	A_2	A_3	A_4	A_5	A_6
A_1		0.50	2.00	2.00	0.50	
A_2				0.50	2.00	3.00
A_3				0.33	0.25	
A_4					1.60	2.00
A_5						2.00
A_6						

Option 1
Reverse $r_{1,4}$
(suggested by the System)

Option 2
Revere $r_{1,2}$ and $r_{1,5}$
(suggested by the System)

Option 3
Direct adjustment
(by the DM)

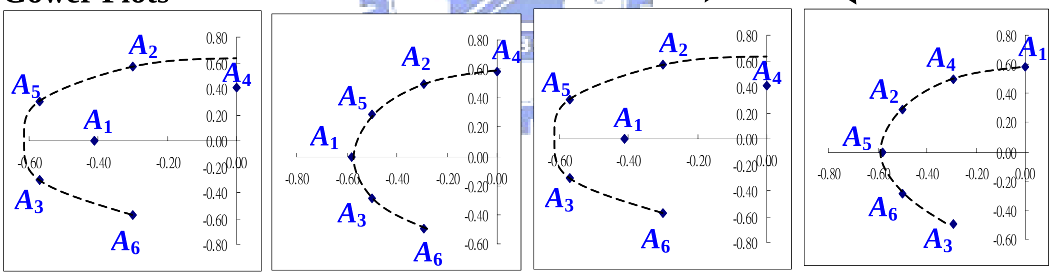
R_1'

$r_{i,j}$	A_1	A_2	A_3	A_4	A_5	A_6
A_1		0.50	2.00	~~2~~ 0.68	0.50	1.59
A_2			1.58	0.50	2.00	3.00
A_3				0.33	0.25	1.17
A_4					1.60	2.00
A_5						2.00
A_6						

$r_{i,j}$	A_1	A_2	A_3	A_4	A_5	A_6
A_1		1.04	2.00	2.00	1.38	1.84
A_2			2.65	0.50	2.00	3.00
A_3				0.33	0.25	0.67
A_4					1.60	2.00
A_5						2.00
A_6						

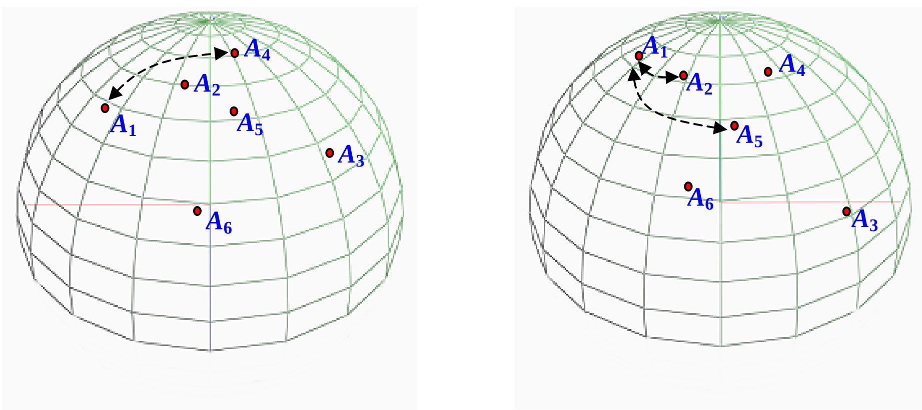
$r_{1,4}>1$ $r_{1,4}<1$ $r_{1,2}<1$ & $r_{1,5}<1$ $r_{1,2}>1$ & $r_{1,5}>1$

Gower Plots



(a) (b) (c) (d)

Decision Balls



ball (b) ball (d)

Figure 6.2 Decision Process of Example 6.1

<Example 2> Selection of universities

Consider a student who needs to enroll in a university. He would like to choose from a list of eight candidate universities. The student sets four criteria for choosing a university: (c_1) rich campus life estimated by size, (c_2) high average salary after graduation, (c_3) high entrance score, and (c_4) low tuition. Low tuition, c_4, is considered to be a cost criterion. An additive score function is used to rank the alternatives. The data set $\mathbf{C_2}$ and an incomplete preference matrix $\mathbf{R_2}$ are listed as in Figure 6.3.

After applying Model 6.1 to the example, three possible ordinally consistent solutions are found. Those are, $u_{1,7}$ = 1 (Option 1), $u_{3,7}$ = 1 (Option 2), and $u_{1,3}$ = 1 (Option 3). The corresponding Gower Plots and Decision Balls (with Stress 7.2%, 6.7%, and 4.5% respectively) are depicted in Figure 6.3.

Option 1 yields $\mathbb{Z}_1 = 1$, $\mathbb{Z}_2 = 3.51$, $u_{1,7} = 1$, $(w_1, w_2, w_3, w_4) = (0.31, 0.59, 0.05, 0.05)$, $(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8) = (0.62, 0.37, 0.52, 0.12, 0.20, 0.34, 0.45, 0.75)$. Examining Gower Plot (a) where $r_{1,7} < 1$ to know it is ordinally inconsistent because there are some angles between consecutive points not equal to 180/n degrees. Alternatives A_1, A_3, and A_7 are the ordinally inconsistent alternatives. Reversing $r_{1,7}$ as $r_{1,7} > 1$ (means A_1 better than A_7) generates an ordianlly consistent situation with $A_8 \succ A_1 \succ A_3 \succ A_7 \succ A_2 \succ A_6 \succ A_5 \succ A_4$. The related Decision Ball (b) illustrates that considering A_8, A_1, and A_3, $A_8 \succ A_1 \succ A_3$. However, A_3

is more similar to A_8. Therefore, if the student is not accepted by A_8, A_3 may be a better choice than A_1.

Suppose the student chooses to reverse $r_{3,7}$ as $r_{3,7} < 1$ (means A_7 better than A_3) (Option 2), the related Decision Ball (d) illustrates the ranks of alternatives are $A_8 \succ A_7 \succ A_1 \succ A_3 \succ A_2 \succ A_6 \succ A_5 \succ A_4$. A_1 and A_7 are very close. Thus, if university A_8 is impossible to candidate for enrollment then A_1 as well as A_7 could be a good choice.

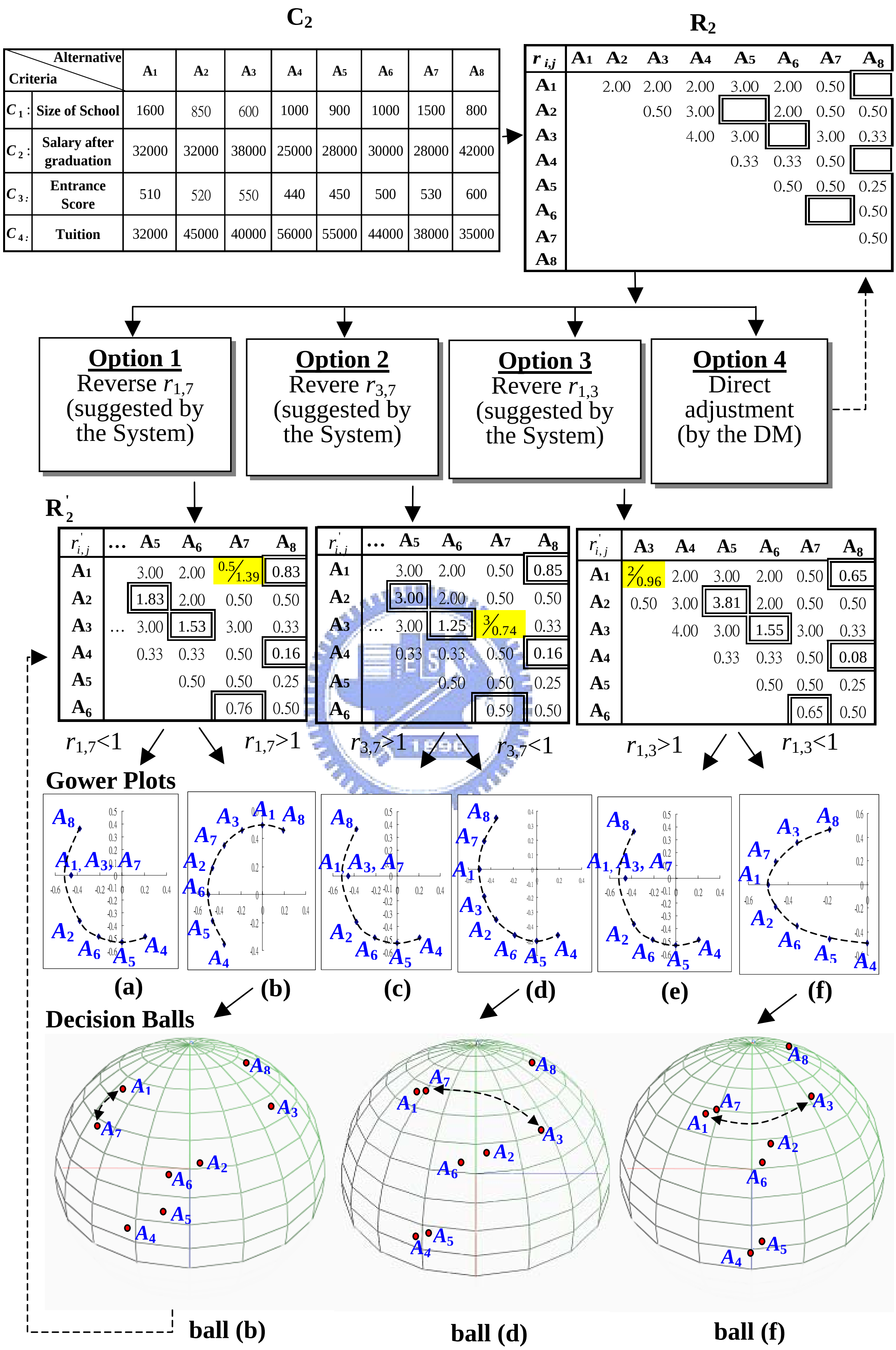


Figure 6.3 Decision Process of Example 6.2

6.4 Summary

This study develops Pairwise Comparison Decision Ball models to help a decision maker improve the quality of decision-making by iteratively reducing ordinal and cardinal inconsistencies. Gower Plots are adopted to detect the alternatives causing ordinal inconsistencies. An optimization model is then developed to provide suggestions for adjusting these inconsistencies conveniently.

Decision Ball techniques are used to provide a useful visual representation of ranks and similarities among alternatives, which are more flexible and easier to observe than traditional 2-dimensional plane and 3-dimensional cube models respectively. In addition, suggestions about how to improve inconsistencies are also shown on the Decision Ball. The decision maker can observe the suggested solutions and choose the most acceptable change to reduce inconsistencies and to rank alternatives more confidently.

The proposed approach assists a decision maker make a more reliable decision by improving inconsistencies. The improvements in inconsistency can be measured by the consistency ratio (CR)(Saaty, 1980), which is briefly illustrated as follows. Given a matrix **R** of rank n, the *consistency index* (*CI*) is first calculated to measure the deviation from a consistent matrix:

$$CI = (\lambda_{\max} - n)/(n-1),$$

where $\lambda_{\max}$ is the maximal eigenvalue of **R**. Then, the *consistency ratio* (*CR*) is computed as the ratio of the *CI* to the so-called *random index* (*RI*) which is a *CI* of randomly generated matrices:

$$CR = CI/RI.$$

The CR = 0 indicates perfect consistency. The CR before and after adjustments for Example 6.1 and 6.2 are listed in Table 6.1. The CR in both examples can be improved significantly. For instance, in Option 1 of Example 6.1, the CR can be significantly improved from 0.087 to 0.047. In Option 1 of Example 6.2, the CR can be improved from 0.064 to 0.049.

Table 6.1 Improvements in inconsistency measured by consistency ratio (*CR*)

Examples	Options	*CR* before Adjustment	*CR* after Adjustment
Example 6.1 Investment in Mutual Funds	Option 1	0.087	0.047
	Option 2	0.078	0.053
Example 6.2 Selection of Universities	Option 1	0.064	0.049
	Option 2	0.070	0.053
	Option 3	0.060	0.055

Chapter 7 Model 4 : Classification Decision Ball Models

This chapter presents Model 4 – Classification Decision Ball models for Type IV decision pattern. The major decision problem solved here is to classify alternatives. That is, the decision makers are interested in the grouping relationships of alternatives more than individual rank of alternatives. In this decision pattern, no personal preferences are taken into account.

Sueyoshi (1999) first proposed a DEA-DA analysis incorporating the non-parametric feature of DEA (Data Envelopment Analysis) into the DA (Discriminant Analysis). However, previous DEA-DA methods cannot display the relationships among alternatives. This study develops Classification Decision Ball models to visualize the grouping results and relationships among alternatives.

The structure of this chapter is organized as follows. Section 7.1 briefly introduces the DEA-DA analysis. Section 7.2 constructs Classification Decision Ball models. Section 7.3 uses two examples to demonstrate the classification processes. Summary of this chapter is made in Section 7.4.

7.1 Introduction to DEA-DA Analysis

Discriminant Analysis (DA) is a statistical technique for predicting group membership.

The GP (Goal Programming)-based DA, first proposed by Freed and Glover (1981), can estimate weights of criteria by minimizing sum of deviations (MSD, Freed and Glover, 1986) or minimizing misclassified alternatives (MMO, Banks and Abad, 1991). Those weights yield an evaluation score, which is compared with a threshold value for classifying alternatives.

The Data Envelopment Analysis (DEA) first proposed by Charnes et al. (1978) is a popular technique for evaluating efficiency of Decision Making Units (DMUs). Sueyoshi (1999) first proposed a DEA-DA analysis incorporating the non-parametric feature of DEA into the DA. He proposed a two-stage DEA-DA analysis: first identifies the existence of an overlap between two groups, and then determines the group membership of an alternative. Sueyoshi (2001) extended DEA-DA analysis, known as extended DEA-DA analysis, which can deal with the negative value in data. Both approaches estimate weights by minimizing the sum of deviation of misclassified alternatives. However, the classification performance is not good enough by both methods.

In order to improve hit rates, Sueyoshi (2004a) developed a mixed integer programming (MIP) approach, referred as two-stage MIP approach, which estimates weights by minimizing the total number of misclassified alternatives. Sueyoshi (2004b) and Sueyoshi and Hwang (2004) dropped the first stage of the two-stage MIP approach, called standard MIP approach, to simplify the estimation process of the two-stage MIP approach. Both MIP approaches can efficiently improve hit rates.

This section briefly illustrates the formulations of standard and two-stage MIP approaches. Let **A** = (A_1, A_2, …, A_n) be a set of n alternatives for solving a classification problem, where each alternative contains m independent factors. An alternative A_i is expressed as A_i ($c_{i,1}$, …, $c_{i,m}$). Each alternative belongs to one of two groups ($\mathbf{G_1}$ and $\mathbf{G_2}$), and the group membership is required to be known before computation. Denote t_i as a binary variable, which is used to indicate whether A_i is correctly classified or not. If A_i is correctly classified, then t_i = 0; otherwise, t_i = 1. The standard MIP approach is formulated as follows:

Standard MIP approach (Sueyoshi, 2004b; Sueyoshi and Hwang, 2004)

$$\text{Min} \quad \sum_{i\in G_1} t_i + \sum_{i\in G_2} t_i \tag{7.1}$$

$$\text{s.t.} \quad \sum_{k=1}^{m} w_k c_{i,k} - d + Mt_i \geq 0, \quad \forall i \in \mathbf{G_1}, \tag{7.2}$$

$$\sum_{k=1}^{m} w_k c_{i,k} - d - Mt_i \leq -\varepsilon, \quad \forall i \in \mathbf{G_2}, \tag{7.3}$$

$$\sum_{k=1}^{m} |w_k| = 1, \tag{7.4}$$

w_k, d: unrestricted, t_i = 0/1.

M and ε are given large and small numbers. The objective is to minimize the total number of incorrect classifications. d is the discriminant score and $\sum_{k=1}^{m} w_k c_{i,k}$ represents the discriminant function. w_k is the weight of factor k, which is unrestricted in sign. Because $|w_k|$ is a nonlinear term, the whole model can be reformulated as following linear one (Sueyoshi, 2004b)

$$\text{Min} \quad \sum_{i \in G_1} t_i + \sum_{i \in G_2} t_i \tag{7.5}$$

$$\text{s.t.} \quad \sum_{k=1}^{m} (w_k^+ - w_k^-) c_{i,k} - d + M t_i \geq 0, \quad \forall i \in \mathbf{G_1}, \tag{7.6}$$

$$\sum_{k=1}^{m} (w_k^+ - w_k^-) c_{i,k} - d - M t_i \leq -\varepsilon, \quad \forall i \in \mathbf{G_2}, \tag{7.7}$$

$$\sum_{k=1}^{m} (w_k^+ + w_k^-) = 1, \tag{7.8}$$

$$\delta_k^+ \geq w_k^+ \geq \varepsilon \delta_k^+ \text{ and } \delta_k^- \geq w_k^- \geq \varepsilon \delta_k^-, \quad k = 1, \ldots, m, \tag{7.9}$$

$$\delta_k^+ + \delta_k^- \leq 1, \quad k = 1, \ldots, m, \tag{7.10}$$

$$\sum_{k=1}^{m} (\delta_k^+ + \delta_k^-) = m, \tag{7.11}$$

d: unrestricted, t_i = 0/1, δ_k^+ = 0/1, δ_k^- = 0/1, and all other variables ≥ 0.

Each weight w_k is separated as ($w_k^+ - w_k^-$), where $w_k^+ = (|w_k| + w_k)/2$ and $w_k^- = (|w_k| - w_k)/2$. Based on Glen (1999), two binary variables δ_k^+ and δ_k^- are incorporated into (7.9) ~ (7.11). Denote d^* and w_k^* as the optimal solutions obtained from optimality of above model. A new alternative sample A_r can be classified by following rule: if $\sum_{k=1}^{m} w_k^* c_{r,k} \geq d^*$, then $A_r \in \mathbf{G_1}$; if $\sum_{k=1}^{m} w_k^* c_{r,k} \leq d^* - \varepsilon$, then $A_r \in \mathbf{G_2}$. The visual structure of the standard MIP approach is depicted in Figure 7.1.

In order to increase the number of correct classifications, Sueyoshi (2004a) proposed a two-stage MIP approach: the first stage is to identify and minimize the overlap distance; the second stage is to minimize the number of incorrect classifications. The two-stage MIP approach is formulated as follows.

Two-stage MIP approach (Sueyoshi, 2004a)

Stage 1 : Classification and overlap identification

$$\text{Min} \quad p \tag{7.12}$$

$$\text{s.t.} \quad \sum_{k=1}^{m}(w_k^+ - w_k^-)c_{i,k} - d + p \geq 0, \quad \forall i \in \mathbf{G_1}, \tag{7.13}$$

$$\sum_{k=1}^{m}(w_k^+ - w_k^-)c_{i,k} - d - p \leq 0, \quad \forall i \in \mathbf{G_2}, \tag{7.14}$$

(7.8) ~ (7.11),

d, p: unrestricted, δ_i^+ = 0/1, δ_i^- = 0/1, and all other variables ≥ 0.

Let $\mathbf{C_1} = \{i \in \mathbf{G_1} \mid \sum_{k=1}^{m} w_k^* c_{i,k} > d^* + p^*\}$, $\mathbf{C_2} = \{i \in \mathbf{G_2} \mid \sum_{k=1}^{m} w_k^* c_{i,k} < d^* - p^*\}$, $\mathbf{D_1} = \mathbf{G_1} - \mathbf{C_1}$ and $\mathbf{D_2} = \mathbf{G_2} - \mathbf{C_2}$. $p^* \geq 0$ indicates the existence of an overlap ($\mathbf{D_1} \cup \mathbf{D_2}$); otherwise, $p^* < 0$ indicates no overlap. A new sample A_r is classified by the following rule: if $A_r \in \mathbf{C_1}$, then $A_r \in \mathbf{G_1}$; if $A_r \in \mathbf{C_2}$, then $A_r \in \mathbf{G_2}$; otherwise, A_r belongs to an overlap. To handle overlap, the second stage is expressed as follows.

Stage 2 : Handling overlap

$$\text{Min} \quad \sum_{i \in D_1} t_i + \sum_{i \in D_2} t_i \tag{7.15}$$

$$\textit{s.t.} \quad \sum_{k=1}^{m}(w_k^+ - w_k^-)c_{i,k} - g + Mt_i \geq 0, \quad \forall i \in \mathbf{D_1}, \tag{7.16}$$

$$\sum_{k=1}^{m}(w_k^+ - w_k^-)c_{i,k} - g - Mt_i \leq -\varepsilon, \quad \forall i \in \mathbf{D_2}, \tag{7.17}$$

$$\sum_{k=1}^{m}(\delta_k^+ + \delta_k^-) = h, \tag{7.18}$$

(7.8) ~ (7.10),

where g: unrestricted, t_i = 0/1, δ_i^+ = 0/1, δ_i^- = 0/1, and all other variables ≥ 0. h is a

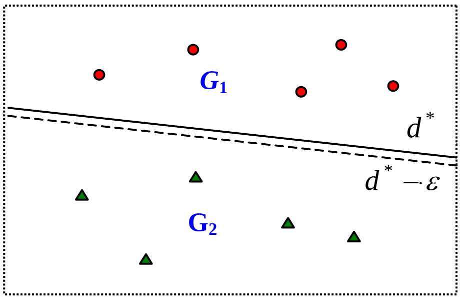


Figure 7.1 The visual structure of the standard MIP approach

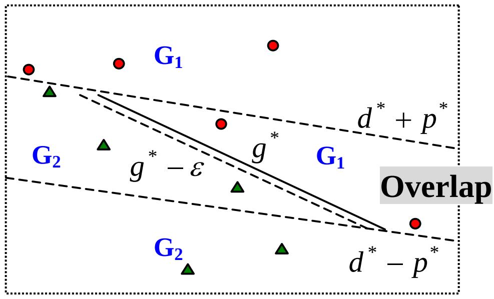


Figure 7.2 The visual structure of the two-stage MIP approach

prescribed positive number with $h \le m$.

The alternatives in the overlap are classified at the second stage as follows: if $\sum_{k=1}^{m} w_k^* c_{r,k} \ge g^*$, then $A_r \in \mathbf{G_1}$; if $\sum_{k=1}^{m} w_k^* c_{r,k} \le g^* - \varepsilon$, then $A_r \in \mathbf{G_2}$. The visual structure of the standard MIP approach is depicted in Figure 7.2.

Although the two-stage MIP approach has better classification performance than standard MIP approach, the former produces two sets of weights (at *Stage* 1 and 2 respectively), which are difficult to make comparisons among all alternatives. Besides, both approaches may result in multiple solutions. For instance, there may exist several sets of weight in the standard MIP approach to achieve the same optimal objective value in (7.5). Moreover, both approaches include too many binary variables.

Based on the Two-stage MIP approach, this study proposes Classification Decision Ball models to group alternatives by a multi-stage MIP approach. Decision Ball techniques are used to display alternatives on the surface of a ball, on which the decision maker can observe grouping relationships among alternatives layer by layer. In addition, the number of binary variables can also be reduced significantly.

7.2 Construction of Classification Decision Ball Models

The conceptual diagram of Classification Decision Ball models is shown in Figure 7.3. Alternatives are displayed on the surface of the Decision Ball. The alternatives with higher scores are located closer to the North Pole. The distance between two alternatives indicates the similarity between them: the higher the similarity, the shorter the distance.

The proposed Classification Decision Ball models, based on the concepts of the Two-stage MIP approach (Sueyoshi, 2004a), are a multi-stage MIP approach, which can be represented as multi-layer Decision Balls. As shown in Figure 7.3(a), the proposed approach first finds an overlap region on a first-layer Decision Ball, and then extends this overlap region to a second-layer Decision Ball as depicted in Figure 7.3(b). The cutting plane, which indicates a discriminant score in the overlap of the first layer Decision Ball (Figure 7.3(a)), is rotated into a horizontal plane in the second-layer (Figure 7.3 (b)).

The proposed multi-stage MIP approach includes two models, which are formulated as follows:

Model 7.1 (Standard Classification Model)

Min $\mathbb{Z} = \sum_{i \in \mathbf{G_1}} t_i + \sum_{i \in \mathbf{G_2}} t_i$ (7.19)

s.t. $\sum_{k=1}^{m} w_k c_{i,k} + Mt_i \geq g, \quad \forall i \in \mathbf{G_1}$, (7.20)

$\sum_{k=1}^{m} w_k c_{i,k} - Mt_i \leq g - \varepsilon, \quad \forall i \in \mathbf{G_2}$, (7.21)

$\sum_{k=1}^{m} (w_k + 2e_k) = 1$, (7.22)

$w_k + e_k \geq 0, \forall k$, (7.23)

$M(u_k - 1) \leq w_k \leq Mu_k - \varepsilon , \forall k$, (7.24)

$-w_k \leq e_k \leq -w_k + Mu_k , \forall k$, (7.25)

$e_k \leq M(1-u_k), \ \forall k$, (7.26)

g: unrestricted, u_k , t_i = 0/1, and all other variables ≥ 0 .

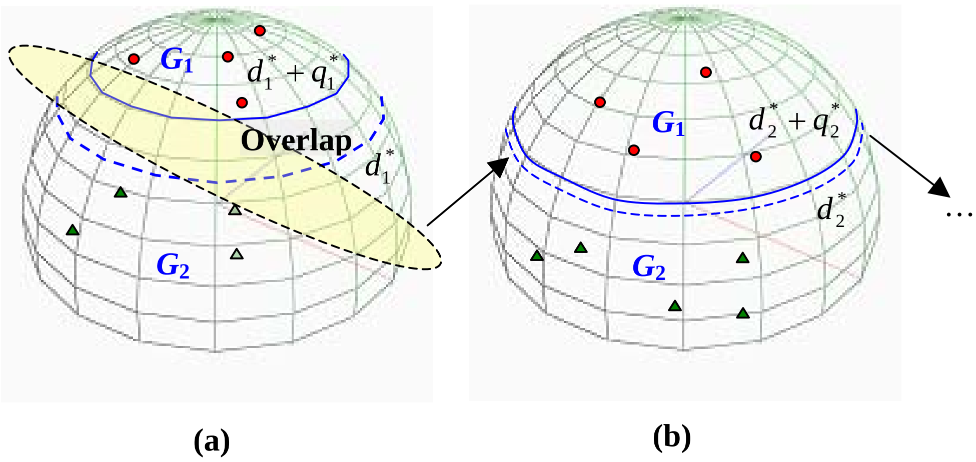


Figure 7.3 The conceptual diagram of the multi-layer Classification Decision Ball models (a) The first layer (b) The second layer

The objective of Model 7.1 is to minimize the total number of incorrect classifications. g is the discriminant score and $\sum_{k=1}^{m} w_k c_{i,k}$ represents the discriminant function. Model 7.1 is improved from the Standard MIP approach (Sueyoshi, 2004b) by reducing the number of binary variables used for $|w_k|$. Based on the study of Li (1996), we can let $|w_k| = w_k + 2e_k$. From expressions (7.22) ~ (7.26), if $w_k < 0$, then a binary variable $u_k = 0$ and $e_k = -w_k$, thus $|w_k| = w_k + 2e_k = -w_k$; otherwise, if $w_k \geq 0$, then $u_k = 1$ and $e_k = 0$, thus $|w_k| = w_k + 2e_k = w_k$. The total number of binary variables used for $|w_k|$ is just half of that in Sueyoshi (2004b).

<u>Model 7.2 (Overlap Identification Model)</u>

$$Min \quad p \tag{7.27}$$

$$s.t. \quad \sum_{k=1}^{m} w_k c_{i,k} \geq d, \quad \forall i \in \mathbf{G_1}, \tag{7.28}$$

$$\sum_{k=1}^{m} w_k c_{i,k} \leq d + p, \quad \forall i \in \mathbf{G_2}, \tag{7.29}$$

(7.22) ~ (7.26)

d, p: unrestricted.

Model 7.2 is improved from the Stage 1 of Two-stage MIP approach (Sueyoshi, 2004a) by reducing the number of binary variables used for $|w_k|$. The objective of this model is to minimize the overlap region between two groups. Let $\mathbf{C_1} = \{i \in \mathbf{G_1} \mid \sum_{k=1}^{m} w_k^* c_{i,k} > d^* + p^*\}$,

$\mathbf{C}_2 = \{i \in \mathbf{G}_2 \mid \sum_{k=1}^{m} w_k^* c_{i,k} < d^*\}$, $\mathbf{D}_1 = \mathbf{G}_1 - \mathbf{C}_1$ and $\mathbf{D}_2 = \mathbf{G}_2 - \mathbf{C}_2$. $p^* \geq 0$ indicates the existence of an overlap ($\mathbf{D}_1 \cup \mathbf{D}_2$); otherwise, $p^* < 0$ indicates no overlap.

The multi-stage classifying processes are shown in Figure 7.4. At the first stage, the decision maker inputs the data set and specifies the minimal hit rate, denoted as $\underline{h}$. Model 7.1 is applied to the data set first. If the hit rate is greater than $\underline{h}$, then the corresponding Decision Ball is displayed to the decision maker, and the processes are terminated. Otherwise, Model 7.2 is applied to the data set to find the overlap area. The corresponding Decision Ball with an overlap area is depicted. Based on the alternatives in the overlap areas, applying Model 7.1 again to find the new discriminate score and hit rate. Model 7.1 and 7.2 are applied iteratively

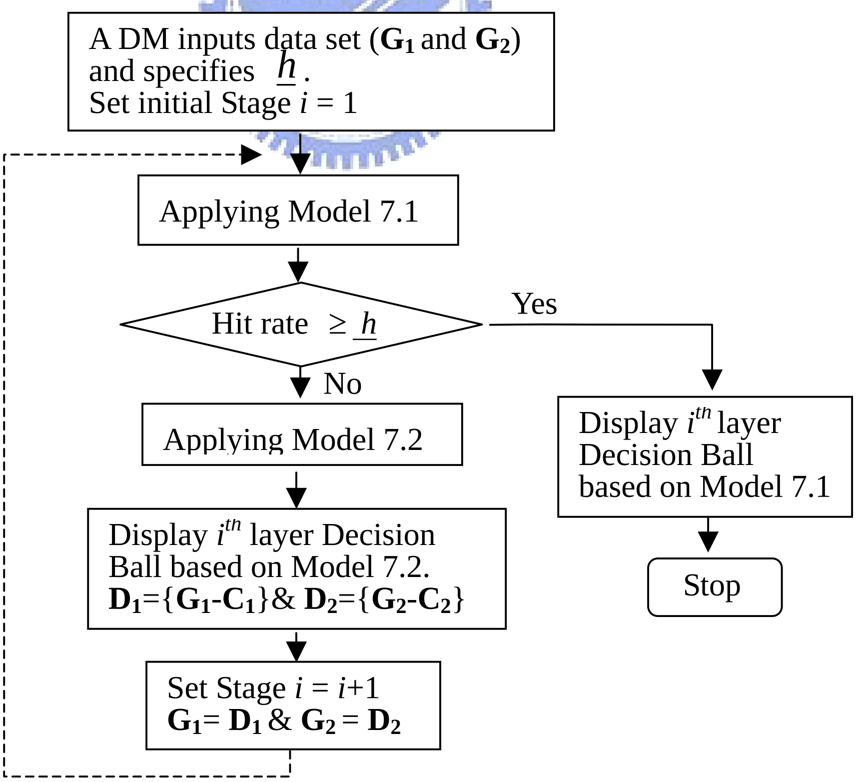


Figure 7.4 The multi-stage classifying processes of the proposed Classification Decision Ball models

until the hit rate is greater then $\underline{h}$. The Decision Ball for the i^{th} stage is called the i^{th} layer Decision Ball.

The Decision Ball techniques used here are slightly different from those in Chapter 3 because the weight of criteria could be negative. Only additive score functions are discussed in this chapter. For simplicity, all factors are normalized to [0,1] scale in advance. Before applying Decision Ball Model 3.1, the score and dissimilarity function should be modified as follows.

Then, the additive score function of A_i is redefined as

$$S_i(\mathbf{w}) = \sum_{\substack{k=1 \\ w_k \geq 0}}^{m} |w_k| c_{i,k} + \sum_{\substack{k=1 \\ w_k < 0}}^{m} |w_k| (1 - c_{i,k}), \tag{7.30}$$

where $0 \leq S_i \leq 1$. S_i here is called the transferred score.

The dissimilarity between A_i and A_j is redefined as

$$\delta_{i,j}(\mathbf{w}) = \sum_{k=1}^{m} |w_k| |c_{i,k} - c_{j,k}|, \tag{7.31}$$

where $0 \leq \delta_{i,j}(\mathbf{w}) \leq 1$ and $\delta_{i,j}(\mathbf{w}) = \delta_{j,i}(\mathbf{w})$.

7.3 Illustrative Examples

Two examples are used to illustrate the processes of the proposed Classification Decision Ball models.

<Example 7.1> A corporate bankruptcy example

This section takes corporate bankruptcy in US electric power industry (Sueyoshi, 2006) as an example to illustrate the proposed approach. As listed in Table 7.1, Alternatives 1~61 are 61 non-default firms ($\mathbf{G_1}$) and alternatives 62~83 are 22 default firms ($\mathbf{G_2}$). The performance of all the firms is measured by 13 financial ratios. Suppose the decision maker sets the minimal hit rate as $\underline{h}$ = 99%.

First, the data set in Table 7.1 is normalized to [0,1] scale. Then, let ε = 0.0001 and M = 1000, applying Model 7.1 to the normalized data set yields $\mathbb{Z} = 0$ and g^* = 0.406. $\mathbb{Z} = 0$ implies that there are no alternative misclassified (i.e. hit rate = 100%). The weights of 13 factors are $\mathbf{w} = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}\}$ = {0.077, -0.112, 0.069, 0.047, -0.040, 0, 0.441, 0, 0, -0.029, 0.014, 0.146, 0.024}, where the weights of 6^{th}, 8^{th} and 9^{th} factors are equal to 0.

Table 7.1 Financial performance of 83 firms in US electric power industry. 1~61 are non-default firms and 62~83 are default firms. (Sueyoshi, 2006)

Obs.	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0.03	-0.01	0.25	0.53	0.15	0.01	0.08	8.43	1.7	1.77	16.35	0.88	16.35
2	0	-0.16	0.93	0.29	0.25	0.04	0.1	16.57	1.67	0.32	9.08	3.74	36.22
3	0.07	0.06	0.47	0.28	0.35	0.04	0.13	11.24	1.86	-0.11	15.09	1.7	25.2
4	0.02	-0.02	0.44	0.39	0.31	0.03	0.13	9.66	1.5	-0.02	12.65	2.3	30.36
5	0.01	-0.07	0.43	0.27	0.34	0.05	0.17	14.2	1.73	0.02	12.82	3.46	42.3
6	0.01	-0.12	1.43	0.21	0.17	0.02	0.07	12.19	1.67	-0.11	19.26	3.11	43.53
7	0.04	0.05	2.43	0.29	0.5	0	0.03	7.94	0.86	0.38	4.72	1.21	13.26
8	0.1	0.13	3.43	0	0.42	0.12	-0.11	29.49	1.06	0.64	3.42	2.6	9.4
9	0.02	-0.22	4.43	0.25	0.31	0.05	0.15	17.18	1.79	0.2	8.12	3.45	33.84
10	0.07	0.04	5.43	0.33	0.14	0.03	0.06	21.29	1.77	0.73	7.33	2.11	16.79
11	0.09	0.09	6.43	0.31	0.4	0	0.13	0.49	1.02	0.26	17.04	0.08	16.7
12	0.11	0.14	7.43	0.18	0.46	0.04	0.14	10.24	1.46	0.14	17.11	3.11	43.47
13	0.01	-0.09	8.43	0.29	0.24	0.04	0.11	15.04	1.83	-0.19	12.57	2.78	33.43
14	0.01	-0.11	9.43	0.35	0.29	0.04	0.19	14.31	2.03	0.08	15.47	1.56	21.97
15	0.01	-0.05	10.43	0.4	0.11	-0.03	-0.06	-17.51	1.61	0.42	70.68	-2.53	24.03
16	0.01	-0.21	11.43	0.22	0.21	0.04	0.03	29.87	1.72	0.15	5.34	4.27	24.49
17	0.02	-0.05	0.57	0.32	0.35	0.04	0.31	12.49	1.35	-0.03	12.6	3.21	40.36
18	0.01	-0.12	0.28	0.19	0.29	0.01	0.11	2.14	1.07	0.21	9.76	0.52	26.55
19	0.01	-0.06	0.31	0.34	0.3	0.02	0.03	6.5	1.98	0.1	19.33	2.17	60.1
20	0	0.01	0.28	0.51	0.2	0.05	0.21	26.2	3.71	0.12	14.15	1.81	24.08
21	0	-0.02	0.4	0.37	0.16	-0.05	0.24	-30.27	1.78	0.22	-16.75	-2.75	18.93
22	0.02	-0.02	0.41	0.39	0.24	0.02	0.1	7.17	1.51	-0.16	24.53	2.15	41.94
23	0.01	-0.02	1.23	0.25	0.27	0.04	0.13	15.6	2.44	-0.06	14.65	2.58	39.26
24	0.11	-0.06	0.31	0.34	0.09	0.03	0.04	73.41	4.49	-0.17	-1.34	7.37	15.1
25	0.02	0.02	0.47	0.36	0.27	0.04	0.16	14.63	1.65	0.26	11.07	1.3	14.5
26	0.01	-0.08	0.3	0.35	0.3	0.01	0.05	3.88	1.59	-0.14	27.27	0.59	21
27	0.06	0.05	0.52	0.34	0.3	0.03	0.14	10.53	1.27	0.25	12.75	1.61	18.99
28	0.01	-0.02	0.37	0.28	0.3	0.03	0.14	9.43	1.15	-0.02	11.5	3.18	39.11
29	0.02	-0.02	0.43	0.37	0.24	0.04	0.03	17.21	1.98	-0.06	13.6	4.42	47.88
30	0.01	-0.06	0.21	0.31	0.21	0.02	0.04	8.85	1.64	-0.06	12.49	2.85	34.98
31	0	-0.12	0.49	0.28	0.36	0.04	0.18	12.98	1.66	-0.01	13.06	4.63	56.4
32	0.02	-0.01	0.86	0.23	0.34	0.03	0.02	10.72	1.05	-0.31	13.32	1.93	18.65
33	0.05	-0.45	0.2	0.13	0.11	0.01	0.02	11.59	1.55	0.04	12.91	3.19	40.28
34	0.04	0	0.35	0.24	0.3	0.04	0.25	12.26	1.13	-0.09	8.8	3.33	29.56
35	0.04	-0.06	1.38	0.26	0.24	0.02	0.03	10.24	1.16	0.13	9	1.65	16.02
36	0.01	-0.12	0.54	0.33	0.2	0.01	0.05	6.1	1.37	0.17	22.81	1.03	23.06
37	0.01	0.01	0.67	0.22	0.22	0.02	0.07	12.56	1.1	0.45	9.09	1.97	17.63
38	0.02	-0.12	1.62	0.32	0.22	0.02	0.04	11.2	1.45	-0.11	13.72	2.03	21.05
39	0.03	-0.06	0.23	0.61	0.17	0.02	0.05	11.86	1.42	0.13	11.07	1.36	15.5
40	0.01	-0.11	0.6	0.26	0.25	0	0.06	-0.19	1.88	0.13	186.88	-0.05	44.85
41	0.16	0.14	0.64	0.2	0.43	0.03	-0.03	25.22	1.89	-0.09	-1.98	3	19.24
42	0	-0.05	0.57	0.33	0.31	0.04	0.13	13.1	1.42	-0.07	10.62	3.86	41.85
43	0.01	0.01	0.8	0.33	0.34	0.05	0.14	14.81	1.05	0.39	6.85	3.83	27.95
44	0	-0.02	0.78	0.37	0.34	0.04	0.14	8.96	1.3	-0.03	7.23	1.51	22.57
45	0.08	0.04	0.46	0.44	0.15	0.01	0.08	9.1	2.19	0.35	8.34	1.16	34.85
46	0	0	0.41	0.46	0.29	0.03	0.1	9.02	1.59	-0.03	14.72	2.65	45.03
47	0.01	-0.12	0.33	0.41	0.16	0.03	0.07	18.44	2.13	-0.01	11.37	3.67	42.19
48	0.02	-0.03	0.61	0.38	0.27	0.02	0.01	8.31	1.38	0.1	10.84	1.31	21.89
49	0.01	-0.11	1.51	0.19	0.22	0.03	0.1	13.39	1.17	0.16	13.46	3.17	26.52
50	0.01	0	0.61	0.38	0.34	0.03	0.07	8.91	1.67	0.07	16.35	2	37.6
51	0.03	0	0.44	0.34	0.3	0.07	0.16	24.57	1.37	-0.02	4.97	5.15	27.83
52	0.01	-0.04	0.56	0.41	0.21	0.01	0	1.76	0.89	0.22	376.25	0.34	15.05
53	0.01	-0.05	0.34	0.28	0.27	0.04	0.15	14.02	2.21	-0.35	16.36	1.62	25.35
54	0.02	-0.17	0.39	0.27	0.32	0.05	0.19	15.4	1.9	0.02	11.66	2.26	26.24
55	0.03	-0.08	0.66	0.38	0.19	0.02	0.04	10.57	1.55	-0.19	13.71	3.12	47.15
56	0.13	0.09	0.2	0.37	0.3	0.03	-0.04	9.48	0.95	1.41	11.69	0.57	4.56
57	0.05	-0.02	0.58	0.27	0.27	0.03	0.12	11.87	1.47	0.18	12.89	4.21	51.3
58	0.08	-0.04	0.53	0.29	0.16	0.02	-0.09	14.33	1.48	0.3	9.83	1.83	18.19
59	0.01	0.01	0.47	0.39	0.25	0.03	0.15	10.14	1.27	-0.01	13.19	1.78	22.56
60	0.02	0.03	0.93	0.25	0.27	0.03	0.13	10.84	1.69	-0.03	14.28	2.75	36.55
61	0.02	-0.07	0.52	0.42	0.22	0.03	0.09	12.6	1.54	-0.01	12.06	2.28	27.74
62	0.03	-0.1	0.96	0.72	0.27	-0.28	-0.28	-91.47	0.22	1.13	-0.46	-1.93	0.84
63	0.03	-0.1	0.19	0.01	0.2	0.01	-0.73	-6.36	0.01	-0.22	0.08	-0.23	0.04
64	0.02	-0.05	0.69	0.17	0.33	-0.41	-1.1	-125.38	0.77	0.69	-2.31	-1.07	1.5
65	0.12	0.05	0.53	1.4	-0.57	-0.1	-2.26	20.17	-0.18	0.92	30.83	-1.54	0.93
66	0.02	-0.28	0.43	0.86	-0.22	-0.48	-0.67	220.16	1.99	0.29	-0.67	-10.21	4
67	0.03	-0.08	0.34	0.5	0	-0.07	-0.06	-3720	0.91	0.82	-4	-4.65	4.52
68	0.12	0.15	0.49	1.01	-0.15	-0.41	-0.32	237.02	0.23	-0.71	5.4	-9.67	0.69
69	0.06	0.02	0.45	0.39	0.33	0.01	0.02	4.47	0.37	0.69	5.03	0.47	3.88
70	0.02	0.45	2.89	0.13	0.18	0.01	0.05	8.66	6.12	0.58	61.57	1.22	83.13
71	0	-0.18	7.82	0.22	0.03	-0.01	-0.55	-22.62	5.05	0.04	20.62	-0.54	15.05
72	0	0.05	0.4	0.99	-0.15	-0.47	-0.66	313.7	0.33	0.86	-0.35	-12.65	2.63
73	0.03	-0.17	0.71	0.18	0.26	-0.13	-0.11	-48.87	0.61	1.29	-1.55	-2.3	3.56
74	0.04	0.14	0.48	0.72	-0.53	0.02	-1.02	-2.9	-0.16	1.33	-0.1	0.15	0.81
75	0.19	0.32	1.19	0	0.63	-1.21	-1.76	-193.51	0.12	2.15	-0.19	-6.38	0.74
76	0.02	0.22	1.49	0.06	0.49	-0.21	-0.59	-42.86	0.8	0.69	-0.97	-0.86	0.81
77	0.06	0.31	0.74	0.15	0.07	-0.16	-0.09	-314.34	1.78	0.69	5.4	-9.29	18.25
78	0.04	-0.17	0.55	0.93	-0.2	-0.29	-0.68	142.04	-2.01	0.25	-1.05	-1.74	0.86
79	0.03	-1.1	0.13	0	-0.21	-0.12	-0.74	59.18	-0.01	0.69	-0.02	-1.08	0.02
80	0.01	0.01	0.41	0.5	0.11	-0.13	-0.11	-111.42	0.35	0.51	-0.93	-6.6	3.6
81	0.02	-0.47	1.08	0.35	-0.01	-0.06	-0.09	166.68	-12.31	1.72	-1.36	-3.01	3.19
82	0.01	-0.45	0.43	0.23	-0.08	-0.22	-0.33	274.42	0.85	0.1	-1.05	-1.95	0.88
83	0.01	-0.71	0.14	0	0.15	0.01	-0.1	9.91	0.23	0.72	4.83	0.24	0.58

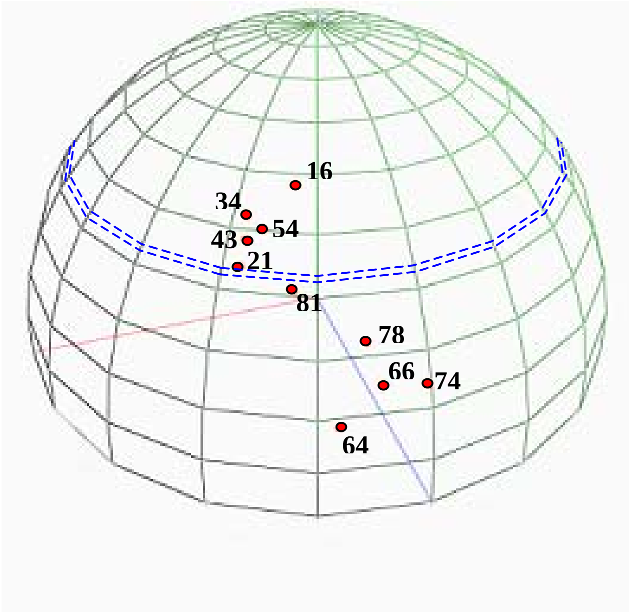


Figure 7.5 The Decision Ball of 10 target alternatives

Next, we will depict alternatives on the surface of the Decision Ball. In order to reduce the complexity of visual presentation (it is easy to get confused and costs lots of computational time if too many alternatives are display simultaneously), the decision maker can choose some alternatives as target alternatives to be a basis for comparison. Suppose the decision maker selects 5 alternatives (16, 21, 34, 43 and 54) from non-default firms and 5 alternatives (64, 66, 74, 78 and 81) from default firms as target alternatives. From (7.30) and (7.31), the transferred scores and dissimilarities of target alternatives can be calculated.

Applying Model 3.1 to the target alternatives yields coordinates of these alternative with Stress = 8.99%, as graphed in Figure 7.5. The area above the dash curves belongs to $\mathbf{G_1}$ and below the dash curves belongs to $\mathbf{G_2}$. The alternative with higher final score is located at the higher latitude. In Figure 7.5, the order of 10 target alternatives is 16, 34, 54, 43, 21, 81, 78,

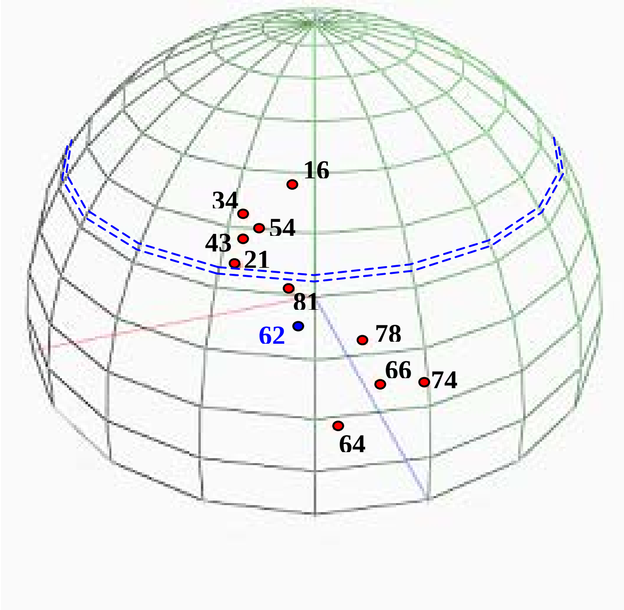


Figure 7.6 The Decision Ball of alternative 62 based on the target alternatives

66, 74 and 64 respectively. Moreover, although alternatives 81, 78, 66, 74 and 64 belongs to the same group **G$_2$**, alternative 81 has higher possibility to become **G$_1$** because its location is very closer to **G$_1$**. The processes are terminated because the hit rate is greater than minimal hit rate $\underline{h}$.

After plotting 10 target alternatives as a basis of comparison, we can classify and compare any alternative by checking the Decision Ball. For example, if a decision maker would like to check alternative 62, the system computes the dissimilarities among alternative 62 and 10 selected target alternatives and applies Model 3.1 to the alternative yields the location of alternative 62, as shown in Figure 7.6. By checking Figure 7.6, alternative 62 belongs to **G$_2$**, which is the most similar to alternative 81. If alternative 62 would like to upgrade itself to **G$_1$**, the benchmark alternatives are suggested as 21, 54 and 16. Alternative 43 and 34 are skipped as a benchmark because 54 and 16 have similar similarities but higher scores.

<Example 7.2>   Classifying Japanese banks

The second example is related to 100 Japanese banks, extracted from Sueyoshi (2001). All banks are listed by their corporate ranks. The first group $\mathbf{G_1}$ contains 50 banks whose ranks are from the 1 to 50. The second group $\mathbf{G_2}$ contains the remaining 50 banks whose ranks are from 51 to 100. Seven financial performance data are chosen as evaluation criteria, including return on total assets, equity to total assets, cost-profit rate, return on total domestic assets, bad loan ratio, loss ratio of bad loan, and return on equity. Suppose the decision maker sets the minimal hit rate = 99 %.

Table 7.2 Financial performance of 100 Japanese Banks. G_1 contains ranks 1~50 banks and G_2 contains ranks 51~100 banks. (Sueyoshi, 2001)

	1	2	3	4	5	6	7
	Return on total assets	Equity to total assets	Cost-Profit rate	Return on total domestic assets	Bad loan ratio	Loss ratio of bad loan	Return on equity
1	0.67	9.28	43.28	0.36	2.08	46.99	18.68
2	1	4.78	61.98	0.94	0.31	155.2	27.26
3	0.65	9.1	50.07	0.52	2.41	48.35	20.9
4	1.67	4.21	51.69	0.88	2.97	80.1	51.24
5	0.91	10.64	59.72	0.72	1.08	78.64	21.44
6	0.62	8.75	51.9	0.35	2.49	46.18	19.83
7	0.71	8.75	53.75	0.52	3.64	51.52	24.87
8	1.06	6.58	59.31	0.93	1.37	47.59	19.56
9	1.02	10.15	58.59	0.81	1.34	65.4	25.79
10	0.87	9.16	61.21	0.84	1.55	77.93	21.45
11	1.13	6.23	58.33	0.95	3.48	37.44	23.78
12	0.82	11.28	61	0.69	1.52	45.85	17.1
13	0.64	9.22	53.15	0.19	4.64	55.55	19.93
14	0.88	11.14	61.84	0.63	1.7	52.2	14.19
15	0.77	9.54	58.62	0.63	0.61	49.62	15.15
16	0.95	9.45	58.77	0.84	3.48	19.53	14.96
17	0.73	9.24	59.93	0.59	1.26	61.95	15.56
18	0.8	10.04	62.9	0.6	2.18	65.03	19.41
19	0.83	9.6	57.83	0.63	2.66	47.66	19.46
20	0.53	8.92	64.13	0.38	3.88	50.26	18.16
21	1.05	6.38	66.56	0.98	3.7	48.67	20.39
22	0.91	10.17	60.03	0.75	2.13	36.63	19.94
23	1.05	8.44	57.93	0.67	2.48	48.67	30.51
24	0.8	9.86	67.12	0.68	1.33	69.61	19.04
25	1.05	4.53	59.6	0.79	3.09	50.72	29.15
26	0.9	9.77	61.37	0.71	2.26	62.25	110
27	0.95	4.41	63.71	0.84	2.8	56.11	35
28	0.69	9.64	70.12	0.49	0.64	85.12	12.77
29	0.78	11.83	67.01	0.49	0.36	58.41	19.4
30	0.56	13.61	66.89	0.34	0.81	51.89	9.43
31	0.79	9.35	68.02	0.66	0.79	59.32	18.46
32	0.73	9.18	66.07	0.58	1.15	66.93	18.82
33	0.86	9.09	61.1	0.6	1.95	47.28	16.97
34	1	11.33	64.34	0.46	1.19	46.54	18.44
35	0.96	5.06	63.42	0.88	4.66	37.96	26.09
36	0.76	9.19	59.35	0.48	1.8	22.89	15.28
37	1.72	4.41	52.91	0.59	6.94	40.62	49.3
38	1.09	9.03	63.24	0.63	4.19	42.17	26.44
39	0.66	9.06	68.22	0.73	1.5	37.58	16.17
40	1.57	4.28	52.91	0.65	6.87	33.21	45.93
41	0.68	10.19	68.76	0.52	0.97	57.65	15.21
42	0.53	8.7	63.42	0.35	2.9	62.7	18.48
43	0.66	9.63	68.17	0.49	1.1	57.2	17.4
44	0.93	4.35	65.66	0.81	1.76	39.3	28.58
45	1.18	8.69	52.83	0.54	6.8	43.16	35
46	0.7	4.72	71.29	0.9	2.39	49.78	18.33
47	0.71	9.35	69.69	0.64	1.84	38.16	16.5
48	0.65	9.11	65.96	0.56	1.42	44.16	16.72
49	0.71	10.54	66.37	0.47	1.43	43.62	14.23
50	0.72	9.47	69.35	0.48	0.86	56.36	15.34

Table 7.2 (Continued)

	1	2	3	4	5	6	7
	Return on total assets	Equity to total assets	Cost-Profit rate	Return on total domestic assets	Bad loan ratio	Loss ratio of bad loan	Return on equity
51	0.71	9.31	66.43	0.51	1.66	52.79	16.04
52	0.69	8.42	67.94	0.55	0.38	45.38	18.42
53	0.55	9.44	70.29	0.43	1.24	74.95	12.59
54	0.51	9.09	61.15	0.24	4.7	68.15	20.74
55	0.6	10.69	66.77	0.41	2.42	40.55	12.41
56	1.1	3.75	62	0.5	2.21	49.9	39.62
57	0.72	4.2	66.99	0.73	1.53	45.88	23
58	0.64	9.63	67.67	0.47	2.82	50.61	18.09
59	0.62	9.26	68.59	0.49	1.51	45.25	16.72
60	0.58	9.82	71.67	0.42	0.89	64.84	14.43
61	0.83	4.96	65.84	0.74	3.65	39.98	23.54
62	0.66	10.85	71.15	0.34	1.19	62.33	13.44
63	0.88	4.28	68.67	0.78	3.93	46.31	26.11
64	0.99	4.37	64.49	0.87	5.48	21.93	26.11
65	0.72	4.54	72.42	0.59	1.54	71.95	21.39
66	0.87	4.25	66.45	0.59	2.18	56.39	28.51
67	0.96	4.02	64.27	0.75	3.59	80.51	44.48
68	0.67	6.06	70.75	0.39	1.42	69.34	13.82
69	0.68	8.76	70.38	0.49	1.34	49.27	16.19
70	0.85	4.24	66.33	0.64	3.03	48.97	28.41
71	0.6	9.2	71.87	0.44	1.09	59.29	16.26
72	0.52	9.08	69.43	0.34	1.52	62.74	14.09
73	0.65	9.22	67.79	0.4	1.47	67.74	16.7
74	0.76	8.13	67.71	0.47	1.71	47.88	19.42
75	0.57	9.63	70.41	0.39	2.72	52.91	12.36
76	0.56	9.43	74.88	0.39	0.54	59.3	15.04
77	0.59	10.55	71.99	0.35	0.89	35.83	11.86
78	0.75	8.42	64.59	0.48	3.1	50.94	20.39
79	0.6	9.02	66.75	1.13	4.65	34.23	18.6
80	0.8	4.16	69.81	0.69	2.62	33.13	26.04
81	0.65	8.29	71.15	0.52	2.23	52.1	16.41
82	0.64	4.11	72.33	0.49	0.86	57.3	20.6
83	0.49	9.05	74.25	0.43	1.61	51.57	10.16
84	1.12	4.24	64.41	0.42	2.71	31.45	32.12
85	0.64	9.03	71.71	0.34	1.3	57.4	16.89
86	0.45	10.43	79.26	0.4	1.28	65.05	10.72
87	0.63	9.7	71.05	0.21	0.68	46.2	13.57
88	0.6	8.4	65.9	0.4	3.09	33.48	20.68
89	0.87	4.77	64.63	0.44	3.16	35.94	25.03
90	0.6	4.83	74.85	0.38	0.27	51.76	15.28
91	0.6	4.11	73.99	0.37	1.12	80.03	17.97
92	0.73	3.46	71.56	0.48	2.18	67.48	24.89
93	0.72	6.88	76.31	0.26	0.56	48.12	14.29
94	0.75	4.47	72.36	0.42	2.46	63.08	20.8
95	0.83	4.13	60.99	0.37	2.63	32.01	28.67
96	0.88	4.13	59.07	0.45	5.88	29	2.49
97	0.82	4.65	65.99	0.47	3.26	28.62	23.28
98	1.01	4.53	63.39	0.3	4.46	43.69	29.04
99	0.4	9.1	74.18	0.33	3.83	54.38	13.98
100	0.54	4.47	78.35	0.49	0.96	46.59	14.63

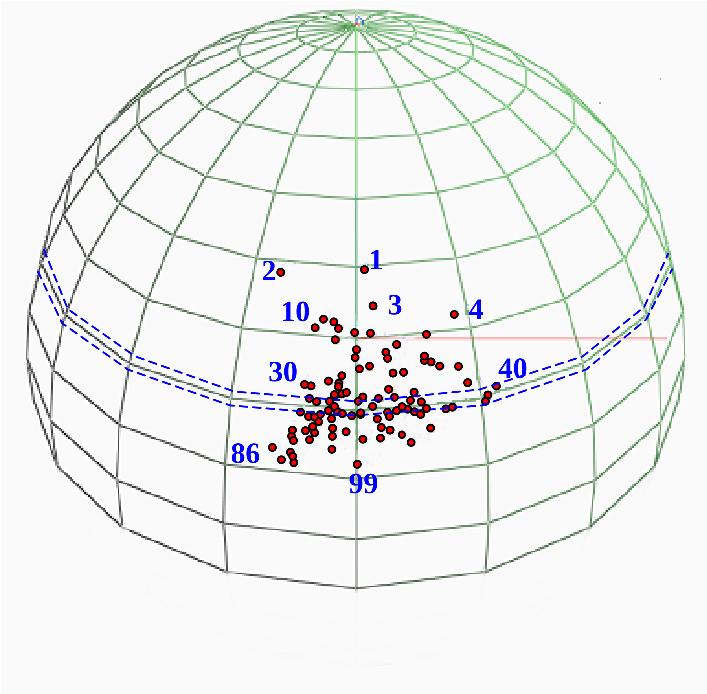


Figure 7.7 The first layer Decision Ball of Example 7.2, where 33 alternatives in the overlap

At the first stage, applying Model 7.1 yields a hit rate 96%. Because the hit rate is less than 99%, applying Model 7.2 yields a set of weights **w** = {w_1, w_2, w_3, w_4, w_5, w_6, w_7} = {-0.03, 0.037, -0.398, 0.127, -0.130, 0.145, -0.134}, d^* = -0.245, and p^* = 0.038. There are 33 alternatives located in the overlap. The first layer Decision Ball is shown in Figure 7.7. At the second stage, applying Model 7.1 to the alternatives in the overlap in Figure 7.7 yields a hit rate 98%. Since the hit rate is still less than 99%, Model 7.2 is applied again to yield a set of weights **w** = {w_1, w_2, w_3, w_4, w_5, w_6, w_7} = {-0.018, 0.019, 0.346, -0.077, 0.118, -0.130, 0.292}, d^* = 0.237, and p^* = 0.026. The second layer Decision Ball is depicted in Figure 7.8, where 25 alternatives are located in the overlap region.

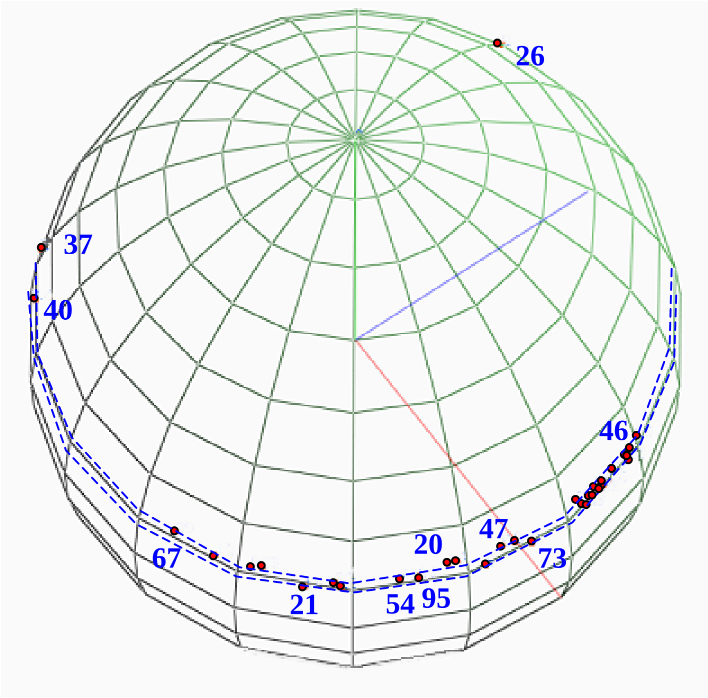


Figure 7.8 The second layer Decision Ball of Example 7.2, where 25 alternatives in the overlap.

At the third stage, applying Model 7.1 to the remaining 25 alternatives yields a hit rate 100%, with **w** = {w_1, w_2, w_3, w_4, w_5, w_6, w_7} = {0.069, -0.031, -0.274, 0.063, -0.075, 0.047, -0.442} and g^* = -0.232. The third layer Decision Ball is graphed in Figure 7.9. The classifying processes are terminated since the hit rate is greater than 99%.

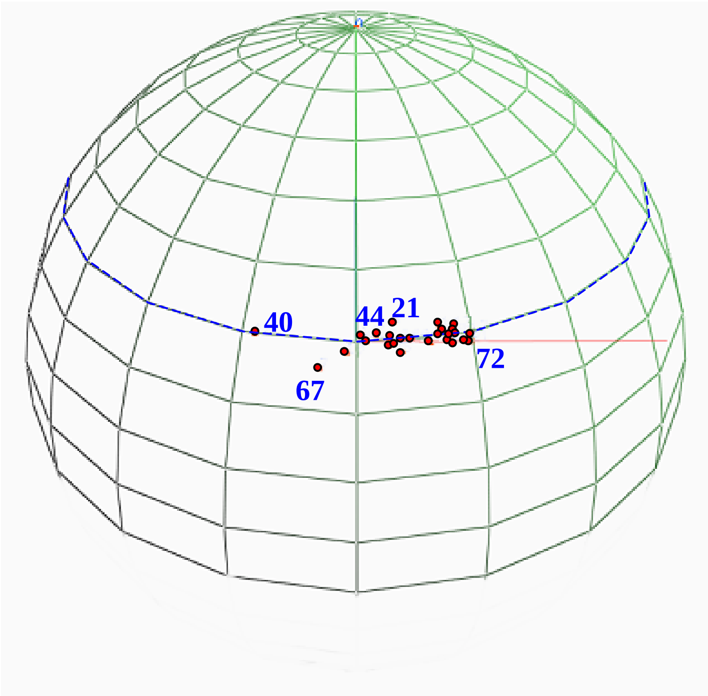


Figure 7.9 The third layer Decision Ball of Example 7.2

7.4 Summary

This study proposes Classification Decision Ball models, which can assist the decision makers visualize the grouping relationships among alternatives.

The major advantages of the Classification Decision Ball models are summarized as follows:

(i) Classifying alternatives on Decision Balls layer-by-layer.

(ii) Visualizing the relationships among alternatives, including ranks of, grouping of and similarities among alternatives on Decision Balls.

(iii) Providing benchmark alternatives if an alternative would like to upgrade its performance from one group to another.

(iv) Reducing the number of binary variables significantly.

(v) Comparing with traditional 2-dimensional plane and 3-dimensional cube models, the proposed approach is more flexible and easier to observe because all alternatives are displayed on the surface of a sphere.

Chapter 8 Concluding Remarks

This study proposes four Decision Ball models to display alternatives on spheres, thus to make a more knowledgeable decision. Four types of Decision Ball models, including Moral Algebra Decision Ball models, Even Swap Decision Ball models, Pairwise Comparison Decision Ball models, and Classification Decision Ball models, are constructed to meet decision makers with different decision preferences.

Some future directions for further research are described below.

(i) Construction of more Decision Ball models: Different decision makers may have various types of decision preferences and requirements. More Decision Ball models can be constructed for the decision makers with other types of decision patterns. For instance, Decision Ball models for the Data Envelopment Analysis (DEA, Charnes et al., 1978), decision preferences involving fuzzy concepts, and decision preferences including psychological factors, etc.

(ii) Global optimization of Decision Ball techniques: As shown in Section 3.3, the proposed Decision Ball techniques are still non-linear. Hence, how to linearize the Decision Ball techniques is an important direction for further research. In addition, because the computational time of the proposed models will increase significantly when the number of alternatives grows, a distributed computing algorithm could be developed to improve the computational efficiency.

(iii) Investigation of decision preferences: deeper study of decision behavior to provide a customized and visualized decision environment is another important direction for future research.

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Appendix

A letter from Benjamin Franklin to Joseph Priestly

London, Sept 19, l772

Dear Sir,

In the affair of so much importance to you, wherein you ask my advice, I cannot, for want of sufficient premises, advise you what to determine, but if you please I will tell you how. When those difficult cases occur, they are difficult, chiefly because while we have them under consideration, all the reasons pro and con are not present to the mind at the same time; but sometimes one set present themselves, and at other times another, the first being out of sight. Hence the various purposes or inclinations that alternatively prevail, and the uncertainty that perplexes us. To get over this, my way is to divide half a sheet of paper by a line into two columns; writing over the one Pro, and over the other Con. Then, during three or four days consideration, I put down under the different heads short hints of the different motives, that at different times occur to me, for or against the measure. When I have thus got them all together in one view, I endeavor to estimate their respective weights; and where I find two, one on each side, that seem equal, I strike them both out. If I find a reason pro equal to some two reasons con, I strike out the three. If I judge some two reasons con, equal to three reasons pro, I strike out the five; and thus proceeding I find at length where the balance lies; and if, after a day or two of further consideration, nothing new that is of importance occurs on either side, I come to a determination accordingly. And, though the weight of the reasons cannot be taken with the precision of algebraic quantities, yet when each is thus considered, separately and comparatively, and the whole lies before me, I think I can judge better, and am less liable to make a rash step, and in fact I have found great advantage from this kind of equation, and what might be called moral or prudential algebra.

Wishing sincerely that you may determine for the best, I am ever, my dear friend, yours most affectionately.

B. Franklin

From: “Letter to Joseph Priestly”, Benjamin Franklin Sampler, (1956).