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The monotone boundary property and the full coverage property of confidence intervals for a binomial proportion

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ABSTRACT

The methodology for deriving the exact confidence coefficient of some confidence intervals for a binomial proportion is proposed in Wang [2007. Exact confidence coefficients of confidence intervals for a binomial proportion. Statist. Sinica 17, 361–368]. The methodology requires two conditions of confidence intervals: the monotone boundary property and the full coverage property. In this paper, we show that for some confidence intervals of a binomial proportion, the two properties hold for any sample size. Based on results presented in this paper, the procedure in Wang [2007. Exact confidence coefficients of confidence intervals for a binomial proportion. Statist. Sinica 17, 361–368] can be directly used to calculate the exact confidence coefficients of these confidence intervals for any fixed sample size.

1. Introduction

The binomial distribution is a very useful distribution in many real application areas. The asymptotic behavior for several confidence intervals of a binomial proportion has been investigated by Brown et al. (2001, 2002). For small sample size behaviors of these confidence intervals, Wang (2007, 2009a) proposes calculation algorithms to derive their exact minimum coverage probabilities and average coverage probabilities. Some of these confidence intervals are successfully adopted in the quality control area (Wang, 2009b) and can potentially be developed to estimate the nucleotide substitution rate in an important biological evolutionary model (Wang et al., 2008).

One of the algorithms proposed in Wang (2007, 2009a) is to calculate the minimum coverage probability, also known as confidence coefficient, of a confidence interval $(U(X), L(X))$ for a binomial proportion p, where X follows a binomial distribution $B(n,p)$. The coverage probability of a confidence interval of p is defined as the probability that the random interval covers the true parameter p. In this case of the binomial distribution, the coverage probability is a variable function of p. For a $1-\alpha$ confidence interval of p that is constructed from the large sample approximation, the exact confidence coefficient may be far away from $1-\alpha$. One example is the $1-\alpha$ Wald interval $(\hat{p}-z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}, \hat{p}+z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n})$, where $\hat{p}=X/n$ and $z_{\alpha/2}$ is the upper $\alpha/2$ cutoff point of the standard normal distribution. It is well known that the confidence coefficient of the Wald interval is zero (see Lehmann, 1986; Blyth and Still, 1983).

Usually, the exact confidence coefficient is unknown because we do not know at which point in the parameter space the infimum coverage probability occurs. Adopting Wang's (2007) procedure, the infimum coverage probability and the maximum coverage probability of a confidence interval for a binomial proportion can be derived if two specific conditions for the confidence interval are satisfied, which are the monotone boundary property and the full coverage property. When applying the procedure for a given sample, we need to check if these two properties hold for the sample size of this sample. Note that the minimum

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coverage probability of a confidence interval only depends on the sample size and the form of the confidence interval. For a confidence interval, if the two properties can be shown to be satisfied for all sample sizes, the procedure can be directly utilized to calculate the confidence coefficient without the necessity of checking the two conditions for any sample size. In this paper, we show that for some confidence intervals, the two properties are satisfied for any sample size.

Beside being used in the coverage probability inference for confidence intervals, the monotone boundary property is also an essential condition in deriving the exact minimum coverage probability of tolerance intervals or simultaneous confidence intervals for discrete distributions (Cai and Wang, 2009; Wang and Tsung, 2009; Wang, 2008). Therefore, the monotone boundary property is an important condition in interval estimation for discrete distributions.

The paper is organized as follows. Section 2 describes the conditions and procedure proposed in Wang (2007). The main results, that the two conditions hold for the Wilson, Agresti–Coull and likelihood ratio intervals with any sample size or when the sample size is greater than 1, are given in Section 3. Finally, a conclusion is given in Section 4.

2. Procedure

We briefly describe the procedure of computing exact confidence intervals proposed in Wang (2007) as well as the conditions for the confidence intervals. The conditions required of a confidence interval $(L(X), U(X))$ for a binomial proportion in the procedure of computing exact confidence coefficients are given in Assumption 1.

Assumption 1. Confidence interval $(L(X), U(X))$ of a binomial proportion p satisfies:

(i) $L(X_1) < L(X_2)$ if $X_1 < X_2$;
(ii) $U(X_1) < U(X_2)$ if $X_1 < X_2$;
(iii) for any fixed $p \in (0,1)$, there exists an x_0 such that $p \in (L(x_0), U(x_0))$.

Remark 1. The conditions in Assumption 1 can be extended to calculate confidence coefficients of confidence intervals for other discrete distributions (Wang, 2009a).

In this paper, condition (i) is called the monotone lower boundary property, condition (ii) is called the monotone upper boundary property, and condition (iii) is called the full coverage property. If a confidence interval has the monotone lower boundary property and the monotone upper boundary property, we say that the confidence interval has the monotone boundary property. By Wang (2007), if a confidence interval satisfies Assumption 1, the exact confidence coefficient of the confidence interval can be derived by applying the following procedure.

Procedure for computing exact confidence coefficient.

Step 1: Check if the union of $(n+1)$ intervals $(L(X), U(X))$, $X = 0, \ldots, n$, covers all $p \in (0,1)$ and if (i) and (ii) in Assumption 1 are satisfied. If it does not cover all $p \in (0,1)$, the confidence coefficient is zero. We do not need to go to step 2.

Step 2: If Assumption 1 is satisfied, list the endpoints of the intervals that are greater than zero and smaller than 1.

Step 3: Calculate the coverage probability of each endpoint in step 2. The minimum value of these coverage probabilities is the exact confidence coefficient.

Note that all endpoints of the confidence interval based on $x = 0, \ldots, n$ are

$$(L(0), L(1), \ldots, L(n), U(0), U(1), \ldots, U(n)).$$

3. The main results

In this section, the monotone boundary property and the full coverage property are examined for any fixed sample size for the three confidence intervals which are discussed in Brown et al. (2002) and Wang (2007).

1. The $1-\alpha$ Wilson interval. Denote $\tilde{X} = X + k^2/2$ and $\tilde{n} = n + k^2$. Let $\tilde{p} = \tilde{X}/\tilde{n}$, $\tilde{q} = 1 - \tilde{p}$, $\hat{p} = X/n$, $\hat{q} = 1 - \hat{p}$ and k be the upper $\alpha/2$ cutoff point of the standard normal distribution. The $1-\alpha$ Wilson interval has the form

$$CI_W(X) = \left(\tilde{p} - \frac{kn^{1/2}}{n+k^2}\left(\hat{p}\hat{q} + \frac{k^2}{4n}\right)^{1/2},\ \tilde{p} + \frac{kn^{1/2}}{n+k^2}\left(\hat{p}\hat{q} + \frac{k^2}{4n}\right)^{1/2} \right).$$

2. The Agresti–Coull interval. The $1-\alpha$ Agresti–Coull interval is

$$CI_{AC}(X) = (\tilde{p} - k(\tilde{p}\tilde{q})^{1/2}\tilde{n}^{-1/2},\ \tilde{p} + k(\tilde{p}\tilde{q})^{1/2}\tilde{n}^{-1/2}).$$

3. The likelihood ratio interval. The $1-\alpha$ interval is

$$CI_{\Lambda_n}(X) = \left\{ p : \frac{p^X(1-p)^{n-X}}{(X/n)^X(1-X/n)^{n-X}} > e^{-k^2/2} \right\}.$$

For the Wilson interval, the monotone boundary property and full coverage property are shown to hold for any sample size in Propositions 1 and 2.

Proposition 1. *The Wilson interval $CI_W(X)$ has the monotone boundary property*.

Proof. Let $L_W(x)$ and $U_W(x)$ denote the lower bound and the upper bound of $CI_W(X)$ corresponding to $X=x$. To prove the monotone boundary property, it is necessary to show that the two functions, $L_W(x+1)-L_W(x)$ and $U_W(x+1)-U_W(x)$ are greater than zero for $x=0,\dots,n-1$.

We have

$$L_W(x+1)-L_W(x)=\frac{1}{2(k^2+n)}\left(2+k\sqrt{n}\left(\sqrt{\frac{k^2n+4(n-x)x}{n^2}}-\sqrt{\frac{-4(1+x)^2+n(4+k^2+4x)}{n^2}}\right)\right). \tag{1}$$

Note that $(1)>0$ is equivalent to

$$\left(2\sqrt{n}+k\sqrt{k^2n+4(n-x)x}\right)^2-\left(k\sqrt{-4(1+x)^2+n(4+k^2+4x)}\right)^2>0. \tag{2}$$

The left hand side of (2) is equal to

$$4(n+k^2(1-n+2x)+k\sqrt{n}\sqrt{k^2n+4(n-x)x})>4(n+k^2(1-n+2x)+k^2n)>0,$$

Therefore, $L_W(x)$ is an increasing function of x.

We have

$$U_W(x+1)-U_W(x)=\frac{1}{2(k^2+n)}\left(2+k\sqrt{n}\left(-\sqrt{\frac{k^2n+4(n-x)x}{n^2}}+\sqrt{\frac{-4(1+x)^2+n(4+k^2+4x)}{n^2}}\right)\right). \tag{3}$$

Note that (3)>0 is equivalent to

$$\left(2\sqrt{n}+k\sqrt{-4(1+x)^2+n(4+k^2+4x)}\right)^2-\left(k\sqrt{k^2n+4(n-x)x}\right)^2>0. \tag{4}$$

The left hand side of (4) is equal to

$$\begin{aligned}4(n+k^2(-1+n-2x)+k\sqrt{n}\sqrt{4(1+x)(n-1-x)+nk^2}&>4(n+k^2(-1+n-2x)+k^2n)\\&>4(n+k^2(2n-2x-1))>0,\end{aligned} \tag{5}$$

because $x\le n-1$. Therefore, $U_W(x)$ is an increasing function of x. □

Proposition 2. *The Wilson interval $CI_W(X)$ has the full coverage property for all n and $k>1$.*

Proof. By Proposition 1, $CI_W(x)$ has the monotone boundary property. According to the result, to show the full coverage property, it is only necessary to show that $L_W(x+1)$ is less than $U_W(x)$ for $x=0,\dots,n-1$, and $U_W(n)$ is not less than 1 and $L_W(0)$ is not larger than 0.

$$\begin{aligned}L_W(x+1)-U_W(x)&=\frac{x+1+k^2/2}{n+k^2}-\frac{kn^{1/2}}{n+k^2}\left(\frac{(x+1)(n-x-1)}{n^2}+\frac{k^2}{4n}\right)^{1/2}-\frac{x+k^2/2}{n+k^2}-\frac{kn^{1/2}}{n+k^2}\left(\frac{x(n-x)}{n^2}+\frac{k^2}{4n}\right)^{1/2}\\&<\frac{1}{n+k^2}-\left\{\frac{kn^{1/2}}{n+k^2}\left[\left(\frac{k^2}{4n}\right)^{1/2}+\left(\frac{k^2}{4n}\right)^{1/2}\right]\right\}\\&=\frac{1}{n+k^2}(1-k^2)<0,\end{aligned}$$

which leads to $L_W(x+1)$ is less than $U_W(x)$ for $k>1$. By straightforward calculation, we have $U_W(n)=1$ and $L_W(0)=0$. Therefore, the proof is complete. □

For the Agresti-Coull interval, the monotone boundary property and full coverage property are shown to hold for any sample size in Propositions 3 and 4.

Proposition 3. *The Agresti–Coull confidence interval $CI_{AC}(X)$ has the monotone boundary property*.

Proof. Let $L_{AC}(x)$ and $U_{AC}(x)$ denote the lower bound and the upper bound of $CI_{AC}(X)$ corresponding to $X=x$.

$$L_{AC}(x+1)-L_{AC}(x)=1/\tilde{n}+k/(2(\tilde{n})^{3/2})\left(\sqrt{(k^2+2n-2x)(k^2+2x)}-\sqrt{(-2+k^2+2n-2x)(2+k^2+2x))}\right). \tag{6}$$

Note that (6) larger than zero is equivalent to

$$\left(2\sqrt{k^2+n}+k\sqrt{(k^2+2n-2x)(k^2+2x)}\right)^2-\left(k\sqrt{(-2+k^2+2n-2x)(2+k^2+2x)}\right)^2>0. \tag{7}$$

By straightforward calculation, (7) is equal to

$$\begin{aligned}4(n+k\sqrt{k^2+n}\sqrt{k^4+2k^2n+4nx-4x^2}+k^2(2-n+2x)) &\geq 4(n+k\sqrt{k^2+n}\sqrt{k^4+2k^2n}+k^2(2-n+2x))\\ &\geq 4(n+k\sqrt{(k^2+n)k^2(k^2+2n)}+k^2(2-n+2x))\\ &\geq 4(n+k^2(k^2+n)+k^2(2-n+2x))>0,\end{aligned} \tag{8}$$

which implies that $L_{AC}(x)$ is an increasing function of x.

$$U_{AC}(x+1)-U_{AC}(x)=1/\tilde{n}+k/(2\tilde{n})^{3/2}\left(-\sqrt{(k^2+2n-2x)(k^2+2x)}+\sqrt{(-2+k^2+2n-2x)(2+k^2+2x)}\right). \tag{9}$$

Note that (9) larger than zero is equivalent to

$$(2\sqrt{k^2+n}+k\sqrt{(-2+k^2+2n-2x)(2+k^2+2x)})^2-(k\sqrt{(k^2+2n-2x)(k^2+2x)})^2>0. \tag{10}$$

By straightforward calculation, (10) is equal to

$$\begin{aligned}&4\left(n+k\sqrt{k^2+n}\sqrt{k^4+2k^2n+4nx+4n-4(x+1)^2}+k^2n+2k^2x\right)\\ &\quad\geq 4\left(n+k\sqrt{k^2+n}\sqrt{k^4+2k^2n+4(x+1)(n-x-1)}+k^2n-2k^2x\right)\\ &\quad\geq 4\left(n+k\sqrt{n}\sqrt{k^2n}+k^2n-2k^2x\right)\\ &\quad\geq 4(n+2k^2n-2k^2x)>0,\end{aligned} \tag{11}$$

which implies that $U_{AC}(x)$ is an increasing function of x. □

Proposition 4. *The Agresti–Coull interval $CI_{AC}(X)$ has the full coverage property for all $n\geq 2$ and $k>1$.*

Proof. By a similar argument as that in Proposition 2, we need to show that $L_{AC}(x+1)$ is less than $U_{AC}(x)$, and $U_{AC}(n)$ is not less than 1 and $L_{AC}(0)$ is not larger than 0.

$$L_{AC}(x+1)-U_{AC}(x)=\frac{x+1+k^2/2}{n+k^2}-kh_1(n+k^2)^{-1/2}-\frac{x+k^2/2}{n+k^2}-kh_2(n+k^2)^{-1/2}, \tag{12}$$

where $h_1=((x+1+k^2/2)(n+k^2/2-x-1)/(n+k^2)^2)^{1/2}$ and $h_2=((x+k^2/2)(n+k^2/2-x)/(n+k^2)^2)^{1/2}$. Thus,

$$(12)=\frac{1}{n+k^2}-k(n+k^2)^{-1/2}(h_1+h_2). \tag{13}$$

Since h_1 and h_2 are greater than $((x+k^2/2)(n+k^2/2-x-1)/(n+k^2)^2)^{1/2}$, (13) is less than

$$\frac{1}{n+k^2}-2k(n+k^2)^{-1/2}(n+k^2)^{-1}\left[\left(x+\frac{k^2}{2}\right)\left(n+\frac{k^2}{2}-x-1\right)\right]^{1/2}. \tag{14}$$

Note that the minimum value of the term

$$\left(x+\frac{k^2}{2}\right)\left(n+\frac{k^2}{2}-x-1\right)$$

in (14) for $x=0,\dots,n-1$ is

$$\frac{k^2}{2}\left(\frac{2n+k^2-2}{2}\right).$$

Thus (14) is less than

$$\frac{1}{n+k^2}\left(1-2k(n+k^2)^{-1/2}\left(\frac{k^2}{2}\left(\frac{2n+k^2-2}{2}\right)\right)^{1/2}\right)<\frac{1}{n+k^2}(1-k^2)<0.$$

The second to last inequality holds because $n \ge 2$ and the last inequality holds because $k>1$. Thus, $L_{AC}(x+1)$ is less than $U_{AC}(x)$.

Moreover, $U_{AC}(n)$ is

$$\begin{aligned}\frac{n+k^2/2}{n+k^2}+\frac{k/2\sqrt{k^4+2k^2n}}{(n+k^2)^{3/2}} &= \frac{n+k^2/2}{n+k^2}+\frac{k^2/2\sqrt{k^2+2n}}{(n+k^2)^{3/2}}\\ &> \frac{n+k^2/2}{n+k^2}+\frac{k^2/2}{n+k^2}=1.\end{aligned}$$

Note that $L_{AC}(0)$ is

$$\frac{k^2}{2(n+k^2)}\left(1-\left(\frac{2n+k^2}{n+k^2}\right)^{1/2}\right)<0.$$

Thus, the proof is complete. □

The Wilson interval and the Agresti–Coull interval have closed forms. However, the likelihood ratio interval does not have a closed form. It is more difficult to show the two properties for the likelihood ratio interval. Before giving the results for the likelihood ratio interval, we need the following lemma.

Lemma 1. *The two functions* $(t/(t-1))^{t-1}$ *and* $(t/(t+1))^{t+1}$ *are increasing functions of* t, *and*

(i) $(t/(t-1))^{t-1}<e$ *for* $t\ge 1$;
(ii) $(t/(t+1))^{t+1}<1/e$ *for* $t\ge 0$.

Proof. First, we show $\log(t/(t-1))^{t-1}$ is an increasing function of t.

$$\frac{\partial}{\partial t}\left(\log\left(\frac{t}{t-1}\right)^{t-1}\right)=\log\frac{t}{t-1}-\log e^{1/t}. \tag{15}$$

To establish (15) greater than zero, we need to show

$$\frac{t}{t-1}>e^{1/t}. \tag{16}$$

The expansion of the left hand side of (16) is $1+1/t+1/t^2+\cdots$. The expansion of the right hand side of (16) is $1+1/t+1/(2!t^2)+\cdots$. Therefore, (16) holds, which implies that $\log(t/(t-1))^{t-1}$ is an increasing function and $(t/(t-1))^{t-1}$ is also an increasing function. Since $\lim_{t\to\infty}(t/(t-1))^{t-1}=e$, we have $(t/(t-1))^{t-1}<e$ for $t\ge 1$. The proof of (i) is complete.

For the second part, we need to show $(t/(t+1))^{t+1}$ is an increasing function of t.

$$\begin{aligned}\frac{\partial}{\partial t}\left(\log\left(\frac{t}{t+1}\right)^{t+1}\right) &= \log\frac{t}{t+1}+\log e^{1/t}\\ &= \log\frac{e^{1/t}}{(t+1)/t}\\ &= \log\frac{1+1/t+1/(2!t^2)+\cdots}{1+1/t},\end{aligned} \tag{17}$$

which is greater than zero. Therefore, $(t/(t+1))^{t+1}$ is an increasing function of t. Since $\lim_{t\to\infty}(t/(t+1))^{t+1}=e^{-1}$, thus $(t/(t+1))^{t+1}<e^{-1}$ for $t>0$. □

Proposition 5. $CI_{\Lambda_n}(x)$ *has the monotone boundary property*.

Proof. Let $U_\Lambda(x)$ and $L_\Lambda(x)$ denote the lower bound and the upper bound of $CI_{\Lambda_n}(X)$, respectively. First we show that CI_{Λ_n} has the monotone lower boundary property. Let

$$L(x,p)=\frac{p^x(1-p)^{n-x}}{(x/n)^x(1-x/n)^{n-x}}.$$

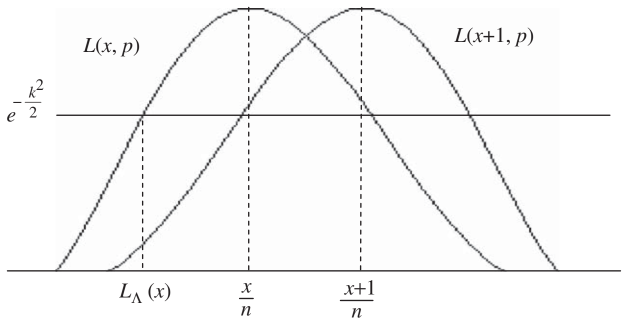

Fig. 1. The plot of $L(x,p)$ and $L(x+1,p)$. The lower endpoint of $CI_{\Lambda_n}(x)$ is $L_\Lambda(x)$. $L_\Lambda(x+1)$ is greater than $L_\Lambda(x)$ if $L(x+1,L_\Lambda(x)) < e^{-k^2/2}$.

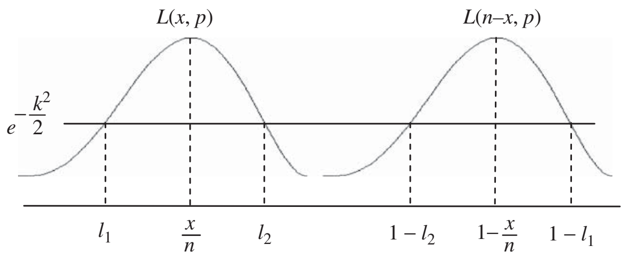

Fig. 2. The plot of $L(x,p)$ and $L(n-x,p)$.

Note that $L(x,p)$ is a unimodal function of p. For a fixed x, we have

$$\frac{L_\Lambda(x)^x(1-L_\Lambda(x))^{(n-x)}}{(x/n)^x(1-x/n)^{(n-x)}} = e^{-k^2/2}.$$

Note that $L_\Lambda(x) < x/n$ because $L(x,x/n) = 1 > e^{-k^2/2}$. If we can demonstrate

$$\frac{L_\Lambda(x)^{(x+1)}(1-L_\Lambda(x))^{(n-x-1)}}{((x+1)/n)^{(x+1)}(1-(x+1)/n)^{(n-x-1)}} < e^{-k^2/2}, \tag{18}$$

then $L_\Lambda(x+1)$ is greater than $L_\Lambda(x)$, see Fig. 1.

The left hand side of (18) can be rewritten as

$$L(x,L_\Lambda(x))\frac{L_\Lambda(x)}{1-L_\Lambda(x)}\frac{(x/n)^x(1-x/n)^{(n-x)}}{((x+1)/n)^{x+1}(1-(x+1)/n)^{n-x-1}}.$$

By the fact $L(x,L_\Lambda(x)) = e^{-k^2/2}$ and $L_\Lambda(x)/(1-L_\Lambda(x)) < (x/n)/(1-x/n)$, to prove (18), we only need to show

$$\frac{(x/n)^{x+1}(1-x/n)^{(n-x-1)}}{((x+1)/n)^{x+1}(1-(x+1)/n)^{n-x-1}} < 1, \tag{19}$$

which is equivalent to

$$\left(\frac{x}{x+1}\right)^{x+1}\left(\frac{n-x}{n-x-1}\right)^{n-x-1} < 1. \tag{20}$$

By Lemma 1, we have $((n-x)/(n-x-1))^{(n-x-1)} < e$ and $(x/(x+1))^{(x+1)} < 1/e$ for all $x = 0,\ldots,n-1$. Therefore (20) holds, which implies that $CI_{\Lambda_n}(x)$ has the monotone lower boundary property.

Note that $L(x,p) = L(n-x,1-p)$. We have $U_\Lambda(n-x) = 1-L_\Lambda(x)$, see Fig. 2.

For $x_2 > x_1$,

$$U_\Lambda(x_2) - U_\Lambda(x_1) = L_\Lambda(n-x_1) - L_\Lambda(n-x_2) > 0$$

because $L_\Lambda(x)$ is an increasing function. Therefore, $CI_{\Lambda_n}(x)$ has the monotone upper boundary property. □

Proposition 6. *The likelihood ratio interval $CI_{\Lambda_n}(X)$ has full coverage property for all n and $k>\sqrt{-2\log\xi}$, where ξ is $\min_{x=0,\ldots,n-1}$ $(d_x^x(1-d_x)^{n-x}/(x/n)^x(1-x/n)^{n-x})$ and*

$$d_x=\frac{\left(\frac{x+1}{n}\right)^{x+1}\left(1-\frac{x+1}{n}\right)^{n-x}}{\left(\frac{x+1}{n}\right)^{x+1}\left(1-\frac{x+1}{n}\right)^{n-x}+\left(\frac{x}{n}\right)^{x}\left(1-\frac{x}{n}\right)^{n-x}\left(1-\frac{x+1}{n}\right)}$$

for $0\le x\le n-1$.

Proof. To show the full coverage property, we need to show that $L_\Lambda(x+1)$ is less than $U_\Lambda(x)$, $U_\Lambda(n)$ is not less than 1 and $L_\Lambda(0)$ is not larger than 0. Note that the function $L(x,p)$ has a maximum value 1 at $p=x/n$. By straightforward calculation, the equation $L(x,p)=L(x+1,p)$ for a fixed x has only one root at $p=d_x$. If k satisfies

$$e^{-k^2/2}<\min_{x=0,\ldots,n-1}(d_x^x(1-d_x)^{n-x}/(x/n)^x(1-x/n)^{n-x}), \tag{21}$$

then $L_\Lambda(x+1)$ is less than $U_\Lambda(x)$ for $x=0,\ldots,n-1$. The condition of k in (21) is equivalent to $k>\sqrt{-2\log\xi}$. $U_\Lambda(n)$ is greater than 1 because $L(n,1)=1$. $L_\Lambda(0)$ is less than 0 because $L(0,0)=1$. Therefore, the proof is complete. □

Remark 2. I have done numerical calculations to approach ξ in Proposition 6 and found that the minimum value always happens at $x=0$ and ξ is an increasing function in n. When n is 2, ξ is 0.64. By the numerical calculations, the condition of k in Proposition 6 is $k>0.945$ for all $n\ge 2$. The lower bound of k can be smaller if n increases.

4. Conclusion

In this paper, the monotone boundary property and the full coverage property for the Wilson, Agresti–Coull and the likelihood ratio confidence intervals of a binomial proportion are shown to hold for any sample size or sample size greater than 1. Although the algorithm proposed in Wang (2007) has been used in Wang (2007) and Wang (2009a) to calculate the minimum coverage probability for the three intervals for some sample sizes, there were no explicit demonstrations that the algorithm can be used for most sample sizes before until this study. With the results in this paper, the procedure can be directly used to calculate the minimum coverage probabilities of the three important confidence intervals without the necessity of checking the conditions in Assumption 1.

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