國立交通大學

機械工程學系

博士論文

藉變強度耦合、適應控制及部份區域穩定理論

之渾沌系統同步與非同步

Unsynchronizability and Synchronization of Chaotic System

by Variable Strength Coupling, by Adaptive Control

and by Partial Stability Theory

研 究 生：鄭普建

指導教授：戈正銘　教授

中華民國九十六年十二月

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研 究 生：鄭普建　　Student：Pu-Chien Tsen

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摘要

本論文由三部分構成(1) 兩個耦合渾沌系統不可能完全同步定理及一個渾沌同步的充份必要定理，兩個耦合系統不可能廣義同步定理。(2) 藉由變強度線性耦合及級數形式之 Lyapunov 函數導數相同系統之渾沌同步及不相同系統之適應渾沌同步。(3) 應用部分區域穩定性理論之廣義渾沌同步及渾沌控制。

Unsynchronizability and Synchronization of Chaotic System by Variable Strength Coupling, by Adaptive Control and by Partial Stability Theory

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Abstract

This thesis consists of three parts：(1) two theorems of unsynchronizability and synchronization for coupled chaotic systems and two theorems of generalized unsynchronization for coupled chaotic systems. (2) chaos synchronization by variable strength linear coupling and Lyapunov function derivative in series form and adaptive chaos synchronization by variable strength linear coupling. (3) chaos generalized synchronization and chaos control by partial region stability theory.

CONTENTS

List of Figures

Chapter 1

Introduction

In recent years, synchronization in chaotic dynamic system is a very interesting problem and has been widely studied [1-8]. Synchronization means that the state variables of a response system approach eventually to that of a drive system. There are many control techniques to synchronize chaotic systems, such as linear error feedback control, adaptive control, active control [9-19]. Besides, generalized synchronization also has been investigated in various fields. Generalized synchronization means that there is a functional relation between the states of driving system and response system.

Recently, the synchronization criteria of unidirectional coupled chaotic systems by partial stability theory are presented [20]. In Chapter 2 of this thesis, we propose two theorems which give the criteria of unsynchronizability for two different chaotic dynamic systems. Chen system, Rössler system and Duffing system with corresponding new chaotic systems proposed are presented as simulated examples for these two theorems [21-22].

In Chapter 3, we propose two theorems which give the criteria of generalized unsynchronization for two different chaotic dynamic systems with whatever large strength of linear coupling. Chen system and Rössler system with corresponding new chaotic systems proposed are presented as simulated examples for these two theorem .

In Chapter 4, a new general strategy to achieve chaos synchronization by variable strength linear coupling without another active control is proposed, in which Lyapunov function derivative in series form is first used. This method can give either local synchronization which is usually good enough or global synchronization which is usually an unnecessary high demand [23-25]. Lorenz system, Duffing system, Rössler system and Hyper-Rössler system are presented as simulated examples.

There are many control techniques to synchronize chaotic systems, but most of them are based on the exact knowledge of the system structure and parameters. In practice, some or all of the system parameters are uncertain, adaptive control method in used. In Chapter 5, we propose a general strategy to achieve adaptive chaos synchronization by variable strength linear coupling solely without using another active control which is usually rather complex. Furthermore, Lyapunov function derivative in series form is first used, which is easier to be obtained than the traditional negative sum of the square of error variables. Lorenz system, Duffing system and Rössler system are presented as simulation examples.

In Chapter 6, a new chaos generalized synchronization strategy by partial region stability theory is proposed [26-27]. By using the theory of stability on partial region the Lyapunov function is a simple linear homogeneous function of states and the controllers are simpler and have less simulation error because they are in lower order than that of traditional controllers where the stability of solutions on the whole neighborhood region of the origin is demanded. Lorenz system and Rössler system are used as simulated examples.

In Chapter 7, a new scheme to achieve chaos control by partial region stability theory is proposed [28]. By using the theory of stability on partial region the Lyapunov function is a simple linear homogeneous function of states and the controllers are simpler and have less simulation error.

In Chapter 8, conclusions of this thesis are given.

Chapter 2

The Theorems of Unsynchronizability and Synchronization for Coupled Chaotic Systems

2.1 Preliminary

Synchronization in chaotic dynamic system is a very interesting problem and has been widely studied in these years. Synchronization means that the state variables of a response system approach eventually to that of a drive system. There are many control techniques to synchronize chaotic systems, such as linear error feedback control, adaptive control, active control. Recently, the synchronization criteria of unidirectional coupled chaotic systems by partial stability theory are presented. In this Chapter, we propose two theorems which give the criteria of unsynchronizability for two different chaotic dynamic systems. Chen system and a new chaotic system which we proposed are presented as simulated examples for the first theorem. Rössler system and Duffing system with two corresponding new chaotic systems proposed are presented as simulated examples for the second theorem.

2.2 Two Theorems of Unsynchronizability

Consider the following nonautomonous systems

$$\dot{\mathbf{x}}_1 = \mathbf{f}(t,\ \mathbf{x}_1) \tag{2.1}$$

where $\mathbf{x}_1 \in \mathbf{R}^n$, $\mathbf{f}:\Omega_1 \subset \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$. Eq. (2.1) is considered as a master system. A slave system is given by

$$\dot{\mathbf{x}}_2 = \mathbf{g}(t,\ \mathbf{x}_2) \tag{2.2}$$

where $\mathbf{x}_2 \in \mathbf{R}^n$, $\mathbf{g}:\Omega_2 \subset \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$. Both $\mathbf{f}$ and $\mathbf{g}$ satisfy Lipschitz condition and $\mathbf{f}(t,\ \mathbf{0})=\mathbf{g}(t,\mathbf{0})=\mathbf{0}$. Ω_1, Ω_2 are domains containing the origin. Assume that the solutions of Eqs. (2.1) and (2.2) are bounded then they must exist for infinite time. That is, for given $(t_0,\ \mathbf{x}_{10},\ \mathbf{x}_{20})\in\Omega_1\bigcap\Omega_2$ the solutions $\mathbf{x}_1(t,\ t_0,\ \mathbf{x}_{10},\ \mathbf{x}_{20})$, $\mathbf{x}_2(t,\ t_0,\ \mathbf{x}_{10},\ \mathbf{x}_{20})$ of Eqs. (2.1) and (2.2) exist for $t\geq t_0$. If $\mathbf{f}(t,\ \mathbf{x})=\mathbf{g}(t,\ \mathbf{x})$, system (2.1) and (2.2) are two identical systems. When $\mathbf{f}(t,\ \mathbf{x})\neq\mathbf{g}(t,\ \mathbf{x})$, they are two different systems.

Now we consider the following unidirectional nonautonomous coupled system

$$\begin{aligned}\dot{\mathbf{x}}_1&=\mathbf{f}(t,\ \mathbf{x}_1)\\ \dot{\mathbf{x}}_2&=\mathbf{g}(t,\ \mathbf{x}_2)+\mathbf{U}(t,\ \mathbf{x}_1,\mathbf{x}_2)\end{aligned}\qquad(2.3)$$

where $\mathbf{U}(t,\ \mathbf{x}_1,\mathbf{x}_2)$ is a coupled term. In order to discuss the synchronization of $\mathbf{x}_1$ and $\mathbf{x}_2$, define $\mathbf{e}=\mathbf{x}_2-\mathbf{x}_1$ as the state error. Error equation can be written as

$$\dot{\mathbf{e}}=\mathbf{g}(t,\ \mathbf{e}+\mathbf{x}_1)-\mathbf{f}(t,\ \mathbf{x}_1)+\mathbf{U}(t,\ \mathbf{x}_1,\mathbf{e}+\mathbf{x}_1)\qquad(2.4)$$

Now the first theorem will be given for a special case of Eqs. (2.3). Consider unidirectional coupled nonautonomous systems as

$$\begin{aligned}\dot{\mathbf{x}}_1&=\mathbf{f}(t,\mathbf{x}_1)\\ \dot{\mathbf{x}}_2&=\mathbf{g}(t,\mathbf{x}_2)+\mathbf{\Gamma}(\mathbf{x}_1-\mathbf{x}_2)\end{aligned}\qquad(2.5)$$

where $\mathbf{f}$ and $\mathbf{g}$ satisfy Lipschitz condition, and the Lipschitz constant of $\mathbf{g}$ is L. $\mathbf{\Gamma}\in M_{n\times n}$ is a constant diagonal matrix with positive entries, represents the strength of the linear coupling term $\mathbf{x}_1-\mathbf{x}_2$. Since $\mathbf{e}=\mathbf{x}_2-\mathbf{x}_1$, the error dynamic equation can be obtained as

$$\dot{\mathbf{e}}=\mathbf{g}(t,\mathbf{e}+\mathbf{x}_1(t))-\mathbf{f}(t,\mathbf{x}_1(t))-\ \mathbf{\Gamma}\mathbf{e}\qquad(2.6)$$

which is a nonautonomous system of differential equations for state $\mathbf{e}$ and has a null solution $\mathbf{e}=0,\mathbf{x}_1=0$. Now we give a definition of unsynchronizability:

Definition If no positive constant C can be found such that $\mathbf{e}\to 0$ as $t\to\infty$ for all $\|\mathbf{e}(t_0)\|<C$, systems (2.5) are unsynchronizable.

Theorem 2.1 Two different dynamic systems in Eq. (2.5) are unsynchronizable for however large coupling strength Γ with positive entries, if $f_i(t,\mathbf{x}_1) \le g_i(t,\mathbf{x}_1)$ $(i=1,\ldots,n)$ in $\Omega_1 \cap \Omega_2$, and $\|\mathbf{f}(t,\mathbf{x}_1)-\mathbf{g}(t,\mathbf{x}_1)\|>0$ except at origin, for any solution $\mathbf{x}_1(t)$.

Proof. Choose a Lyapunov function $V(\mathbf{e}) = e_1 e_2 \cdots e_n$ which is positive in quadrant $e_1>0, e_2>0, \cdots, e_n>0$, then $\dot{V}$ along any state trajectory of system (2.6) [25] becomes:

$$
\begin{aligned}
\dot{V} &= e_2 e_3 \cdots e_n \dot{e}_1 + e_1 e_3 \cdots e_n \dot{e}_2 + \cdots + e_1 e_2 \cdots e_{n-1} \dot{e}_n \\
&= e_2 e_3 \cdots e_n [g_1(t,\mathbf{e}+\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + e_1 e_3 \cdots e_n [g_2(t,\mathbf{e}+\mathbf{x}_1) - f_2(t,\mathbf{x}_1) - \Gamma_2 e_2] \\
&\quad \cdots + e_1 e_2 \cdots e_{n-1} [g_n(t,\mathbf{e}+\mathbf{x}_1) - f_n(t,\mathbf{x}_1) - \Gamma_n e_n] \\
&= e_2 e_3 \cdots e_n [g_1(t,\mathbf{e}+\mathbf{x}_1) - g_1(t,\mathbf{x}_1) + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots + e_1 e_2 \cdots e_{n-1} [\\
&\quad g_n(t,\mathbf{e}+\mathbf{x}_1) - g_n(t,\mathbf{x}_1) + g_n(t,\mathbf{x}_1) - f_n(t,\mathbf{x}_1) - \Gamma_n e_n]
\end{aligned}
$$

When $e_1>0, e_2>0, \cdots, e_n>0$, we have

$$
\begin{aligned}
\dot{V} &\le e_2 e_3 \cdots e_n [|g_1(t,\mathbf{e}+\mathbf{x}_1) - g_1(t,\mathbf{x}_1)| + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots + e_1 e_2 \cdots e_{n-1} [\\
&\quad |g_n(t,\mathbf{e}+\mathbf{x}_1) - g_n(t,\mathbf{x}_1)| + g_n(t,\mathbf{x}_1) - f_n(t,\mathbf{x}_1) - \Gamma_n e_n] \\
&\le e_2 e_3 \cdots e_n [L\|\mathbf{e}\| + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots
\end{aligned}
\tag{2.7}
$$

where $|g_1(t,\mathbf{e}+\mathbf{x}_1) - g_1(t,\mathbf{x}_1)| \le L\|\mathbf{e}\|$ follows the Lipschitz condition. When $\|\mathbf{e}\| \gg 1$, the terms of lower degree of error components $e_2 e_3 \cdots e_n [g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1)]$, $e_1 e_3 \cdots e_n [g_2(t,\mathbf{x}_1) - f_2(t,\mathbf{x}_1)]$, $\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \le e_2 e_3 \cdots e_n [L\|\mathbf{e}\| - \Gamma_1 e_1] + e_1 e_3 \cdots e_n [L\|\mathbf{e}\| - \Gamma_2 e_2] + \cdots \tag{2.8}$$

For sufficient large Γ_i, $\dot{V}$ can be negative in the quadrant $e_1>0$, $e_2>0$, $\cdots$, $e_n>0$. So the state point tends to decrease $\|\mathbf{e}(t)\|$ with time when $\|\mathbf{e}_0\|$ is sufficiently large. When $\|\mathbf{e}\| \ll 1$, the proof is as follows. Now when

$e_1 > 0, e_2 > 0, \cdots, e_n > 0$, $\dot{V}$ is expressed as

$$\begin{aligned}\dot{V} &\geq e_2 e_3 \cdots e_n[-|g_1(t,\mathbf{e}+\mathbf{x}_1) - g_1(t,\mathbf{x}_1)| + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots \\ &\geq e_2 e_3 \cdots e_n[-L\|\mathbf{e}\| + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots\end{aligned} \tag{2.9}$$

When $\|\mathbf{e}\| \ll 1$, the terms of higher degree $e_2 e_3 \cdots e_n[-L\|\mathbf{e}\| - \Gamma_1 e_1], \cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \geq e_2 e_3 \cdots e_n[g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1)] + e_1 e_3 \cdots e_n[g_2(t,\mathbf{x}_1) - f_2(t,\mathbf{x}_1)] + \cdots \tag{2.10}$$

By the condition $\|\mathbf{f}(t,\mathbf{x}_1) - \mathbf{g}(t,\mathbf{x}_1)\| > 0$, $f_i(t,\mathbf{x}_1) = g_i(t,\mathbf{x}_1)$ $(i = 1,\cdots,n)$ do not occur simultaneously except at the origin $\mathbf{x}_1 = \mathbf{0}$. Therefore the right-hand side of above inequality is positive, i.e. $\dot{V}$ is positive in region D of Fig. 2.1, which is the quadrant $e_1 > 0$, $e_2 > 0$, $\cdots$, $e_n > 0$ of the neighborhood of the origin.

Choose $r > 0$ such that for the ball $B_r = \{\mathbf{e} \in R^n \mid \|\mathbf{e}\| \leq r\}$, we have

$$D = \{\mathbf{e} \in B_r | V(\mathbf{e}) > 0\} \tag{2.11}$$

of which the boundary is the surface $V(\mathbf{e}) = 0$ and the sphere $\|\mathbf{e}\| = r$. Since $V(\mathbf{0}) = 0$, the origin lies on the boundary of D inside B_r. The point $\mathbf{e}_0$ is in the interior of D and $V(\mathbf{e}_0) = b > 0$. Now we prove that the trajectory $\mathbf{e}(t)$ started at $\mathbf{e}(0) = \mathbf{e}_0$ must leave the set D, i.e. the trajectory must leave the neighborhood of origin, $\mathbf{e}$ cannot approach zero. To see this point, notice that as long as $\mathbf{e}(t)$ is inside D, $V(\mathbf{e}(t)) \geq b$ since $\dot{V}(\mathbf{e}) > 0$ in D. Let

$$\beta = \min\{\dot{V}(\mathbf{e}) |\ \mathbf{e} \in D \ and \ V(\mathbf{e}) \geq b\} \tag{2.12}$$

which exists since the continuous function $\dot{V}(\mathbf{e})$ has a minimum over the compact set $\{\mathbf{e} \in D \ and \ V(\mathbf{e}) \geq b\} = \{\mathbf{e} \in B_r, and \ V(\mathbf{e}) \geq b\}$ [29]. Then, $\beta > 0$ and

$$V(\mathbf{e}(t)) = V(\mathbf{e}_0) + \int_0^t \dot{V}(\mathbf{e}(s))ds \geq b + \int_0^t \beta ds = b + \beta t \tag{2.13}$$

This inequality shows that $\mathbf{e}(t)$ cannot stay forever in D because $V(\mathbf{e})$ is bounded on D. Now, $\mathbf{e}(t)$ cannot leave D through the surface $V(\mathbf{e})=0$ since $V(\mathbf{e}(t))\geq b$. Hence, it must leave D through the sphere $\|\mathbf{e}\|=r$, i.e. it must leave the neighborhood of the origin, $\mathbf{e}$ can never approach zero. Two different dynamic systems in Eq.(2.5) are unsynchronizable for however large Γ.

Theorem 2.2 Two different dynamic systems in Eq. (2.5) is unsynchronizable for however large coupling strength Γ, if $f_i(t,\mathbf{x}_1)\geq g_i(t,\mathbf{x}_1)$ $(i=1,\ldots,n)$ in $\Omega_1\cap\Omega_2$, and $\|f(t,\mathbf{x}_1)-g(t,\mathbf{x}_1)\|>0$ except at origin, for any solution $\mathbf{x}_1(t)$.

Proof. Choose a Lyapunov function $V(\mathbf{e})=e_1e_2\cdots e_n$, then $\dot{V}$ along any state trajectory of system (2.6) becomes:

Case 1. When n is odd, $V(\mathbf{e})$ is negative in quadrant $e_1<0, e_2<0,\cdots e_n<0$.

$$
\begin{aligned}
\dot{V} &= e_2e_3\cdots e_n\dot{e}_1 + e_1e_3\cdots e_n\dot{e}_2+\cdots+e_1e_2\cdots e_{n-1}\dot{e}_n\\
&= e_2e_3\cdots e_n[g_1(t,\mathbf{e}+\mathbf{x}_1)-g_1(t,\mathbf{x}_1)+g_1(t,\mathbf{x}_1)-f_1(t,\mathbf{x}_1)-\Gamma_1e_1]+\cdots+e_1e_2\cdots e_{n-1}[\\
&\quad g_n(t,\mathbf{e}+\mathbf{x}_1)-g_n(t,\mathbf{x}_1)+g_n(t,\mathbf{x}_1)-f_n(t,\mathbf{x}_1)-\Gamma_ne_n]
\end{aligned}
$$

When $e_1<0, e_2<0,\cdots e_n<0$, we have

$$
\begin{aligned}
\dot{V} &\geq e_2e_3\cdots e_n[-|g_1(t,\mathbf{e}+\mathbf{x}_1)-g_1(t,\mathbf{x}_1)|+g_1(t,\mathbf{x}_1)-f_1(t,\mathbf{x}_1)-\Gamma_1e_1]+\cdots+e_1e_2\cdots e_{n-1}[\\
&\quad -|g_n(t,\mathbf{e}+\mathbf{x}_1)-g_n(t,\mathbf{x}_1)|+g_n(t,\mathbf{x}_1)-f_n(t,\mathbf{x}_1)-\Gamma_ne_n]\\
&\geq e_2e_3\cdots e_n[-L\|\mathbf{e}\|+g_1(t,\mathbf{x}_1)-f_1(t,\mathbf{x}_1)-\Gamma_1e_1]+\cdots
\end{aligned}
\tag{2.14}
$$

where $|g_1(t,\mathbf{e}+\mathbf{x}_1)-g_1(t,\mathbf{x}_1)|\leq L\|\mathbf{e}\|$ follows the Lipschitz condition. When $\|\mathbf{e}\|\gg 1$, the terms of lower degree of error components $e_2e_3\cdots e_n[g_1(t,\mathbf{x}_1)-f_1(t,\mathbf{x}_1)]$, $e_1e_3\cdots e_n[g_2(t,\mathbf{x}_1)-f_2(t,\mathbf{x}_1)]$, $\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$
\begin{aligned}
\dot{V} &\geq e_2e_3\cdots e_n[-L\|\mathbf{e}\|-\Gamma_1e_1]+e_1e_3\cdots e_n[-L\|\mathbf{e}\|-\Gamma_2e_2]+\cdots\\
&= -e_2e_3\cdots e_n[L\|\mathbf{e}\|+\Gamma_1e_1]-e_1e_3\cdots e_n[L\|\mathbf{e}\|+\Gamma_2e_2]+\cdots
\end{aligned}
\tag{2.15}
$$

For sufficient large Γ_i, $\dot{V}$ can be positive in the quadrant $e_1 < 0$, $e_2 < 0$, $\cdots$, $e_n < 0$. So the state point tends to decrease $\|\mathbf{e}(t)\|$ with time when $\|\mathbf{e}_0\|$ is sufficiently large. When $\|\mathbf{e}\| \ll 1$, the proof is as follows. Now when $e_1 < 0$, $e_2 < 0$, $\cdots e_n < 0$, $\dot{V}$ is expressed as

$$\begin{aligned}\dot{V} &\le e_2 e_3 \cdots e_n [|g_1(t, \mathbf{e} + \mathbf{x}_1) - g_1(t, \mathbf{x}_1)| + g_1(t, \mathbf{x}_1) - f_1(t, \mathbf{x}_1) - \Gamma_1 e_1] + \cdots \\ &\le e_2 e_3 \cdots e_n [L\|\mathbf{e}\| + g_1(t, \mathbf{x}_1) - f_1(t, \mathbf{x}_1) - \Gamma_1 e_1] + \cdots\end{aligned} \tag{2.16}$$

When $\|\mathbf{e}\| \ll 1$, the terms of higher degree $e_2 e_3 \cdots e_n [L\|\mathbf{e}\| - \Gamma_1 e_1]$, $\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \le e_2 e_3 \cdots e_n [g_1(t, \mathbf{x}_1) - f_1(t, \mathbf{x}_1)] + e_1 e_3 \cdots e_n [g_2(t, \mathbf{x}_1) - f_2(t, \mathbf{x}_1)] + \cdots \tag{2.17}$$

By the condition $\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{g}(t, \mathbf{x}_1)\| > 0$, $f_i(t, \mathbf{x}_1) = g_i(t, \mathbf{x}_1)$ $(i = 1, \cdots, n)$ do not occur simultaneously except at $\mathbf{x}_1 = \mathbf{0}$. Therefore the right-hand side of above inequality is negative, i.e. $\dot{V}$ is negative in region D of Fig. 2.2, which is the quadrant $e_1 < 0$, $e_2 < 0$, $\cdots$, $e_n < 0$ of the neighborhood of the origin.

Choose $r > 0$ such that for the ball $B_r = \{\mathbf{e} \in R^n \mid \|\mathbf{e}\| \le r\}$, we have

$$D = \{\mathbf{e} \in B_r \mid V(\mathbf{e}) < 0\} \tag{2.18}$$

By the similar reasoning as that in the latter part of the proof for Theorem1, we can prove that the state trajectory started from D must leave the neighborhood of the origin, $\mathbf{e}$ can never approach zero. Two different dynamic systems in Eq. (2.5) are unsynchronizable for however large Γ.

Case 2. When n is even, $V(\mathbf{e})$ is positive in quadrant $e_1 < 0, e_2 < 0, \cdots e_n < 0$.

$$\dot{V} = e_2e_3\cdots e_n\dot{e}_1 + e_1e_3\cdots e_n\dot{e}_2 + \cdots + e_1e_2\cdots e_{n-1}\dot{e}_n$$
$$= e_2e_3\cdots e_n[g_1(t,\mathbf{e}+\mathbf{x}_1) - g_1(t,\mathbf{x}_1) + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots + e_1e_2\cdots e_{n-1}[$$
$$g_n(t,\mathbf{e}+\mathbf{x}_1) - g_n(t,\mathbf{x}_1) + g_n(t,\mathbf{x}_1) - f_n(t,\mathbf{x}_1) - \Gamma_n e_n]$$

When $e_1 < 0, e_2 < 0, \cdots e_n < 0$, we have

$$\dot{V} \le e_2e_3\cdots e_n[-|g_1(t,\mathbf{e}+\mathbf{x}_1) - g_1(t,\mathbf{x}_1)| + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots + e_1e_2\cdots e_{n-1}[$$
$$-|g_n(t,\mathbf{e}+\mathbf{x}_1) - g_n(t,\mathbf{x}_1)| + g_n(t,\mathbf{x}_1) - f_n(t,\mathbf{x}_1) - \Gamma_n e_n]$$
$$\le e_2e_3\cdots e_n[-L\|\mathbf{e}\| + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots$$

(2.21)

where $|g_1(t,\mathbf{e}+\mathbf{x}_1) - g_1(t,\mathbf{x}_1)| \le L\|\mathbf{e}\|$ follows the Lipschitz condition. When $\|\mathbf{e}\| \gg 1$, the terms of lower degree of error components $e_2e_3\cdots e_n[g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1)]$, $e_1e_3\cdots e_n[g_2(t,\mathbf{x}_1) - f_2(t,\mathbf{x}_1)]$, $\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\begin{aligned}\dot{V} &\le e_2e_3\cdots e_n[-L\|\mathbf{e}\| - \Gamma_1 e_1] + e_1e_3\cdots e_n[-L\|\mathbf{e}\| - \Gamma_2 e_2] + \cdots \\ &= -e_2e_3\cdots e_n[L\|\mathbf{e}\| + \Gamma_1 e_1] - e_1e_3\cdots e_n[L\|\mathbf{e}\| + \Gamma_2 e_2] + \cdots\end{aligned} \tag{2.22}$$

For sufficient large Γ_i, $\dot{V}$ can be negative in the quadrant $e_1 < 0$, $e_2 < 0$, $\cdots$, $e_n < 0$. So the state point tends to decrease $\|\mathbf{e}(t)\|$ with time when $\|\mathbf{e}_0\|$ is sufficiently large. When $\|\mathbf{e}\| \ll 1$, the proof is as follows. Now when $e_1 < 0$, $e_2 < 0$, $\cdots$, $e_n < 0$, $\dot{V}$ is expressed as

$$\begin{aligned}\dot{V} &\ge e_2e_3\cdots e_n[|g_1(t,\mathbf{e}+\mathbf{x}_1) - g_1(t,\mathbf{x}_1)| + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots \\ &\ge e_2e_3\cdots e_n[L\|\mathbf{e}\| + g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1) - \Gamma_1 e_1] + \cdots\end{aligned} \tag{2.23}$$

When $\|\mathbf{e}\| \ll 1$, the terms of higher degree $e_2e_3\cdots e_n[L\|\mathbf{e}\| - \Gamma_1 e_1]$, $\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \ge e_2e_3\cdots e_n[g_1(t,\mathbf{x}_1) - f_1(t,\mathbf{x}_1)] + e_1e_3\cdots e_n[g_2(t,\mathbf{x}_1) - f_2(t,\mathbf{x}_1)] + \cdots \tag{2.24}$$

By the condition $\|\mathbf{f}(t,\mathbf{x}_1)-\mathbf{g}(t,\mathbf{x}_1)\|>0$, $f_i(t,\mathbf{x}_1)=g_i(t,\mathbf{x}_1)$ $(i=1,\cdots,n)$ do not occur simultaneously except at $\mathbf{x}_1=\mathbf{0}$. Therefore the right hand side of above inequality is positive, i.e. $\dot{V}$ is positive in region D of Fig. 2.2 which is the quadrant $e_1<0$, $e_2<0$, $\cdots$, $e_n<0$ of the neighborhood of the origin.

By the same reasoning as that in the latter part of the proof for Theorem 1, we can prove that the state trajectory started from the neighborhood of the origin in the quadrant $e_1<0$, $e_2<0$, $\cdots$, $e_n<0$ must leave the neighborhood and can never approach zero. Two different dynamic systems in Eq. (2.5) are unsynchronizable for however large Γ_i.

It was proved that for sufficient large Γ, $\mathbf{f}(t,\ \mathbf{x})=\mathbf{g}(t,\ \mathbf{x})$ is the sufficient condition for synchronization of systems (2.5) [20]. By the above two theorems, $\mathbf{f}(t,\ \mathbf{x})=\mathbf{g}(t,\ \mathbf{x})$ is enhanced as the necessary and sufficient condition for synchronization of systems (2.5):

Theorem 2.3 If in $\Omega_1\cap\Omega_2$, $\|f(t,\mathbf{x})-g(t,\mathbf{x})\|>0$ except at $\mathbf{x}_1=\mathbf{0}$, we have $f_i(t,\mathbf{x})\geq g_i(t,\mathbf{x})$, $f_i(t,\mathbf{x})\leq g_i(t,\mathbf{x})$ or $f_i(t,\mathbf{x})=g_i(t,\mathbf{x})$ $(i=1,\ldots,n)$. With sufficient large Γ, the necessary and sufficient condition for synchronization of systems (2.5) is $f_i(t,\mathbf{x})=g_i(t,\mathbf{x})$, $(i=1,\ldots,n)$ in $\Omega=\Omega_1=\Omega_2$.

2.3 Simulated Examples

An example for the first theorem is Chen system with a new chaotic system proposed. Consider the following unidirectional coupled systems with linear coupling in the form of Eq. (2.5):

$$\begin{aligned}\dot{x}&=a(y-x)\\ \dot{y}&=(c-a)x-xz+cy\\ \dot{z}&=xy-bz\end{aligned}\qquad(2.25a)$$

$$\begin{aligned}\dot{\tilde{x}} &= a(\tilde{y}-\tilde{x})+\sin^2\tilde{x}+\gamma(x-\tilde{x})\\ \dot{\tilde{y}} &= (c-a)\tilde{x}-\tilde{x}\tilde{z}+c\tilde{y}+\tilde{x}^2+\gamma(y-\tilde{y})\\ \dot{\tilde{z}} &= \tilde{x}\tilde{y}-b\tilde{z}+\tilde{x}^2+\gamma(z-\tilde{z})\end{aligned} \tag{2.25b}$$

where $\gamma = 300$ which is sufficiently large. Eq. (2.25a) is Chen system and Eq. (2.25b) is a new chaotic system which we proposed. The chaotic attractor and Lyapunov exponent diagrams for system (2.25a) and (2.25b) without coupling term are shown in Fig. 2.3, Fig. 2.4, Fig. 2.5 and Fig. 2.6. For initial states (0.5,1,5), (30,20,18) and system parameters $a = 35, b = 3$ and $c = 28$, three state errors versus time are shown in Fig. 2.7 and Fig. 2.8. Fig. 2.7 shows that state errors decreases with time when state error is large, but one can clearly find in Fig. 2.8 that the errors cannot approach zero as time evolves.

An example for the second theorem is Rössler system with a new chaotic system proposed. Consider the following unidirectional coupled systems with linear coupling in the form of Eq. (2.5):

$$\begin{aligned}\dot{x} &= -y-z\\ \dot{y} &= x+ay\\ \dot{z} &= b+z(x-c)\end{aligned} \tag{2.26a}$$

$$\begin{aligned}\dot{\tilde{x}} &= -\tilde{y}-\tilde{z}-\sin^2\tilde{y}+\gamma(x-\tilde{x})\\ \dot{\tilde{y}} &= \tilde{x}+a\tilde{y}-\sin^2\tilde{y}+\gamma(y-\tilde{y})\\ \dot{\tilde{z}} &= b+\tilde{z}(\tilde{x}-c)-\sin^2\tilde{z}+\gamma(z-\tilde{z})\end{aligned} \tag{2.26b}$$

where $\gamma = 300$. The Lyapunov exponent diagrams for system (2.26a) and (2.26b) without coupling term are shown in Fig. 2.9 and Fig. 2.10. For initial states (20,10,25), (2.5,2,2.5) and system parameter $a = 0.2, b = 0.2$ and $c = 5.7$, three state errors versus time are shown in Fig. 2.11. Fig. 2.11 shows that state errors decreases with time when state error is large, but one can clearly find that the errors cannot approach zero as time evolves.

Finally, second example for the second theorem is Duffing system with a new chaotic system proposed for $n = 2$. Consider the following unidirectional coupled systems with linear coupling in the form of Eq. (2.5):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + \alpha x_1 - \beta x_1^3 + a\cos\omega t \end{aligned} \tag{2.27a}$$

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 + \gamma(x_1 - \tilde{x}_1) \\ \dot{\tilde{x}}_2 &= -\delta \tilde{x}_2 + \alpha \tilde{x}_1 - \beta \tilde{x}_1^3 + a\cos\omega t - 0.05 x_2^2 + \gamma(x_2 - \tilde{x}_2) \end{aligned} \tag{2.27b}$$

where $\gamma = 30$. The chaotic attractor and Lyapunov exponent diagrams for system (2.27a) and (2.27b) without coupling term are shown in Fig. 2.12 and Fig. 2.13. For initial states (2,2), (0.1,0) and system parameters $\delta = 0.15$, $\alpha = \beta = \omega = 1$ and $a = 3$, three state errors versus time are shown in Fig. 2.14. Fig. 2.14 shows that state errors decreases with time when state error is large, but one can clearly find that the errors cannot approach zero as time evolves.

2.4 Summary

In this Chapter two theorems which give the criteria of unsynchronizability for two different chaotic dynamic systems are presented. A sufficient criterion for synchronization is enhanced to necessary and sufficient one. Three simulated examples are given to illustrate the theory.

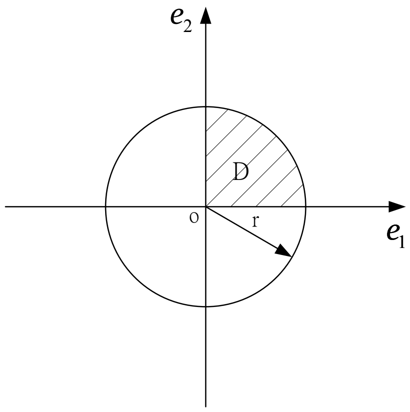

Fig. 2.1 D region for n=2.

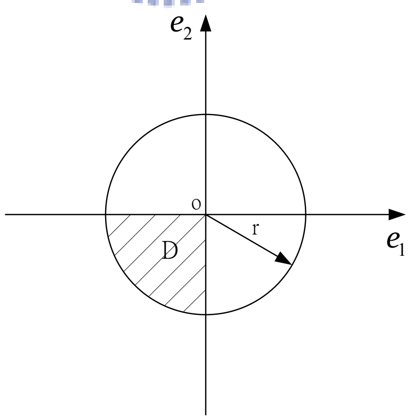

Fig. 2.2 D region for n=2.

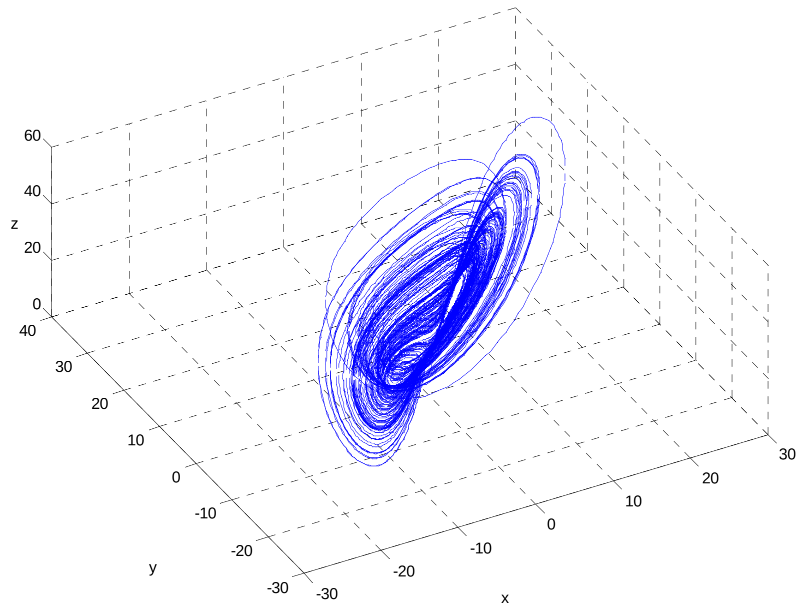

Fig. 2.3 Chaotic attractor for Chen system (2.25a), with $a = 35, b = 3$ and $c = 28$, initial condition (0.5,1,5).

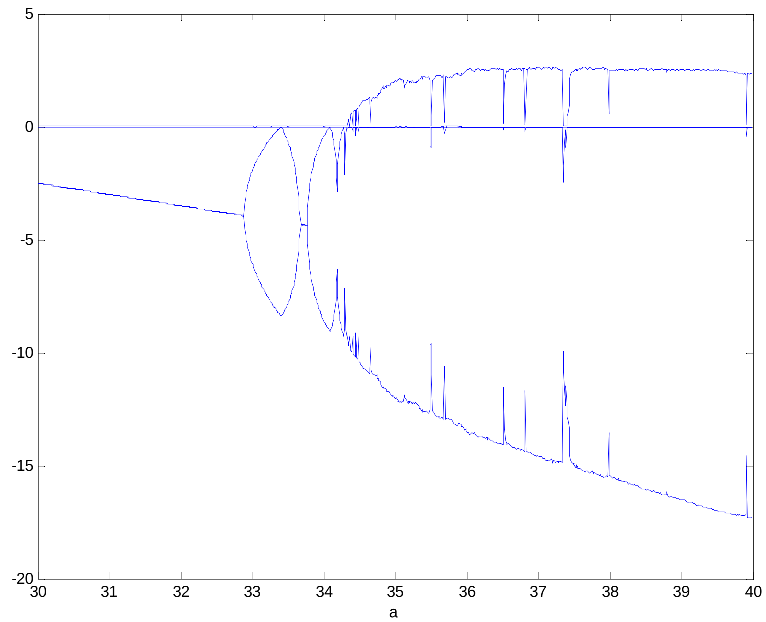

Fig. 2.4 Lyapunov exponents for Chen system (2.25a), with $b = 3$ and $c = 28$, initial condition (0.5,1,5).

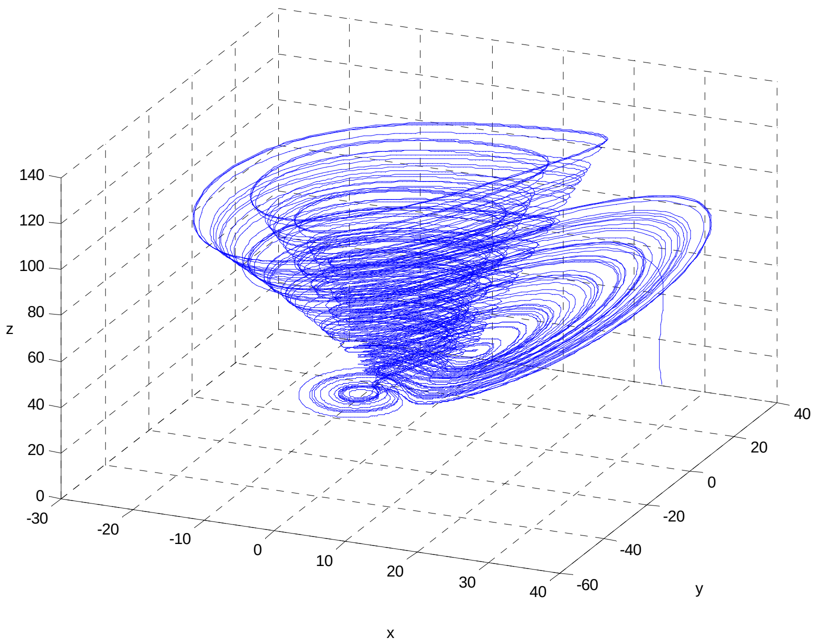

Fig. 2.5 Chaotic attractor for chaotic system (2.25b) , with $a = 35, b = 3$ and $c = 28$, initial condition (30,20,18).

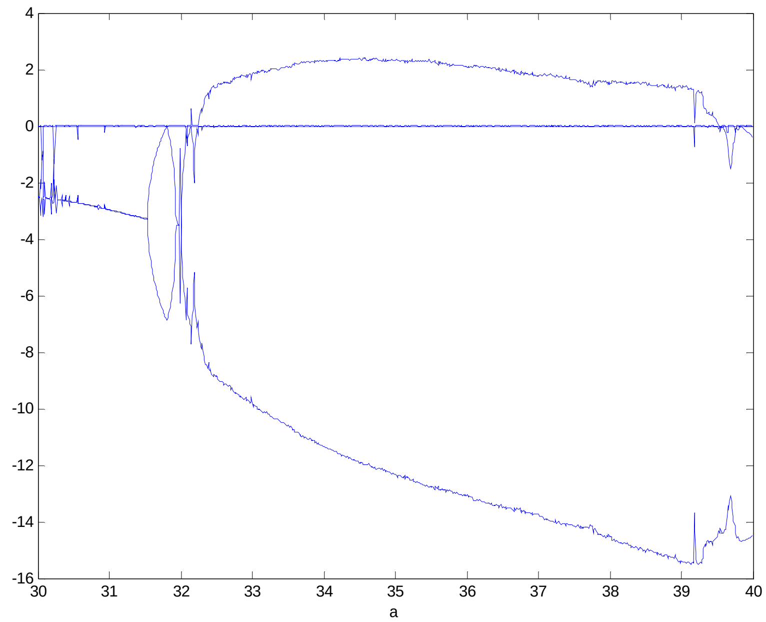

Fig. 2.6 Lyapunov exponents for chaotic system (2.25b) , with $b = 3$ and $c = 28$, initial condition (30,20,18).

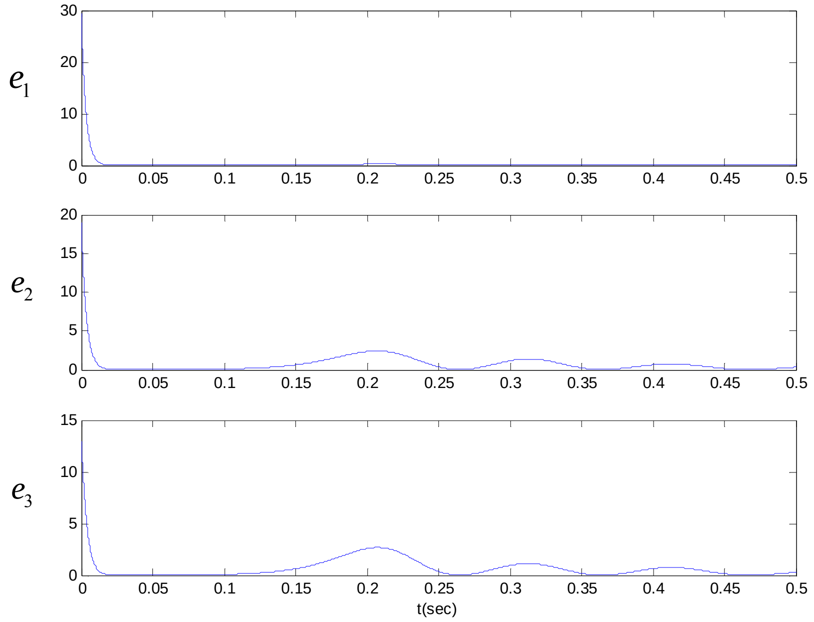

Fig. 2.7 State errors versus time for unidirectional coupled systems (2.25) , with $a = 35, \quad b = 3$ and $c = 28$, initial conditions (0.5,1,5), (30,20,18).

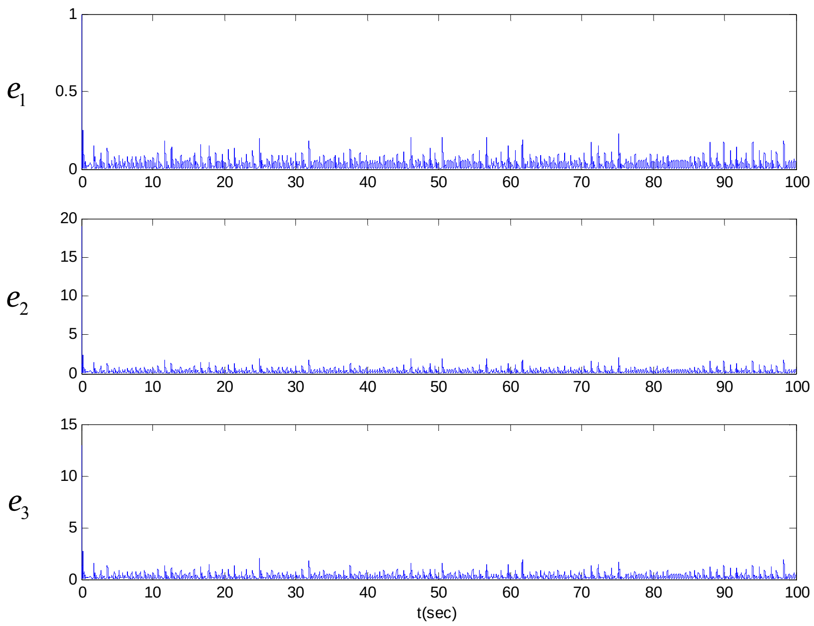

Fig. 2.8 State errors versus time for unidirectional coupled systems (2.25) , with $a = 35, b = 3$ and $c = 28$, initial conditions (0.5,1,5), (30,20,18).

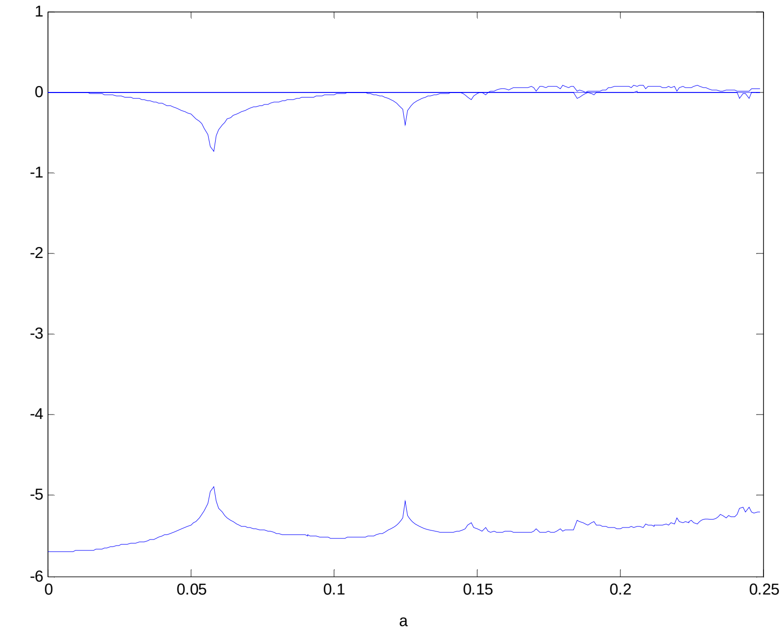

Fig. 2.9 Lyapunov exponents for Rössler system (2.26a), with $b = 0.2$ and $c = 5.7$, initial condition (20,10,25).

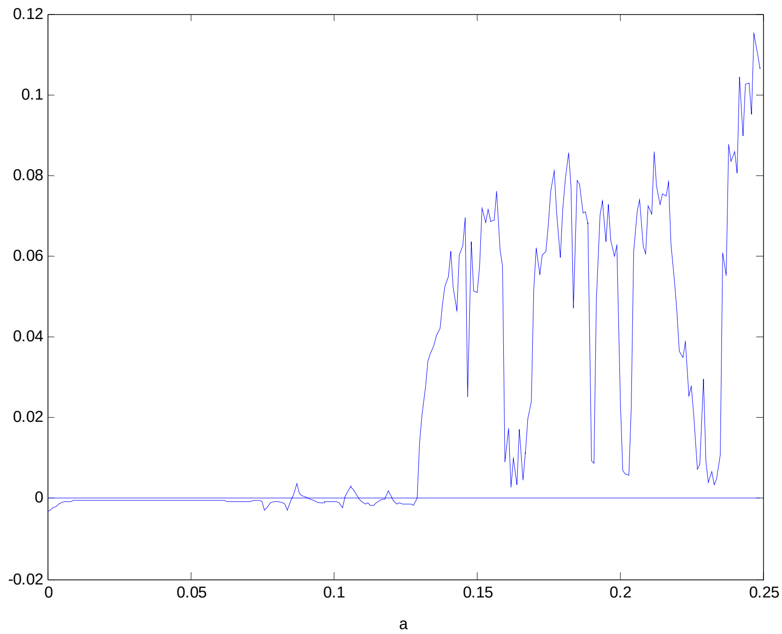

Fig. 2.10 Lyapunov exponent for chaotic system (2.26b) , with $b = 0.2$ and $c = 5.7$, initial condition (2.5,2,2.5).

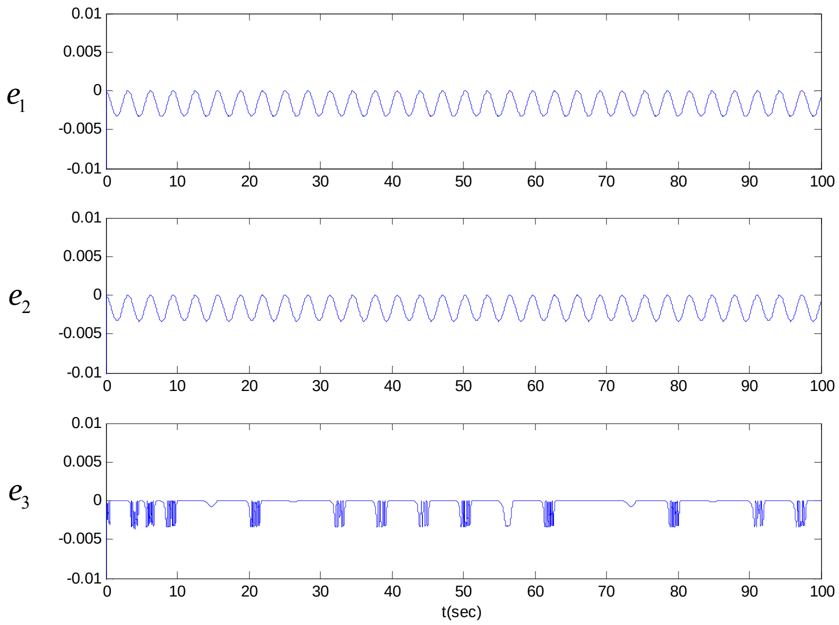

Fig. 2.11 State errors versus time for unidirectional coupled systems (2.26) , with $a = 0.2, b = 0.2$ and $c = 5.7$, initial conditions (20,10,25), (2.5,2,2.5).

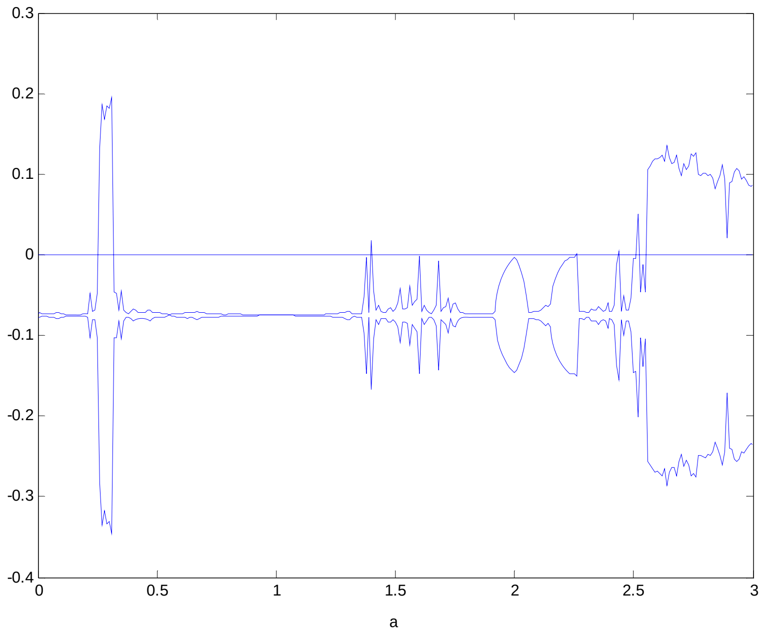

Fig. 2.12 Lyapunov exponents for Duffing system, with $\alpha = \beta = \omega = 1$ and $\delta = 0.15$, initial condition (2,2).

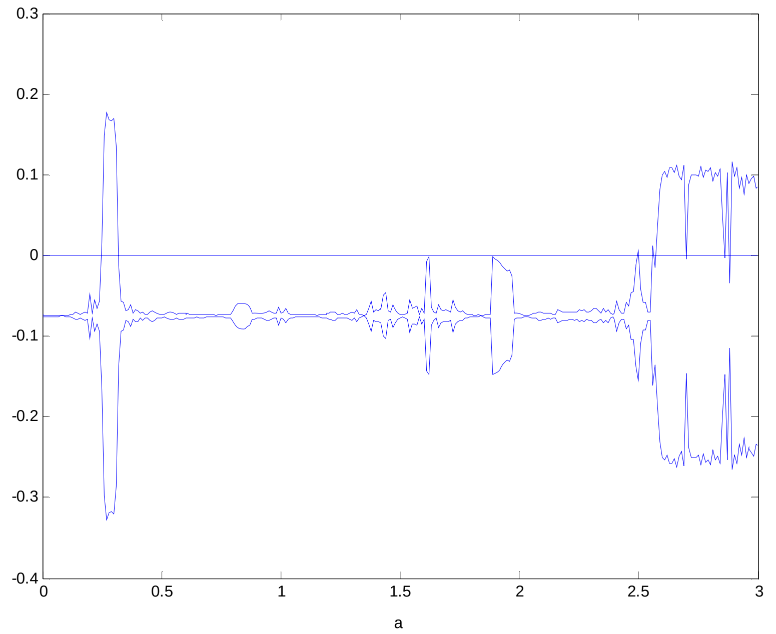

Fig. 2.13 Lyapunov exponents for chaotic system (2.27b) , with $\alpha = \beta = \omega = 1$ and $\delta = 0.15$, initial condition (0.1,0).

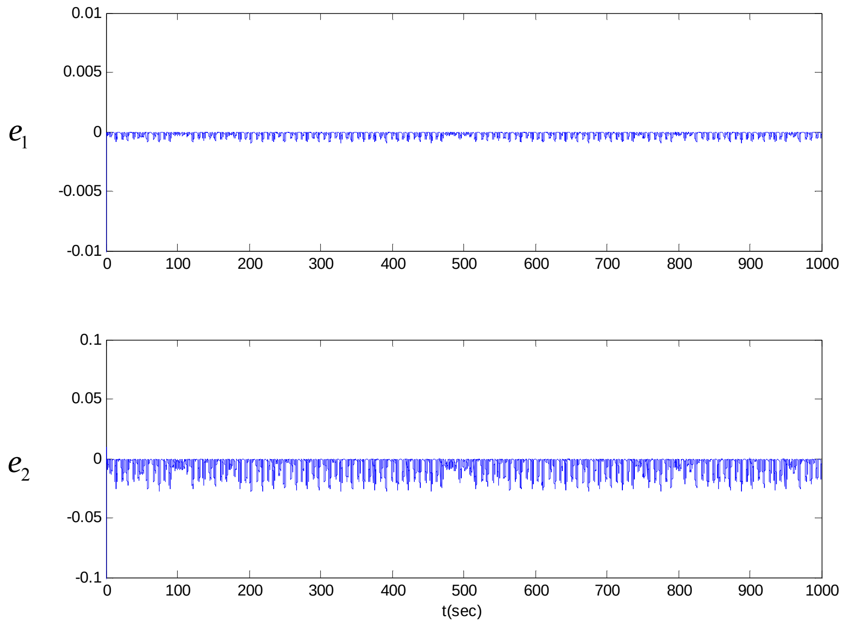

Fig. 2.14 State errors versus time for unidirectional coupled systems (2.27) , with $\delta = 0.15$, $\alpha = \beta = \omega = 1$ and $a = 3$, initial conditions (2,2), (0.1,0).

Chapter 3

Two Theorems of Generalized Unsynchronization for Coupled Chaotic Systems

3.1 Preliminary

In this Chapter, we propose two theorems which give the criteria of generalized unsynchronization for two different chaotic dynamic systems with whatever large strength of linear coupling. Chen system and a new chaotic system which we proposed are presented as a simulated example for the first theorem. Rössler system with corresponding new chaotic system proposed are presented as simulated examples for the second theorem.

3.2 Two Theorems of Generalized Unsynchronizability

Consider the following nonautomonous systems

$$\dot{\mathbf{x}} = \mathbf{f}(t,\ \mathbf{x}) \tag{3.1}$$

where $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{f} : \Omega_1 \subset \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$. Eq. (3.1) is considered as a master system. A slave system is given by

$$\dot{\mathbf{y}} = \mathbf{g}(t,\ \mathbf{y}) \tag{3.2}$$

where $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{g} : \Omega_2 \subset \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$. Both $\mathbf{f}$ and $\mathbf{g}$ satisfy Lipschitz condition. Ω_1, Ω_2 are domains containing the origin. Assume that the solutions of Eqs. (3.1) and (3.2) have bounds then they must exist for infinite time.

Now we consider the following unidirectional nonautonomous coupled system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t,\ \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{g}(t,\ \mathbf{y}) + \mathbf{U}(t, \mathbf{x}, \mathbf{y}) \end{aligned} \tag{3.3}$$

where $\mathbf{U}(t, \mathbf{x}, \mathbf{y})$ is a coupled term.

Definition The system (3.3) is generalized synchronized if there is a continuous function $H(\mathbf{x})$ and let error $\mathbf{e} = \mathbf{y} - H(\mathbf{x})$ s.t. $\lim_{t \to \infty} \|\mathbf{e}\| = 0$. But, if no positive constant C can be found such that $\mathbf{e} \to 0$ as $t \to \infty$ for all $\|\mathbf{e}(t_0)\| < C$, systems (3.3) are generalized unsynchronizable.

In order to discuss the generalized synchronization of $\mathbf{x}$ and $\mathbf{y}$, define $\mathbf{z} = H(\mathbf{x})$ and error $\mathbf{e} = \mathbf{y} - \mathbf{z}$. Error equation can be written as

$$\dot{\mathbf{e}} = \dot{\mathbf{y}} - \dot{\mathbf{z}} = \mathbf{g}(t,\ \mathbf{e} + \mathbf{z}) - \frac{\partial H}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}) + \mathbf{U}(t,\ \mathbf{z}, \mathbf{e} + \mathbf{z}) \tag{3.4}$$

Now the first theorem will be given for a special case of Eqs. (3.3). Consider unidirectional coupled nonautonomous systems as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{g}(t, \mathbf{y}) + \mathbf{\Gamma}(\mathbf{z} - \mathbf{y}) \end{aligned} \tag{3.5}$$

where $\mathbf{f}$ and $\mathbf{g}$ satisfy Lipschitz condition, and the Lipschitz constant of $\mathbf{g}$ is L. $\mathbf{\Gamma} \in M_{n \times n}$ is a constant diagonal matrix with positive entries which represents the strength of the linear coupling term $\mathbf{z} - \mathbf{y}$. Since $\mathbf{e} = \mathbf{y} - H(\mathbf{x}) = \mathbf{y} - \mathbf{z}$, the error dynamic equation can be obtained as

$$\dot{\mathbf{e}} = \dot{\mathbf{y}} - \dot{\mathbf{z}} = \mathbf{g}(t, \mathbf{e} + \mathbf{z}) - \frac{\partial H}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}) - \mathbf{\Gamma} \mathbf{e} \tag{3.6}$$

Let $\mathbf{h}(t, \mathbf{z}) = \frac{\partial H}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x})$, system (3.6) can be written as

$$\dot{\mathbf{e}} = \dot{\mathbf{y}} - \dot{\mathbf{z}} = \mathbf{g}(t, \mathbf{e} + \mathbf{z}) - \mathbf{h}(t, \mathbf{z}) - \mathbf{\Gamma} \mathbf{e} \tag{3.7}$$

which is a nonautonomous system of differential equations for state $\mathbf{e}$.

Theorem 3.1 Two different dynamic systems in Eq. (3.5) are of generalized unsynchronizability for however large coupling strength Γ with positive entries, if $h_i(t, \mathbf{z}) \le g_i(t, \mathbf{z})$ $(i = 1, \ldots, n)$ in $\Omega_1 \cap \Omega_2$, and $\|\mathbf{h}(t, \mathbf{z}) - \mathbf{g}(t, \mathbf{z})\| > 0$ except at origin, for any solution $\mathbf{z}(t)$.

Proof. Choose a Lyapunov function $V(\mathbf{e}) = e_1 e_2 \cdots e_n$ which is positive in quadrant $e_1 > 0, e_2 > 0, \cdots, e_n > 0$, then $\dot{V}$ along any state trajectory of system (3.6) becomes [25]:

$$\begin{aligned}
\dot{V} &= e_2 e_3 \cdots e_n \dot{e}_1 + e_1 e_3 \cdots e_n \dot{e}_2 + \cdots + e_1 e_2 \cdots e_{n-1} \dot{e}_n \\
&= e_2 e_3 \cdots e_n [g_1(t, \mathbf{e}+\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + e_1 e_3 \cdots e_n [g_2(t,\mathbf{e}+\mathbf{z}) - h_2(t,\mathbf{z}) - \Gamma_2 e_2] \\
&\quad \cdots + e_1 e_2 \cdots e_{n-1}[g_n(t,\mathbf{e}+\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n] \\
&= e_2 e_3 \cdots e_n [g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z}) + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots + e_1 e_2 \cdots e_{n-1}[\\
&\quad g_n(t,\mathbf{e}+\mathbf{z}) - g_n(t,\mathbf{z}) + g_n(t,\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n]
\end{aligned}$$

When $e_1 > 0, e_2 > 0, \cdots, e_n > 0$, we have

$$\begin{aligned}
\dot{V} &\le e_2 e_3 \cdots e_n [|g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots + e_1 e_2 \cdots e_{n-1}[\\
&\quad |g_n(t,\mathbf{e}+\mathbf{z}) - g_n(t,\mathbf{z})| + g_n(t,\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n] \\
&\le e_2 e_3 \cdots e_n [L\|\mathbf{e}\| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots
\end{aligned} \tag{3.8}$$

where $|g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| \le L\|\mathbf{e}\|$ follows the Lipschitz condition. When $\|\mathbf{e}\| \gg 1$, the terms of lower degree of error components $e_2 e_3 \cdots e_n [g_1(t,\mathbf{z}) - h_1(t,\mathbf{z})]$, $e_1 e_3 \cdots e_n [g_2(t,\mathbf{z}) - h_2(t,\mathbf{z})]$, $\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \le e_2 e_3 \cdots e_n [L\|\mathbf{e}\| - \Gamma_1 e_1] + e_1 e_3 \cdots e_n [L\|\mathbf{e}\| - \Gamma_2 e_2] + \cdots \tag{3.9}$$

For sufficient large Γ_i, $\dot{V}$ can be negative in the quadrant $e_1 > 0$, $e_2 > 0$, $\cdots$, $e_n > 0$. So the state point tends to decrease $\|\mathbf{e}(t)\|$ with time when $\|\mathbf{e}_0\|$ is sufficiently large. When $\|\mathbf{e}\| \ll 1$, the proof is as follows. Now when $e_1 > 0, e_2 > 0, \cdots, e_n > 0$, $\dot{V}$ is expressed as

$$\begin{aligned}
\dot{V} &\ge e_2 e_3 \cdots e_n [-|g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots \\
&\ge e_2 e_3 \cdots e_n [-L\|\mathbf{e}\| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots
\end{aligned} \tag{3.10}$$

When $\|\mathbf{e}\| \ll 1$, the terms of higher degree $e_2 e_3 \cdots e_n[-L\|\mathbf{e}\| - \Gamma_1 e_1], \cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \geq e_2 e_3 \cdots e_n[g_1(t,\mathbf{z}) - h_1(t,\mathbf{z})] + e_1 e_3 \cdots e_n[g_2(t,\mathbf{z}) - h_2(t,\mathbf{z})] + \cdots \tag{3.11}$$

By the condition $h_i(t,\mathbf{z}) \leq g_i(t,\mathbf{z}) \quad (i = 1,\ldots,n)$ in $\Omega_1 \cap \Omega_2$, $\|\mathbf{h}(t,\mathbf{z}) - \mathbf{g}(t,\mathbf{z})\| > 0$, $f_i(t,\mathbf{z}) = g_i(t,\mathbf{z})$ $(i = 1,\cdots,n)$ do not occur simultaneously. Therefore the right-hand side of above inequality is positive, i.e. $\dot{V}$ is positive in region D of Fig. 3.1, which is the quadrant $e_1 > 0$, $e_2 > 0$, $\cdots$, $e_n > 0$ of the neighborhood of the origin.

Choose $r > 0$ such that for the ball $B_r = \{\mathbf{e} \in R^n \mid \|\mathbf{e}\| \leq r\}$, we have

$$D = \{\mathbf{e} \in B_r \,|\, V(\mathbf{e}) > 0\} \tag{3.12}$$

of which the boundary is the surface $V(\mathbf{e}) = 0$ and the sphere $\|\mathbf{e}\| = r$. Since $V(\mathbf{0}) = 0$, the origin lies on the boundary of D inside B_r. The point $\mathbf{e}_0$ is in the interior of D and $V(\mathbf{e}_0) = b > 0$. Now we prove that the trajectory $\mathbf{e}(t)$ started at $\mathbf{e}(0) = \mathbf{e}_0$ must leave the set D, i.e. the trajectory must leave the neighborhood of origin, $\mathbf{e}$ cannot approach zero. To see this point, notice that as long as $\mathbf{e}(t)$ is inside D, $V(\mathbf{e}(t)) \geq b$ since $\dot{V}(\mathbf{e}) > 0$ in D. Let

$$\beta = \min\{\dot{V}(\mathbf{e}) \mid \mathbf{e} \in D \; and \; V(\mathbf{e}) \geq b\} \tag{3.13}$$

which exists since the continuous function $\dot{V}(\mathbf{e})$ has a minimum over the compact set $\{\mathbf{e} \in D \; and \; V(\mathbf{e}) \geq b\} = \{\mathbf{e} \in B_r, and \; V(\mathbf{e}) \geq b\}$ [29]. Then, $\beta > 0$ and

$$V(\mathbf{e}(t)) = V(\mathbf{e}_0) + \int_0^t \dot{V}(\mathbf{e}(s))ds \geq b + \int_0^t \beta ds = b + \beta t \tag{3.14}$$

This inequality shows that $\mathbf{e}(t)$ cannot stay forever in D because $V(\mathbf{e})$ is bounded on D. Now, $\mathbf{e}(t)$ cannot leave D through the surface $V(\mathbf{e}) = 0$ since $V(\mathbf{e}(t)) \geq b$. Hence, it must leave D through the sphere $\|\mathbf{e}\| = r$, i.e. it must leave the neighborhood of the origin, $\mathbf{e}$ can never approach zero. Two different dynamic systems in Eq.(3.5) are unsynchronizable for however large Γ.

Theorem 3.2 Two different dynamic systems in Eq. (3.5) are of generalized unsynchronizability for however large coupling strength Γ, if $h_i(t,\mathbf{z}) \ge g_i(t,\mathbf{z})$ $(i=1,\ldots,n)$ in $\Omega_1 \cap \Omega_2$, and $\|\mathbf{h}(t,\mathbf{z})-\mathbf{g}(t,\mathbf{z})\| > 0$ except at origin, for any solution $\mathbf{z}(t)$.

Proof. Choose a Lyapunov function $V(\mathbf{e}) = e_1 e_2 \cdots e_n$, then $\dot{V}$ along any state trajectory of system (3.6) becomes:

Case 1. When n is odd, $V(\mathbf{e})$ is negative in quadrant $e_1<0, e_2<0, \cdots e_n<0$.

$$\begin{aligned}\dot{V} &= e_2 e_3 \cdots e_n \dot{e}_1 + e_1 e_3 \cdots e_n \dot{e}_2 + \cdots + e_1 e_2 \cdots e_{n-1}\dot{e}_n \\ &= e_2 e_3 \cdots e_n [g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z}) + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots + e_1 e_2 \cdots e_{n-1}[\\ &\quad g_n(t,\mathbf{e}+\mathbf{z}) - g_n(t,\mathbf{z}) + g_n(t,\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n]\end{aligned}$$

When $e_1<0, e_2<0, \cdots e_n<0$, we have

$$\begin{aligned}\dot{V} &\ge e_2 e_3 \cdots e_n [-|g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots + e_1 e_2 \cdots e_{n-1}[\\ &\quad -|g_n(t,\mathbf{e}+\mathbf{z}) - g_n(t,\mathbf{z})| + g_n(t,\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n] \\ &\ge e_2 e_3 \cdots e_n [-L\|\mathbf{e}\| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots\end{aligned} \tag{3.15}$$

where $|g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| \le L\|\mathbf{e}\|$ follows the Lipschitz condition. When $\|\mathbf{e}\| \gg 1$, the terms of lower degree of error components $e_2 e_3 \cdots e_n [g_1(t,\mathbf{z}) - h_1(t,\mathbf{z})]$, $e_1 e_3 \cdots e_n [g_2(t,\mathbf{z}) - h_2(t,\mathbf{z})]$, $\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\begin{aligned}\dot{V} &\ge e_2 e_3 \cdots e_n [-L\|\mathbf{e}\| - \Gamma_1 e_1] + e_1 e_3 \cdots e_n [-L\|\mathbf{e}\| - \Gamma_2 e_2] + \cdots \\ &= -e_2 e_3 \cdots e_n [L\|\mathbf{e}\| + \Gamma_1 e_1] - e_1 e_3 \cdots e_n [L\|\mathbf{e}\| + \Gamma_2 e_2] + \cdots\end{aligned} \tag{3.16}$$

For sufficient large Γ_i, $\dot{V}$ can be positive in the quadrant $e_1<0$, $e_2<0$, $\cdots$, $e_n<0$. So the state point tends to decrease $\|\mathbf{e}(t)\|$ with time when $\|\mathbf{e}_0\|$ is sufficiently large. When $\|\mathbf{e}\| \ll 1$, the proof is as follows. Now when $e_1 < 0$, $e_2 < 0$, $\cdots e_n < 0$, $\dot{V}$ is expressed as

$$\begin{aligned}\dot{V} &\le e_2 e_3 \cdots e_n [|g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z})| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots \\ &\le e_2 e_3 \cdots e_n [L\|\mathbf{e}\| + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots\end{aligned} \tag{3.17}$$

When $\|\mathbf{e}\| \ll 1$, the terms of higher degree $e_2 e_3 \cdots e_n [L\|\mathbf{e}\| - \Gamma_1 e_1]$, $\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \le e_2 e_3 \cdots e_n [g_1(t,\mathbf{z}) - h_1(t,\mathbf{z})] + e_1 e_3 \cdots e_n [g_2(t,\mathbf{z}) - h_2(t,\mathbf{z})] + \cdots \tag{3.18}$$

By the condition $\|\mathbf{h}(t,\mathbf{z}) - \mathbf{g}(t,\mathbf{z})\| > 0$, $h_i(t,\mathbf{z}) = g_i(t,\mathbf{z})$ $(i = 1,\cdots,n)$ do not occur simultaneously. Therefore the right-hand side of above inequality is negative, i.e. $\dot{V}$ is negative in region D of Fig. 3.2, which is the quadrant $e_1 < 0$, $e_2 < 0$, $\cdots$, $e_n < 0$ of the neighborhood of the origin.

Choose $r > 0$ such that for the ball $B_r = \{\mathbf{e} \in R^n \mid \|\mathbf{e}\| \le r\}$, we have

$$D = \{\mathbf{e} \in B_r | V(\mathbf{e}) < 0\} \tag{3.19}$$

By the similar reasoning as that in the latter part of the proof for Theorem1, we can prove that the state trajectory started from D must leave the neighborhood of the origin, $\mathbf{e}$ can never approach zero. Two different dynamic systems in Eq. (3.5) are unsynchronizable for however large Γ.

Case 2. When n is even, $V(\mathbf{e})$ is positive in quadrant $e_1 < 0, e_2 < 0, \cdots e_n < 0$.

$$\begin{aligned}\dot{V} &= e_2 e_3 \cdots e_n \dot{e}_1 + e_1 e_3 \cdots e_n \dot{e}_2 + \cdots + e_1 e_2 \cdots e_{n-1} \dot{e}_n \\ &= e_2 e_3 \cdots e_n [g_1(t,\mathbf{e}+\mathbf{z}) - g_1(t,\mathbf{z}) + g_1(t,\mathbf{z}) - h_1(t,\mathbf{z}) - \Gamma_1 e_1] + \cdots + e_1 e_2 \cdots e_{n-1} [\\ &\quad g_n(t,\mathbf{e}+\mathbf{z}) - g_n(t,\mathbf{z}) + g_n(t,\mathbf{z}) - h_n(t,\mathbf{z}) - \Gamma_n e_n]\end{aligned}$$

When $e_1 < 0, e_2 < 0, \cdots e_n < 0$, we have

$$\dot{V} \le e_2e_3\cdots e_n[-|g_1(t,\mathbf{e}+\mathbf{z})-g_1(t,\mathbf{z})|+g_1(t,\mathbf{z})-h_1(t,\mathbf{z})-\Gamma_1 e_1]+\cdots+e_1e_2\cdots e_{n-1}[$$
$$-|g_n(t,\mathbf{e}+\mathbf{z})-g_n(t,\mathbf{z})|+g_n(t,\mathbf{z})-h_n(t,\mathbf{z})-\Gamma_n e_n]$$
$$\le e_2e_3\cdots e_n[-L\|\mathbf{e}\|+g_1(t,\mathbf{z})-h_1(t,\mathbf{z})-\Gamma_1 e_1]+\cdots \qquad (3.20)$$

where $|g_1(t,\mathbf{e}+\mathbf{z})-g_1(t,\mathbf{z})| \le L\|\mathbf{e}\|$ follows the Lipschitz condition. When $\|\mathbf{e}\| \gg 1$, the terms of lower degree of error components $e_2e_3\cdots e_n[g_1(t,\mathbf{z})-h_1(t,\mathbf{z})]$, $e_1e_3\cdots e_n[g_2(t,\mathbf{z})-h_2(t,\mathbf{z})]$, $\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\begin{aligned}\dot{V} &\le e_2e_3\cdots e_n[-L\|\mathbf{e}\|-\Gamma_1 e_1]+e_1e_3\cdots e_n[-L\|\mathbf{e}\|-\Gamma_2 e_2]+\cdots \\ &= -e_2e_3\cdots e_n[L\|\mathbf{e}\|+\Gamma_1 e_1]-e_1e_3\cdots e_n[L\|\mathbf{e}\|+\Gamma_2 e_2]+\cdots\end{aligned} \qquad (3.21)$$

For sufficient large Γ_i, $\dot{V}$ can be negative in the quadrant $e_1<0$, $e_2<0$, $\cdots$, $e_n<0$. So the state point tends to decrease $\|\mathbf{e}(t)\|$ with time when $\|\mathbf{e}_0\|$ is sufficiently large. When $\|\mathbf{e}\| \ll 1$, the proof is as follows. Now when $e_1<0$, $e_2<0$, $\cdots$, $e_n<0$, $\dot{V}$ is expressed as

$$\begin{aligned}\dot{V} &\ge e_2e_3\cdots e_n[|g_1(t,\mathbf{e}+\mathbf{z})-g_1(t,\mathbf{z})|+g_1(t,\mathbf{z})-h_1(t,\mathbf{z})-\Gamma_1 e_1]+\cdots \\ &\ge e_2e_3\cdots e_n[L\|\mathbf{e}\|+g_1(t,\mathbf{z})-h_1(t,\mathbf{z})-\Gamma_1 e_1]+\cdots\end{aligned} \qquad (3.22)$$

When $\|\mathbf{e}\| \ll 1$, the terms of higher degree $e_2e_3\cdots e_n[L\|\mathbf{e}\|-\Gamma_1 e_1],\cdots$ can be neglected when the sign of $\dot{V}$ is considered, then

$$\dot{V} \ge e_2e_3\cdots e_n[g_1(t,\mathbf{z})-h_1(t,\mathbf{z})]+e_1e_3\cdots e_n[g_2(t,\mathbf{z})-h_2(t,\mathbf{z})]+\cdots \qquad (3.23)$$

By the condition $\|\mathbf{h}(t,\mathbf{z})-\mathbf{g}(t,\mathbf{z})\|>0$, $h_i(t,\mathbf{z})=g_i(t,\mathbf{z})$ $(i=1,\cdots,n)$ do not occur simultaneously. Therefore the right hand side of above inequality is positive, i.e. $\dot{V}$ is positive in region D of Fig. 3.2 which is the quadrant $e_1<0$, $e_2<0$, $\cdots$, $e_n<0$ of the neighborhood of the origin.

By the same reasoning as that in the latter part of the proof for Theorem 1, we can prove that the state trajectory started from the neighborhood of the origin in the quadrant $e_1 < 0$, $e_2 < 0$, $\cdots$, $e_n < 0$ must leave the neighborhood and can never approach zero. Two different dynamic systems in Eq. (3.5) are unsynchronizable for however large Γ_i.

3.3 Simulated Examples

An example for the first theorem is Chen system with a new chaotic system proposed. Consider the following unidirectional coupled systems:

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= (c-a)x - xz + cy \\ \dot{z} &= xy - bz \end{aligned} \tag{3.24a}$$

$$\begin{aligned} \dot{\tilde{x}} &= a(\tilde{y}-\tilde{x}) + \sin^2\tilde{x} - \gamma e_1 \\ \dot{\tilde{y}} &= (c-a)\tilde{x} - \tilde{x}\tilde{z} + c\tilde{y} + \tilde{x}^2 - \gamma e_2 \\ \dot{\tilde{z}} &= \tilde{x}\tilde{y} - b\tilde{z} + \tilde{x}^2 - \gamma e_3 \end{aligned} \tag{3.24b}$$

$$\mathbf{e} = \mathbf{y} - \mathbf{z}, \quad \mathbf{z} = H(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

where $\gamma = 1000$ which is sufficiently large. Eq. (3.24a) is Chen system and Eq. (3.24b) without coupling is a new chaotic system which we proposed. The chaotic attractor and Lyapunov exponent diagrams for system (3.24a) and (3.24b) without coupling term are shown in Figs. 3.3 ~ 3.6. For initial states (0.5,1,5), (30,20,18) and system parameters $a = 35, b = 3$ and $c = 28$, three state errors and error versus time are shown in Figs. 3.7 ~ 3.9. Fig. 3.8 shows that errors decreases with time when error is large, but one can clearly find in Fig. 3.9 that the errors cannot approach zero as time evolves.

An example for the second theorem is Rössler system with a new chaotic system proposed. Consider the following unidirectional coupled systems with linear coupling in the form of Eq. (3.5):

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \tag{3.25a}$$

$$\begin{aligned} \dot{\tilde{x}} &= -\tilde{y} - \tilde{z} - \sin^2 y - \gamma e_1 \\ \dot{\tilde{y}} &= \tilde{x} + a\tilde{y} - \sin^2 y - \gamma e_2 \\ \dot{\tilde{z}} &= b + \tilde{z}(\tilde{x} - c) - \sin^2 \tilde{z} - \gamma e_3 \end{aligned} \tag{3.25b}$$

$$\mathbf{e} = \mathbf{y} - \mathbf{z}, \quad \mathbf{z} = H(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

where $\gamma = 300$. The Lyapunov exponent diagrams for system (3.25a) and (3.25b) without coupling term are shown in Figs. 3.10 ~ 3.11. For initial states (20,10,25), (2.5,2,2.5) and system parameter $a = 0.2, b = 0.2$ and $c = 5.7$, three state errors and errors versus time are shown in Figs. 3.12 ~ 3.14. Fig. 3.13 shows that errors decreases with time when error is large, but one can clearly find in Fig. 3.14 that the errors cannot approach zero as time evolves.

3.4 Summary

In this Chapter, two theorems are proposed. They give the criteria of generalized unsynchronization for two different chaotic dynamic systems with whatever large strength of linear coupling. Chen system and Rössler system with two corresponding new chaotic systems proposed are used as simulation examples which effectively confirm the theorems.

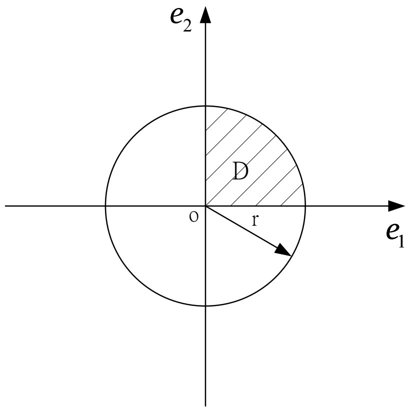

Fig. 3.1 D region for n=2.

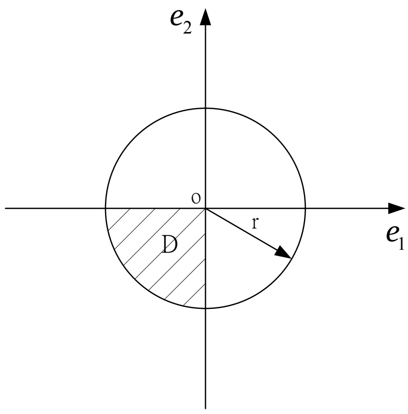

Fig. 3.2 D region for n=2.

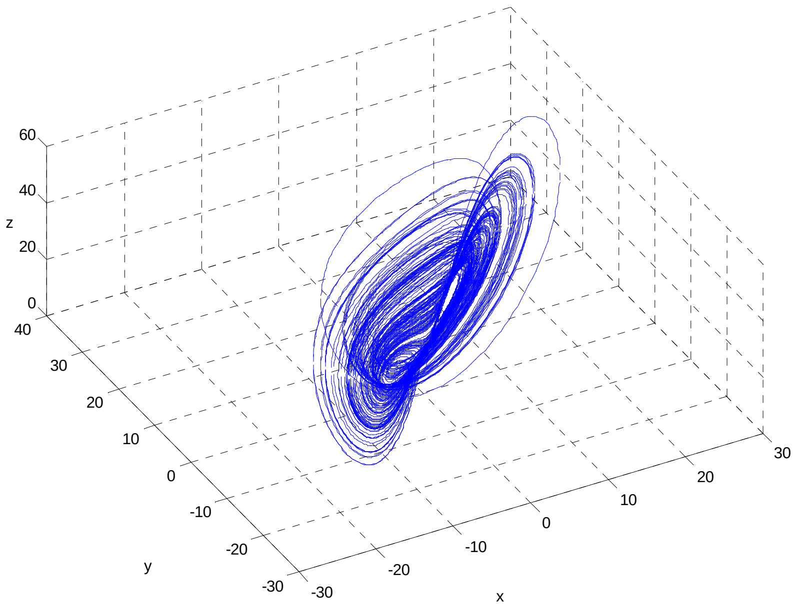

Fig. 3.3 Chaotic attractor for Chen system (3.24a), with $a = 35, b = 3$ and $c = 28$, initial condition (0.5,1,5).

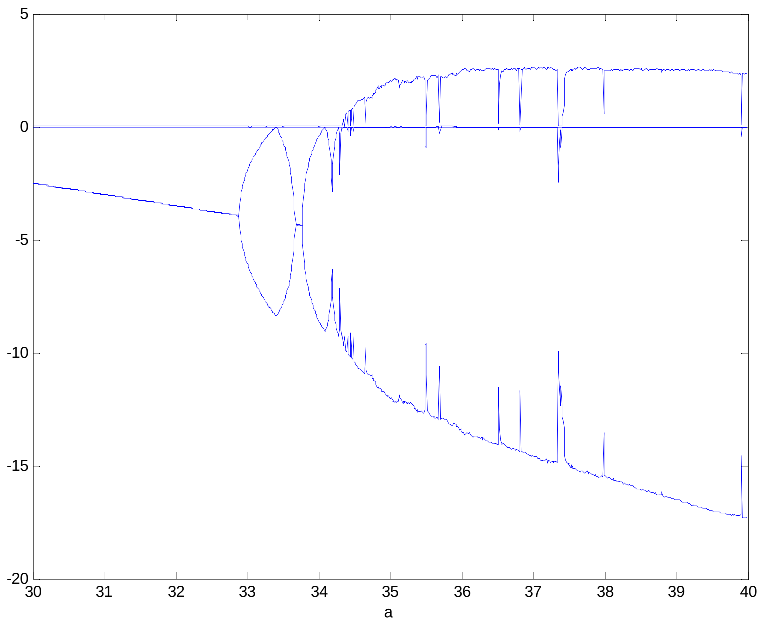

Fig. 3.4 Lyapunov exponents for Chen system (3.24a), with $b = 3$ and $c = 28$, initial condition (0.5,1,5).

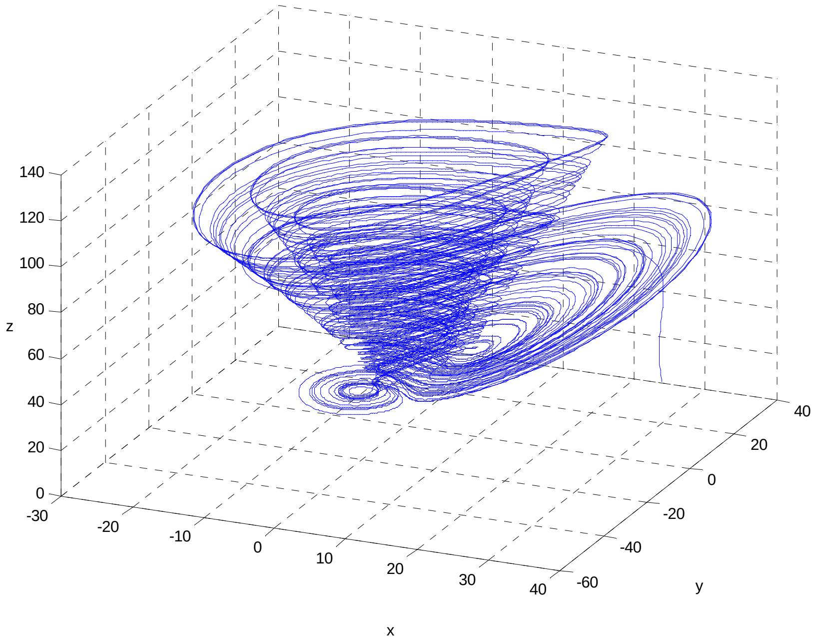

Fig. 3.5 Chaotic attractor for chaotic system (3.24b), with $a = 35, b = 3$ and $c = 28$, initial condition (30,20,18).

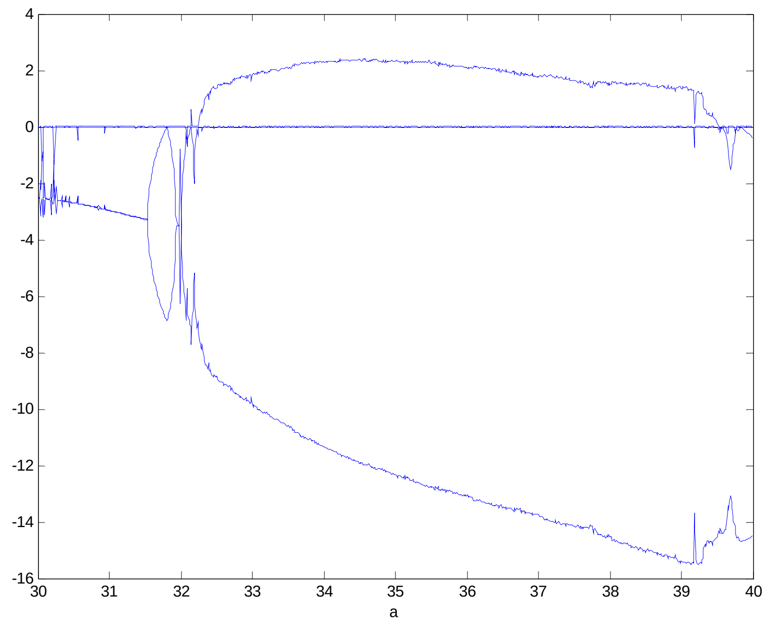

Fig. 3.6 Lyapunov exponents for chaotic system (3.24b), with $b = 3$ and $c = 28$, initial condition (30,20,18).

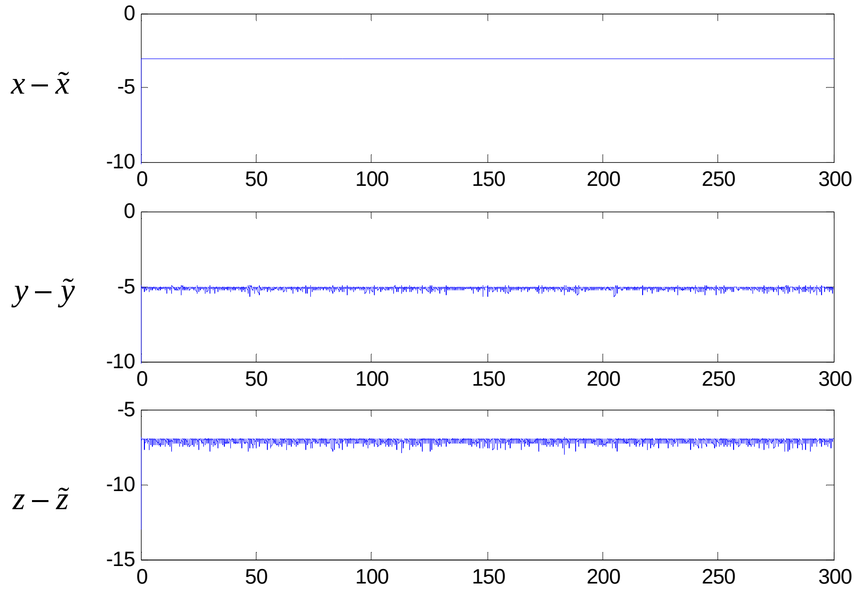

Fig. 3.7 State errors versus time for unidirectional coupled systems (3.24) , with $a = 35, b = 3$ and $c = 28$, initial conditions (0.5,1,5), (30,20,18).

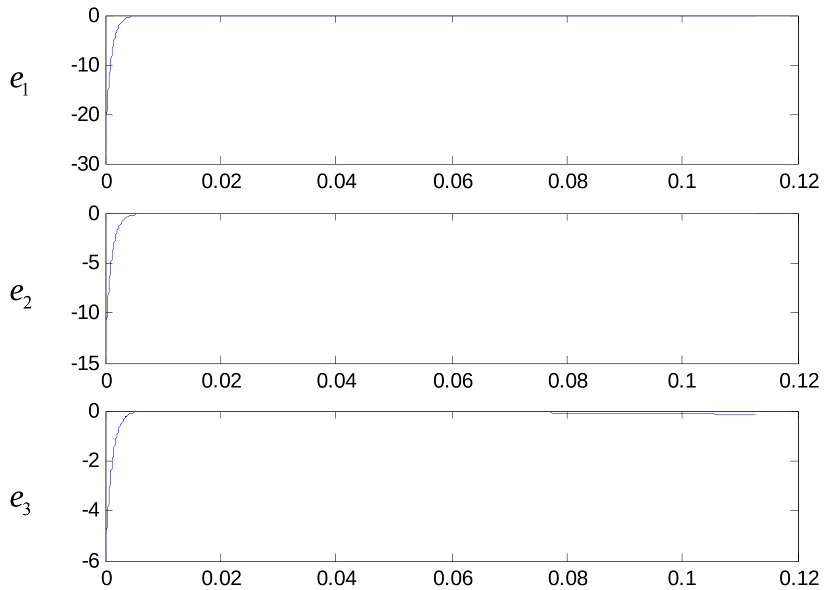

Fig. 3.8 Errors versus time for unidirectional coupled systems (3.24) , with $a = 35$, $b = 3$ and $c = 28$, initial conditions (0.5,1,5), (30,20,18).

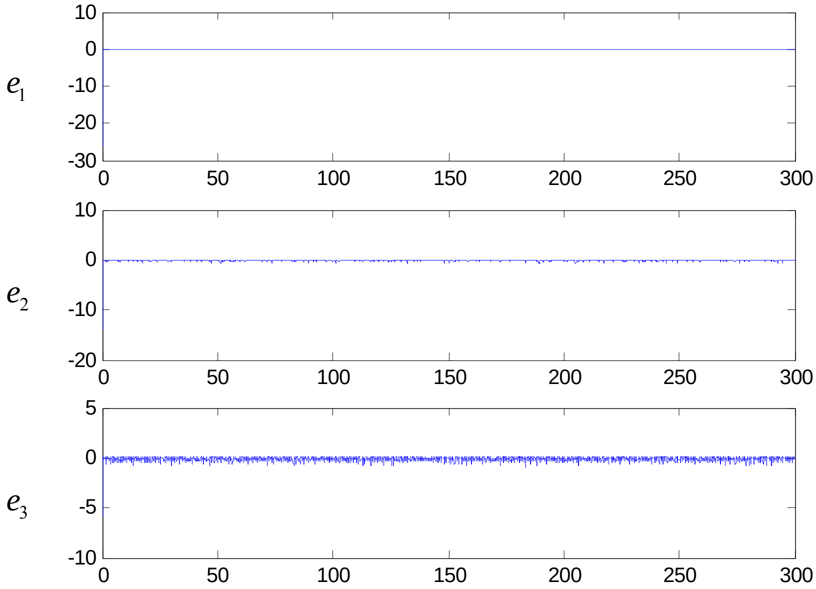

Fig. 3.9 Errors versus time for unidirectional coupled systems (3.24) , with $a = 35, b = 3$ and $c = 28$, initial conditions (0.5,1,5), (30,20,18).

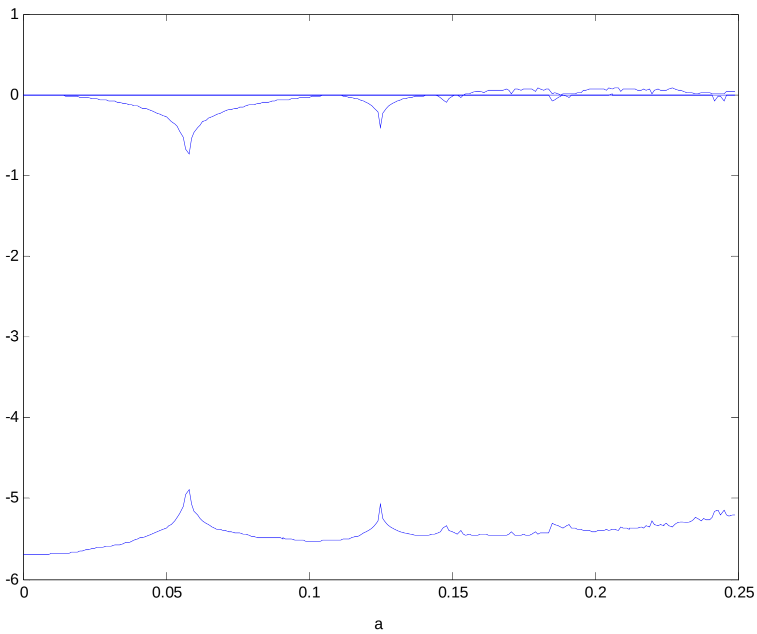

Fig. 3.10 Lyapunov exponents for Rössler system (3.25a), with $b = 0.2$ and $c = 5.7$, initial condition (20,10,25).

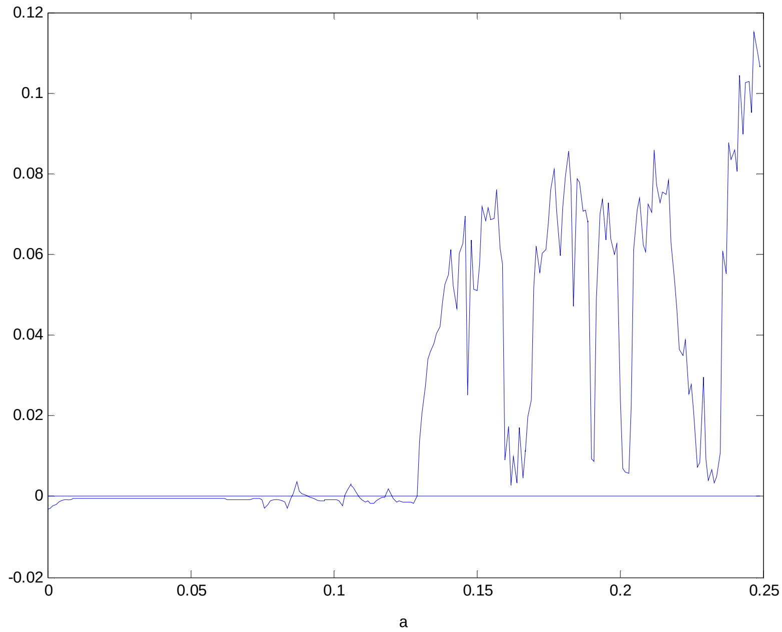

Fig. 3.11 Lyapunov exponent for chaotic system (3.25b) , with $b = 0.2$ and $c = 5.7$, initial condition (2.5,2,2.5).

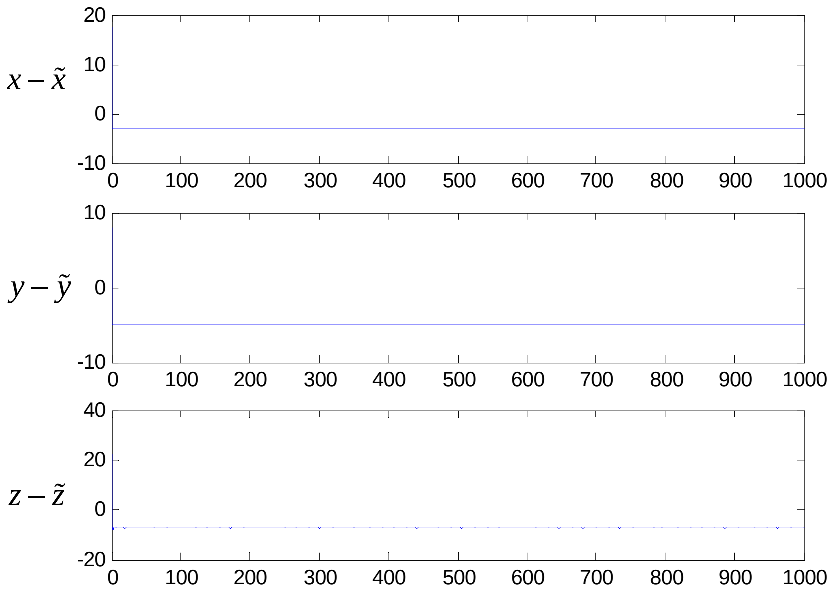

Fig. 3.12 State errors versus time for unidirectional coupled systems (3.25) , with $a = 0.2, b = 0.2$ and $c = 5.7$, initial conditions (20,10,25), (2.5,2,2.5).

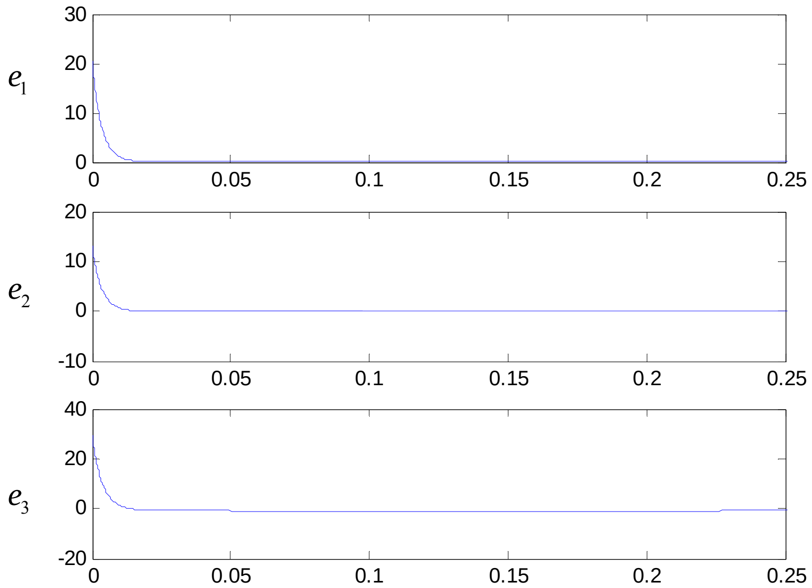

Fig. 3.13 Errors versus time for unidirectional coupled systems (3.25) , with $a = 0.2, b = 0.2$ and $c = 5.7$, initial conditions (20,10,25), (2.5,2,2.5).

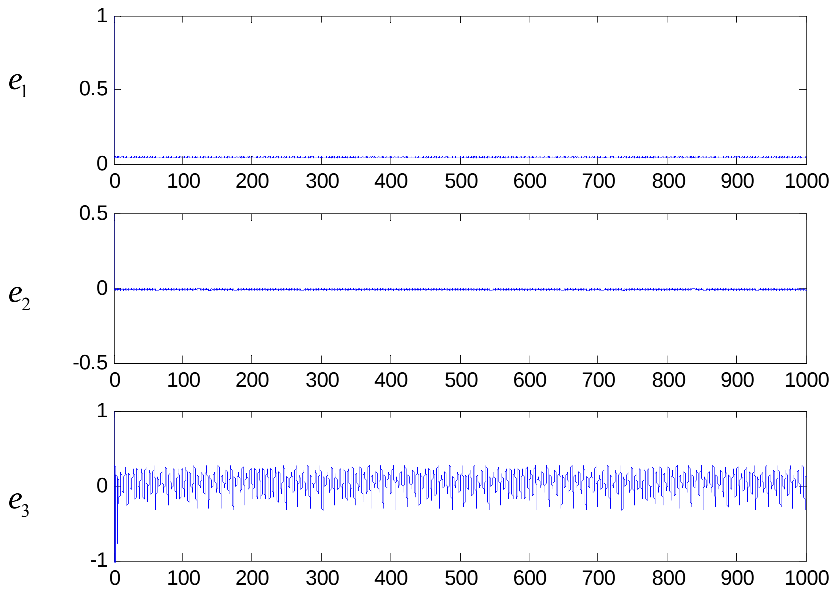

Fig. 3.14 Errors versus time for unidirectional coupled systems (3.25) , with $a = 0.2, b = 0.2$ and $c = 5.7$, initial conditions (20,10,25), (2.5,2,2.5).

Chapter 4

Chaos Synchronization by Variable Strength Linear Coupling and Lyapunov Function Derivative in Series Form

4.1 Preliminary

In this Chapter, a new general strategy to achieve chaos synchronization by variable strength linear coupling is proposed. This method, in which the time derivative of Lyapunov function in series form is firstly used, can give either local synchronization which is usually good enough or global synchronization which is usually an unnecessary high demand.

4.2 Synchronization Strategy by Variable Strength Linear Coupling and Lyapunov Function Derivative in Series Form

(a) Consider the following unidirectional coupled identical chaotic systems

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})\end{aligned} \tag{4.1}$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in R^n$, $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T \in R^n$ denote two state vectors, $\mathbf{A}$ is an $n \times n$ constant coefficient matrix, $\mathbf{f}$ is a nonlinear vector function, and $\boldsymbol{\Gamma}$ is an $n \times n$ matrix which gives the variable strength of the linear coupling term $(\mathbf{y}-\mathbf{x})$.

In order to study the synchronization of $\mathbf{x}$ and $\mathbf{y}$, define $\mathbf{e} = \mathbf{y} - \mathbf{x}$ as the state error. Error equation can be written as

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x}) - \mathbf{A}\mathbf{x} - \mathbf{f}(\mathbf{x}) \tag{4.2}$$

By Taylor expansion

$$\begin{aligned}\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) &= \mathbf{f}(\mathbf{x}+\mathbf{e}) - \mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})\mathbf{e} + \text{HOT of } \mathbf{e} \\ &= \mathbf{F}(\mathbf{x})\mathbf{e} + \text{HOT of } \mathbf{e}\end{aligned} \tag{4.3}$$

where $\mathbf{f}'(\mathbf{x})$ is the time derivative $\mathbf{f}(\mathbf{x})$, and $\mathbf{F}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})$.

Theorem 4.1. The chaotic systems in Eq. (4.1) can be locally completely synchronized, if $\|\mathbf{e}\|^2$ is smaller than a bounded value and $\mathbf{\Gamma}$ is chosen such that $\mathbf{A} + \mathbf{\Gamma} + \mathbf{F} = \mathbf{-C}$ where $\mathbf{C}$ is positive definite diagonal matrix.

Proof. Choose a positive definite function as

$$V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^T\mathbf{e} \tag{4.3}$$

Then

$$\begin{aligned}\dot{V}(\mathbf{e}) &= \mathbf{e}^T\dot{\mathbf{e}} \\ &= \mathbf{e}^T(\mathbf{Ay} + \mathbf{f}(\mathbf{y}) + \mathbf{\Gamma}(\mathbf{y}-\mathbf{x}) - \mathbf{Ax} - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{e}^T(\mathbf{Ae} + \mathbf{\Gamma e} + \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{e}^T(\mathbf{A} + \mathbf{\Gamma} + \mathbf{F})\mathbf{e} + \text{HOT of } \mathbf{e}\end{aligned} \tag{4.4}$$

Since $\|\mathbf{e}\|^2$ is smaller than a bounded value and $\mathbf{\Gamma}$ is chosen such that $\mathbf{A} + \mathbf{\Gamma} + \mathbf{F} = \mathbf{-C}$, Eq. (4.4) becomes $\dot{V}(\mathbf{e}) = -\mathbf{e}^T\mathbf{Ce} + \text{HOT of } \mathbf{e} < 0$, since $-\mathbf{e}^T\mathbf{Ce}$ is a definite form, the higher order terms of **e** have no influence on the definiteness of $\dot{V}$, provided that $\|\mathbf{e}\|^2$ is smaller than a bounded value. The proof of this theorem can be found in [24, 25], which is used extensively in the theory of stability of motion. By Lyapunov asymptotical stability theorem, the origin of error equation (4.2) is locally asymptotically stable and the chaotic systems in Eq. (4.1) are locally completely synchronized. □

Corollary 1. If $\mathbf{f}(\mathbf{x}+\mathbf{e}) - \mathbf{f}(\mathbf{x})$ is a linear function of **e, De,** Eq. (4.4) become $\dot{V}(\mathbf{e}) = \mathbf{e}^T(\mathbf{A} + \mathbf{\Gamma} + \mathbf{D})\mathbf{e}$. Let $\mathbf{A} + \mathbf{\Gamma} + \mathbf{D} = \mathbf{-C}$, then $\dot{V}(\mathbf{e}) = -\mathbf{e}^T\mathbf{Ce} < 0$. By Lyapunov asymptotical stability theorem, the origin of error equation (4.2) is globally asymptotically stable. Hence, the chaotic systems in Eq. (4.1) are globally completely synchronized. □

(b) Consider the following two unidirectional coupled different chaotic systems

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{y}} &= \hat{\mathbf{A}}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \mathbf{u} \end{aligned} \tag{4.5}$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in R^n$, $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T \in R^n$ denote two state vectors, $\mathbf{A}$ and $\hat{\mathbf{A}}$ are two different $n \times n$ constant coefficient matrices, $\mathbf{f}$ is a nonlinear vector function, and $\mathbf{u}$ is the coupling vector of which the elements are functions of $\mathbf{x}$ and $\mathbf{y}$.

In order to study the synchronization of $\mathbf{x}$ and $\mathbf{y}$, define $\mathbf{e} = \mathbf{y} - \mathbf{x}$ as the state error. Error equation can be written as

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \mathbf{u} - \mathbf{A}\mathbf{x} - \mathbf{f}(\mathbf{x}) \tag{4.6}$$

By Taylor expansion

$$\begin{aligned} \mathbf{f(y)} - \mathbf{f(x)} = \mathbf{f(x+e)} - \mathbf{f(x)} &= \mathbf{f'(x)e} + \text{HOT of } \mathbf{e} \\ &= \mathbf{F(x)e} + \text{HOT of } \mathbf{e} \end{aligned} \tag{4.7}$$

Theorem 4.2. Choose $\mathbf{\Gamma} = -\mathbf{C} - \mathbf{A} - \mathbf{F}$ and $\mathbf{B} = -\tilde{\mathbf{A}}$, where $\mathbf{C}$ is positive definite diagonal matrix and $\tilde{\mathbf{A}} = \hat{\mathbf{A}} - \mathbf{A}$. The chaotic systems in Eq. (4.5) can be locally completely synchronized, if $\|\mathbf{e}\|^2$ is smaller than a bounded value and $\mathbf{u} = \mathbf{\Gamma}\mathbf{e} + \mathbf{B}\mathbf{y}$.

Proof. Choose a positive definite function as

$$V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^T\mathbf{e} \tag{4.8}$$

Then

$$\begin{aligned} \dot{V}(\mathbf{e}) &= \mathbf{e}^T\dot{\mathbf{e}} \\ &= \mathbf{e}^T(\hat{\mathbf{A}}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \mathbf{u} - \mathbf{A}\mathbf{x} - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{e}^T(\tilde{\mathbf{A}}\mathbf{y} + \mathbf{A}\mathbf{e} + \mathbf{u} + \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) \end{aligned} \tag{4.9}$$

Let $\mathbf{u} = \mathbf{\Gamma}\mathbf{e} + \mathbf{B}\mathbf{y}$, Eq.(4.9) becomes

$$\begin{aligned} \dot{V}(\mathbf{e}) &= \mathbf{e}^T(\tilde{\mathbf{A}}\mathbf{y} + \mathbf{A}\mathbf{e} + \mathbf{\Gamma}\mathbf{e} + \mathbf{B}\mathbf{y} + \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{e}^T(\mathbf{A} + \mathbf{\Gamma} + \mathbf{F})\mathbf{e} + \mathbf{e}^T(\tilde{\mathbf{A}} + \mathbf{B})\mathbf{y} + \text{HOT of } \mathbf{e} \end{aligned} \tag{4.10}$$

Since $\|\mathbf{e}\|^2$ is smaller than a bounded value, $\mathbf{\Gamma}$ and $\mathbf{B}$ is chosen such that

$\mathbf{A}+\mathbf{\Gamma}+\mathbf{F}=\mathbf{-C}$ and $\mathbf{B}=-\tilde{\mathbf{A}}$, Eq. (4.9) becomes $\dot{V}(\mathbf{e})=-\mathbf{e}^T\mathbf{C}\mathbf{e}+\text{HOT of }\mathbf{e}<0$. By Lyapunov asymptotical stability theorem, the origin of error equation (4.6) is locally asymptotically stable and the chaotic systems in Eq. (4.5) are locally completely synchronized. □

Corollary 2. If $\mathbf{f}(\mathbf{x}+\mathbf{e})-\mathbf{f}(\mathbf{x})$ is a linear function of **e, De,** Eq. (4.10) become $\dot{V}(\mathbf{e})=\mathbf{e}^T(\mathbf{A}+\mathbf{\Gamma}+\mathbf{D})\mathbf{e}+\mathbf{e}^T(\tilde{\mathbf{A}}+\mathbf{B})\mathbf{y}$. Let $\mathbf{A}+\mathbf{\Gamma}+\mathbf{D}=\mathbf{-C}$ and $\mathbf{B}=-\tilde{\mathbf{A}}$, then $\dot{V}(\mathbf{e})=-\mathbf{e}^T\mathbf{C}\mathbf{e}<0$. By Lyapunov asymptotical stability theorem, the origin of error equation (4.6) is globally asymptotically stable, and the chaotic systems in Eq. (4.5) are globally completely synchronized. □

4.3 Numerical Results for Typical Chaotic Systems

First example for Theorem 4.1 is Rössler system. Consider following two unidirectional coupled chaotic Rössler systems:

$$\begin{aligned}
\dot{x}_1 &= -y_1 - z_1\\
\dot{y}_1 &= x_1 + ay_1\\
\dot{z}_1 &= b + z_1(x_1 - c)\\
\dot{x}_2 &= -y_2 - z_2 + \Gamma_{11}e_1 + \Gamma_{12}e_2 + \Gamma_{13}e_3\\
\dot{y}_2 &= x_2 + ay_2 + \Gamma_{21}e_1 + \Gamma_{22}e_2 + \Gamma_{23}e_3\\
\dot{z}_2 &= b + z_2(x_2 - c) + \Gamma_{31}e_1 + \Gamma_{32}e_2 + \Gamma_{33}e_3
\end{aligned} \tag{4.11}$$

where

$$\mathbf{A}=\begin{bmatrix}0 & -1 & -1\\ 1 & a & 0\\ 0 & 0 & -c\end{bmatrix} \tag{4.12}$$

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1,e_2,e_3)=\frac{1}{2}(e_1^{\,2}+e_2^{\,2}+e_3^{\,2}) \tag{4.13}$$

by Taylor Formula

$$\mathbf{f(y)}-\mathbf{f(x)}=\begin{bmatrix}0\\0\\z_1e_1+x_1e_3+e_1e_3\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\z_1&0&x_1\end{bmatrix}\mathbf{e}+\begin{bmatrix}0\\0\\e_1e_3\end{bmatrix} \tag{4.14}$$
$$=\mathbf{Fe}+\cdots$$

Let

$$\mathbf{\Gamma}=-\mathbf{I}-\mathbf{A}-\mathbf{F}=\begin{bmatrix}-1&1&1\\-1&-1-a&0\\-z_1&0&-1+c-x_1\end{bmatrix} \tag{4.15}$$

According to Theorem 4.1, we obtain

$$\dot{V}=-e_1^2-e_2^2-e_3^2+\text{HOT of }\mathbf{e}<0 \tag{4.16}$$

is negative definite when $\|\mathbf{e}\|^2$ is smaller than a bounded value . The Rössler systems in Eq.(4.11) are locally synchronized. For initial states (-20,10,25), (-21,10.5,25) and system parameters $a=0.2$, $b=0.2$, $c=5.7$, the chaotic phase portraits and state errors versus time are shown in Fig. 4.1 and Fig. 4.2.

Second example for Corollary 4.1 is Hyper-Rössler system. Consider following two unidirectional coupled chaotic hyper-Rössler systems:

$$\begin{aligned}
\dot{x}_1&=-x_2-x_3\\
\dot{x}_2&=x_1+ax_2+x_4\\
\dot{x}_3&=b+x_1x_3\\
\dot{x}_4&=cx_4-dx_3\\
\dot{y}_1&=-y_2-y_3+\Gamma_{11}e_1+\Gamma_{12}e_2+\Gamma_{13}e_3+\Gamma_{14}e_4\\
\dot{y}_2&=y_1+ay_2+y_4+\Gamma_{21}e_1+\Gamma_{22}e_2+\Gamma_{23}e_3+\Gamma_{24}e_4\\
\dot{y}_3&=b+y_1y_3+\Gamma_{31}e_1+\Gamma_{32}e_2+\Gamma_{33}e_3+\Gamma_{34}e_4\\
\dot{y}_4&=cy_4-dy_3+\Gamma_{41}e_1+\Gamma_{42}e_2+\Gamma_{43}e_3+\Gamma_{44}e_4
\end{aligned} \tag{4.17}$$

where

$$\mathbf{A}=\begin{bmatrix}0&-1&-1&0\\1&a&0&1\\0&0&0&0\\0&0&-d&c\end{bmatrix} \tag{4.18}$$

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1,e_2,e_3,e_4)=\frac{1}{2}(e_1^{\,2}+e_2^{\,2}+e_3^{\,2}+e_4^{\,2}) \tag{4.19}$$

$$\mathbf{f(y)}-\mathbf{f(x)}=\begin{bmatrix}0\\0\\y_1y_3-x_1x_3\\0\end{bmatrix}=\begin{bmatrix}y_3&0&0&0\\0&0&0&0\\0&0&x_1&0\\0&0&0&0\end{bmatrix}\mathbf{e}=\mathbf{De} \tag{4.20}$$

Let

$$\mathbf{\Gamma}=-\mathbf{C}-\mathbf{A}-\mathbf{D}=\begin{bmatrix}-1-y_3&1&1&0\\-1&-1-a&0&-1\\0&0&-1-x_1&0\\0&0&d&-1-c\end{bmatrix} \tag{4.21}$$

According to Corollary 1, we obtain

$$\dot{V}=-e_1^2-e_2^2-e_3^2-e_4^2<0 \tag{4.22}$$

The Hyper-Rössler systems in Eq.(4.17) are globally synchronized. For initial states (-20,0,0,15), (-20,10.15,15) and system parameters $a=0.25$, $b=3$, $c=0.05$, $d=0.5$, the chaotic phase portraits and state errors versus time are shown in Fig. 4.3 and Fig. 4.4.

Third example for Theorem 4.2 is Duffing system. Consider following two unidirectional coupled chaotic Duffing systems:

$$\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=-\delta x_2+\alpha x_1-\beta x_1^3+a\cos\omega t\end{aligned} \tag{4.23}$$

$$\begin{aligned}\dot{y}_1&=y_2+u_1\\ \dot{y}_2&=-\hat{\delta}y_2+\hat{\alpha}y_1-\beta y_1^3+a\cos\omega t+u_2\end{aligned}$$

where $\mathbf{u}=[u_1,u_2]^T$ is the coupling term.

$$\mathbf{A}=\begin{bmatrix}0&1\\ \alpha&-\delta\end{bmatrix} \tag{4.24}$$

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1,e_2)=\frac{1}{2}({e_1}^2+{e_2}^2) \tag{4.25}$$

by Taylor expansion

$$\begin{aligned}\mathbf{f(y)}-\mathbf{f(x)}&=\begin{bmatrix}0\\-\beta y_1^3+\beta x_1^3\end{bmatrix}=\begin{bmatrix}0&0\\-3\beta x_1^2&0\end{bmatrix}\mathbf{e}+\begin{bmatrix}0\\-6\beta x_1 e_1^2+\cdots\end{bmatrix}\\&=\mathbf{Fe}+\text{H.O.T.of }\mathbf{e}\end{aligned} \tag{4.26}$$

Let $\mathbf{u}=\boldsymbol{\Gamma}\mathbf{e}+\mathbf{By}$

$$\boldsymbol{\Gamma}=-\mathbf{I}-\mathbf{A}-\mathbf{F}=\begin{bmatrix}-1&-1\\-\alpha+3\beta x_1^2&-1+\delta\end{bmatrix} \tag{4.27}$$

$$\mathbf{B}=-\tilde{\mathbf{A}}=\begin{bmatrix}0&0\\\hat{\alpha}-\alpha&-\hat{\delta}+\delta\end{bmatrix} \tag{4.28}$$

According to Theorem 2, we obtain

$$\dot{V}=-e_1^2-e_2^2+\text{HOT of }\mathbf{e}<0 \tag{4.29}$$

is negative definite when $\|\mathbf{e}\|^2$ is smaller than a bounded value . The Duffing systems (4.23) are locally synchronized. For initial states (2,2), (5,5) and system parameters $\alpha=-0.01$, $\delta=0.1$, $\beta=\omega=1$, $a=10$, $\hat{\alpha}=1$ and $\hat{\delta}=0.15$, the chaotic phase portrait and state errors versus time are shown in Fig. 4.5 and Fig. 4.6.

Last example for Corollary 4.2 is Lorenz system. Consider following two unidirectional coupled chaotic Lorenz systems:

$$\begin{aligned}\dot{x}_1&=\sigma(y_1-x_1)\\\dot{y}_1&=\gamma x_1-x_1z_1-y_1\\\dot{z}_1&=x_1y_1-\beta z_1\\[1ex]\dot{x}_2&=\hat{\sigma}(y_2-x_2)+u_1\\\dot{y}_2&=\hat{\gamma}x_2-x_2z_2-y_2+u_2\\\dot{z}_2&=x_2y_2-\hat{\beta}z_2+u_3\end{aligned} \tag{4.30}$$

where $\mathbf{u}=[u_1,u_2,u_3]^T$ is the coupling term.

$$\mathbf{A}=\begin{bmatrix}-\sigma & \sigma & 0\\ \gamma & -1 & 0\\ 0 & 0 & -\beta\end{bmatrix} \tag{4.31}$$

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1,e_2,e_3)=\frac{1}{2}(e_1^{\ 2}+e_2^{\ 2}+e_3^{\ 2}) \tag{4.32}$$

$$\mathbf{f(y)}-\mathbf{f(x)}=\begin{bmatrix}0\\ -x_2z_2+x_1z_1\\ x_2y_2-x_1y_1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ -z_2 & 0 & -x_1\\ y_2 & x_1 & 0\end{bmatrix}\mathbf{e}=\mathbf{De} \tag{4.33}$$

Let $\mathbf{u}=\mathbf{\Gamma e}+\mathbf{By}$

$$\mathbf{\Gamma}=-\mathbf{I}-\mathbf{A}-\mathbf{D}=\begin{bmatrix}\sigma-1 & -\sigma & 0\\ -\gamma+z_2 & 0 & x_1\\ -y_2 & -x_1 & \beta-1\end{bmatrix} \tag{4.34}$$

$$\mathbf{B}=-\tilde{\mathbf{A}}=\begin{bmatrix}6 & -6 & 0\\ -17.92 & 0 & 0\\ 0 & 0 & 4/3\end{bmatrix} \tag{4.35}$$

According to Corollary 2, we obtain

$$\dot{V}=-e_1^2-e_2^2-e_3^2\ <0 \tag{4.36}$$

is negative definite. The Lorenz systems (4.30) are global synchronized. For initial states (0.5,1,5), (0.6,2,5.3) and system parameters $\sigma=10$, $\gamma=28$, $\beta=8/3$, $\hat{\sigma}=16$, $\hat{\gamma}=45.92$ and $\hat{\beta}=4$, the chaotic phase portraits and state errors versus time are shown in Fig. 4.7 and Fig. 4.8.

4.4 Summary

In this Chapter, two theorems for chaos synchronization are proposed by using variable strength linear coupling without another active control, while the time derivative of Lyapunov function in series form is firstly used, which makes the demand for Lyapunov function derivative as negative sum of the square of state variables, lower. They give the criteria of chaos synchronization for two identical chaotic systems and for two different chaotic dynamic systems. Either local synchronization which is mostly good enough or global synchronization which is mostly an unnecessary high demand, can be obtained. Lorenz system, Duffing system, Rössler system and Hyper-Rössler system are used as simulation examples which effectively confirm the scheme.

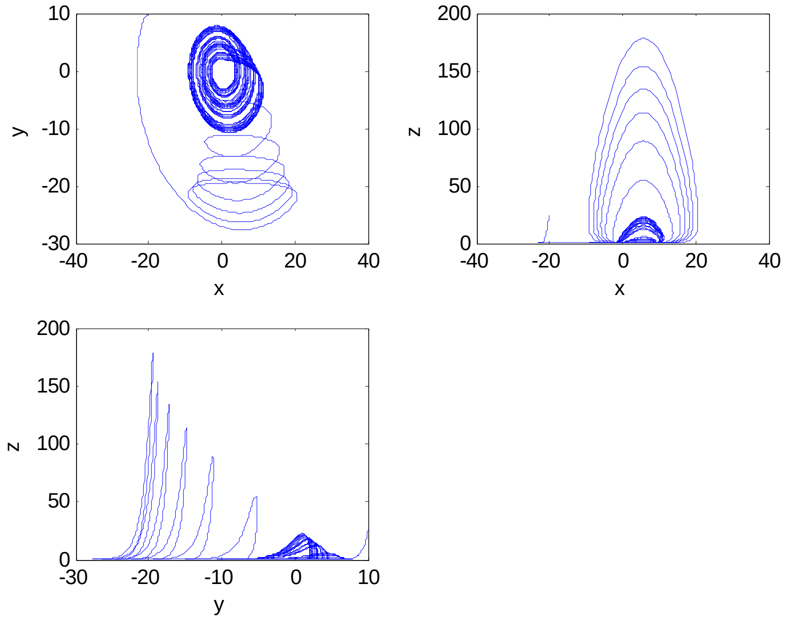

Fig. 4.1 Chaotic phase portraits for Rössler system.

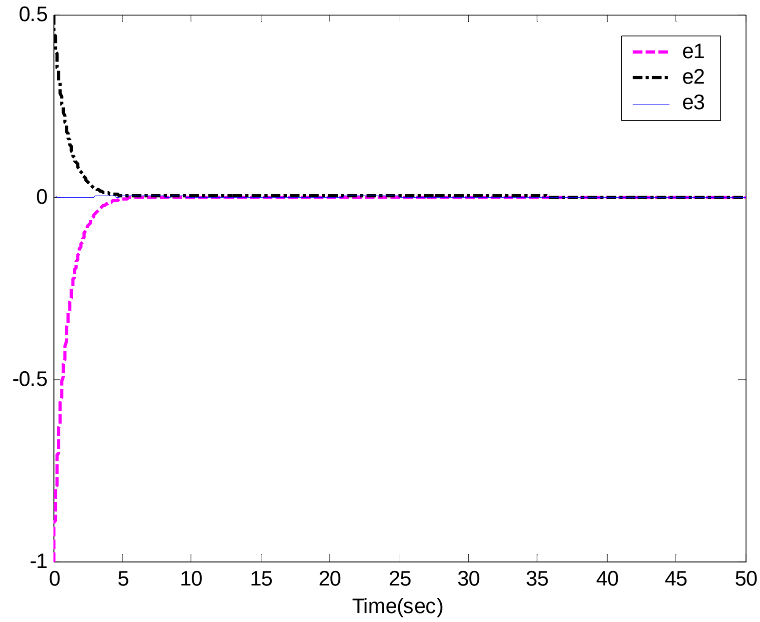

Fig. 4.2 Time histories of errors for two Rössler systems.

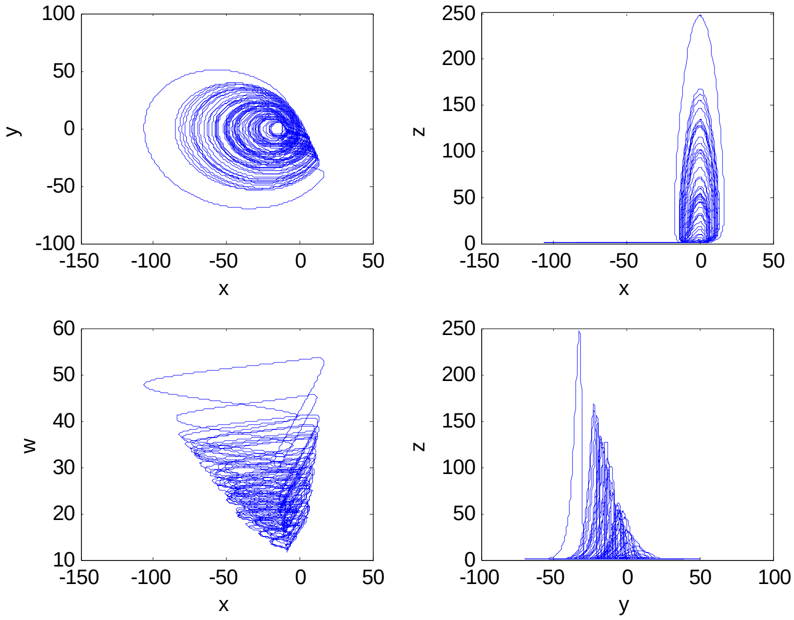

Fig. 4.3　Chaotic phase portraits for Hyper-Rössler system.

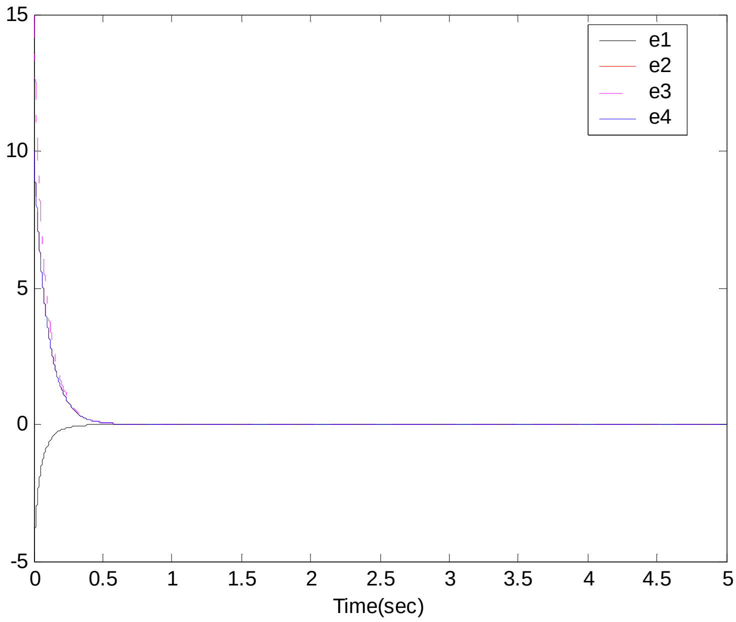

Fig. 4.4　Time histories of errors for two synchronized Hyper-Rössler systems.

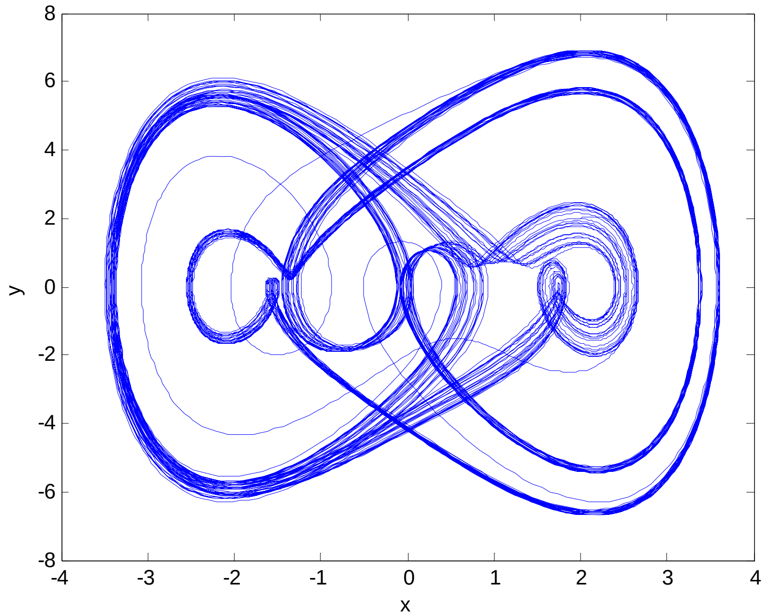

Fig. 4.5 Chaotic phase portrait for Duffing system.

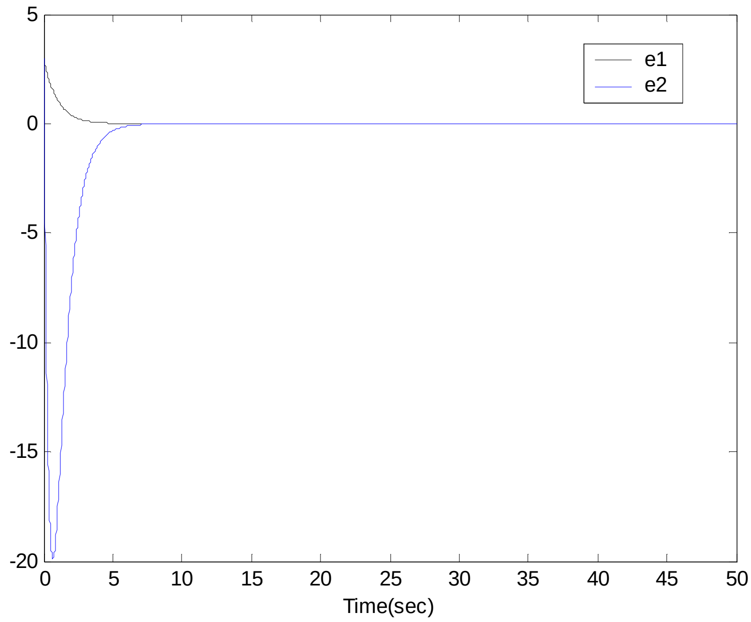

Fig. 4.6 Time histories of errors for two synchronized Duffing systems.

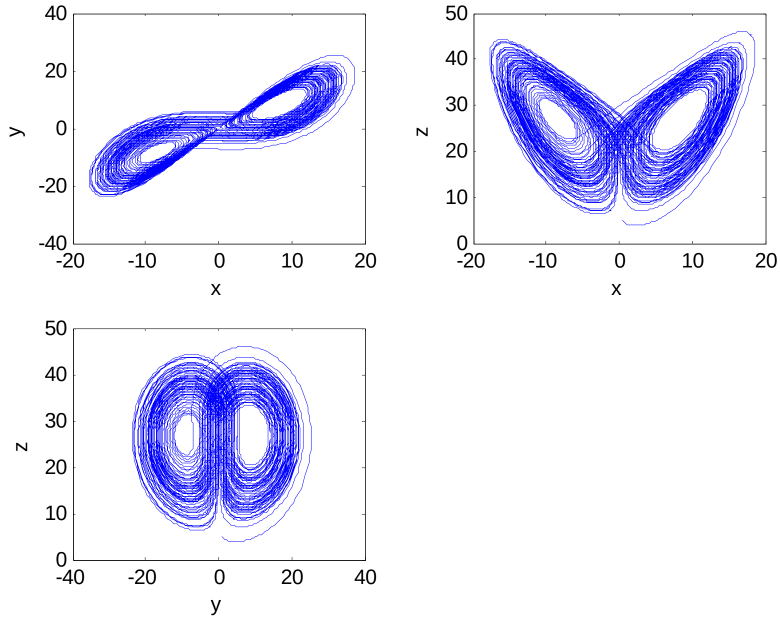

Fig. 4.7 Chaotic phase portraits for Lorenz system.

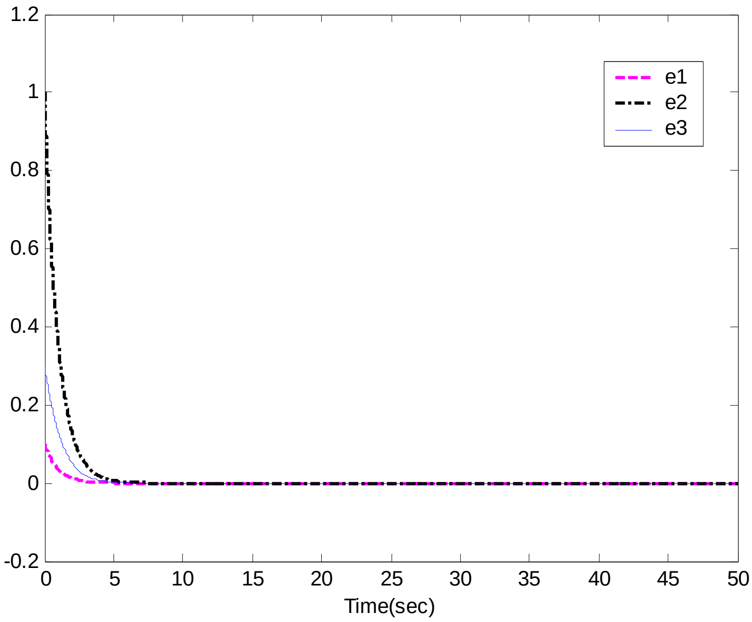

Fig. 4.8 Time histories of errors for two synchronized Lorenz systems.

Chapter 5

Adaptive Chaos Synchronization by Variable Strength Linear Coupling

5.1 Preliminary

Synchronization of chaotic dynamic systems is a very interesting problem and has been widely studied. Most of them are based on the exact knowledge of the system structure and parameters. But in practice, some or all of the system parameters are uncertain, adaptive control method in used. In this Chapter, we propose a general strategy to achieve adaptive chaos synchronization by variable strength linear coupling solely without using another active control which is usually rather complex. Furthermore, Lyapunov function derivative in series form is first used in this Chapter, which is easier to be obtained than the traditional negative sum of the square of error variables. Lorenz system, Duffing system and Rössler system are presented as simulation examples.

5.2 Adaptive Chaos Synchronization Strategy by Variable Strength Linear Coupling

Consider the following two unidirectional coupled system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{y}} &= \hat{\mathbf{A}}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \mathbf{\Gamma}(\mathbf{y} - \mathbf{x}) \end{aligned} \tag{5.1}$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in R^n$, $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T \in R^n$ denote two state vectors, $\mathbf{A}$ is an $n \times n$ uncertain constant coefficient matrix, $\hat{\mathbf{A}}$ is an $n \times n$ estimated coefficient matrix, $\mathbf{f}$ is a nonlinear vector function, and $\mathbf{\Gamma}$ is a matrix which gives the variable strength of the linear coupling term $(\mathbf{y} - \mathbf{x})$.

In order to study the synchronization of $\mathbf{x}$ and $\mathbf{y}$, define $\mathbf{e} = \mathbf{y} - \mathbf{x}$ as the state error. From Eq. (5.1), error equation can be written as

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \mathbf{\Gamma}(\mathbf{y}-\mathbf{x}) - \mathbf{A}\mathbf{x} - \mathbf{f}(\mathbf{x}) \tag{5.2}$$

Choosing an appropriate $\mathbf{\Gamma}$, our goal is that the state error can approach zero and the estimated coefficient matrix $\hat{\mathbf{A}}$ can approach the coefficient matrix $\mathbf{A}$.

Choose a positive definite function as

$$V(\mathbf{e}, \tilde{\mathbf{A}}_c) = \frac{1}{2}\mathbf{e}^T\mathbf{e} + \frac{1}{2}\tilde{\mathbf{A}}_c^T\tilde{\mathbf{A}}_c \tag{5.3}$$

where $\tilde{\mathbf{A}} = \hat{\mathbf{A}} - \mathbf{A}$, $\tilde{\mathbf{A}}_\mathbf{c}$ is a column matrix whose elements are all the elements of matrix $\tilde{\mathbf{A}}$. Then the time derivative of V through Eq. (5.2) is

$$\begin{aligned}\dot{V}(\mathbf{e}, \tilde{\mathbf{A}}_c) &= \mathbf{e}^T\dot{\mathbf{e}} + \tilde{\mathbf{A}}_c^T\dot{\tilde{\mathbf{A}}}_c \\ &= \mathbf{e}^T(\hat{\mathbf{A}}\mathbf{y} + \mathbf{f}(\mathbf{y}) + \mathbf{\Gamma}(\mathbf{y}-\mathbf{x}) - \mathbf{A}\mathbf{x} - \mathbf{f}(\mathbf{x})) + \tilde{\mathbf{A}}_c^T\dot{\tilde{\mathbf{A}}}_c \\ &= \mathbf{e}^T(\tilde{\mathbf{A}}\mathbf{y} + \mathbf{A}\mathbf{e} + \mathbf{\Gamma}\mathbf{e} + \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) + \tilde{\mathbf{A}}_c^T\dot{\tilde{\mathbf{A}}}_c \\ &= \mathbf{e}^T\tilde{\mathbf{A}}\mathbf{y} + \mathbf{e}^T(\hat{\mathbf{A}} - \tilde{\mathbf{A}})\mathbf{e} + \mathbf{e}^T\mathbf{\Gamma}\mathbf{e} + \mathbf{e}^T(\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})) + \tilde{\mathbf{A}}_c^T\dot{\tilde{\mathbf{A}}}_c \\ &= \mathbf{e}^T(\hat{\mathbf{A}} + \mathbf{\Gamma})\mathbf{e} + \mathbf{e}^T\tilde{\mathbf{A}}(\mathbf{y}-\mathbf{e}) + \tilde{\mathbf{A}}_c^T\dot{\tilde{\mathbf{A}}}_c + \mathbf{e}^T[\mathbf{f}(\mathbf{x}+\mathbf{e}) - \mathbf{f}(\mathbf{x})]\end{aligned} \tag{5.4}$$

In general, there are two ways to express $\mathbf{f}(\mathbf{x}+\mathbf{e}) - \mathbf{f}(\mathbf{x})$. Firstly, it is a linear function of **e, Be,** where the elements of **B** are variable. Then Eq. (5.4) become

$$\begin{aligned}\dot{V}(\mathbf{e}, \tilde{\mathbf{A}}_c) &= \mathbf{e}^T(\hat{\mathbf{A}} + \mathbf{\Gamma})\mathbf{e} + \mathbf{e}^T\tilde{\mathbf{A}}(\mathbf{y}-\mathbf{e}) + \tilde{\mathbf{A}}_c^T\dot{\tilde{\mathbf{A}}}_c + \mathbf{e}^T\mathbf{B}\mathbf{e} \\ &= \mathbf{e}^T(\hat{\mathbf{A}} + \mathbf{\Gamma} + \mathbf{B})\mathbf{e} + \mathbf{e}^T\tilde{\mathbf{A}}(\mathbf{y}-\mathbf{e}) + \tilde{\mathbf{A}}_c^T\dot{\tilde{\mathbf{A}}}_c\end{aligned} \tag{5.5}$$

Secondly, it can be developed to a Taylor series:

$$\begin{aligned}\dot{V}(\mathbf{e}, \tilde{\mathbf{A}}_c) = \mathbf{e}^T(\hat{\mathbf{A}} + \mathbf{\Gamma})\mathbf{e} + \mathbf{e}^T\tilde{\mathbf{A}}(\mathbf{y}-\mathbf{e}) + \tilde{\mathbf{A}}_c^T\dot{\tilde{\mathbf{A}}}_c + \mathbf{e}^T\mathbf{f}'(\mathbf{x})\mathbf{e} \\ + \text{HOT of } \mathbf{e}\end{aligned} \tag{5.6}$$

where $\mathbf{f}'(\mathbf{x})$ is time derivative of $\mathbf{f}(\mathbf{x})$.

If we can choose appropriate estimated coefficient matrix $\dot{\tilde{\mathbf{A}}}_c$ and $\mathbf{\Gamma}$ so that $\dot{\mathbf{V}} = -\mathbf{e}^\mathbf{T}\mathbf{C}\mathbf{e} - \tilde{\mathbf{A}}^\mathbf{T}\mathbf{P}\tilde{\mathbf{A}}$ or $\dot{\mathbf{V}} = -\mathbf{e}^\mathbf{T}\mathbf{C}\mathbf{e} - \tilde{\mathbf{A}}^\mathbf{T}\mathbf{P}\tilde{\mathbf{A}} + \text{HOT of } \mathbf{e}$, where $\mathbf{C}$ and $\mathbf{P}$ are a diagonal positive definite matrices. This series form of $\dot{\mathbf{V}}$ is easier to be obtained than the traditional negative sum of error variables. Then $\dot{\mathbf{V}}$ is a negative definite function of $\mathbf{e}$ and is a negative semi-definite function of $\mathbf{e}$ and all the elements of $\tilde{\mathbf{A}}_c$. In general, the elements of $\mathbf{\Gamma}$ may be functions of the state variables which are all bounded for chaotic system. So the elements of $\mathbf{\Gamma}$ are all bounded.

In current scheme of synchronization, traditional Lyapunov asymptotical stability theorem and Babalat lemma are used to prove the error vector approaches zero, as time approaches infinity.

5.3 Numerical Results for Typical Chaotic Systems

The first example is two Lorenz systems, with the following unidirectional coupling:

$$\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1)\\
\dot{y}_1 &= \gamma x_1 - x_1 z_1 - y_1\\
\dot{z}_1 &= x_1 y_1 - \beta z_1\\
\dot{x}_2 &= \hat{\sigma}(y_2 - x_2) + \Gamma_{11}e_1 + \Gamma_{12}e_2 + \Gamma_{13}e_3\\
\dot{y}_2 &= \hat{\gamma} x_2 - x_2 z_2 - y_2 + \Gamma_{21}e_1 + \Gamma_{22}e_2 + \Gamma_{23}e_3\\
\dot{z}_2 &= x_2 y_2 - \hat{\beta} z_2 + \Gamma_{31}e_1 + \Gamma_{32}e_2 + \Gamma_{33}e_3
\end{aligned} \tag{5.9}$$

The error dynamics is

$$\begin{aligned}
\dot{e}_1 &= \dot{x}_2 - \dot{x}_1 = \hat{\sigma}(y_2 - x_2) - \sigma(y_1 - x_1) + \Gamma_{11}e_1 + \Gamma_{12}e_2 + \Gamma_{13}e_3\\
\dot{e}_2 &= \dot{y}_2 - \dot{y}_1 = \hat{\gamma} x_2 - x_2 z_2 - y_2 - (\gamma x_1 - x_1 z_1 - y_1) + \Gamma_{21}e_1 + \Gamma_{22}e_2 + \Gamma_{23}e_3\\
\dot{e}_3 &= \dot{z}_2 - \dot{z}_1 = x_2 y_2 - \hat{\beta} z_2 - (x_1 y_1 - \beta z_1) + \Gamma_{31}e_1 + \Gamma_{32}e_2 + \Gamma_{33}e_3
\end{aligned} \tag{5.10}$$

where

$$\mathbf{A} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \gamma & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \tag{5.11}$$

In this example, we find

$$\mathbf{f(y)} - \mathbf{f(x)} = \begin{bmatrix} 0 \\ -x_2 z_2 + x_1 z_1 \\ x_2 y_2 - x_1 y_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -z_2 & 0 & -x_1 \\ y_2 & x_1 & 0 \end{bmatrix} \mathbf{e} = \mathbf{Be} \tag{5.12}$$

Choose a Lyapunov function in the form of a positive global definite function:

$$V(e_1,e_2,e_3,\tilde{\sigma},\tilde{\gamma},\tilde{\beta}) \\ =\frac{1}{2}(e_1{}^2+e_2{}^2+e_3{}^2+\tilde{\sigma}^2+\tilde{\gamma}^2+\tilde{\beta}^2) \tag{5.13}$$

where $\tilde{\sigma}=\hat{\sigma}-\sigma$, $\tilde{\gamma}=\hat{\gamma}-\gamma$, $\tilde{\beta}=\hat{\beta}-\beta$ and $\hat{\sigma}$, $\hat{\gamma}$, $\hat{\beta}$ are estimates of uncertain parameters σ, γ and β respectively.

Its time derivative through error dynamics (5.9) is

$$\begin{aligned}\dot{V} &= e_1[\hat{\sigma}(y_2-x_2)-\sigma(y_1-x_1)+\Gamma_{11}e_1+\Gamma_{12}e_2+\Gamma_{13}e_3] \\ &+e_2[\hat{\gamma}x_2-x_2z_2-y_2-(\gamma x_1-x_1z_1-y_1)+\Gamma_{21}e_1+\Gamma_{22}e_2+\Gamma_{23}e_3] \\ &+e_3[x_2y_2-\hat{\beta}z_2-(x_1y_1-\beta z_1)+\Gamma_{31}e_1+\Gamma_{32}e_2+\Gamma_{33}e_3] \\ &+\tilde{\sigma}\dot{\tilde{\sigma}}+\tilde{\gamma}\dot{\tilde{\gamma}}+\tilde{\beta}\dot{\tilde{\beta}}\end{aligned} \tag{5.14}$$

Choose

$$\mathbf{\Gamma}=\mathbf{-I-\hat{A}-B}=\begin{bmatrix}\hat{\sigma}-1 & -\hat{\sigma} & 0 \\ -\hat{\gamma}+z_2 & 0 & x_1 \\ -y_2 & -x_1 & \hat{\beta}-1\end{bmatrix} \tag{5.15}$$

$$\begin{aligned}\dot{\tilde{\beta}} &= \dot{\hat{\beta}} = z_2e_3-e_3^2-\tilde{\beta} \\ \dot{\tilde{\sigma}} &= \dot{\hat{\sigma}} = x_2e_1-y_2e_1+e_1e_2-e_1^2-\tilde{\sigma} \\ \dot{\tilde{\gamma}} &= \dot{\hat{\gamma}} = -x_2e_2+e_1e_2-\tilde{\gamma}\end{aligned} \tag{5.16}$$

Introducing Eqs. (5.15) and (5.16) into Eq. (5.14), we obtain

$$\dot{V}=-e_1^2-e_2^2-e_3^2-\tilde{\sigma}^2-\tilde{\gamma}^2-\tilde{\beta}^2\le 0 \tag{5.17}$$

By Lyapunov asymptotical stability theorem, $e_1=e_2=e_3=\tilde{\sigma}=\tilde{\gamma}=\tilde{\beta}=0$ is asymptotically stable. The initial states is (0.5,1,5), (30,20,18) and system parameters $\sigma=10$, $\gamma=28$, $\beta=8/3$ and initial values of estimate for uncertain parameters $\hat{\sigma}=\hat{\gamma}=\hat{\beta}=0$. Three state errors and estimated parameters versus time are shown in Fig. 5.1 and Fig. 5.2. Fig. 5.1 shows that state errors quickly approach zeros. The estimated parameters approach to the uncertain parameters as shown in Fig. 5.2. The synchronization is global synchronization.

The second example is two Duffing systems, with the following unidirectional coupling:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + \alpha x_1 - \beta x_1^3 + a\cos\omega t \end{aligned} \tag{5.18}$$

$$\begin{aligned} \dot{y}_1 &= y_2 + \Gamma_{11}e_1 + \Gamma_{12}e_2 \\ \dot{y}_2 &= -\hat{\delta} y_2 + \hat{\alpha} y_1 - \beta y_1^3 + a\cos\omega t + \Gamma_{21}e_1 + \Gamma_{22}e_2 \end{aligned} \tag{5.19}$$

The error dynamics is

$$\begin{aligned} \dot{e}_1 &= \dot{y}_1 - \dot{x}_1 = e_2 + \Gamma_{11}e_1 + \Gamma_{12}e_2 \\ \dot{e}_2 &= \dot{y}_2 - \dot{x}_2 = -\hat{\delta} y_2 + \hat{\alpha} y_1 - \beta y_1^3 - (-\delta x_2 + \alpha x_1 - \beta x_1^3) + \Gamma_{21}e_1 + \Gamma_{22}e_2 \end{aligned} \tag{5.20}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \alpha & -\delta \end{bmatrix} \tag{5.21}$$

By Taylor expansion

$$\begin{aligned} \mathbf{f(y)} - \mathbf{f(x)} &= \begin{bmatrix} 0 \\ -\beta y_1^3 + \beta x_1^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -3\beta x_1^2 & 0 \end{bmatrix}\mathbf{e} + \begin{bmatrix} 0 \\ -6\beta x_1 e_1^2 + \cdots \end{bmatrix} \\ &= \mathbf{Be} + \text{H.O.T.of } \mathbf{e} \end{aligned} \tag{5.22}$$

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1, e_2, \tilde{\delta}, \tilde{\alpha}) = \frac{1}{2}(e_1^{\,2} + e_2^{\,2} + \tilde{\delta}^2 + \tilde{\alpha}^2) \tag{5.23}$$

where $\tilde{\delta} = \hat{\delta} - \delta$, $\tilde{\alpha} = \hat{\alpha} - \alpha$ and $\hat{\delta}$, $\hat{\alpha}$ are estimates of uncertain parameters δ and α.

Its time derivative through error dynamics (5.20) is

$$\begin{aligned} \dot{V} = {} & e_1(e_2 + \Gamma_{11}e_1 + \Gamma_{12}e_2) \\ & + e_2(-\hat{\delta} y_2 + \hat{\alpha} y_1 - \beta y_1^3 - (-\delta x_2 + \alpha x_1 - \beta x_1^3) + \Gamma_{21}e_1 + \Gamma_{22}e_2) \\ & + \tilde{\delta}\dot{\tilde{\delta}} + \tilde{\alpha}\dot{\tilde{\alpha}} \end{aligned}$$

$$\tag{5.24}$$

Choose

$$\mathbf{\Gamma} = \mathbf{-I} - \hat{\mathbf{A}} - \mathbf{B} = \begin{bmatrix} -1 & -1 \\ -\hat{\alpha} + 3\beta x_1^2 & -1 + \hat{\delta} \end{bmatrix} \tag{5.25}$$

$$\dot{\tilde{\delta}} = \dot{\hat{\delta}} = y_2 e_2 - e_2^2 - \tilde{\delta}$$

$$\dot{\tilde{\alpha}} = \dot{\hat{\alpha}} = -y_1 e_2 + e_1 e_2 - \tilde{\alpha} \tag{5.26}$$

We obtain

$$\dot{V} = -e_1^2 - e_2^2 - \tilde{\delta}^2 - \tilde{\alpha}^2 + \text{H.O.T.of } e \le 0 \tag{5.27}$$

By Lyapunov asymptotical stability theorem, $e_1 = e_2 = \tilde{\delta} = \tilde{\alpha} = 0$ is asymptotically stable. The initial states is (2,2), (5,5) and system parameters $\delta = 0.15$, $\alpha = \beta = \omega = 1$, $a = 3$, and initial values of estimate for uncertain parameters $\hat{\delta} = \hat{\alpha} = 0$, The state errors and estimated parameters versus time are shown in Fig. 5.3 and Fig. 5.4. Fig. 5.3 shows that state errors quickly approach zeros. The estimated parameters approach the uncertain parameters as shown in Fig. 5.4.

The last example is two Rössler systems, with the following unidirectional coupling:

$$\begin{aligned} \dot{x}_1 &= -y_1 - z_1 \\ \dot{y}_1 &= x_1 + a y_1 \\ \dot{z}_1 &= b + z_1(x_1 - c) \end{aligned} \tag{5.28}$$

$$\begin{aligned} \dot{x}_2 &= -y_2 - z_2 + \Gamma_{11} e_1 + \Gamma_{12} e_2 + \Gamma_{13} e_3 \\ \dot{y}_2 &= x_2 + \hat{a} y_2 + \Gamma_{21} e_1 + \Gamma_{22} e_2 + \Gamma_{23} e_3 \\ \dot{z}_2 &= b + z_2(x_2 - \hat{c}) + \Gamma_{31} e_1 + \Gamma_{32} e_2 + \Gamma_{33} e_3 \end{aligned} \tag{5.29}$$

The error dynamics is

$$\begin{aligned} \dot{e}_1 &= -e_2 - e_3 + \Gamma_{11} e_1 + \Gamma_{12} e_2 + \Gamma_{13} e_3 \\ \dot{e}_2 &= e_1 + \hat{a} y_2 - a y_1 + \Gamma_{21} e_1 + \Gamma_{22} e_2 + \Gamma_{23} e_3 \\ \dot{e}_3 &= z_2(x_2 - \hat{c}) - z_1(x_1 - c) + \Gamma_{31} e_1 + \Gamma_{32} e_2 + \Gamma_{33} e_3 \end{aligned} \tag{5.30}$$

where

$$\mathbf{A}=\begin{bmatrix}0 & -1 & -1\\ 1 & a & 0\\ 0 & 0 & -c\end{bmatrix} \tag{5.31}$$

By Taylor expansion,

$$\begin{aligned}\mathbf{f(y)}-\mathbf{f(x)}&=\begin{bmatrix}0\\ 0\\ z_1e_1+x_1e_3+e_1e_3\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ z_1 & 0 & x_1\end{bmatrix}\mathbf{e}+\begin{bmatrix}0\\ 0\\ e_1e_3\end{bmatrix}\\ &=\mathbf{Be}+\begin{bmatrix}0\\ 0\\ e_1e_3\end{bmatrix}\end{aligned} \tag{5.32}$$

Choose a Lyapunov function in the form of a positive definite function:

$$V(e_1,e_2,e_3,\tilde{a},\tilde{c})=\frac{1}{2}(e_1^{\ 2}+e_2^{\ 2}+e_3^{\ 2}+\tilde{a}^2+\tilde{c}^2) \tag{5.33}$$

where $\tilde{a}=\hat{a}-a$, $\tilde{c}=\hat{c}-c$ and $\hat{a}$, $\hat{c}$ are estimates of uncertain parameters a and c.

Its derivative through the error dynamics is

$$\begin{aligned}\dot{V}&=e_1[-e_2-e_3+\Gamma_{11}e_1+\Gamma_{12}e_2+\Gamma_{13}e_3]\\ &\quad+e_2[e_1+\hat{a}y_2-ay_1+\Gamma_{21}e_1+\Gamma_{22}e_2+\Gamma_{23}e_3]\\ &\quad+e_3[z_2(x_2-\hat{c})-z_1(x_1-c)+\Gamma_{31}e_1+\Gamma_{32}e_2+\Gamma_{33}e_3]\\ &\quad+\tilde{a}\dot{\tilde{a}}+\tilde{c}\dot{\tilde{c}}\end{aligned} \tag{5.34}$$

Choose

$$\mathbf{\Gamma}=\mathbf{-I}-\hat{\mathbf{A}}-\mathbf{B}=\begin{bmatrix}-1 & 1 & 1\\ -1 & -1-\hat{a} & 0\\ -z_1 & 0 & -1+\hat{c}-x_1\end{bmatrix} \tag{5.35}$$

$$\dot{\tilde{a}}=\dot{\hat{a}}=-y_2e_2+e_2^2-\tilde{a}$$

$$\dot{\tilde{c}}=\dot{\hat{c}}=z_2e_3-e_3^2-\tilde{c} \tag{5.36}$$

We obtain

$$\dot{V}=-e_1^2-e_2^2-e_3^2-\tilde{a}^2-\tilde{c}^2+\text{H.O.T.of } e\le 0 \tag{5.37}$$

By Lyapunov asymptotical stability theorem, $e_1 = e_2 = e_3 = \tilde{a} = \tilde{c} = 0$ is asymptotically stable. The initial states is (-20,10,25), (-20.1,10.5,25) and system parameters $a = 0.2$, $b = 0.2$, $c = 5.7$, and initial values of estimate for uncertain parameters $\hat{a} = \hat{c} = 0$. The state errors and estimated parameters versus time are shown in Fig. 5.5 and Fig. 5.6. Fig. 5.5 shows that state errors quickly approach zeros. The estimated parameters approach the uncertain parameters as shown in Fig. 5.6. In the second and third examples, the synchronizations are local synchronization.

5.4 Summary

In this Chapter a general strategy to achieve adaptive chaos synchronization by variable strength linear coupling is studied. Lyapunov function derivative in series form is first used in this Chapter, which is easier to be obtained than the traditional negative sum of the square of error variables. In most cases, local synchronization is good enough, while global synchronization is an unnecessary high demand. Lorenz system, Duffing system and Rössler system are used as simulation examples which effectively confirm the scheme and our opinion.

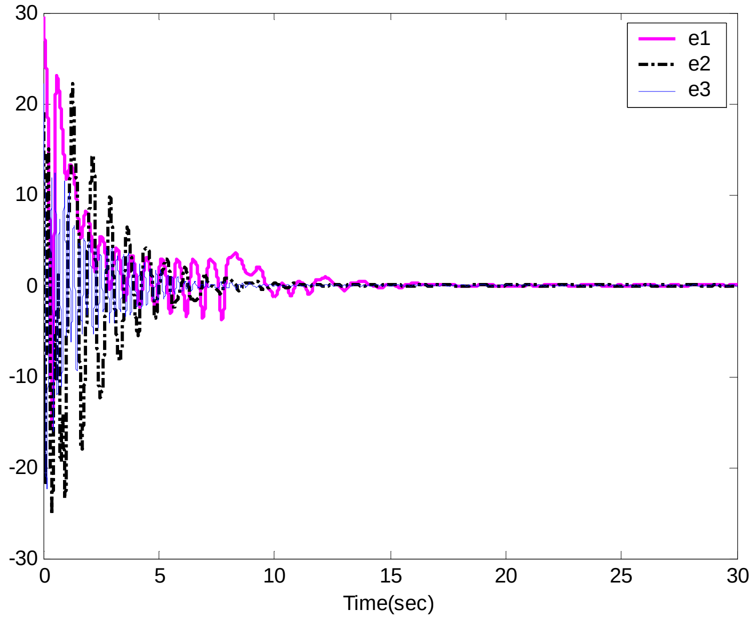

Fig. 5.1 Time histories of errors for Lorenz system.

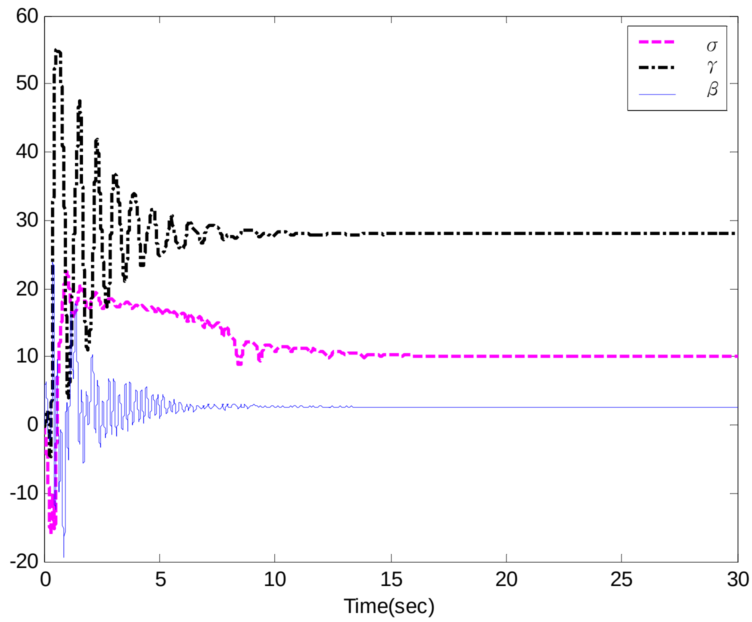

Fig. 5.2 Time histories of estimated parameters for Lorenz system.

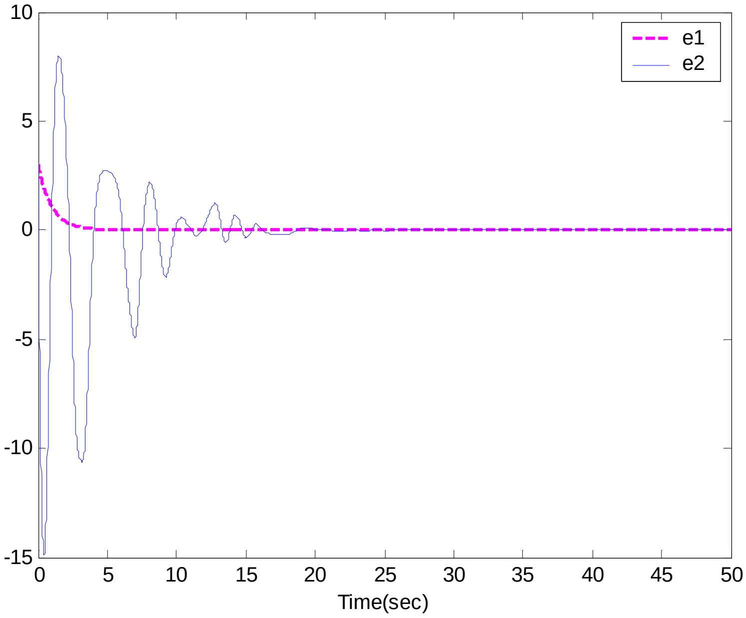

Fig. 5.3 Time histories of errors for Duffing system.

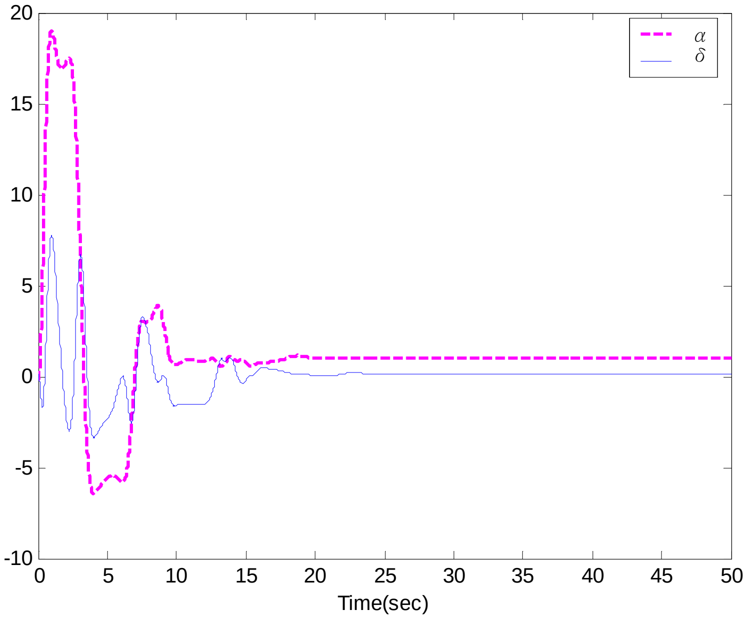

Fig. 5.4 Time histories of estimated parameters for Duffing system.

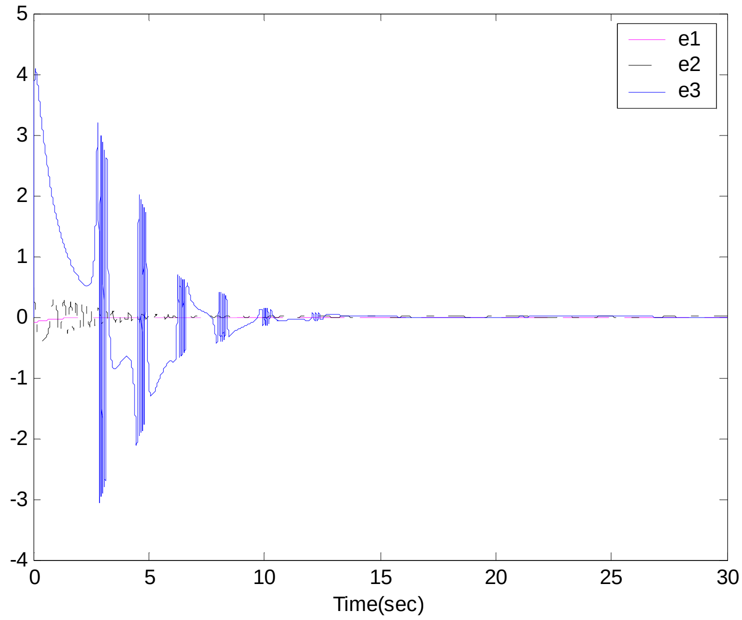

Fig. 5.5 Time histories of errors for Rössler system.

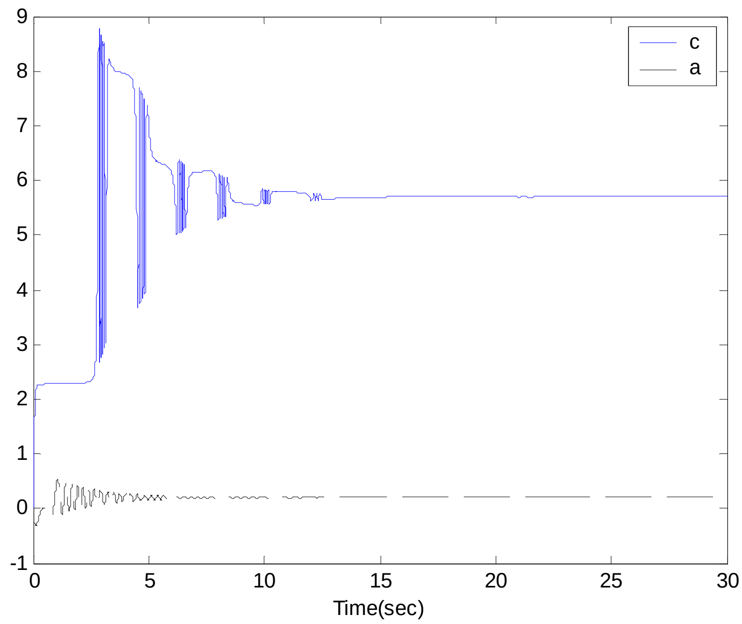

Fig. 5.6 Time histories of estimated parameters for Rössler system.

Chapter 6

Chaos Generalized Synchronization by Partial Region Stability Theory

6.1 Preliminary

In this Chapter, a new chaos generalized synchronization strategy by partial region stability theory is proposed. By using the theory of stability on partial region the Lyapunov function is a simple linear homogeneous function of states and the controllers are simpler and have less simulation error because they are in lower order than that of traditional controllers. Lorenz system and Rössler system are used as simulated examples.

6.2 Chaos Generalized Synchronization Strategy by Partial Region Stability Theory

Consider the following unidirectional coupled chaotic systems

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{h}(t, \mathbf{y}) + \mathbf{u}\end{aligned} \tag{6.1}$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in R^n$, $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T \in R^n$ denote two state vectors, $\mathbf{f}$ and $\mathbf{h}$ are nonlinear vector functions, and $\mathbf{u} = [u_1, u_2, \cdots, u_n]^T \in R^n$ is a control input vector.

The generalized synchronization can be accomplished when $t \to \infty$, the limit of the error vector $\mathbf{e} = [e_1, e_2, \cdots, e_n]^T$ approaches zero:

$$\lim_{t \to \infty} \mathbf{e} = 0 \tag{6.2}$$

where

$$\mathbf{e} = \mathbf{G}(\mathbf{x}) - \mathbf{y} \tag{6.3}$$

By using the theory of stability on partial region, the Lyapunov function is easier to find, since the terms of first degree can be used to construct the definite Lyapunov function and the controller can be designed in lower order.

6.2.1 Definition of the Stability on Partial Region

Consider the differential equations of disturbed motion of a nonautonomous system in the normal form

$$\frac{dx_s}{dt} = X_s(t, x_1, \cdots, x_n), \qquad (s = 1, \cdots, n) \tag{6.4}$$

where the function X_s is defined on the intersection of the partial region Ω (shown in Fig. 6.1) and

$$\sum_s x_s^2 \le H \tag{6.5}$$

and $t > t_0$, where t_0 and H are certain positive constants. X_s which vanishes when the variables x_s are all zero, is a real valued function of t, $x_1, \cdots, x_n$. It is assumed that X_s is smooth enough to ensure the existence, uniqueness of the solution of the initial value problem. When X_s does not contain t explicitly, the system is autonomous.

Obviously, $x_s = 0 \quad (s = 1, \cdots n)$ is a solution of Eq. (6.4). We are interested to the asymptotical stability of this zero solution on partial region Ω (including the boundary) of the neighborhood of the origin which in general may consist of several subregions (Fig. 6.1).

Definition 1:

For any given number $\varepsilon > 0$, if there exists a $\delta > 0$, such that on the closed given partial region Ω when

$$\sum_s x_{s0}^2 \le \delta, \quad (s = 1, \cdots, n) \tag{6.6}$$

for all $t \ge t_0$, the inequality

$$\sum_s x_s^2 \le \varepsilon, \quad (s = 1, \cdots, n) \tag{6.7}$$

is satisfied for the solutions of Eq.(6.4) on Ω, then the disturbed motion $x_s = 0 \quad (s = 1, \cdots n)$ is stable on the partial region Ω.

Definition 2:

If the undisturbed motion is stable on the partial region Ω, and there exists a $\delta^{'} > 0$, so that on the given partial region Ω when

$$\sum_s x_{s0}^2 \leq \delta^{'}, \quad (s = 1, \cdots, n) \tag{6.8}$$

The equality

$$\lim_{t \to \infty} \left(\sum_s x_s^2 \right) = 0 \tag{6.9}$$

is satisfied for the solutions of Eq.(6.4) on Ω, then the undisturbed motion $x_s = 0 \quad (s = 1, \cdots n)$ is asymptotically stable on the partial region Ω.

The intersection of Ω and region defined by Eq.(6.8) is called the region of attraction.

Definition of Functions $V(t, x_1, \cdots, x_n)$:

Let us consider the functions $V(t, x_1, \cdots, x_n)$ given on the intersection Ω_1 of the partial region Ω and the region

$$\sum_s x_s^2 \leq h, \quad (s = 1, \cdots, n) \tag{6.10}$$

for $t \geq t_0 > 0$, where t_0 and h are positive constants. We suppose that the functions are single-valued and have continuous partial derivatives and become zero when $x_1 = \cdots = x_n = 0$.

Definition 3:

If there exists $t_0 > 0$ and a sufficiently small $h > 0$, so that on partial region Ω_1 and $t \geq t_0$, $V \geq 0$ (or ≤ 0), then V is a positive (or negative) semidefinite, in general semidefinite, function on the Ω_1 and $t \geq t_0$.

Definition 4:

If there exists a positive (negative) definitive function $W(x_1 \ldots x_n)$ on Ω_1, so that on the partial region Ω_1 and $t \geq t_0$

$$V - W \geq 0 \; (or \; -V - W \geq 0), \tag{6.11}$$

then $V(t, x_1, \ldots, x_n)$ is a positive definite function on the partial region Ω_1 and $t \geq t_0$.

Definition 5:

If $V(t, x_1, \ldots, x_n)$ is neither definite nor semidefinite on Ω_1 and $t \geq t_0$, then

$V(t, x_1, \ldots, x_n)$ is an indefinite function on partial region Ω_1 and $t \geq t_0$. That is, for any small $h > 0$ and any large $t_0 > 0$, $V(t, x_1, \ldots, x_n)$ can take either positive or negative value on the partial region Ω_1 and $t \geq t_0$.

Definition 6: Bounded function *V*

If there exist $t_0 > 0$, $h > 0$, so that on the partial region Ω_1, we have

$$|V(t, x_1, \ldots, x_n)| < L$$

where *L* is a positive constant, then *V* is said to be bounded on Ω_1.

Definition 7: Function with infinitesimal upper bound

If *V* is bounded, and for any $\lambda > 0$, there exists $\mu > 0$, so that on Ω_1 when $\sum_s x_s^2 \leq \mu$, and $t \geq t_0$, we have

$$|V(t, x_1, \ldots, x_n)| \leq \lambda$$

then *V* admits an infinitesimal upper bound on Ω_1.

6.2.2 Theorem of stability and of asymptotical stability on partial region

Theorem 6.1

If there can be found for the differential equations of the disturbed motion (Eq. 6.4) a definite function $V(t, x_1, \ldots, x_n)$ on the partial region, and for which the derivative with respect to time based on these equations as given by the following :

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} X_s \tag{6.12}$$

is a semidefinite function on the paritial region whose sense is opposite to that of V, or if it becomes zero identically, then the undisturbed motion is stable on the partial region.

Proof:

Let us assume for the sake of definiteness that *V* is a positive definite function. Consequently, there exists a sufficiently large number t_0 and a sufficiently small number $h < H$, such that on the intersection Ω_1 of partial region Ω and

$$\sum_s x_s^2 \leq h, \quad (s = 1, \ldots, n)$$

and $t \geq t_0$, the following inequality is satisfied

$$V(t, x_1, \ldots, x_n) \geq W(x_1, \ldots, x_n),$$

where W is a certain positive definite function which does not depend on t. Besides that, Eq. (6.12) may assume only negative or zero value in this region.

Let ε be an arbitrarily small positive number. We shall suppose that in any case $\varepsilon < h$. Let us consider the aggregation of all possible values of the quantities $x_1, \ldots, x_n$, which are on the intersection ω_2 of Ω_1 and

$$\sum_s x_s^2 = \varepsilon, \tag{6.13}$$

and let us designate by $l > 0$ the precise lower limit of the function W under this condition. by virtue of Eq. (6.11), we shall have

$$V(t, x_1, \ldots, x_n) \geq l \quad \text{for} \quad (x_1, \ldots, x_n) \quad \text{on} \quad \omega_2. \tag{6.14}$$

We shall now consider the quantities x_s as functions of time which satisfy the differential equations of disturbed motion. We shall assume that the initial values x_{s0} of these functions for $t = t_0$ lie on the intersection Ω_2 of Ω_1 and the region

$$\sum_s x_s^2 \leq \delta, \tag{6.15}$$

where δ is so small that

$$V(t_0, x_{10}, \ldots, x_{n0}) < l \tag{6.16}$$

By virtue of the fact that $V(t_0, 0, \ldots, 0) = 0$, such a selection of the number δ is obviously possible. We shall suppose that in any case the number δ is smaller than ε.Then the inequality

$$\sum_s x_s^2 \leq \varepsilon,$$

being satisfied at the initial instant will be satisfied, in the very least, for a sufficiently small $t - t_0$, since the functions $x_s(t)$ very continuously with time. We shall show that these inequalities will be satisfied for all values $t > t_0$. Indeed, if these inequalities were not satisfied at some time, there would have to exist such an instant t=T for which this inequality would become an equality. In other words, we would have

$$\sum_s x_s^2(T) = \varepsilon,$$

and consequently, on the basis of Eq. (6.14)

$$V(T, x_1(T), \ldots, x_n(T)) \geq l \tag{6.17}$$

On the other hand, since $\varepsilon < h$, the inequality (Eq.(10)) is satisfied in the entire interval of time [t_0, T], and consequently, in this entire time interval $\frac{dV}{dt} \leq 0$. This yields

$$V(T, x_1(T), \ldots, x_n(T)) \leq V(t_0, x_{10}, \ldots, x_{n0}),$$

which contradicts Eq. (6.17) on the basis of Eq. (6.16). Thus, the inequality (Eq.(6.7)) must be satisfied for all values of $t > t_0$, hence follows that the motion is stable.

Finally, we must point out that from the view-point of mathematics, the stability on partial region in general does not be related logically to the stability on whole region. If an undisturbed solution is stable on a partial region, it may be either stable or unstable on the whole region and vice versa. From the viewpoint of dynamics, we wre not interesting to the solution starting from Ω_2 and going out of Ω.

Theorem 6.2

If in satisfying the conditions of Theorem 1, the derivative $\frac{dV}{dt}$ is a definite function on the partial region with opposite sign to that of V and the function V itself permits an infinitesimal upper limit, then the undisturbed motion is asymptotically stable on the partial region.

Proof:

Let us suppose that V is a positive definite function on the partial region and that consequently, $\frac{dV}{dt}$ is negative definite. Thus on the intersection Ω_1 of Ω and the region defined by Eq. (6.10) and $t \geq t_0$ there will be satisfied not only the inequality

(Eq.(6.11)), but the following inequality as will:

$$\frac{dV}{dt} \le -W_1(x_1, \dots x_n), \tag{6.18}$$

where W_1 is a positive definite function on the partial region independent of t.

Let us consider the quantities x_s as functions of time which satisfy the differential equations of disturbed motion assuming that the initial values $x_{s0} = x_s(t_0)$ of these quantities satisfy the inequalities (Eq. (6.15)). Since the undisturbed motion is stable in any case, the magnitude δ may be selected so small that for all values of $t \ge t_0$ the quantities x_s remain within Ω_1. Then, on the basis of Eq. (6.18) the derivative of function $V(t, x_1(t), \dots, x_n(t))$ will be negative at all times and, consequently, this function will approach a certain limit, as *t* increases without limit, remaining larger than this limit at all times. We shall show that this limit is equal to some positive quantity different from zero. Then for all values of $t \ge t_0$ the following inequality will be satisfied:

$$V(t, x_1(t), \dots, x_n(t)) > \alpha \tag{6.19}$$

where $\alpha > 0$.

Since *V* permits an infinitesimal upper limit, it follows from this inequality that

$$\sum_s x_s^2(t) \ge \lambda, \quad (s = 1, \dots, n), \tag{6.20}$$

where λ is a certain sufficiently small positive number. Indeed, if such a number λ did not exist, that is , if the quantity $\sum_s x_s(t)$ were smaller than any preassigned number no matter how small, then the magnitude $V(t, x_1(t), \dots, x_n(t))$, as follows from the definition of an infinitesimal upper limit, would also be arbitrarily small, which contradicts (6.19).

If for all values of $t \ge t_0$ the inequality (Eq. (6.20)) is satisfied, then Eq. (6.18) shows that the following inequality will be satisfied at all times:

$$\frac{dV}{dt} \le -l_1,$$

where l_1 is positive number different from zero which constitutes the precise lower limit of the function $W_1(t, x_1(t), \ldots, x_n(t))$ under condition (Eq. (6.20)). Consequently, for all values of $t \ge t_0$ we shall have:

$$V(t, x_1(t), \ldots, x_n(t)) = V(t_0, x_{10}, \ldots, x_{n0}) + \int_{t_0}^{t} \frac{dV}{dt} dt \le V(t_0, x_{10}, \ldots, x_{n0}) - l_1(t - t_0),$$

which is, obviously, in contradiction with Eq.(6.19). The contradiction thus obtained shows that the function $V(t, x_1(t), \ldots, x_n(t))$ approached zero as t increase without limit. Consequently, the same will be true for the function $W(x_1(t), \ldots, x_n(t))$ as well, from which it follows directly that

$$\lim_{t\to\infty} x_s(t) = 0, \quad (s = 1, \ldots, n),$$

which proves the theorem.

6.3 Numerical Simulations

The following example is two Lorenz systems **x** and **y**, with the unidirectional coupling:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \gamma x_1 - x_1 x_3 - x_2 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \\ \dot{y}_1 &= \sigma(y_2 - y_1) + u_1 \\ \dot{y}_2 &= \gamma y_1 - y_1 y_3 - y_2 + u_2 \\ \dot{y}_3 &= y_1 y_2 - \beta y_3 + u_3 \end{aligned} \tag{6.21}$$

CASE I. The generalized synchronization error function is $\mathbf{e} = \mathbf{x} - \mathbf{y} + 100$.

Our goal is $\mathbf{y} = \mathbf{x} + 100$, i.e. $\lim_{t\to\infty} \mathbf{e} = \lim_{t\to\infty} (\mathbf{x} - \mathbf{y} + 100) = 0$ (6.22)

The error dynamics becomes

$$\begin{aligned} \dot{e}_1 &= \dot{x}_1 - \dot{y}_1 = \sigma(x_2 - x_1) - \sigma(y_2 - y_1) - u_1 \\ \dot{e}_2 &= \dot{x}_2 - \dot{y}_2 = \gamma x_1 - x_1 x_3 - x_2 - (\gamma y_1 - y_1 y_3 - y_2) - u_2 \\ \dot{e}_3 &= \dot{x}_3 - \dot{y}_3 = x_1 x_2 - \beta x_3 - (y_1 y_2 - \beta y_3) - u_3 \end{aligned} \tag{6.23}$$

Let initial states is (x_{10}, x_{20}, x_{30})= (0.5, 1, 5), (y_{10}, y_{20}, y_{30})= (0.6, 3, 10) and system parameters $\sigma = 10$, $\gamma = 28$, $\beta = 8/3$, we find the error dynamic always exist in first quadrant showed in Fig. 6.2 By partial region stability, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3$$

Its time derivative is

$$\begin{aligned} \dot{V} &= e_1 + e_2 + e_3 \\ &= \left(\sigma(x_2 - x_1) - \sigma(y_2 - y_1) - u_1\right) + \left(\gamma x_1 - x_1 x_3 - x_2 - (\gamma y_1 - y_1 y_3 - y_2) - u_2\right) \\ &\quad + \left(x_1 x_2 - \beta x_3 - (y_1 y_2 - \beta y_3) - u_3\right) \end{aligned} \tag{6.24}$$

Choose

$$\begin{aligned} u_1 &= \sigma(x_2 - x_1) - \sigma(y_2 - y_1) + e_1 \\ u_2 &= \gamma x_1 - x_1 x_3 - x_2 - (\gamma y_1 - y_1 y_3 - y_2) + e_2 \\ u_3 &= x_1 x_2 - \beta x_3 - (y_1 y_2 - \beta y_3) + e_3 \end{aligned} \tag{6.25}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 < 0 \tag{6.26}$$

which is negative definite function. Three state errors versus time and time histories of states are shown in Fig. 6.3 and Fig. 6.4.

CASE II. The generalized synchronization error function is $e_i = x_i - y_i + F\sin\omega t + 100$, i=1, 2, 3.

Our goal is $\mathbf{y} = \mathbf{x} + \mathbf{F}\sin\omega t + 100$, i.e. $\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + F\sin\omega t + 100) = 0$, $i = 1, 2, 3$

The error dynamics become

$$\begin{aligned} \dot{e}_1 &= \sigma(x_2 - x_1) - \sigma(y_2 - y_1) - u_1 + F\omega\cos\omega t \\ \dot{e}_2 &= \gamma x_1 - x_1 x_3 - x_2 - (\gamma y_1 - y_1 y_3 - y_2) - u_2 + F\omega\cos\omega t \\ \dot{e}_3 &= x_1 x_2 - \beta x_3 - (y_1 y_2 - \beta y_3) - u_3 + F\omega\cos\omega t \end{aligned} \tag{6.27}$$

Let initial states is (x_{10}, x_{20}, x_{30})= (0.5, 1, 5), (y_{10}, y_{20}, y_{30})= (0.6, 3, 10) and system parameters $\sigma=10$, $\gamma=28$, $\beta=8/3$, $F=10$ and $\omega=0.1$, we find the error dynamic always exists in first quadrant showed in Fig. 6.5. By partial region stability, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V=e_1+e_2+e_3$$

Its time derivative is

$$\begin{aligned}\dot{V}=&\left(\sigma(x_2-x_1)-\sigma(y_2-y_1)-u_1+\omega\cos\omega t\right)+\left(\gamma x_1-x_1x_3-x_2-(\gamma y_1-y_1y_3-y_2)\right.\\&\left.-u_2+\omega\cos\omega t\right)+\left(x_1x_2-\beta x_3-(y_1y_2-\beta y_3)-u_3+\omega\cos\omega t\right)\end{aligned} \tag{6.28}$$

Choose

$$\begin{aligned}u_1&=\sigma(x_2-x_1)-\sigma(y_2-y_1)+\omega\cos\omega t+e_1\\u_2&=\gamma x_1-x_1x_3-x_2-(\gamma y_1-y_1y_3-y_2)+\omega\cos\omega t+e_2\\u_3&=x_1x_2-\beta x_3-(y_1y_2-\beta y_3)+\omega\cos\omega t+e_3\end{aligned} \tag{6.29}$$

We obtain

$$\dot{V}=-e_1-e_2-e_3<0 \tag{6.30}$$

which is negative definite function. Three state errors versus time and time histories of x_i-y_i+100 are shown in Fig. 6.6 and Fig. 6.7.

CASE III. The generalized synchronization error function is $e_i=\frac{1}{2}x_i^2-y_i+100$, i=1, 2, 3.

Our goal is $\mathbf{y}=\frac{1}{2}\mathbf{x}^2+100$, i.e. $\lim_{t\to\infty}\mathbf{e}=\lim_{t\to\infty}(\frac{1}{2}\mathbf{x}^2-\mathbf{y}+100)=0$

The error dynamics become

$$\begin{aligned}\dot{e}_1&=x_1\dot{x}_1-\dot{y}_1=x_1\sigma(x_2-x_1)-\sigma(y_2-y_1)-u_1\\\dot{e}_2&=x_2\dot{x}_2-\dot{y}_2=x_2(\gamma x_1-x_1x_3-x_2)-(\gamma y_1-y_1y_3-y_2)-u_2\\\dot{e}_3&=x_3\dot{x}_3-\dot{y}_3=x_3(x_1x_2-\beta x_3)-(y_1y_2-\beta y_3)-u_3\end{aligned} \tag{6.31}$$

Let initial states is (x_{10},x_{20},x_{30})= (0.5, 1, 5), (y_{10},y_{20},y_{30})= (0.6, 3, 10) and system parameters $\sigma=10$, $\gamma=28$, $\beta=8/3$, we find the error dynamic always exist in first quadrant showed in Fig. 6.8. By partial region stability, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3$$

Its time derivative is

$$\dot{V} = \left(x_1\sigma(x_2 - x_1) - \sigma(y_2 - y_1) - u_1\right) + \left(x_2(\gamma x_1 - x_1x_3 - x_2) - (\gamma y_1 - y_1y_3 - y_2) - u_2\right) \\ + \left(x_3(x_1x_2 - \beta x_3) - (y_1y_2 - \beta y_3) - u_3\right) \tag{6.32}$$

Choose

$$\begin{aligned} u_1 &= x_1\sigma(x_2 - x_1) - \sigma(y_2 - y_1) + e_1 \\ u_2 &= x_2(\gamma x_1 - x_1x_3 - x_2) - (\gamma y_1 - y_1y_3 - y_2) + e_2 \\ u_3 &= x_3(x_1x_2 - \beta x_3) - (y_1y_2 - \beta y_3) + e_3 \end{aligned} \tag{6.33}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 < 0 \tag{6.34}$$

which is negative definite function. Three state errors versus time are shown in Fig. 6.9.

CASE IV. The generalized synchronization error function is $\mathbf{e} = \mathbf{x} - \mathbf{y} + \mathbf{z} + 100$, $\mathbf{z}$ is the state vector of Rössler system.

The goal system for synchronization is Rössler system and initial states is (20, 10, 25), system parameters $a = 0.2$, $b = 0.2$, $c = 5.7$.

$$\begin{aligned} \dot{z}_1 &= -z_2 - z_3 \\ \dot{z}_2 &= z_1 + az_2 \\ \dot{z}_3 &= b + z_3(z_1 - c) \end{aligned}$$

We have $\lim_{t\to\infty} \mathbf{e} = \lim_{t\to\infty}(\mathbf{x} - \mathbf{y} + \mathbf{z} + 100) = 0$

The error dynamics become

$$\begin{aligned} \dot{e}_1 &= \sigma(x_2 - x_1) - \sigma(y_2 - y_1) - u_1 - z_2 - z_3 \\ \dot{e}_2 &= \gamma x_1 - x_1x_3 - x_2 - (\gamma y_1 - y_1y_3 - y_2) - u_2 + z_1 + az_2 \\ \dot{e}_3 &= x_1x_2 - \beta x_3 - (y_1y_2 - \beta y_3) - u_3 + b + z_3(z_1 - c) \end{aligned} \tag{6.35}$$

Let initial states is (x_{10}, x_{20}, x_{30})= (0.5, 1, 5), (y_{10}, y_{20}, y_{30})= (0.6, 3, 10) and system parameters $\sigma = 10$, $\gamma = 28$, $\beta = 8/3$, we find the error dynamic always exist in first quadrant showed in Fig. 6.10. By partial region stability, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3$$

Its time derivative is

$$\begin{aligned} \dot{V} &= \left(\sigma(x_2 - x_1) - \sigma(y_2 - y_1) - u_1 - z_2 - z_3\right) \\ &\quad + \left(\gamma x_1 - x_1 x_3 - x_2 - (\gamma y_1 - y_1 y_3 - y_2) - u_2 + z_1 + a z_2\right) \\ &\quad + \left(x_1 x_2 - \beta x_3 - (y_1 y_2 - \beta y_3) - u_3 + b + z_3(z_1 - c)\right) \end{aligned} \tag{6.36}$$

Choose

$$\begin{aligned} u_1 &= \sigma(x_2 - x_1) - \sigma(y_2 - y_1) - z_2 - z_3 + e_1 \\ u_2 &= \gamma x_1 - x_1 x_3 - x_2 - (\gamma y_1 - y_1 y_3 - y_2) + z_1 + a z_2 + e_2 \\ u_3 &= x_1 x_2 - \beta x_3 - (y_1 y_2 - \beta y_3) + b + z_3(z_1 - c) + e_3 \end{aligned} \tag{6.37}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 < 0 \tag{6.38}$$

which is negative definite function. Three state errors versus time and time histories of $x_i - y_i + 100$ are shown in Fig. 6.11 and Fig. 6.12.

6.4 Summary

In this Chapter, a new strategy to achieve chaos generalized synchronization by partial region stability is proposed. By using the theory of stability on partial region the Lyapunov function is a simple linear homogeneous function of states and the controllers are simpler and have less simulation error because they are in lower order than that of traditional controllers. The Lorenz system and Rössler system are used as simulation examples which effectively confirm the scheme.

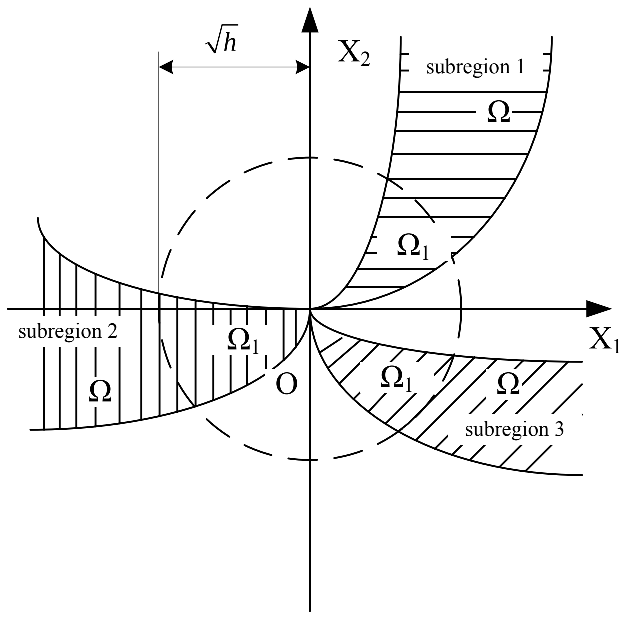

Fig. 6.1 Partial region Ω and Ω_1.

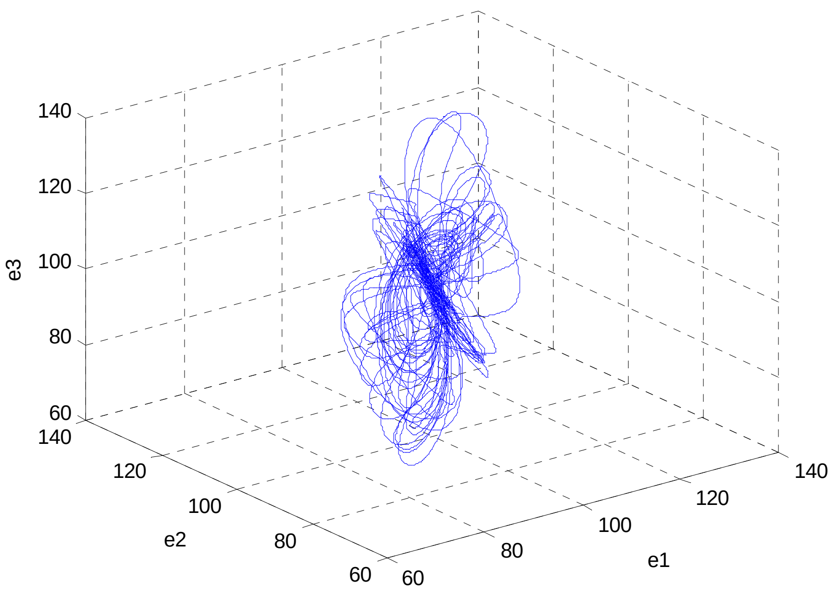

Fig. 6.2 Phase portrait of error dynamics for Case I.

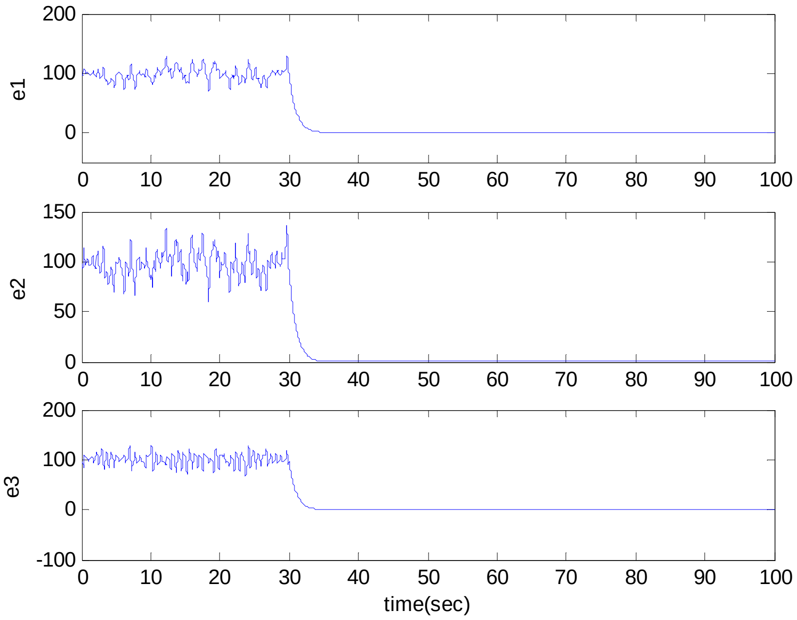

Fig. 6.3 Time histories of errors for Case I.

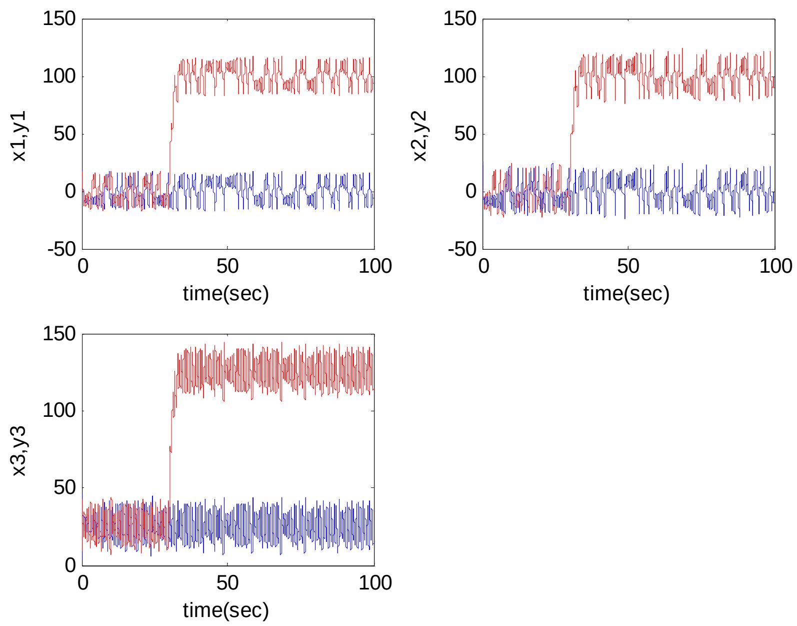

Fig. 6.4 Time histories of x_1, x_2, x_3, y_1, y_2, y_3 for Case I.

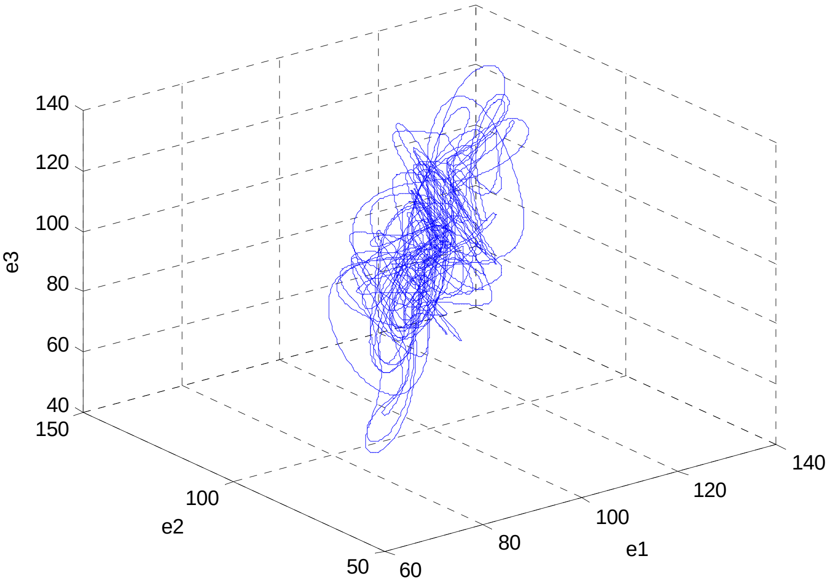

Fig. 6.5 Phase portrait of error dynamics for Case II.

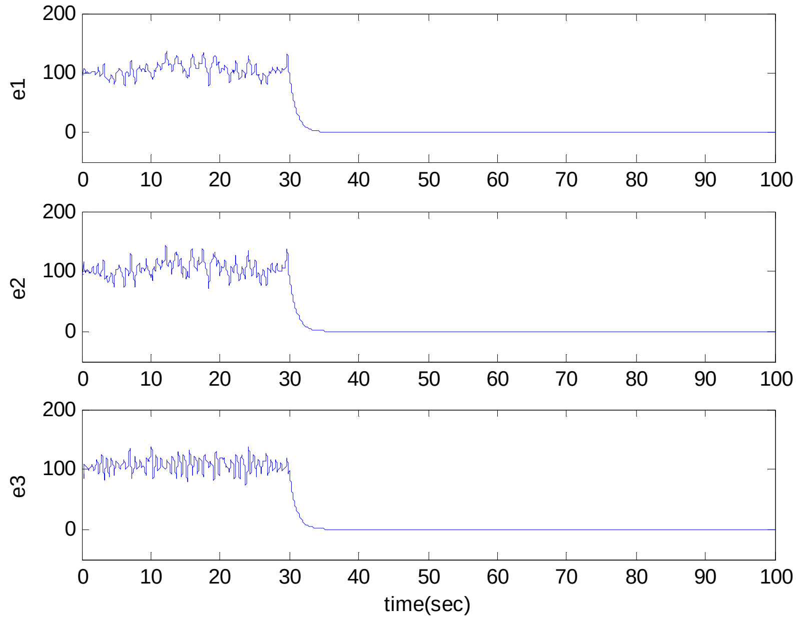

Fig. 6.6 Time histories of errors for Case II.

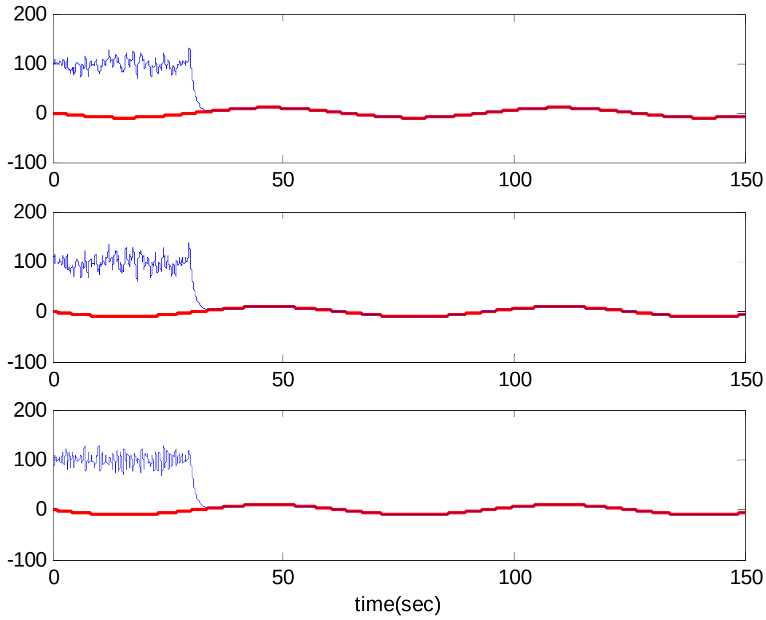

Fig. 6.7 Time histories of $x_i - y_i + 100$ and $-F \sin \omega t$ for Case II.

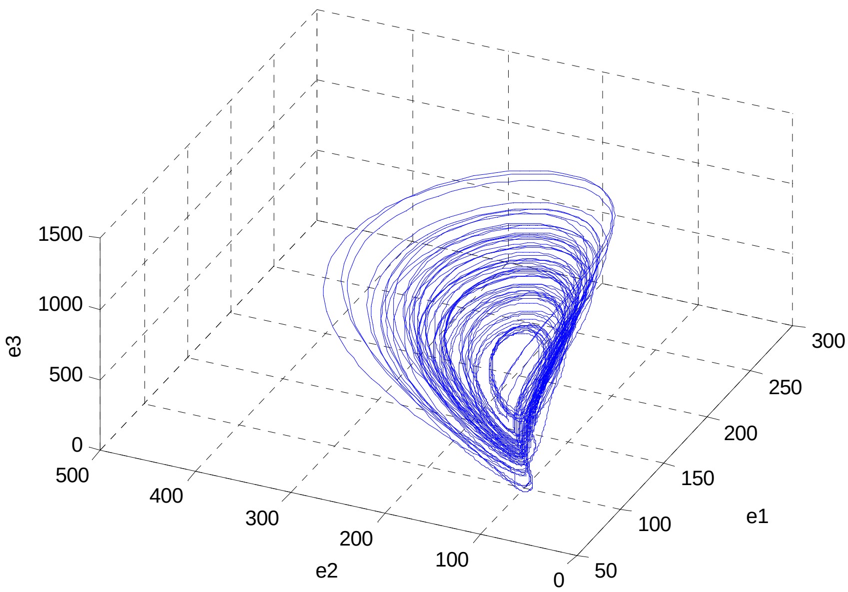

Fig. 6.8 Phase portrait of error dynamics for Case III.

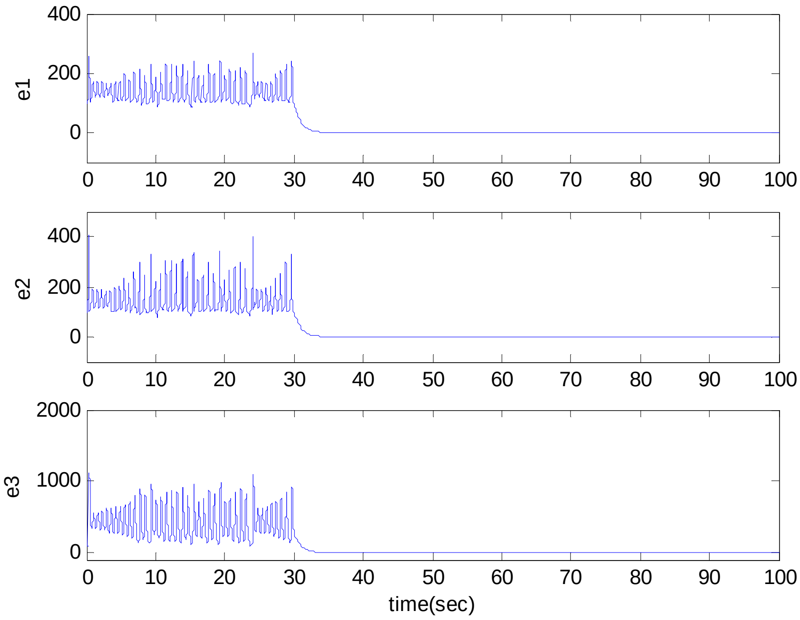

Fig. 6.9 Time histories of errors for Case III.

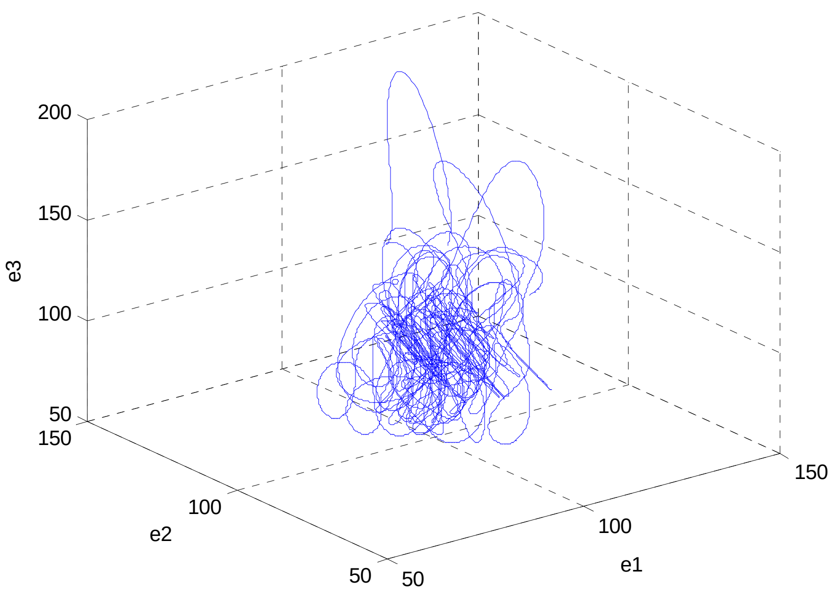

Fig. 6.10 Phase portrait of error dymanics for Case IV.

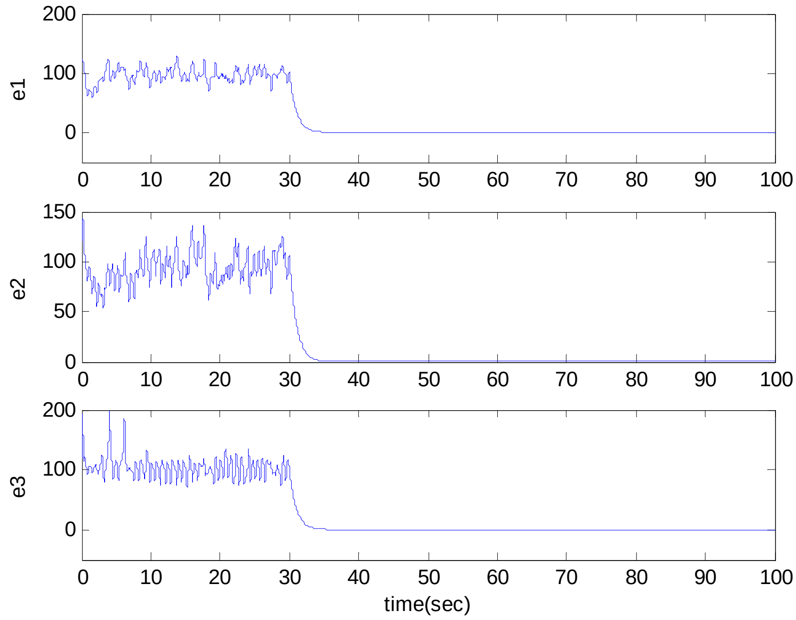

Fig. 6.11 Time histories of errors for Case IV.

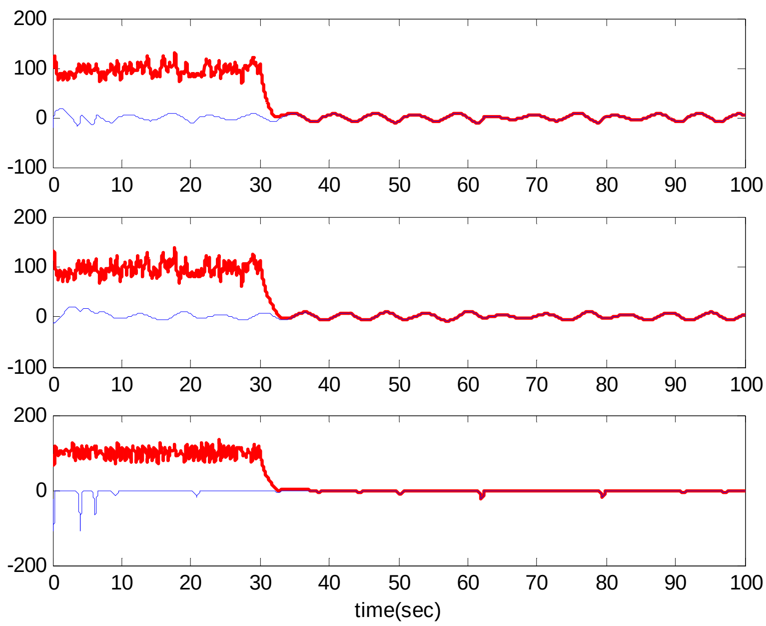

Fig. 6.12 Time histories of $\mathbf{x}-\mathbf{y}+100$ and $-\mathbf{z}$ for Case IV.

Chapter 7

Chaos Control by Partial Region Stability Theory

7.1 Preliminary

In this Chapter, a new scheme to achieve chaos control by partial region stability is proposed. By using the theory of stability on partial region the Lyapunov function is a simple linear homogeneous function of states and the controllers are simpler and have less simulation error because they are in lower order than that of traditional controllers. The Lorenz system is used as simulation examples.

7.2 Chaos Control Scheme

Consider the following chaotic systems

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{7.1}$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in R^n$ is a the state vector, $\mathbf{f} : R_+ \times R^n \to R^n$ is a vector function.

The goal system which can be either chaotic or regular is

$$\dot{\mathbf{y}} = \mathbf{g}(t, \mathbf{y}) \tag{7.2}$$

where $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T \in R^n$ is a state vector, $\mathbf{g} : R_+ \times R^n \to R^n$ is a vector function.

In order to make the chaos state $\mathbf{x}$ approaching the goal state $\mathbf{y}$, define $\mathbf{e} = \mathbf{x} - \mathbf{y}$ as the state error. The chaos control is accomplished in the sense that:

$$\lim_{t\to\infty} \mathbf{e} = \lim_{t\to\infty} (\mathbf{x} - \mathbf{y}) = 0 \tag{7.3}$$

In this Chapter, we will use examples in which the e state is placed in the first quadrant of coordinate system and use the theory of stability on partial region, the Lyapunov function is a simple linear homogeneous function of states and the controllers are simpler because they are in lower order than that of traditional controllers

7.3 Numerical Simulations

The following chaotic system is the Lorenz system of which the old origin is translated to $(x_1, x_2, x_3) = (100,100,100)$ and the chaotic motion always happens in the first quadrant of coordinate system (x_1, x_2, x_3). This special Lorenz system is presented as simulated examples which the initial conditions is x1(0) =80, x2(0) =100, x3(0) =90, $\sigma = 10$, $\gamma = 28$, $\beta = 8/3$. The chaotic motion showed in Fig. 7.1.

$$\begin{aligned}\dot{x}_1 &= \sigma((x_2 - 100) - (x_1 - 100))\\ \dot{x}_2 &= \gamma(x_1 - 100) - (x_1 - 100)(x_3 - 100) - (x_2 - 100)\\ \dot{x}_3 &= (x_1 - 100)(x_2 - 100) - \beta(x_3 - 100)\end{aligned} \tag{7.4}$$

In order to lead (x_1, x_2, x_3) to the goal, we add control terms u_1, u_2 and u_3 to each equation of Eq. (7.4), respectively.

$$\begin{aligned}\dot{x}_1 &= \sigma((x_2 - 100) - (x_1 - 100)) + u_1\\ \dot{x}_2 &= \gamma(x_1 - 100) - (x_1 - 100)(x_3 - 100) - (x_2 - 100) + u_2\\ \dot{x}_3 &= (x_1 - 100)(x_2 - 100) - \beta(x_3 - 100) + u_3\end{aligned} \tag{7.5}$$

CASE I. Control the chaotic motion to zero.

In this case we will control the chaotic motion of the Lorenz system (7.4) to zero. The goal is $\mathbf{y} = 0$. The state error is $\mathbf{e} = \mathbf{x} - \mathbf{y} = \mathbf{x}$ and error dynamics becomes

$$\begin{aligned}\dot{e}_1 &= \dot{x}_1 = \sigma((x_2 - 100) - (x_1 - 100)) + u_1\\ \dot{e}_2 &= \dot{x}_2 = \gamma(x_1 - 100) - (x_1 - 100)(x_3 - 100) - (x_2 - 100) + u_2\\ \dot{e}_3 &= \dot{x}_3 = (x_1 - 100)(x_2 - 100) - \beta(x_3 - 100) + u_3\end{aligned} \tag{7.6}$$

In Fig. 7.2, we see that the error dynamics always exists in first quadrant.

By partial region stability, one can easy choose a Lyapunov function in the form of a positive definite function in first quadrant as:

$$V = e_1 + e_2 + e_3 \tag{7.7}$$

Its time derivative through error dynamics (7.6) is

$$\begin{aligned}\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3\\ &= \sigma((e_2 - 100) - (e_1 - 100)) + u_1 + \gamma(e_1 - 100) - (e_1 - 100)(e_3 - 100)\\ &\quad - (e_2 - 100) + u_2 + (e_1 - 100)(e_2 - 100) - \beta(e_3 - 100) + u_3\end{aligned} \tag{7.8}$$

Choose

$$\begin{aligned} u_1 &= -\sigma((e_2-100)-(e_1-100))-e_1 \\ u_2 &= -(\gamma(e_1-100)-(e_1-100)(e_3-100)-(e_2-100))-e_2 \\ u_3 &= -((e_1-100)(e_2-100)-\beta(e_3-100))-e_3 \end{aligned} \tag{7.9}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 < 0$$

which is negative definite function. The numerical results are shown in Fig. 7.3. After 30 sec, the motion trajectories approach the origin.

CASE II. Control the chaotic motion to a sine function.

In this case we will control the chaotic motion of the Lorenz system (7.4) to sine function of time. The goal is $\mathbf{y} = \mathbf{F}\sin\omega t$. The error equation

$$\mathbf{e} = \mathbf{x} - \mathbf{y} = \mathbf{x} - \mathbf{F}\sin\boldsymbol{\omega}t \tag{7.10}$$

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - F_i\sin\omega_i t) = 0, \qquad i = 1,2,3$$

and $\dot{e}_i = \dot{x}_i - \omega_i F_i\cos\omega_i t, \qquad i = 1,2,3$ and $F_1 = F_2 = F_3$.

and the error dynamics is

$$\begin{aligned} \dot{e}_1 &= \dot{x}_1 - \omega_1 F\cos\omega_1 t = \sigma((x_2-100)-(x_1-100)) - \omega_1 F\cos\omega_1 t + u_1 \\ \dot{e}_2 &= \dot{x}_2 - \omega_2 F\cos\omega_2 t = \gamma(x_1-100)-(x_1-100)(x_3-100)-(x_2-100) - \omega_2 F\cos\omega_2 t + u_2 \\ \dot{e}_3 &= \dot{x}_3 - \omega_3 F\cos\omega_3 t = (x_1-100)(x_2-100)-\beta(x_3-100) - \omega_3 F\cos\omega_3 t + u_3 \end{aligned}$$

(7.11)

In Fig. 7.4, we see that the error dynamics always exists in first quadrant.

By partial region stability, one can easy choose a Lyapunov function in the form of a positive definite function in first quadrant as:

$$V = e_1 + e_2 + e_3$$

Its time derivative is

$$\begin{aligned} \dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 \\ &= \sigma((x_2-100)-(x_1-100)) - \omega_1 F\cos\omega_1 t + u_1 + \gamma(x_1-100) \\ &\quad -(x_1-100)(x_3-100)-(x_2-100) - \omega_2 F\cos\omega_2 t + u_2 \\ &\quad +(x_1-100)(x_2-100)-\beta(x_3-100) - \omega_3 F\cos\omega_3 t + u_3 \end{aligned} \tag{7.12}$$

Choose

$$\begin{aligned} u_1 &= -(\sigma((x_2-100)-(x_1-100)) - \omega_1 F\cos\omega_1 t) - e_1 \\ u_2 &= -(\gamma(x_1-100)-(x_1-100)(x_3-100)-(x_2-100)-\omega_2 F\cos\omega_2 t) - e_2 \\ u_3 &= -((x_1-100)(x_2-100)-\beta(x_3-100)-\omega_3 F\cos\omega_3 t) - e_3 \end{aligned} \tag{7.13}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 < 0$$

which is negative definite function. The numerical results are shown in Fig. 7.5 and Fig. 7.6, where $F = 10$, $\omega_1 = 0.7$, $\omega_2 = 0.5$ and $\omega_3 = 0.3$. After 30 sec., the errors approach zero and the motion trajectories approach to sine functions.

CASE III. Control the chaotic motion of Lorenz system to chaotic motion of a Rössler system.

In this case we will control chaotic motion of Lorenz system (7.4) to that of a Rössler system. The goal system is Rössler system:

$$\begin{aligned} \dot{z}_1 &= -z_2 - z_3 \\ \dot{z}_2 &= z_1 + a z_2 \\ \dot{z}_3 &= b + z_3(z_1 - c) \end{aligned} \tag{7.14}$$

The error equation is $\mathbf{e} = \mathbf{x} - \mathbf{z}$, $\lim_{t\to\infty} \mathbf{e} = 0$. The error dynamics become

$$\begin{aligned} \dot{e}_1 &= \dot{x}_1 - \dot{z}_1 = \sigma((x_2-100)-(x_1-100)) - (-z_2 - z_3) + u_1 \\ \dot{e}_2 &= \dot{x}_2 - \dot{z}_2 = \gamma(x_1-100)-(x_1-100)(x_3-100)-(x_2-100)-(z_1 + a z_2) + u_2 \\ \dot{e}_3 &= \dot{x}_3 - \dot{z}_3 = (x_1-100)(x_2-100)-\beta(x_3-100)-(b + z_3(z_1 - c)) + u_3 \end{aligned} \tag{7.15}$$

By Fig. 7.7, we know the error dynamic always exist in first quadrant.

By partial region stability, one can easy choose a Lyapunov function in the form of a positive definite function in first quadrant as:

$$V = e_1 + e_2 + e_3$$

Its time derivative is

$$\begin{aligned}\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 \\ &= \left(\sigma((x_2 - 100) - (x_1 - 100)) - (-z_2 - z_3) + u_1\right) \\ &\quad + \left(\gamma(x_1 - 100) - (x_1 - 100)(x_3 - 100) - (x_2 - 100) - (z_1 + az_2) + u_2\right) \\ &\quad + \left((x_1 - 100)(x_2 - 100) - \beta(x_3 - 100) - (b + z_3(z_1 - c)) + u_3\right)\end{aligned} \tag{7.16}$$

Choose

$$\begin{aligned}u_1 &= -\sigma((x_2 - 100) - (x_1 - 100)) - z_2 - z_3 - e_1 \\ u_2 &= -\left(\gamma(x_1 - 100) - (x_1 - 100)(x_3 - 100) - (x_2 - 100) - (z_1 + az_2)\right) - e_2 \\ u_3 &= -\left((x_1 - 100)(x_2 - 100) - \beta(x_3 - 100) - (b + z_3(z_1 - c))\right) - e_3\end{aligned} \tag{7.17}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 < 0$$

which is negative definite function. The numerical results are shown in Fig.7.8 and Fig. 7.9 where $a = 0.2$, $b = 0.2$ and $c = 5.7$. After 30 sec., the errors approach zero and the chaotic trajectories of Lorenz system approach to that of the Rössler system.

7.4 Summary

In this Chapter, a new strategy to achieve chaos control by partial region stability is proposed. By using theory of stability on partial region the Lyapunov function is a simple linear homogeneous function of error states and the controller is simpler because they are of lower order. The Lorenz system in the first quadrant is used as simulation examples which effectively confirm the scheme.

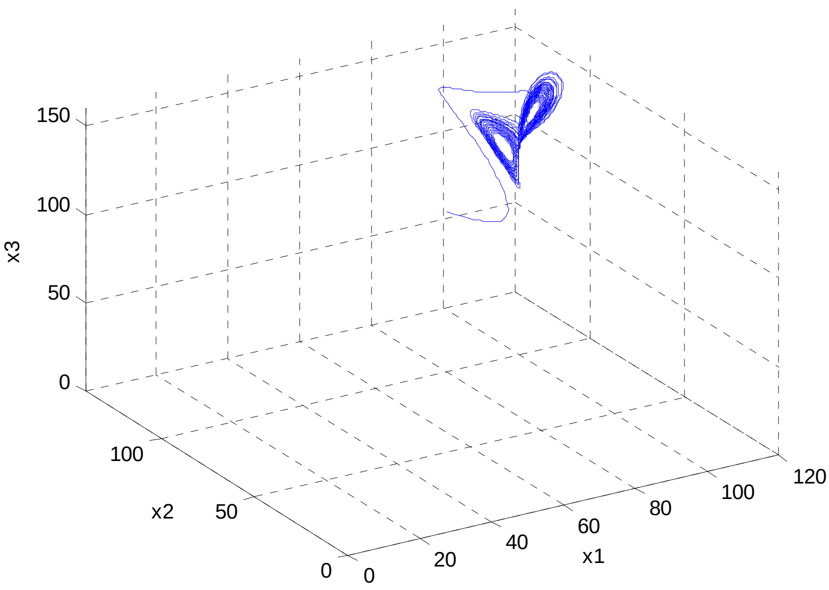

Fig. 7.1 Chaotic phase portrait for Lorenz system in the first quadrant.

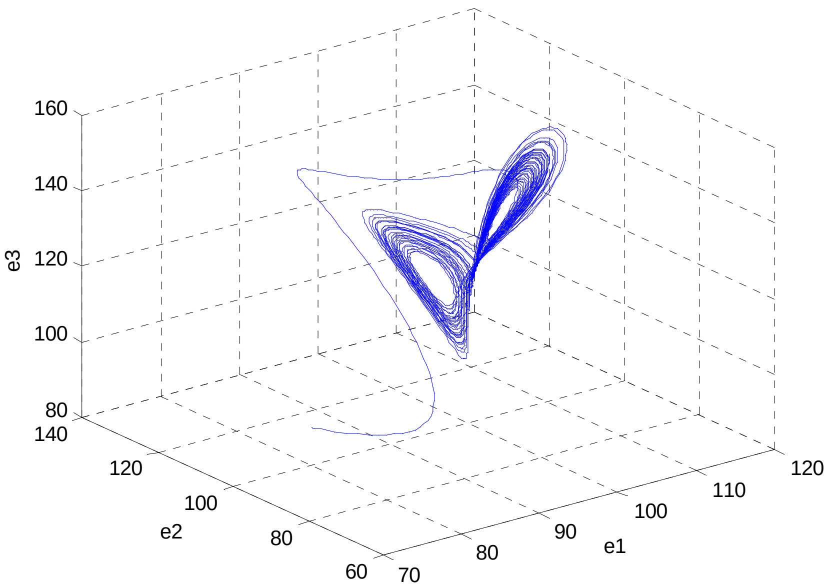

Fig. 7.2 Phase portrait of error dynamics for Case I.

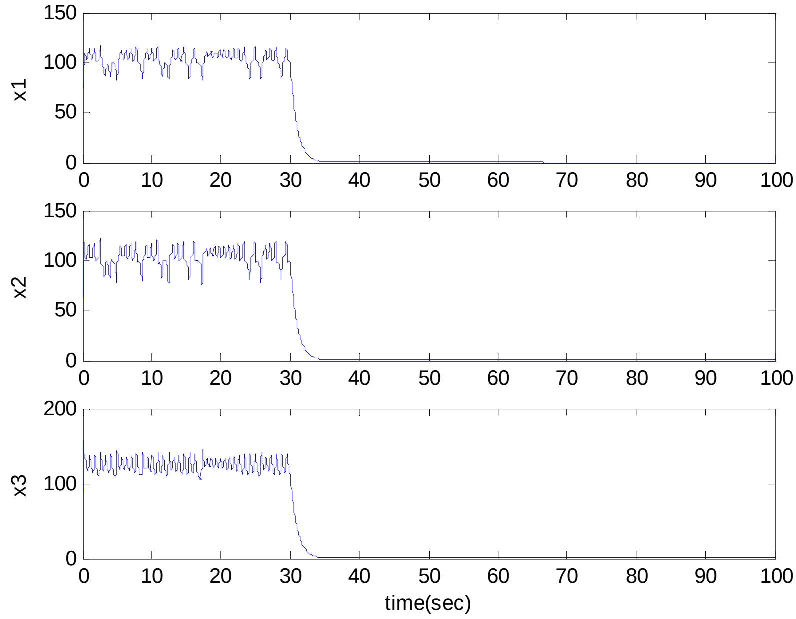

Fig. 7.3 Time histories of x_1, x_2, x_3 for Case I.

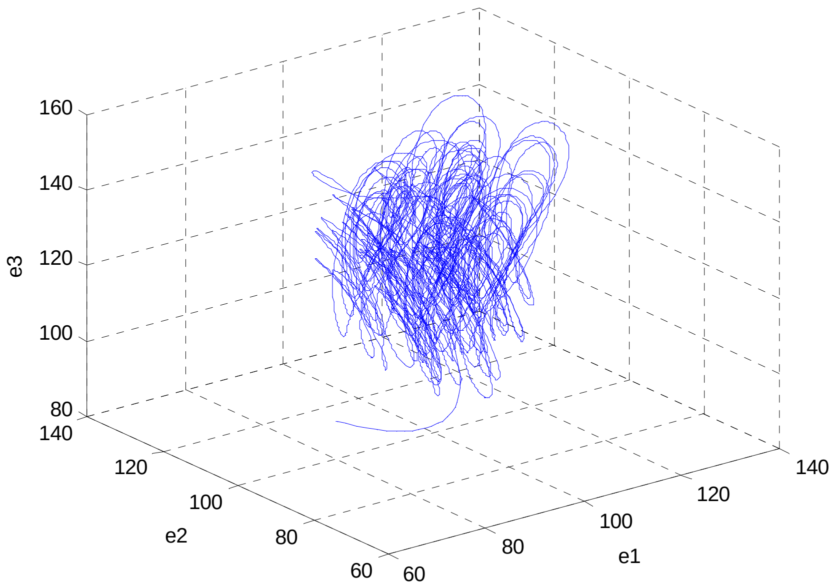

Fig. 7.4 Phase portrait of error dynamics for Case II.

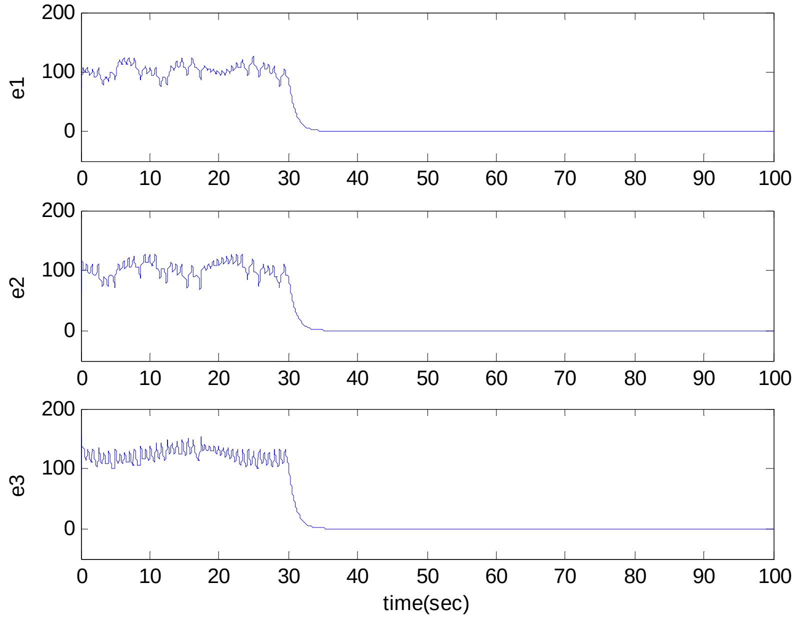

Fig. 7.5 Time histories of errors for Case II.

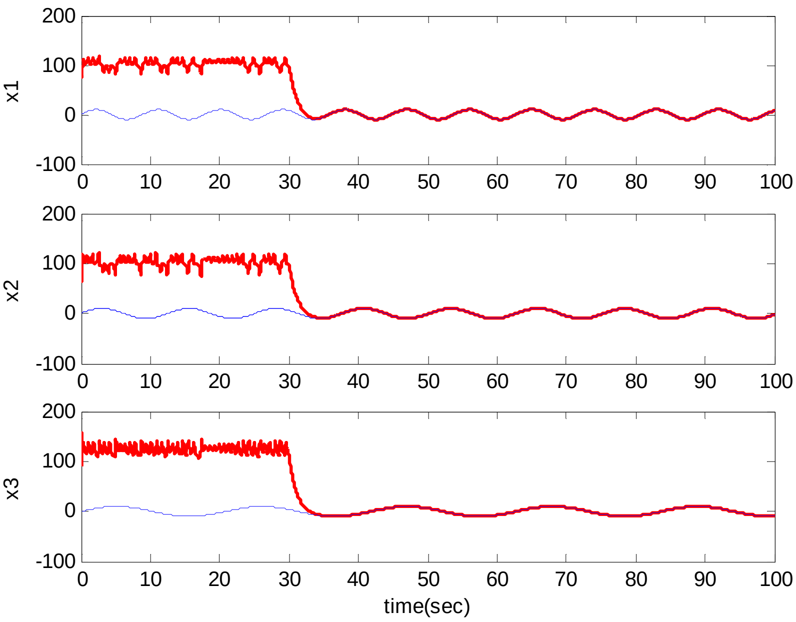

Fig. 7.6 Time histories of x_1, x_2, x_3 for Case II.

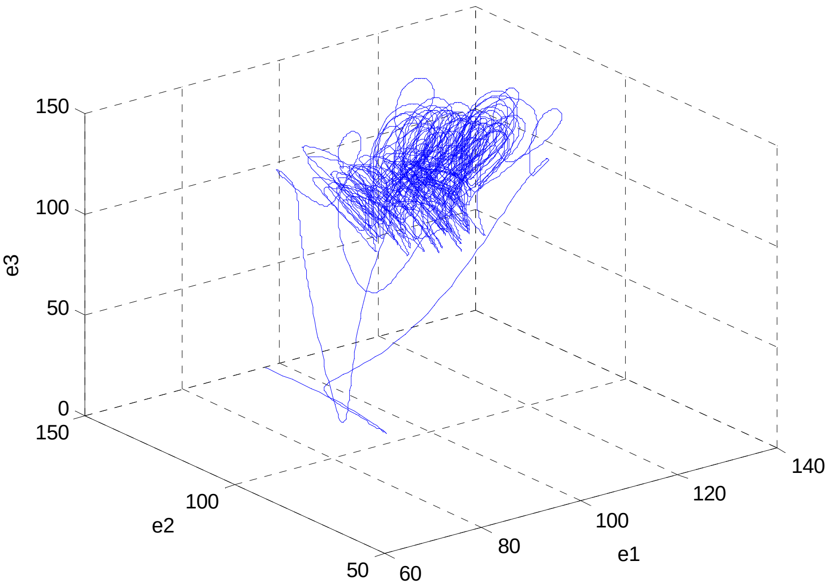

Fig. 7.7 Phase portrait of error dynamics for Case III.

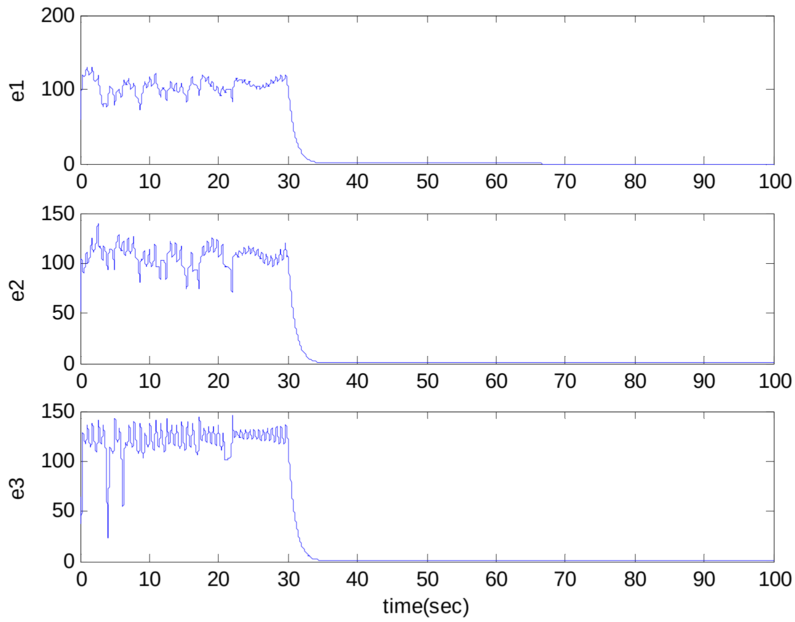

Fig. 7.8 Time histories of errors for Case III.

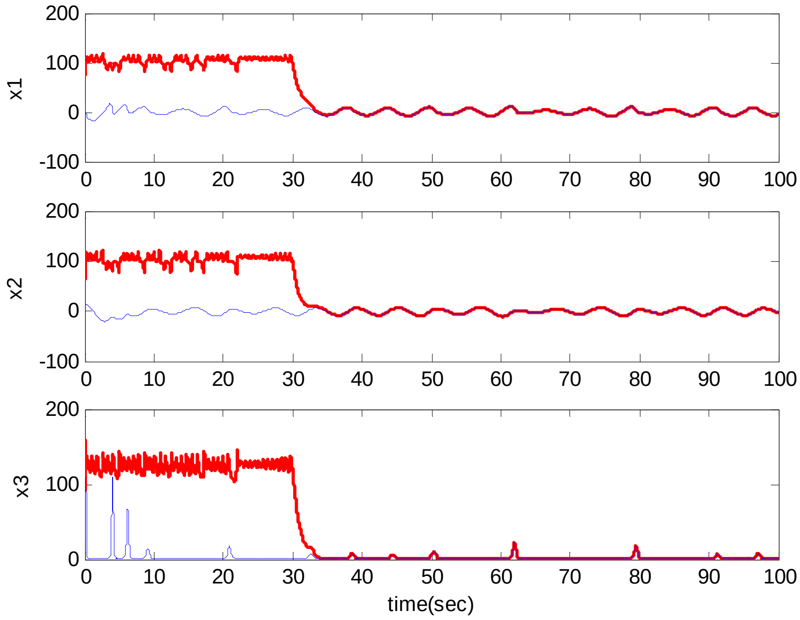

Fig. 7.9 Time histories of x_1, x_2, x_3 for Case III.

Chapter 8

Conclusions

Chaotic systems exhibit sensitive dependence on initial conditions. Slight errors of initial conditions will lead to completely different trajectories. Because of this property, chaotic systems are thought difficult to be synchronized or controlled. There are many control techniques which are presented to synchronize and control chaotic systems. In this thesis, the theorems of unsynchronizability, synchronization and generalized unsynchronization for coupled chaotic systems, chaos synchronization and adaptive chaos synchronization by variable strength linear coupling and chaos synchronization and control by partial region stability are presented.

In Chapter 2, two theorems which give the criteria of unsynchronizability for two different chaotic dynamic systems are presented. A sufficient criterion for synchronization is enhanced to necessary and sufficient one. Three simulated examples are given to illustrate the theory.

In Chapter 3, two theorems are proposed. They give the criteria of generalized unsynchronization for two different chaotic dynamic systems with whatever large strength of linear coupling. Chen system and Rössler system with two corresponding new chaotic systems proposed are used as simulation examples which effectively confirm the theorems.

In Chapter 4, two theorems for chaos synchronization are proposed by using variable strength linear coupling without another active control, while the time derivative of Lyapunov function in series form is firstly used, which makes the demand for Lyapunov function derivative as negative sum of the square of state variables, lower. They give the criteria of chaos synchronization for two identical chaotic systems and for two different chaotic dynamic systems. Either local synchronization which is mostly good enough or global synchronization which is mostly an unnecessary high demand can be obtained. Lorenz system, Duffing system, Rössler system and Hyper-Rössler system are used as simulation examples which effectively confirm the scheme.

In Chapter 5, adaptive chaos synchronization by variable strength linear coupling is studied. By using adaptive synchronization, we not only obtain the synchronization of chaotic states only by variable strength linear coupling without using another active control which is usually rather complex, but also obtain parameter pursuance. Furthermore, Lyapunov function derivative in series form is used in this paper. In most cases, local synchronization is good enough, while global synchronization is an unnecessary high demand. Lorenz system, Duffing system and Rössler system are used as simulation examples which effectively confirm the scheme and our opinion.

In Chapter 6, a new strategy to achieve chaos generalized synchronization by partial region stability is proposed. By using the theory of stability on partial region the Lyapunov function is a simple linear homogeneous function of states and the controllers are simpler and have less simulation error because they are in lower order than that of traditional controllers. The Lorenz system and Rössler system are used as simulation examples which effectively confirm the scheme.

In Chapter 7, a new strategy to achieve chaos control by partial region stability is proposed. By using theory of stability on partial region the Lyapunov function is a simple linear homogeneous function of error states and the controller is simpler because they are of lower order. The Lorenz system in the first quadrant is used as simulation examples which effectively confirm the scheme.

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