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Elliptic equations in highly heterogeneous porous media

Li-Ming Yeh[*†]

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Uniform estimate and convergence for highly heterogeneous elliptic equations are concerned. The domain considered consists of a connected fractured subregion (with high permeability) and a disconnected matrix block subregion (with low permeability). Let ε denote the size ratio of one matrix block to the whole domain and let the permeability ratio of the matrix block region to the fractured region be of the order ε^2. In the fractured region, uniform Hölder and uniform Lipschitz estimates in ε of the elliptic solutions are derived; the convergence of the solutions in L^∞ norm is obtained as well. Copyright © 2009 John Wiley & Sons, Ltd.

Keywords: highly heterogeneous elliptic equations; fractured region; permeability

1. Introduction

Uniform estimate and convergence for highly heterogeneous elliptic equations are concerned. The domain $\Omega \subset R^3$ contains two subregions: a connected fractured region (with high permeability) and a disconnected matrix block region (with low permeability). Let $Y \equiv [0,1]^3$ be a cell consisting of a sub-domain Y_m completely surrounded by another connected sub-domain Y_f ($\equiv Y \setminus Y_m$), $\mathscr{X}$ be the characteristic function of Y_m and be extended Y-periodically to R^3, ε (<1) be the size ratio of one matrix block to the whole medium, and $\Omega(2\varepsilon) \equiv \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 2\varepsilon\}$. The disconnected matrix block region is $\Omega_m^\varepsilon \equiv \{x : x \in \varepsilon(Y_m + j) \subset \Omega(2\varepsilon) \text{ for } j \in Z^3\}$, the connected fractured region is $\Omega_f^\varepsilon \equiv \Omega \setminus \Omega_m^\varepsilon$, and the boundary of Ω (resp. Ω_m^ε) is represented by $\partial\Omega$ (resp. $\partial\Omega_m^\varepsilon$). The elliptic equations in Ω are (see [1])

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\varepsilon(\nabla U_\varepsilon + Q_\varepsilon)) + \omega U_\varepsilon = F_\varepsilon & \text{in } \Omega_f^\varepsilon \\ -\varepsilon\nabla\cdot(\mathbf{k}_\varepsilon(\varepsilon\nabla u_\varepsilon + q_\varepsilon)) + \omega u_\varepsilon = \varepsilon f_\varepsilon & \text{in } \Omega_m^\varepsilon \\ \mathbf{K}_\varepsilon(\nabla U_\varepsilon + Q_\varepsilon)\cdot\vec{\mathbf{n}}^\varepsilon = \varepsilon\mathbf{k}_\varepsilon(\varepsilon\nabla u_\varepsilon + q_\varepsilon)\cdot\vec{\mathbf{n}}^\varepsilon & \text{on } \partial\Omega_m^\varepsilon \\ U_\varepsilon = u_\varepsilon & \text{on } \partial\Omega_m^\varepsilon \\ U_\varepsilon = U_{b_\varepsilon} & \text{on } \partial\Omega \end{cases} \tag{1}$$

where ω is a non-negative constant, $\mathbf{K}_\varepsilon(x) = \mathbf{K}(x/\varepsilon)$, $\mathbf{k}_\varepsilon(x) = \mathbf{k}(x/\varepsilon)$, $\mathbf{K}(1-\mathscr{X}) + \mathbf{k}\mathscr{X}$ is a periodic positive function in R^3 with period Y, and $\vec{\mathbf{n}}^\varepsilon$ is the unit outward normal vector on $\partial\Omega_m^\varepsilon$. It is known that if $\mathbf{K}, \mathbf{k}, Q_\varepsilon, q_\varepsilon, F_\varepsilon, f_\varepsilon, U_{b_\varepsilon}$ are smooth, a piecewise regular solution of (1) exists uniquely and the H^1 estimate of the solution is bounded uniformly in ε [2]. Convergence problem for (1) was considered in L^2 space in [1]. Uniform estimate in ε in Hilbert space for elliptic diffraction equation was considered in [3]. Convergence problem for uniform elliptic equations in perforated domains was considered in L^2 space in [4]. Uniform Lipschitz estimate in ε for Laplace equation in perforated domains was given in [5], and the uniform L^p estimate of the same problem was claimed in [6]. Lipschitz estimate for uniform elliptic equations was studied in [7]. Hölder and Lipschitz estimates uniform in ε for uniform elliptic equations in periodic domains were obtained in [8]. This work presents uniform estimate and convergence for the non-uniform elliptic equations (1). In the fractured region, uniform Hölder and Lipschitz estimates in ε of the solutions are derived; the convergence of the solutions in L^∞ norm is also obtained.

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30050, Taiwan, Republic of China
Correspondence to: Li-Ming Yeh, Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30050, Taiwan, Republic of China.
†*E-mail: liming@math.nctu.edu.tw*

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The rest of the work is organized as follows: Notation and main results are stated in Section 2. Some auxiliary lemmas are given in Section 3. Uniform Hölder estimate of the solutions with Lipschitz boundary condition in the fractured region is derived in Section 4. Lipschitz estimate of the solutions in fractured region is obtained in Section 5. By the results in Sections 4 and 5, uniform Hölder estimate in fractured region for the solutions with Hölder boundary condition is given in Section 6. L^∞ convergence for solutions in fractured region is presented in Section 7.

2. Notation and main result

Let $C^{k,\alpha}$ denote the Hölder space and $L^s, H^i, W^{i,s}$ denote the Sobolev spaces for $k\geqslant 0, \alpha\in(0,1), i\geqslant 1$, and $s>0$ [9]. Define $\|f_1,f_2,\ldots,f_m\|\equiv\|f_1\|+\|f_2\|+\cdots+\|f_m\|$ and $B(x,r)$ represents a ball centered at x with radius r. For any domain D, $\overline{D}$ is the closure of D, $D/r\equiv\{x: rx\in D\}$, $|D|$ is the volume of D, $\mathscr{X}_D$ is the characteristic function on D, and $D(x,r)\equiv D\cap B(x,r)$. For any $g\in L^1(D)$, we define

$$(g)_{x,r}\equiv \fint_{D(x,r)} g(y)\mathrm{d}y\equiv\frac{1}{|D(x,r)|}\int_{D(x,r)} g(y)\mathrm{d}y$$

Next we recall an extension result.

Lemma 2.1 (*Acerbi et al.* [*10*])

(1) For $1\leqslant s<\infty$, there is a constant $\gamma_1(Y_f,s)$ and a linear continuous extension operator $\Pi_\varepsilon: W^{1,s}(\Omega_f^\varepsilon)\cap L^\infty(\Omega_f^\varepsilon)\to W^{1,s}(\Omega)\cap L^\infty(\Omega)$ such that (1) If $g\in W^{1,s}(\Omega_f^\varepsilon)\cap L^\infty(\Omega_f^\varepsilon)$ and $\gamma_2\leqslant g\leqslant\gamma_3$, then

$$\begin{cases}\Pi_\varepsilon g=g & \text{in } \Omega_f^\varepsilon \text{ almost everywhere}\\ \|\Pi_\varepsilon g\|_{L^s(\Omega)}\leqslant\gamma_1\|g\|_{L^s(\Omega_f^\varepsilon)} & \\ \|\nabla\Pi_\varepsilon g\|_{L^s(\Omega)}\leqslant\gamma_1\|\nabla g\|_{L^s(\Omega_f^\varepsilon)} & \\ \gamma_2\leqslant\Pi_\varepsilon g\leqslant\gamma_3 & \\ \Pi_\varepsilon g=\zeta & \text{in } \Omega \text{ if } g=\zeta|_{\Omega_f^\varepsilon} \text{ for some linear function } \zeta \text{ in } \Omega\end{cases}$$

(2) For any constant $r>0$, $\Pi_{\varepsilon/r}\zeta(x)=(\Pi_\varepsilon g)(rx)$, where $\zeta(x)\equiv g(rx)$.

It is known that if $0<\mathbf{K},\mathbf{k}\in C^{0,1}(R^3)$, $Q_\varepsilon\mathscr{X}_{\Omega_f^\varepsilon}$ converges to Q in $L^2(\Omega)$ strongly, and $\|U_{b_\varepsilon}\|_{H^1(\Omega)}+\|Q_\varepsilon,F_\varepsilon\|_{L^2(\Omega_f^\varepsilon)}+\|q_\varepsilon,\varepsilon f_\varepsilon\|_{L^2(\Omega_m^\varepsilon)}\leqslant c_1$ (independent of ε), then the solution of (1) exists uniquely and $\|U_\varepsilon\|_{H^1(\Omega_f^\varepsilon)}+\|\varepsilon\nabla u_\varepsilon\|_{L^2(\Omega_m^\varepsilon)}+\|u_\varepsilon\|_{L^2(\Omega_m^\varepsilon)}\leqslant c_2$ (independent of ε). By compactness principle [1, 4, 9]

$$\begin{cases}\Pi_\varepsilon U_\varepsilon\to U & \text{in } L^2(\Omega) \text{ strongly}\\ \mathbf{K}_\varepsilon(\nabla U_\varepsilon+Q_\varepsilon)\mathscr{X}_{\Omega_f^\varepsilon}\to\check{\mathbf{K}}(\nabla U+Q) & \text{in } L^2(\Omega) \text{ weakly}\\ F_\varepsilon\mathscr{X}_{\Omega_f^\varepsilon}+\varepsilon f_\varepsilon\mathscr{X}_{\Omega_m^\varepsilon}\to F & \text{in } L^2(\Omega) \text{ weakly} \qquad \text{as } \varepsilon\to 0\\ u_\varepsilon\mathscr{X}_{\Omega_m^\varepsilon},\varepsilon f_\varepsilon\mathscr{X}_{\Omega_m^\varepsilon},q_\varepsilon\mathscr{X}_{\Omega_m^\varepsilon}\to u,f,q & \text{in } L^2(\Omega\times Y_m) \text{ in two scale}\\ U_{b_\varepsilon}\to U_b & \text{in } H^1(\Omega) \text{ weakly}\end{cases}$$

where $\check{\mathbf{K}}$ is a constant positive-definite matrix. Moreover, the functions U,u satisfy

$$\begin{cases}-\nabla\cdot(\check{\mathbf{K}}(\nabla U+Q))+\omega|Y_f|U+\omega\bar{u}=F & \text{in } \Omega\\ -\nabla_y\cdot(\mathbf{k}(\nabla_y u+q))+\omega u=f & \text{in } \Omega\times Y_m\\ u=U & \text{on } \Omega\times\partial Y_m\\ U=U_b & \text{on } \partial\Omega\end{cases}\tag{2}$$

where $\bar{u}\equiv\int_{Y_m}u(x,y)\mathrm{d}y$ and $|Y_f|$ is the volume of Y_f.

Lemma 2.2
The solution of (2) satisfies

$$\|U\|_{W^{1,r}(\Omega)}\leqslant c(\|Q,F\|_{L^r(\Omega)}+\|q,f\|_{L^r(\Omega\times Y_m)}+\|U_b\|_{W^{1,r}(\Omega)})\tag{3}$$

where $r\geqslant 2$ and c is a constant depending on $\mathbf{K},\mathbf{k},\omega$.

Proof

By energy method, we see

$$\|U\|_{H^1(\Omega)} \leqslant c(\|Q,F\|_{L^2(\Omega)} + \|q,f\|_{L^2(\Omega\times Y_m)} + \|U_b\|_{H^1(\Omega)}) \tag{4}$$

For any fixed point $x \in \Omega$, we consider the following problem:

$$\begin{cases} -\nabla_y \cdot (\mathbf{k}(\nabla_y u + q)) + \omega u = f & \text{in } Y_m \\ u = U(x) & \text{on } \partial Y_m \end{cases}$$

By [9]

$$\|u(x,\cdot)\|_{W^{1,r}(Y_m)} \leqslant c(\|q(x,\cdot), f(x,\cdot)\|_{L^r(Y_m)} + U(x)) \tag{5}$$

where c is a constant depending on $\mathbf{k}, \omega$. If $2 \leqslant r < 6$, then, by (4) and (5)

$$\|\bar{u}\|_{L^r(\Omega)} \leqslant c\|u\|_{L^r(\Omega\times Y_m)} \leqslant c(\|Q,F\|_{L^2(\Omega)} + \|q,f\|_{L^r(\Omega\times Y_m)} + \|U_b\|_{H^1(\Omega)}) \tag{6}$$

Employing (4) and (6) to the following problem:

$$\begin{cases} -\nabla \cdot (\check{\mathbf{K}}(\nabla U + Q)) = -\omega|Y_f|U - \omega\bar{u} + F & \text{in } \Omega \\ U = U_b & \text{on } \partial\Omega \end{cases}$$

we obtain (3) for $2 \leqslant r < 6$. For $r \geqslant 6$, Equation (3) can obtained by repeating the above process. □

For any $\delta \in (0,3)$ and $\alpha \in (\mu, 1)$, where $\mu \equiv 1 - 3/(3+\delta)$, we assume that

A1. Domain Ω is $C^{1,\alpha}$ and Y_m is a smooth, simply connected domain.
A2. $\mathbf{K}, \mathbf{k} \in C^{0,1}(R^3)$ are positive periodic functions with period Y and $\|\nabla \mathbf{K}\|_{L^\infty(Y_f)} + \|\nabla \mathbf{k}\|_{L^\infty(Y_m)}$ is small compared with $\min_{x\in R^3}\{\mathbf{K}, \mathbf{k}\}$.
A3. $Q_\varepsilon, \nabla\cdot Q_\varepsilon, F_\varepsilon \in L^{3+\delta}(\Omega_f^\varepsilon), q_\varepsilon, \nabla\cdot q_\varepsilon, f_\varepsilon \in L^{3+\delta}(\Omega_m^\varepsilon),\ U_{b_\varepsilon} \in C^{1,\alpha}(\bar{\Omega})$.

The main results are:

Theorem 2.1

Under A1–A3, the solution of (1) satisfies

$$[U_\varepsilon]_{C^{0,\mu}(\Omega_f^\varepsilon)} \leqslant c(\|Q_\varepsilon, \varepsilon\nabla\cdot Q_\varepsilon, F_\varepsilon\|_{L^{3+\delta}(\Omega_f^\varepsilon)} + \|q_\varepsilon, \varepsilon^2\nabla\cdot q_\varepsilon, \varepsilon f_\varepsilon\|_{L^{3+\delta}(\Omega_m^\varepsilon)} + [U_{b_\varepsilon}]_{C^{0,1}(\Omega)})$$

where $\mu \equiv 1 - 3/(3+\delta)$ and constant c is independent of ε.

Theorem 2.2

Under A1–A3, the solution of (1) with $\omega = Q_\varepsilon = q_\varepsilon = 0$ satisfies

$$\|\nabla U_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)} \leqslant c(\|F_\varepsilon \mathscr{X}_{\Omega_f^\varepsilon} + f_\varepsilon \mathscr{X}_{\Omega_m^\varepsilon}\|_{L^{3+\delta}(\Omega)} + [U_{b_\varepsilon}]_{C^{1,\alpha}(\Omega)})$$

where constant c is independent of ε.

Indeed, the Dirichlet boundary of (1) can be Hölder continuous, that is,

Theorem 2.3

Under A1–A3, the solution of (1) with $\omega = 0$ satisfies

$$[U_\varepsilon]_{C^{0,\mu}(\Omega_f^\varepsilon)} \leqslant c(\|Q_\varepsilon, \varepsilon\nabla\cdot Q_\varepsilon, F_\varepsilon\|_{L^{3+\delta}(\Omega_f^\varepsilon)} + \|q_\varepsilon, \varepsilon^2\nabla\cdot q_\varepsilon, \varepsilon f_\varepsilon\|_{L^{3+\delta}(\Omega_m^\varepsilon)} + [U_{b_\varepsilon}]_{C^{0,\mu}(\Omega)})$$

where $\mu \equiv 1 - 3/(3+\delta)$ and constant c is independent of ε.

For given $F \in L^2(\Omega)$, we consider the following problem:

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\varepsilon \nabla U_\varepsilon) = F & \text{in } \Omega_f^\varepsilon \\ -\varepsilon^2 \nabla\cdot(\mathbf{k}_\varepsilon \nabla u_\varepsilon) = 0 & \text{in } \Omega_m^\varepsilon \\ \mathbf{K}_\varepsilon \nabla U_\varepsilon \cdot \vec{\mathbf{n}}^\varepsilon = \varepsilon^2 \mathbf{k}_\varepsilon \nabla u_\varepsilon \cdot \vec{\mathbf{n}}^\varepsilon & \text{on } \partial\Omega_m^\varepsilon \\ U_\varepsilon = u_\varepsilon & \text{on } \partial\Omega_m^\varepsilon \\ U_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

It is known that (7) is solvable, $\Pi_\varepsilon U_\varepsilon$ converges to U_0 in $L^2(\Omega)$ strongly, and

$$\begin{cases} -\nabla\cdot(\check{\mathbf{K}}\nabla U_0)=|Y_f|F & \text{in } \Omega \\ U_0=0 & \text{on } \partial\Omega \end{cases} \tag{8}$$

We have the following convergence result.

Theorem 2.4
Under A1–A2, domain Ω is in $C^{3,\alpha}$, and $F\in W^{2,3+\delta}(\Omega)$, the solutions of (7) and (8) satisfy

$$\|U_\varepsilon-U_0\|_{L^\infty(\Omega_f^\varepsilon)}\leqslant c\varepsilon^{1-\mu}\|F\|_{W^{2,3+\delta}(\Omega)}$$

where $\mu\equiv 1-3/(3+\delta)$ and constant c is independent of ε.

3. Some auxiliary result

First we recall a result.

Lemma 3.1 (*Giaquinta* [*11, p. 68*])
For any $\beta\in(0,1)$, there are γ_4,γ_5 depending on β and the geometry of domain D so that for any Hölder continuous function g in D,

$$\gamma_4[g]_{C^{0,\beta}(\overline{D})}\leqslant \sup_{x\in D}\sup_{\ell>0}\left(\frac{1}{\ell^{2\beta}}\fint_{D(x,\ell)}|g-(g)_{x,\ell}|^2\mathrm{d}y\right)^{1/2}\leqslant\gamma_5[g]_{C^{0,\beta}(\overline{D})}$$

Let $G(x-y)$ denote the fundamental solution of Laplace equation, see Section 6.2 [12]. Define single-layer and double-layer potentials as for any smooth function g on the boundary ∂D of a smooth bounded domain D

$$\begin{cases} \mathscr{E}_{\partial D}(g)(x)\equiv\int_{\partial D}G(x-y)g(y)\mathrm{d}\sigma_y \\ \mathscr{T}_{\partial D}(g)(x)\equiv\int_{\partial D}\partial_y G(x-y)\vec{\mathbf{n}}_y\, g(y)\mathrm{d}\sigma_y \end{cases} \quad \text{for } x\in\partial D$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂D.

Lemma 3.2
For any bounded smooth domain D,

1. $\mathscr{E}_{\partial D},\mathscr{T}_{\partial D}$ are pseudo-differential operators of order -1 on ∂D.
2. For any $|\beta|>\frac{1}{2}$ and $s\in(2,\infty)$, the linear operators

$$\begin{cases} \mathscr{E}_{\partial D}:W^{-1/s,s}(\partial D)\to W^{1-1/s,s}(\partial D) \\ \beta I-\mathscr{T}_{\partial D}:W^{1-1/s,s}(\partial D)\to W^{1-1/s,s}(\partial D) \end{cases} \tag{9}$$

are bounded and $\beta I-\mathscr{T}_{\partial D}$ is invertible in $W^{1-1/s,s}(\partial D)$.
3. For any $|\beta|>\frac{1}{2}$ and $\gamma\in(0,1)$, the linear operators

$$\begin{cases} \mathscr{E}_{\partial D}:C^{0,\gamma}(\partial D)\to C^{1,\gamma}(\partial D) \\ \beta I-\mathscr{T}_{\partial D}:C^{1,\gamma}(\partial D)\to C^{1,\gamma}(\partial D) \end{cases} \tag{10}$$

are bounded and $\beta I-\mathscr{T}_{\partial D}$ is invertible in $C^{1,\gamma}(\partial D)$.

Proof
By [3, 12], both $\mathscr{E}_{\partial D},\mathscr{T}_{\partial D}$ are pseudo-differential operators of order -1. By theorem in Section 2.3.4 [13] and following the proof of Theorem 2.5, Chapter XI [14], $\mathscr{E}_{\partial D}$ of (9) is a bounded linear operator. From Section 1.6 and Proposition 5.2, Chapter XI [14], $\beta I-\mathscr{T}_{\partial D}$ of (9) is a bounded linear operator. Tracing the proof of Theorem 4.6.5 [12], we see that $\beta I-\mathscr{T}_{\partial D}$ is a Fredholm operator. Since $\beta I-\mathscr{T}_{\partial D}$ is invertible in L^2 [15], it is one-to-one and bounded, and has closed range. By inverse mapping theorem [16], $\beta I-\mathscr{T}_{\partial D}$ is invertible in $W^{1-1/s,s}(\partial D)$. That $\mathscr{E}_{\partial D}$ and $\beta I-\mathscr{T}_{\partial D}$ of (10) are bounded linear operators is due to Theorem 2.5, Chapter XI of [14]. By an analogous argument as that for (9), we see that $\beta I-\mathscr{T}_{\partial D}$ is invertible in $C^{1,\gamma}(\partial D)$. □

Now we consider the following problem:

$$\begin{cases} -\nabla\cdot(\mathbf{K}(\nabla P_\varepsilon+\hat{Q}_\varepsilon))+\varepsilon^2\omega P_\varepsilon=\hat{F}_\varepsilon & \text{in } Y_f\\ -\varepsilon\nabla\cdot(\mathbf{k}(\varepsilon\nabla p_\varepsilon+\hat{q}_\varepsilon))+\varepsilon^2\omega p_\varepsilon=\hat{f}_\varepsilon & \text{in } Y_m\\ \mathbf{K}(\nabla P_\varepsilon+\hat{Q}_\varepsilon)\cdot\vec{\mathbf{n}}_y=\varepsilon\mathbf{k}(\varepsilon\nabla p_\varepsilon+\hat{q}_\varepsilon)\cdot\vec{\mathbf{n}}_y & \text{on } \partial Y_m\\ P_\varepsilon=p_\varepsilon & \text{on } \partial Y_m \end{cases} \tag{11}$$

where $\vec{\mathbf{n}}_y$ is the unit vector normal to ∂Y_m. Let $Y_m\subset\mathbf{D}\subset Y=Y_f\cup Y_m$ satisfy

$$\ell\equiv\min\{\text{dist}(Y_m,\partial\mathbf{D}),\text{dist}(\mathbf{D},\partial Y)\}>0$$

Define $\mathbf{D}_1\equiv\{x\in Y_f|\text{dist}(x,Y_m)>\ell/4,\text{dist}(x,\partial Y)>\ell/4\}$.

Lemma 3.3
If $\omega<\omega_0$ and $\|P_\varepsilon\|_{L^2(Y_f)}+\|\varepsilon^2 p_\varepsilon\|_{L^2(Y_m)}+\|\hat{Q}_\varepsilon,\nabla\cdot\hat{Q}_\varepsilon,\hat{F}_\varepsilon\|_{L^r(Y_f)}+\|\varepsilon\hat{q}_\varepsilon,\varepsilon\nabla\cdot\hat{q}_\varepsilon,\hat{f}_\varepsilon\|_{L^r(Y_m)}$ is bounded independently of ε for $r\in(3,\infty)$, then the solutions of (11) satisfy

$$\|P_\varepsilon\|_{W^{1,r}(\mathbf{D}\setminus Y_m)}\leqslant c_1 \tag{12}$$

where c_1 is a constant independent of ε,ω. In addition to $\omega=\hat{Q}_\varepsilon=\hat{q}_\varepsilon=0$

$$\|P_\varepsilon\|_{C^{1,1-3/r}(\mathbf{D}\setminus Y_m)}\leqslant c_2 \tag{13}$$

where c_2 is a constant independent of ε.

Proof
Theorem 8.17 of [9] implies that for $r\in(3,\infty)$,

$$\|P_\varepsilon\|_{W^{1,r}(\mathbf{D}_1)}\leqslant c \tag{14}$$

where the constant c is independent of ε,ω. We assume that the coefficients and solution of (11) are smooth in Y_f and Y_m. Let $\widehat{p}$ be a solution of

$$\begin{cases} -\nabla\cdot(\varepsilon^2\widehat{\mathbf{k}}\nabla\widehat{p}+\varepsilon^2(\mathbf{k}-\widehat{\mathbf{k}})\nabla p_\varepsilon+\varepsilon\mathbf{k}\hat{q}_\varepsilon)+\varepsilon^2\omega p_\varepsilon=\hat{f}_\varepsilon & \text{in } Y_m\\ \widehat{p}|_{\partial Y_m}=0 \end{cases} \tag{15}$$

and $\widehat{P}$ a solution of

$$\begin{cases} -\nabla\cdot(\widehat{\mathbf{K}}\nabla\widehat{P}+(\mathbf{K}-\widehat{\mathbf{K}})\nabla P_\varepsilon+\mathbf{K}\hat{Q}_\varepsilon)+\varepsilon^2\omega P_\varepsilon=\hat{F}_\varepsilon & \text{in } \mathbf{D}\setminus Y_m\\ \widehat{P}|_{\partial Y_m}=0\\ \widehat{P}-P_\varepsilon|_{\partial\mathbf{D}}=0 \end{cases} \tag{16}$$

where $\widehat{\mathbf{K}},\widehat{\mathbf{k}}$ are two constants in $(\min_{R^3}\{\mathbf{K},\mathbf{k}\},\max_{R^3}\{\mathbf{K},\mathbf{k}\})$. Then, by (14)

$$\begin{cases} \|\widehat{p}\|_{W^{1,r}(Y_m)}\leqslant c(1/\varepsilon^2+\|\omega p_\varepsilon\|_{L^r(Y_m)}+\|(\mathbf{k}-\widehat{\mathbf{k}})\nabla p_\varepsilon\|_{L^r(Y_m)})\\ \|\widehat{P}\|_{W^{1,r}(\mathbf{D}\setminus Y_m)}\leqslant c(1+\|\varepsilon^2\omega P_\varepsilon\|_{L^r(\mathbf{D}\setminus Y_m)}+\|(\mathbf{K}-\widehat{\mathbf{K}})\nabla P_\varepsilon\|_{L^r(\mathbf{D}\setminus Y_m)}) \end{cases} \tag{17}$$

where c is a constant independent of ε,ω. Define $\check{p}\equiv p_\varepsilon-\widehat{p}$ in Y_m and $\check{P}\equiv P_\varepsilon-\widehat{P}$ in $\mathbf{D}\setminus Y_m$. Equations (11) and (15)–(16) imply

$$\begin{cases} \Delta\check{p}=0 & \text{in } Y_m\\ \Delta\check{P}=0 & \text{in } \mathbf{D}\setminus Y_m\\ \check{P}|_{\partial Y_m}=\check{p}|_{\partial Y_m}\\ \nabla\check{P}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}-\check{\varepsilon}\nabla\check{p}\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}=\mathscr{F}/\widehat{\mathbf{K}}\\ \check{P}|_{\partial\mathbf{D}}=0 \end{cases} \tag{18}$$

where $\check{\varepsilon}\equiv\varepsilon^2\widehat{\mathbf{k}}/\widehat{\mathbf{K}}$ and by (14), (17), and [17]

$$\begin{aligned} \|\mathscr{F}\|_{W^{-1/r,r}(\partial Y_m)} \leqslant\ & c(1+\varepsilon^2\|(\mathbf{k}-\widehat{\mathbf{k}})\nabla p_\varepsilon\|_{L^r(Y_m)}+\|(\mathbf{K}-\widehat{\mathbf{K}})\nabla P_\varepsilon\|_{L^r(\mathbf{D}\setminus Y_m)}\\ &+\|\varepsilon^2\omega p_\varepsilon\|_{L^r(Y_m)}+\|\varepsilon^2\omega P_\varepsilon\|_{L^r(\mathbf{D}\setminus Y_m)}) \end{aligned} \tag{19}$$

where c is a constant independent of ε,ω. By Green's formula, (18), and Theorem 6.5.1 [12], we see that

$$\begin{cases} \check{p}/2+\mathscr{T}_{\partial Y_m}(\check{p})=\mathscr{E}_{\partial Y_m}(\partial_{\mathbf{n}_y}\check{p}) \\ \check{P}/2-\mathscr{T}_{\partial Y_m}(\check{P})=-\mathscr{E}_{\partial Y_m}(\partial_{\mathbf{n}_y}\check{P})+\mathscr{E}_{\partial \mathbf{D}}(\partial_{\mathbf{n}_y}\check{P}|_{\partial \mathbf{D}}) \end{cases} \quad \text{on } \partial Y_m$$

where $\partial_{\mathbf{n}_y}\check{P}|_{\partial\mathbf{D}}$ is the normal derivative of $\check{P}$ on $\partial\mathbf{D}$. Therefore,

$$\left(\frac{\check{\varepsilon}+1}{2(1-\check{\varepsilon})}I-\mathscr{T}_{\partial Y_m}\right)\check{p}=\frac{\mathscr{E}_{\partial\mathbf{D}}(\partial_{\mathbf{n}_y}\check{P}|_{\partial\mathbf{D}})}{1-\check{\varepsilon}}-\frac{\mathscr{E}_{\partial Y_m}(\mathscr{F})}{(1-\check{\varepsilon})\widehat{\mathbf{K}}} \quad \text{on } \partial Y_m \tag{20}$$

Equations (14), (17), (20), Lemma 3.2, and [14] imply

$$\|\check{p}\|_{W^{1-1/r,r}(\partial Y_m)}\leqslant c(\|\mathscr{F}\|_{W^{-1/r,r}(\partial Y_m)}+\|\partial_{\mathbf{n}_y}\check{P}\|_{W^{-1/r,r}(\partial\mathbf{D})}) \tag{21}$$

$$\|\partial_{\mathbf{n}_y}\check{P}\|_{W^{-1/r,r}(\partial\mathbf{D})}\leqslant c(1+\|(\mathbf{K}-\widehat{\mathbf{K}})\nabla P_\varepsilon\|_{L^r(\mathbf{D}\setminus Y_m)}) \tag{22}$$

By Equations (14), (17), (19), (21), and (22), we obtain

$$\|P_\varepsilon\|_{W^{1,r}(\mathbf{D}\setminus Y_m)}+\varepsilon^2\|p_\varepsilon\|_{W^{1,r}(Y_m)}\leqslant c(1+\varepsilon^2\|(\mathbf{k}-\widehat{\mathbf{k}})\nabla p_\varepsilon\|_{L^r(Y_m)}$$
$$+\|(\mathbf{K}-\widehat{\mathbf{K}})\nabla P_\varepsilon\|_{L^r(\mathbf{D}\setminus Y_m)}+\|\varepsilon^2\omega p_\varepsilon\|_{L^r(Y_m)}+\|\varepsilon^2\omega P_\varepsilon\|_{L^r(\mathbf{D}\setminus Y_m)})$$

By interpolation theorem [9], we obtain (12) for the smooth coefficient and solution case.

The estimate (12) for non-smooth coefficient case is directly from above result, Section 16, Chapter 3 [2], energy method, and approximation method. Equation (13) can be proved by a similar argument as (12). We omit it. □

Though Lemma 3.3 is just a result for one sub-domain Y_m case, an analogous argument proves the same conclusion if Y contains several non-overlapping sub-domains. Moreover, an analogous argument also works for the following boundary-value problem. Define a boundary portion of Y by $\Gamma\equiv\{y\in Y: y=(y_1,y_2,0)\}$, and consider the following problem:

$$\begin{cases} -\nabla\cdot(\mathbf{K}(\nabla P_\varepsilon+\hat{Q}_\varepsilon))+\varepsilon^2\omega P_\varepsilon=\hat{F}_\varepsilon & \text{in } Y_f \\ -\varepsilon\nabla\cdot(\mathbf{k}(\varepsilon\nabla p_\varepsilon+\hat{q}_\varepsilon))+\varepsilon^2\omega p_\varepsilon=\hat{f}_\varepsilon & \text{in } Y_m \\ \mathbf{K}(\nabla P_\varepsilon+\hat{Q}_\varepsilon)\cdot\vec{\mathbf{n}}_y=\varepsilon\mathbf{k}(\varepsilon\nabla p_\varepsilon+\hat{q}_\varepsilon)\cdot\vec{\mathbf{n}}_y & \text{on } \partial Y_m \\ P_\varepsilon=p_\varepsilon & \text{on } \partial Y_m \\ P_\varepsilon=P_{b_\varepsilon} & \text{on } \Gamma \end{cases} \tag{23}$$

where $\vec{\mathbf{n}}_y$ is the unit vector normal to ∂Y_m. Let $Y_m\subset\widetilde{\mathbf{D}}\subset Y$ satisfy

$$\min\{\text{dist}(Y_m,\partial\widetilde{\mathbf{D}}),\text{dist}(\widetilde{\mathbf{D}},\partial Y\setminus\Gamma)\}>0$$

Lemma 3.4

If $\omega<\omega_0$ and $\|P_\varepsilon\|_{L^2(Y_f)}+\|\varepsilon^2 p_\varepsilon\|_{L^2(Y_m)}+\|\hat{Q}_\varepsilon,\nabla\cdot\hat{Q}_\varepsilon,\hat{F}_\varepsilon\|_{L^r(Y_f)}+\|\varepsilon\hat{q}_\varepsilon,\varepsilon\nabla\cdot\hat{q}_\varepsilon,\hat{f}_\varepsilon\|_{L^r(Y_m)}+\|P_{b_\varepsilon}\|_{W^{1,r}(Y_f)}$ is bounded independently of ε for $r\in(3,\infty)$, then the solutions of (23) satisfy

$$\|P_\varepsilon\|_{W^{1,r}(\widetilde{\mathbf{D}}\setminus Y_m)}\leqslant c_1$$

where c_1 is independent of ε,ω. In addition, $\|P_{b_\varepsilon}\|_{C^{1,1-3/r}(Y_f)}$ is bounded independently of ε and $\omega=\hat{Q}_\varepsilon=\hat{q}_\varepsilon=0$

$$\|P_\varepsilon\|_{C^{1,1-3/r}(\widetilde{\mathbf{D}}\setminus Y_m)}\leqslant c_2$$

where c_2 is a constant independent of ε.

4. Uniform Hölder estimate

In this section we prove Theorem 2.1. It includes two Sections 4.1 and 4.2. The Hölder estimate in the interior region is derived in Section 4.1, and the Hölder estimate around the boundary is in Section 4.2.

4.1. Interior estimate

For convenience we assume $\overline{B(0,1)}\subset\Omega$.

Lemma 4.1

For any $\delta>0$ and $\gamma\in[0,\omega_0]$, there are constants $\theta\in(0,1)$ (depending on $\delta,\omega_0,\mathbf{K},\mathbf{k}$) and $\varepsilon_0\in(0,1)$ (depending on $\theta,\omega_0,\delta,\mathbf{k}$) such that if $U_{\varepsilon,\nu},u_{\varepsilon,\nu},Q_{\varepsilon,\nu},q_{\varepsilon,\nu},F_{\varepsilon,\nu},f_{\varepsilon,\nu}$ satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\nu(\nabla U_{\varepsilon,\nu}+Q_{\varepsilon,\nu}))+\gamma U_{\varepsilon,\nu}=F_{\varepsilon,\nu} & \text{in } B(0,1)\cap\Omega_f^\nu \\ -\varepsilon\nabla\cdot(\mathbf{k}_\nu(\varepsilon\nabla u_{\varepsilon,\nu}+q_{\varepsilon,\nu}))+\gamma u_{\varepsilon,\nu}=\varepsilon f_{\varepsilon,\nu} & \text{in } B(0,1)\cap\Omega_m^\nu \\ \mathbf{K}_\nu(\nabla U_{\varepsilon,\nu}+Q_{\varepsilon,\nu})\cdot\vec{\mathbf{n}}^\nu=\varepsilon\mathbf{k}_\nu(\varepsilon\nabla u_{\varepsilon,\nu}+q_{\varepsilon,\nu})\cdot\vec{\mathbf{n}}^\nu & \text{on } B(0,1)\cap\partial\Omega_m^\nu \\ U_{\varepsilon,\nu}=u_{\varepsilon,\nu} & \text{on } B(0,1)\cap\partial\Omega_m^\nu \end{cases} \tag{24}$$

and if

$$\max\left\{\|U_{\varepsilon,\nu}\|_{L^2(B(0,1)\cap\Omega_f^\nu)},\varepsilon\|u_{\varepsilon,\nu}\|_{L^2(B(0,1)\cap\Omega_m^\nu)}\right.$$
$$\left.\|\frac{Q_{\varepsilon,\nu}}{\varepsilon_0}\mathscr{X}_{\Omega_f^\nu}+q_{\varepsilon,\nu}\mathscr{X}_{\Omega_m^\nu},\frac{F_{\varepsilon,\nu}}{\varepsilon_0}\mathscr{X}_{\Omega_f^\nu}+\frac{\nu}{\varepsilon_0}f_{\varepsilon,\nu}\mathscr{X}_{\Omega_m^\nu}\|_{L^{3+\delta}(B(0,1))}\right\}\leqslant 1 \tag{25}$$

then, for any $\varepsilon\leqslant\nu\leqslant\varepsilon_0$ and $\mu=1-3/(3+\delta)$

$$\begin{cases} \displaystyle⨍_{B(0,\theta)}|\Pi_\nu U_{\varepsilon,\nu}-(\Pi_\nu U_{\varepsilon,\nu})_{0,\theta}|^2\mathrm{d}x\leqslant\theta^{2\mu} \\ \displaystyle⨍_{B(0,\theta)\cap\Omega_m^\nu}\varepsilon^2|u_{\varepsilon,\nu}-(\Pi_\nu U_{\varepsilon,\nu})_{0,\theta}|^2\mathrm{d}x\leqslant\theta^{2\mu} \end{cases} \tag{26}$$

Proof

Consider the following problem:

$$\begin{cases} -\nabla\cdot(\check{\mathbf{K}}\nabla U)+\gamma|Y_f|U+\gamma\bar{u}=0 & \text{in } B(0,1) \\ -\nabla_y\cdot(\mathbf{k}\nabla_y u)+\gamma u=0 & \text{in } B(0,1)\times Y_m \\ U=u & \text{on } B(0,1)\times\partial Y_m \end{cases} \tag{27}$$

where $\bar{u}(x)\equiv\int_{Y_m}u(x,y)\mathrm{d}y$ and $\check{\mathbf{K}}$ is the positive-definite matrix in (2). If (U,u) is a solution of (27), then by following the argument of Lemma 2.2, we have

$$\|U\|_{W^{1,r}(B(0,1/2)\cap\Omega)}\leqslant c\|U\|_{L^2(B(0,1)\cap\Omega)}$$

where $r>2$ and c depends on $\check{\mathbf{K}},\mathbf{k},\omega_0$. If μ' satisfies $\mu<\mu'<1$, then, by Lemma 3.1

$$⨍_{B(0,\theta)}|U-(U)_{0,\theta}|^2\mathrm{d}x\leqslant\theta^{2\mu'}⨍_{B(0,1)}U^2\mathrm{d}x \tag{28}$$

for θ (depending on $\delta,\check{\mathbf{K}},\mathbf{k},\omega_0$) sufficiently small. Fix a value θ, and we claim $(26)_1$. If not, there is a sequence $\{U_{\varepsilon,\nu},u_{\varepsilon,\nu},Q_{\varepsilon,\nu},q_{\varepsilon,\nu},F_{\varepsilon,\nu},f_{\varepsilon,\nu}\}$ satisfying (24) and

$$\begin{cases} \max\{\|U_{\varepsilon,\nu}\|_{L^2(B(0,1)\cap\Omega_f^\nu)},\varepsilon\|u_{\varepsilon,\nu}\|_{L^2(B(0,1)\cap\Omega_m^\nu)},\|q_{\varepsilon,\nu}\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^\nu)}\}\leqslant 1 \\ \displaystyle\lim_{\varepsilon\leqslant\nu\to 0}\|Q_{\varepsilon,\nu},F_{\varepsilon,\nu}\|_{L^{3+\delta}(B(0,1)\cap\Omega_f^\nu)}+\|\nu f_{\varepsilon,\nu}\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^\nu)}=0 \\ \displaystyle⨍_{B(0,\theta)}|\Pi_\nu U_{\varepsilon,\nu}-(\Pi_\nu U_{\varepsilon,\nu})_{0,\theta}|^2\mathrm{d}x>\theta^{2\mu} \end{cases} \tag{29}$$

By Lemma 2.1 and compactness principle, we extract a subsequence (same notation for subsequence) such that, as $\varepsilon\leqslant\nu\to 0$

$$\begin{cases} \Pi_\nu U_{\varepsilon,\nu}\to U & \text{in } L^2(B(0,\theta)) \text{ strongly} \\ u_{\varepsilon,\nu}\mathscr{X}_{\Omega_m^\nu}\to u & \text{in } L^2(B(0,\theta)\times Y_m) \text{ in two scale} \\ \mathbf{K}_\nu\mathscr{X}_{\Omega_f^\nu}\nabla U_{\varepsilon,\nu}\to\check{\mathbf{K}}\nabla U & \text{in } L^2(B(0,\theta)) \text{ weakly} \\ Q_{\varepsilon,\nu}\mathscr{X}_{\Omega_f^\nu},F_{\varepsilon,\nu}\mathscr{X}_{\Omega_f^\nu},\nu f_{\varepsilon,\nu}\mathscr{X}_{\Omega_m^\nu}\to 0 & \text{in } L^2(B(0,1)) \text{ strongly} \end{cases} \tag{30}$$

We also see that (U, u) is a solution of (27). Equations (28)–(30) then imply

$$\theta^{2\mu} \leqslant \lim_{\varepsilon\leqslant v\to 0} \fint_{B(0,\theta)} |\Pi_v U_{\varepsilon,v} - (\Pi_v U_{\varepsilon,v})_{0,\theta}|^2 dx = \fint_{B(0,\theta)} U^2 dx - \left| \fint_{B(0,\theta)} U dx \right|^2 = \fint_{B(0,\theta)} |U - (U)_{0,\theta}|^2 dx \leqslant \theta^{2\mu'}$$

Thus we get $\theta^{2\mu} \leqslant \theta^{2\mu'}$, which is absurd. Therefore we prove $(26)_1$.

Define $\hat{U} \equiv \theta^{-\mu}(\Pi_v U_{\varepsilon,v} - (\Pi_v U_{\varepsilon,v})_{0,\theta})$ and $\hat{u} \equiv \theta^{-\mu}(u_{\varepsilon,v} - (\Pi_v U_{\varepsilon,v})_{0,\theta})$. Then (24) implies, for any smooth function g with support in $v(Y_m + j) \subset B(0,\theta) \cap \Omega_m^v$ for some $j \in Z^3$,

$$\varepsilon^2 \int_{v(Y_m+j)} (\hat{u} - \hat{U}) \nabla \cdot (\mathbf{k}_v \nabla g) dx - \gamma \int_{v(Y_m+j)} (\hat{u} - \hat{U}) g \, dx$$
$$= \int_{v(Y_m+j)} \mathbf{k}_v (\varepsilon^2 \nabla \hat{U} + \varepsilon \theta^{-\mu} q_{\varepsilon,v}) \nabla g \, dx - \int_{v(Y_m+j)} \theta^{-\mu} (\varepsilon f_{\varepsilon,v} - \gamma \Pi_v U_{\varepsilon,v}) g \, dx$$

If g is the solution of

$$\begin{cases} \nabla \cdot (\mathbf{k}_v \nabla g) = \hat{u} - \hat{U} & \text{in } v(Y_m + j) \\ g = 0 & \text{on } v(\partial Y_m + j) \end{cases}$$

then $c_1 v^{-1} \|g\|_{L^2(v(Y_m+j))} \leqslant \|\nabla g\|_{L^2(v(Y_m+j))} \leqslant c_2 v \|\hat{u} - \hat{U}\|_{L^2(v(Y_m+j))}$, where c_1, c_2 are independent of v. By integration by part and Hölder inequality

$$-\gamma \int_{v(Y_m+j)} (\hat{u} - \hat{U}) g \, dx = \gamma \|\mathbf{k}_v^{1/2} \nabla g\|^2_{L^2(v(Y_m+j))}$$

$$\int_{v(Y_m+j)} \theta^{-\mu} \gamma \Pi_v U_{\varepsilon,v} g \, dx \leqslant c\theta^{-\mu} \gamma^{1/2} v \|\Pi_v U_{\varepsilon,v}\|_{L^2(v(Y_m+j))} \|\gamma^{1/2} \nabla g\|_{L^2(v(Y_m+j))}$$

Equation $(26)_2$ follows from above estimates if ε_0 is small enough. □

Lemma 4.2

For any $\delta \in (0,3)$ and $\omega \in [0, \omega_0]$, there are constants $\theta \in (0,1)$ (depending on $\delta, \omega_0, \mathbf{K}, \mathbf{k}$) and $\varepsilon_0 \in (0,1)$ (depending on $\theta, \omega_0, \delta, \mathbf{k}$) such that if

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\lambda (\nabla U_{\varepsilon,\lambda} + Q_{\varepsilon,\lambda})) + \left| \frac{\varepsilon}{\lambda} \right|^2 \omega U_{\varepsilon,\lambda} = F_{\varepsilon,\lambda} & \text{in } B(0,1) \cap \Omega_f^\lambda \\ -\varepsilon \nabla \cdot (\mathbf{k}_\lambda (\varepsilon \nabla u_{\varepsilon,\lambda} + q_{\varepsilon,\lambda})) + \left| \frac{\varepsilon}{\lambda} \right|^2 \omega u_{\varepsilon,\lambda} = \varepsilon f_{\varepsilon,\lambda} & \text{in } B(0,1) \cap \Omega_m^\lambda \\ \mathbf{K}_\lambda (\nabla U_{\varepsilon,\lambda} + Q_{\varepsilon,\lambda}) \cdot \vec{\mathbf{n}}^\lambda = \varepsilon \mathbf{k}_\lambda (\varepsilon \nabla u_{\varepsilon,\lambda} + q_{\varepsilon,\lambda}) \cdot \vec{\mathbf{n}}^\lambda & \text{on } B(0,1) \cap \partial \Omega_m^\lambda \\ U_{\varepsilon,\lambda} = u_{\varepsilon,\lambda} & \text{on } B(0,1) \cap \partial \Omega_m^\lambda \end{cases}$$

then, for all $\varepsilon \leqslant \lambda \leqslant \varepsilon_0$ and k satisfying $\lambda/\theta^k \leqslant \varepsilon_0$

$$\begin{cases} \fint_{B(0,\theta^k)} |\Pi_\lambda U_{\varepsilon,\lambda} - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,\theta^k}|^2 dx \leqslant \theta^{2k\mu} J^2_{\varepsilon,\lambda} \\ \fint_{B(0,\theta^k) \cap \Omega_m^\lambda} \varepsilon^2 |u_{\varepsilon,\lambda} - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,\theta^k}|^2 dx \leqslant \theta^{2k\mu} J^2_{\varepsilon,\lambda} \end{cases} \tag{31}$$

where

$$J_{\varepsilon,\lambda} \equiv 2(\|U_{\varepsilon,\lambda} \mathscr{X}_{\Omega_f^\lambda} + \varepsilon u_{\varepsilon,\lambda} \mathscr{X}_{\Omega_m^\lambda}\|_{L^2(B(0,1))} + \|\varepsilon_0^{-1} Q_{\varepsilon,\lambda} \mathscr{X}_{\Omega_f^\lambda} + q_{\varepsilon,\lambda} \mathscr{X}_{\Omega_m^\lambda}\|_{L^{3+\delta}(B(0,1))}$$
$$+ \varepsilon_0^{-1} \|F_{\varepsilon,\lambda} \mathscr{X}_{\Omega_f^\lambda} + \lambda f_{\varepsilon,\lambda} \mathscr{X}_{\Omega_m^\lambda}\|_{L^{3+\delta}(B(0,1))})$$

Proof

We only consider $J_{\varepsilon,\lambda} < \infty$ case. For $k = 1$, define

$$\hat{U}_\varepsilon \equiv \frac{U_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}, \quad \hat{u}_\varepsilon \equiv \frac{u_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}, \quad \hat{Q}_\varepsilon \equiv \frac{Q_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}, \quad \hat{q}_\varepsilon \equiv \frac{q_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}, \quad \hat{F}_\varepsilon \equiv \frac{F_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}, \quad \hat{f}_\varepsilon \equiv \frac{f_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}$$

Then these functions satisfy (24) and (25) with $v=\lambda$ and $\gamma=|\varepsilon/\lambda|^2\omega$. By Lemma 4.1

$$\begin{cases} \fint_{B(0,\theta)} |\Pi_\lambda \hat{U}_\varepsilon - (\Pi_\lambda \hat{U}_\varepsilon)_{0,\theta}|^2 dx \leqslant \theta^{2\mu} \\ \fint_{B(0,\theta)\cap\Omega_m^\lambda} \varepsilon^2 |\hat{u}_\varepsilon - (\Pi_\lambda \hat{U}_\varepsilon)_{0,\theta}|^2 dx \leqslant \theta^{2\mu} \end{cases}$$

This implies (31) for $k=1$. If (31) holds for some k satisfying $\lambda/\theta^k \leqslant \varepsilon_0$, we define

$$\begin{cases} \hat{U}_\varepsilon(x) \equiv J_{\varepsilon,\lambda}^{-1} \theta^{-k\mu}(U_{\varepsilon,\lambda}(\theta^k x) - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,\theta^k}) \\ \hat{Q}_\varepsilon(x) \equiv J_{\varepsilon,\lambda}^{-1} \theta^{k(1-\mu)} Q_{\varepsilon,\lambda}(\theta^k x) & \text{in } B(0,1)\cap\Omega_f^\lambda/\theta^k \\ \hat{F}_\varepsilon(x) \equiv J_{\varepsilon,\lambda}^{-1} \theta^{k(2-\mu)} \left(F_{\varepsilon,\lambda}(\theta^k x) - \left|\frac{\varepsilon}{\lambda}\right|^2 \omega (\Pi_\lambda U_{\varepsilon,\lambda})_{0,\theta^k} \right) \end{cases}$$

$$\begin{cases} \hat{u}_\varepsilon(x) \equiv J_{\varepsilon,\lambda}^{-1} \theta^{-k\mu}(u_{\varepsilon,\lambda}(\theta^k x) - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,\theta^k}) \\ \hat{q}_\varepsilon(x) \equiv J_{\varepsilon,\lambda}^{-1} \theta^{k(1-\mu)} q_{\varepsilon,\lambda}(\theta^k x) & \text{in } B(0,1)\cap\Omega_m^\lambda/\theta^k \\ \hat{f}_\varepsilon(x) \equiv J_{\varepsilon,\lambda}^{-1} \theta^{k(2-\mu)} \left(f_{\varepsilon,\lambda}(\theta^k x) - \frac{\left|\frac{\varepsilon}{\lambda}\right|^2 \omega}{\varepsilon} (\Pi_\lambda U_{\varepsilon,\lambda})_{0,\theta^k} \right) \end{cases}$$

Then these functions satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\lambda/\theta^k}(\nabla\hat{U}_\varepsilon + \hat{Q}_\varepsilon)) + \theta^{2k}\left|\frac{\varepsilon}{\lambda}\right|^2 \omega \hat{U}_\varepsilon = \hat{F}_\varepsilon & \text{in } B(0,1)\cap\Omega_f^\lambda/\theta^k \\ -\varepsilon\nabla\cdot(\mathbf{k}_{\lambda/\theta^k}(\varepsilon\nabla\hat{u}_\varepsilon + \hat{q}_\varepsilon)) + \theta^{2k}\left|\frac{\varepsilon}{\lambda}\right|^2 \omega \hat{u}_\varepsilon = \varepsilon\hat{f}_\varepsilon & \text{in } B(0,1)\cap\Omega_m^\lambda/\theta^k \\ \mathbf{K}_{\lambda/\theta^k}(\nabla\hat{U}_\varepsilon + \hat{Q}_\varepsilon)\cdot\vec{\mathbf{n}} = \varepsilon\mathbf{k}_{\lambda/\theta^k}(\varepsilon\nabla\hat{u}_\varepsilon + \hat{q}_\varepsilon)\cdot\vec{\mathbf{n}} & \text{on } B(0,1)\cap\partial\Omega_m^\lambda/\theta^k \\ \hat{U}_\varepsilon = \hat{u}_\varepsilon & \text{on } B(0,1)\cap\partial\Omega_m^\lambda/\theta^k \end{cases}$$

where $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\Omega_m^\lambda/\theta^k$. By induction,

$$\begin{cases} \max\{\|\hat{U}_\varepsilon\|_{L^2(B(0,1)\cap\Omega_f^\lambda/\theta^k)}, \varepsilon\|\hat{u}_\varepsilon\|_{L^2(B(0,1)\cap\Omega_m^\lambda/\theta^k)}\} \leqslant 1 \\ \|\varepsilon_0^{-1}\hat{Q}_\varepsilon \mathscr{X}_{\Omega_f^\lambda/\theta^k} + \hat{q}_\varepsilon \mathscr{X}_{\Omega_m^\lambda/\theta^k}\|_{L^{3+\delta}(B(0,1))} \\ \quad + \varepsilon_0^{-1}\|\hat{F}_\varepsilon \mathscr{X}_{\Omega_f^\lambda/\theta^k} + \lambda\theta^{-k}\hat{f}_\varepsilon \mathscr{X}_{\Omega_m^\lambda/\theta^k}\|_{L^{3+\delta}(B(0,1))} \leqslant 1 \end{cases}$$

By Lemma 4.1 (take $v=\lambda/\theta^k$ and $\gamma=\theta^{2k}|\varepsilon/\lambda|^2\omega$), we obtain

$$\begin{cases} \fint_{B(0,\theta)} |\Pi_{\lambda/\theta^k} \hat{U}_\varepsilon - (\Pi_{\lambda/\theta^k} \hat{U}_\varepsilon)_{0,\theta}|^2 dx \leqslant \theta^{2\mu} \\ \fint_{B(0,\theta)\cap\Omega_m^\lambda/\theta^k} \varepsilon^2 |\hat{u}_\varepsilon - (\Pi_{\lambda/\theta^k} \hat{U}_\varepsilon)_{0,\theta}|^2 dx \leqslant \theta^{2\mu} \end{cases} \tag{32}$$

Note by Lemma 2.1,

$$\fint_{B(0,\theta)} |\Pi_{\lambda/\theta^k}\hat{U}_\varepsilon - (\Pi_{\lambda/\theta^k}\hat{U}_\varepsilon)_{0,\theta}|^2 = \fint_{B(0,\theta^{k+1})} \frac{|\Pi_\lambda U_{\varepsilon,\lambda} - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,\theta^{k+1}}|^2}{J_{\varepsilon,\lambda}^2 \theta^{2k\mu}} \tag{33}$$

$$\fint_{B(0,\theta)\cap\Omega_m^\lambda/\theta^k} |\hat{u}_\varepsilon - (\Pi_{\lambda/\theta^k}\hat{U}_\varepsilon)_{0,\theta}|^2 = \fint_{B(0,\theta^{k+1})\cap\Omega_m^\lambda} \frac{|u_{\varepsilon,\lambda} - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,\theta^{k+1}}|^2}{J_{\varepsilon,\lambda}^2 \theta^{2k\mu}} \tag{34}$$

Equations (32)–(34) imply the inequality (31) for $k+1$ case.

Lemma 4.3
Under the same assumptions as Lemma 4.2,

$$[U_{\varepsilon,\lambda}]_{C^{0,\mu}(B(0,1/2)\cap\Omega_f^\lambda)} \leqslant c(J_{\varepsilon,\lambda} + \lambda\|\nabla\cdot Q_{\varepsilon,\lambda}\mathscr{X}_{\Omega_f^\lambda} + \varepsilon\nabla\cdot q_{\varepsilon,\lambda}\mathscr{X}_{\Omega_m^\lambda}\|_{L^{3+\delta}(B(0,1))})$$

where the constant c is independent of $\varepsilon, \lambda, \omega$.

Proof

Let c denote a constant independent of $\varepsilon, \lambda, \omega$. Lemma 4.2 implies that for $r \geqslant \lambda/\varepsilon_0$

$$\begin{cases} \displaystyle\fint_{B(0,r)} |\Pi_\lambda U_{\varepsilon,\lambda} - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,r}|^2 \mathrm{d}x \leqslant c r^{2\mu} J^2_{\varepsilon,\lambda} \\ \displaystyle\fint_{B(0,r)\cap\Omega^\lambda_m} \varepsilon^2 |u_{\varepsilon,\lambda} - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,r}|^2 \mathrm{d}x \leqslant c r^{2\mu} J^2_{\varepsilon,\lambda} \end{cases} \tag{35}$$

Set $\hat{J}_{\varepsilon,\lambda} \equiv J_{\varepsilon,\lambda} + \lambda \|\nabla\cdot Q_{\varepsilon,\lambda}\mathscr{X}_{\Omega^\lambda_f} + \varepsilon\nabla\cdot q_{\varepsilon,\lambda}\mathscr{X}_{\Omega^\lambda_m}\|_{L^{3+\delta}(B(0,1))}$

$$\begin{cases} \hat{U}_\varepsilon(x) \equiv \hat{J}^{-1}_{\varepsilon,\lambda}\lambda^{-\mu}\left(U_{\varepsilon,\lambda}(\lambda x) - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,2\lambda/\varepsilon_0}\right) & \\ \hat{Q}_\varepsilon(x) \equiv \hat{J}^{-1}_{\varepsilon,\lambda}\lambda^{1-\mu} Q_{\varepsilon,\lambda}(\lambda x) & \text{in } B\left(0,\dfrac{2}{\varepsilon_0}\right)\cap\Omega^\lambda_f/\lambda \\ \hat{F}_\varepsilon(x) \equiv \hat{J}^{-1}_{\varepsilon,\lambda}\lambda^{2-\mu}\left(F_{\varepsilon,\lambda}(\lambda x) - \left|\dfrac{\varepsilon}{\lambda}\right|^2 \omega(\Pi_\lambda U_{\varepsilon,\lambda})_{0,2\lambda/\varepsilon_0}\right) & \end{cases}$$

$$\begin{cases} \hat{u}_\varepsilon(x) \equiv \hat{J}^{-1}_{\varepsilon,\lambda}\lambda^{-\mu}(u_{\varepsilon,\lambda}(\lambda x) - (\Pi_\lambda U_{\varepsilon,\lambda})_{0,2\lambda/\varepsilon_0}) & \\ \hat{q}_\varepsilon(x) \equiv \hat{J}^{-1}_{\varepsilon,\lambda}\lambda^{1-\mu} q_{\varepsilon,\lambda}(\lambda x) & \text{in } B\left(0,\dfrac{2}{\varepsilon_0}\right)\cap\Omega^\lambda_m/\lambda \\ \hat{f}_\varepsilon(x) \equiv \hat{J}^{-1}_{\varepsilon,\lambda}\lambda^{2-\mu}\left(f_{\varepsilon,\lambda}(\lambda x) - \dfrac{\left|\frac{\varepsilon}{\lambda}\right|^2\omega}{\varepsilon}(\Pi_\lambda U_{\varepsilon,\lambda})_{0,2\lambda/\varepsilon_0}\right) & \end{cases}$$

Then those functions satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\lambda/\lambda}(\nabla\hat{U}_\varepsilon + \hat{Q}_\varepsilon)) + \varepsilon^2\omega\hat{U}_\varepsilon = \hat{F}_\varepsilon & \text{in } B\left(0,\dfrac{2}{\varepsilon_0}\right)\cap\Omega^\lambda_f/\lambda \\ -\varepsilon\nabla\cdot(\mathbf{k}_{\lambda/\lambda}(\varepsilon\nabla\hat{u}_\varepsilon + \hat{q}_\varepsilon)) + \varepsilon^2\omega\hat{u}_\varepsilon = \varepsilon\hat{f}_\varepsilon & \text{in } B\left(0,\dfrac{2}{\varepsilon_0}\right)\cap\Omega^\lambda_m/\lambda \\ \mathbf{K}_{\lambda/\lambda}(\nabla\hat{U}_\varepsilon + \hat{Q}_\varepsilon)\cdot\vec{\mathbf{n}} = \varepsilon\hat{\mathbf{k}}_{\lambda/\lambda}(\varepsilon\nabla\hat{u}_\varepsilon + \hat{q}_\varepsilon)\cdot\vec{\mathbf{n}} & \text{on } B\left(0,\dfrac{2}{\varepsilon_0}\right)\cap\partial\Omega^\lambda_m/\lambda \\ \hat{U}_\varepsilon = \hat{u}_\varepsilon & \text{on } B\left(0,\dfrac{2}{\varepsilon_0}\right)\cap\partial\Omega^\lambda_m/\lambda \end{cases}$$

where $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\Omega^\lambda_m/\lambda$. Take $r = 2\lambda/\varepsilon_0$ in (35) to get

$$\begin{aligned} &\|\hat{U}_\varepsilon\mathscr{X}_{\Omega^\lambda_f/\lambda} + \varepsilon\hat{u}_\varepsilon\mathscr{X}_{\Omega^\lambda_m/\lambda}\|_{L^2(B(0,2/\varepsilon_0))} + \|\hat{Q}_\varepsilon\mathscr{X}_{\Omega^\lambda_f/\lambda} + \hat{q}_\varepsilon\mathscr{X}_{\Omega^\lambda_m/\lambda}\|_{L^{3+\delta}(B(0,\frac{2}{\varepsilon_0}))} \\ &+ \|\nabla\cdot\hat{Q}_\varepsilon\mathscr{X}_{\Omega^\lambda_f/\lambda} + \varepsilon\nabla\cdot\hat{q}_\varepsilon\mathscr{X}_{\Omega^\lambda_m/\lambda}\|_{L^{3+\delta}(B(0,2/\varepsilon_0))} + \|\hat{F}_\varepsilon\mathscr{X}_{\Omega^\lambda_f/\lambda} + \hat{f}_\varepsilon\mathscr{X}_{\Omega^\lambda_m/\lambda}\|_{L^{3+\delta}(B(0,2/\varepsilon_0))} \leqslant c \end{aligned}$$

where the constant c is independent of $\varepsilon, \lambda, \omega$. By Lemma 3.3,

$$[\hat{U}_\varepsilon]_{C^{0,\mu}(B(0,1/\varepsilon_0)\cap\Omega^\lambda_f/\lambda)} \leqslant c \tag{36}$$

The lemma follows from Lemma 3.1 and Equations (35) and (36).

4.2. Boundary estimate

Let $\phi: R^2 \to R$ be a function satisfying $[\phi]_{C^{1,\alpha}(R^2)} \leqslant c$, $\phi(0) = |\nabla\phi(0)| = 0$. Define $D_\phi \equiv \{(x', x_3) \in R^3 | x_3 > \phi(x')\}$. For the purpose of proving boundary estimate, we assume

$$0 \in \partial\Omega \quad \text{and} \quad B(0,1)\cap\Omega = B(0,1)\cap D_\phi \tag{37}$$

For $\tau \in R^+$, if $\phi_\tau(x) \equiv \tau^{-1}\phi(\tau x)$, we have

$$[\phi_\tau]_{C^{1,\alpha}(R^2)} \leqslant \tau^\alpha [\phi]_{C^{1,\alpha}(R^2)} \tag{38}$$

Lemma 4.4

For any $\delta>0$ and $\gamma\in[0,\omega_0]$, there are constants $\tilde{\theta}\in(0,1)$ (depending on $\delta,\alpha,\omega_0,\mathbf{K},\mathbf{k}$) and $\tilde{\varepsilon}_0\in(0,1)$ (depending on $\theta,\omega_0,\delta,\mathbf{k}$) satisfying $\tilde{\theta}<\theta$, $\tilde{\varepsilon}_0<\varepsilon_0$ (θ, ε_0 are those in Lemma 4.1) such that if $U_{\varepsilon,v},u_{\varepsilon,v},Q_{\varepsilon,v},q_{\varepsilon,v},F_{\varepsilon,v},f_{\varepsilon,v}$ satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_v(\nabla U_{\varepsilon,v}+Q_{\varepsilon,v}))+\gamma U_{\varepsilon,v}=F_{\varepsilon,v} & \text{in } B(0,1)\cap\Omega_f^v \\ -\varepsilon\nabla\cdot(\mathbf{k}_v(\varepsilon\nabla u_{\varepsilon,v}+q_{\varepsilon,v}))+\gamma u_{\varepsilon,v}=\varepsilon f_{\varepsilon,v} & \text{in } B(0,1)\cap\Omega_m^v \\ \mathbf{K}_v(\nabla U_{\varepsilon,v}+Q_{\varepsilon,v})\cdot\vec{\mathbf{n}}^v=\varepsilon\mathbf{k}_v(\varepsilon\nabla u_{\varepsilon,v}+q_{\varepsilon,v})\cdot\vec{\mathbf{n}}^v & \text{on } B(0,1)\cap\partial\Omega_m^v \\ U_{\varepsilon,v}=u_{\varepsilon,v} & \text{on } B(0,1)\cap\partial\Omega_m^v \\ U_{\varepsilon,v}=U_{b_{\varepsilon,v}} & \text{on } B(0,1)\cap\partial\Omega \end{cases} \tag{39}$$

and if

$$\max\left\{\|U_{\varepsilon,v}\|_{L^2(B(0,1)\cap\Omega_f^v)},\varepsilon\|u_{\varepsilon,v}\|_{L^2(B(0,1)\cap\Omega_m^v)},[U_{b_{\varepsilon,v}}]_{C^{0,1}(B(0,1)\cap\Omega)}\right.$$
$$\left.+\left\|\frac{1}{\tilde{\varepsilon}_0}Q_{\varepsilon,v}\mathscr{X}_{\Omega_f^v}+q_{\varepsilon,v}\mathscr{X}_{\Omega_m^v},\frac{1}{\tilde{\varepsilon}_0}F_{\varepsilon,v}\mathscr{X}_{\Omega_f^v}+\frac{v}{\tilde{\varepsilon}_0}f_{\varepsilon,v}\mathscr{X}_{\Omega_m^v}\right\|_{L^{3+\delta}(B(0,1))}\right\}\leqslant 1$$

then, for all $\varepsilon\leqslant v\leqslant\tilde{\varepsilon}_0$ and $\mu=1-3/(3+\delta)$

$$\begin{cases} \displaystyle\fint_{B(0,\tilde{\theta})\cap\Omega}|\Pi_v U_{\varepsilon,v}-U_{b_{\varepsilon,v}}(0)|^2\mathrm{d}x\leqslant\tilde{\theta}^{2\mu} \\ \displaystyle\fint_{B(0,\tilde{\theta})\cap\Omega_m^v}\varepsilon^2|u_{\varepsilon,v}-U_{b_{\varepsilon,v}}(0)|^2\mathrm{d}x\leqslant\tilde{\theta}^{2\mu} \end{cases} \tag{40}$$

Proof

Consider the following problem:

$$\begin{cases} -\nabla\cdot(\check{\mathbf{K}}\nabla U)+\gamma|Y_f|U+\gamma\bar{u}=0 & \text{in } B(0,1)\cap\Omega \\ -\nabla_y\cdot(\mathbf{k}\nabla_y u)+\gamma u=0 & \text{in } (B(0,1)\cap\Omega)\times Y_m \\ U=u & \text{on } (B(0,1)\cap\Omega)\times\partial Y_m \\ U=U_b & \text{on } B(0,1)\cap\partial\Omega \end{cases} \tag{41}$$

where $\bar{u}(x)\equiv\int_{Y_m}u(x,y)\mathrm{d}y$ and $\check{\mathbf{K}}$ is the positive-definite matrix in (2). If (U,u) is a solution of (41), then by following the argument of Lemma 2.2, we have

$$\|U\|_{W^{1,r}(B(0,1/2)\cap\Omega)}\leqslant c(\|U\|_{L^2(B(0,1)\cap\Omega)}+[U_b]^2_{C^{0,1}(B(0,1)\cap\Omega)})$$

where $r>2$ and c depends on $\alpha,\check{\mathbf{K}},\mathbf{k},\omega_0$. It is possible to pick a small number $\tilde{\theta}$ (depending on $\delta,\alpha,\check{\mathbf{K}},\mathbf{k},\omega_0$) such that $\tilde{\theta}\leqslant\theta$ and

$$\fint_{B(0,\tilde{\theta})\cap\Omega}|U-U_b(0)|^2\mathrm{d}x\leqslant\tilde{\theta}^{2\mu'}\left(\fint_{B(0,1)\cap\Omega}U^2\mathrm{d}x+[U_b]^2_{C^{0,1}(B(0,1)\cap\Omega)}\right) \tag{42}$$

holds for some μ', $\mu\leqslant\mu'\leqslant 1$. Fix a value $\tilde{\theta}$, and the proof of $(40)_1$ is done by contradiction. If not, there is a sequence $\{U_{\varepsilon,v},u_{\varepsilon,v},Q_{\varepsilon,v},q_{\varepsilon,v},F_{\varepsilon,v},f_{\varepsilon,v},U_{b_{\varepsilon,v}}\}$ satisfying (39) and

$$\begin{cases} \max\{\|U_{\varepsilon,v}\|_{L^2(B(0,1)\cap\Omega_f^v)},\varepsilon\|u_{\varepsilon,v}\|_{L^2(B(0,1)\cap\Omega_m^v)},\|q_{\varepsilon,v}\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^v)}, \\ \quad [U_{b_{\varepsilon,v}}]_{C^{0,1}(B(0,1)\cap\Omega)}\}\leqslant 1 \\ \displaystyle\lim_{\varepsilon\leqslant v\to 0}\|Q_{\varepsilon,v},F_{\varepsilon,v}\|_{L^{3+\delta}(B(0,1)\cap\Omega_f^v)}+\|vf_{\varepsilon,v}\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^v)}=0 \\ \displaystyle\fint_{B(0,\tilde{\theta})\cap\Omega}|\Pi_v U_{\varepsilon,v}-U_{b_{\varepsilon,v}}(0)|^2\mathrm{d}x>\tilde{\theta}^{2\mu} \end{cases} \tag{43}$$

After extraction of subsequence (same notation for subsequence), as $\varepsilon \leqslant v \to 0$,

$$\begin{cases} \Pi_v U_{\varepsilon,v} \to U & \text{in } L^2(B(0,\tilde{\theta}) \cap \Omega) \text{ strongly} \\ u_{\varepsilon,v} \mathscr{X}_{\Omega_m^v} \to u & \text{in } L^2((B(0,\tilde{\theta}) \cap \Omega) \times Y_m) \text{ in two scale} \\ \mathbf{K}_v \mathscr{X}_{\Omega_f^v} \nabla U_{\varepsilon,v} \to \check{\mathbf{K}} \nabla U & \text{in } L^2(B(0,\tilde{\theta}) \cap \Omega) \text{ weakly} \\ U_{b_{\varepsilon,v}} \to U_b & \text{in } C^{0,1}(B(0,1) \cap \Omega) \text{ weakly star} \\ Q_{\varepsilon,v} \mathscr{X}_{\Omega_f^v}, F_{\varepsilon,v} \mathscr{X}_{\Omega_f^v}, v f_{\varepsilon,v} \mathscr{X}_{\Omega_m^v} \to 0 & \text{in } L^2(B(0,1) \cap \Omega) \text{ strongly} \end{cases} \tag{44}$$

Note (U,u) is a solution of (41). By (42)–(44), we conclude

$$\tilde{\theta}^{2\mu} \leqslant \fint_{B(0,\tilde{\theta}) \cap \Omega} |U - U_b(0)|^2 dx \leqslant \tilde{\theta}^{2\mu'} \left(\fint_{B(0,1) \cap \Omega} U^2 dx + [U_b]^2_{C^{0,1}(B(0,1) \cap \Omega)} \right) \tag{45}$$

But (45) is impossible if we take $\tilde{\theta}$ small enough. Therefore, there is a $\tilde{\varepsilon}_0$ such that $(40)_1$ holds for $\varepsilon < \tilde{\varepsilon}_0$. Clearly, $\tilde{\varepsilon}_0$ can be chosen so that $\tilde{\varepsilon}_0 < \varepsilon_0$. The proof of $(40)_2$ is similar to that of $(26)_2$. We skip it. □

Lemma 4.5

For any $\delta \in (0,3)$ and $\omega \in [0, \omega_0]$, there are constants $\tilde{\theta} \in (0,1)$ (depending on $\delta, \alpha, \omega_0, \mathbf{K}, \mathbf{k}$) and $\tilde{\varepsilon}_0 \in (0,1)$ (depending on $\theta, \omega_0, \delta, \mathbf{k}$) satisfying $\tilde{\theta} < \theta$, $\tilde{\varepsilon}_0 < \varepsilon_0$ (θ, ε_0 are those in Lemma 4.1) such that if

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\lambda (\nabla U_{\varepsilon,\lambda} + Q_{\varepsilon,\lambda})) + \left| \frac{\varepsilon}{\lambda} \right|^2 \omega U_{\varepsilon,\lambda} = F_{\varepsilon,\lambda} & \text{in } B(0,1) \cap \Omega_f^\lambda \\ -\varepsilon \nabla \cdot (\mathbf{k}_\lambda (\varepsilon \nabla u_{\varepsilon,\lambda} + q_{\varepsilon,\lambda})) + \left| \frac{\varepsilon}{\lambda} \right|^2 \omega u_{\varepsilon,\lambda} = \varepsilon f_{\varepsilon,\lambda} & \text{in } B(0,1) \cap \Omega_m^\lambda \\ \mathbf{K}_\lambda (\nabla U_{\varepsilon,\lambda} + Q_{\varepsilon,\lambda}) \cdot \vec{\mathbf{n}}^\lambda = \varepsilon \mathbf{k}_\lambda (\varepsilon \nabla u_{\varepsilon,\lambda} + q_{\varepsilon,\lambda}) \cdot \vec{\mathbf{n}}^\lambda & \text{on } B(0,1) \cap \partial \Omega_m^\lambda \\ U_{\varepsilon,\lambda} = u_{\varepsilon,\lambda} & \text{on } B(0,1) \cap \partial \Omega_m^\lambda \\ U_{\varepsilon,\lambda} = U_{b_{\varepsilon,\lambda}} & \text{on } B(0,1) \cap \partial \Omega \end{cases}$$

then, for all $\varepsilon \leqslant \lambda \leqslant \tilde{\varepsilon}_0$ and k satisfying $\lambda / \tilde{\theta}^k \leqslant \tilde{\varepsilon}_0$

$$\begin{cases} \fint_{B(0,\tilde{\theta}^k) \cap \Omega} |\Pi_\lambda U_{\varepsilon,\lambda} - U_{b_{\varepsilon,\lambda}}(0)|^2 dx \leqslant \tilde{\theta}^{2k\mu} \tilde{J}^2_{\varepsilon,\lambda} \\ \fint_{B(0,\tilde{\theta}^k) \cap \Omega_m^\lambda} \varepsilon^2 |u_{\varepsilon,\lambda} - U_{b_{\varepsilon,\lambda}}(0)|^2 dx \leqslant \tilde{\theta}^{2k\mu} \tilde{J}^2_{\varepsilon,\lambda} \end{cases} \tag{46}$$

where

$$\begin{aligned} \tilde{J}_{\varepsilon,\lambda} \equiv 2 (& \| U_{\varepsilon,\lambda} \mathscr{X}_{\Omega_f^\lambda} + \varepsilon u_{\varepsilon,\lambda} \mathscr{X}_{\Omega_m^\lambda} \|_{L^2(B(0,1))} + \| \frac{Q_{\varepsilon,\lambda}}{\tilde{\varepsilon}_0} \mathscr{X}_{\Omega_f^\lambda} + q_{\varepsilon,\lambda} \mathscr{X}_{\Omega_m^\lambda} \|_{L^{3+\delta}(B(0,1))} \\ & + \frac{1}{\tilde{\varepsilon}_0} \| F_{\varepsilon,\lambda} \mathscr{X}_{\Omega_f^\lambda} + \lambda f_{\varepsilon,\lambda} \mathscr{X}_{\Omega_m^\lambda} \|_{L^{3+\delta}(B(0,1))} + [U_{b_{\varepsilon,\lambda}}]_{C^{0,1}(B(0,1) \cap \Omega)}) \end{aligned}$$

Proof

The proof is similar to that of Lemma 4.2 and is done by induction on k. For $k=1$, Equation (46) is deduced from Lemma 4.4 with $v = \lambda$ and $\gamma = |\varepsilon / \lambda|^2 \omega$. Suppose (46) holds for some k satisfying $\lambda / \tilde{\theta}^k \leqslant \tilde{\varepsilon}_0$, then we define

$$\left. \begin{aligned} \hat{U}_\varepsilon(x) &\equiv \tilde{J}^{-1}_{\varepsilon,\lambda} \tilde{\theta}^{-k\mu} (U_{\varepsilon,\lambda}(\tilde{\theta}^k x) - U_{b_{\varepsilon,\lambda}}(0)) \\ \hat{Q}_\varepsilon(x) &\equiv \tilde{J}^{-1}_{\varepsilon,\lambda} \tilde{\theta}^{k(1-\mu)} Q_{\varepsilon,\lambda}(\tilde{\theta}^k x) \\ \hat{F}_\varepsilon(x) &\equiv \tilde{J}^{-1}_{\varepsilon,\lambda} \tilde{\theta}^{k(2-\mu)} \left(F_{\varepsilon,\lambda}(\tilde{\theta}^k x) - \left| \frac{\varepsilon}{\lambda} \right|^2 \omega U_{b_{\varepsilon,\lambda}}(0) \right) \end{aligned} \right\} \text{in } B(0,1) \cap \Omega_f^\lambda / \tilde{\theta}^k$$

$$\begin{cases} \hat{u}_\varepsilon(x) \equiv \tilde{J}_{\varepsilon,\lambda}^{-1}\tilde{\theta}^{-k\mu}(u_{\varepsilon,\lambda}(\tilde{\theta}^k x) - U_{b_{\varepsilon,\lambda}}(0)) \\ \hat{q}_\varepsilon(x) \equiv \tilde{J}_{\varepsilon,\lambda}^{-1}\tilde{\theta}^{k(1-\mu)} q_{\varepsilon,\lambda}(\tilde{\theta}^k x) & \text{in } B(0,1)\cap\Omega_m^\lambda/\tilde{\theta}^k \\ \hat{f}_\varepsilon(x) \equiv \tilde{J}_{\varepsilon,\lambda}^{-1}\tilde{\theta}^{k(2-\mu)}\left(f_{\varepsilon,\lambda}(\tilde{\theta}^k x) - \dfrac{|\frac{\varepsilon}{\lambda}|^2\omega}{\varepsilon} U_{b_{\varepsilon,\lambda}}(0)\right) \end{cases}$$

$$\hat{U}_{b_\varepsilon}(x) \equiv \tilde{J}_{\varepsilon,\lambda}^{-1}\tilde{\theta}^{-k\mu}(U_{b_{\varepsilon,\lambda}}(\tilde{\theta}^k x) - U_{b_{\varepsilon,\lambda}}(0)) \qquad \text{in } B(0,1)\cap\Omega/\tilde{\theta}^k$$

By changing scaling and employing Lemma 4.4 with $v = \lambda/\tilde{\theta}^k$ and $\gamma = \tilde{\theta}^{2k}|\varepsilon/\lambda|^2\omega$, we obtain (46) with $k+1$ in place of k. □

Lemma 4.6
Under the same assumptions as Lemma 4.5

$$[U_{\varepsilon,\lambda}]_{C^{0,\mu}(B(0,1/2)\cap\Omega_f^\lambda)} \leqslant c\hat{J}_{\varepsilon,\lambda} \tag{47}$$

where $\hat{J}_{\varepsilon,\lambda} \equiv \tilde{J}_{\varepsilon,\lambda} + \lambda\|\nabla\cdot Q_{\varepsilon,\lambda}\mathscr{X}_{\Omega_f^\lambda} + \varepsilon\nabla\cdot q_{\varepsilon,\lambda}\mathscr{X}_{\Omega_m^\lambda}\|_{L^{3+\delta}(B(0,1))}$ and constant c is independent of $\varepsilon, \lambda, \omega$.

Proof
Take k so that $\lambda/\tilde{\theta}^k \leqslant \tilde{\varepsilon}_0 < \lambda/\tilde{\theta}^{k+1}$. For any $x \in B(0,\frac{1}{2})\cap\Omega_f^\lambda$, define $\eta(x) \equiv |x - x_0|$, where $x_0 \in \partial\Omega$ satisfying $|x - x_0| = \min_{y\in\partial\Omega}|x-y|$. Then we have either case (1) $\tilde{\theta}^j/2 < \eta(x) \leqslant \tilde{\theta}^{j-1}/2$ for $1 \leqslant j \leqslant k$ or case (2) $\eta(x) \leqslant \tilde{\theta}^k/2$.

For case (1), by Lemma 4.5

$$\begin{cases} \displaystyle⨍_{B(x_0,\tilde{\theta}^{j-1})\cap\Omega} |\Pi_\lambda U_{\varepsilon,\lambda} - U_{b_{\varepsilon,\lambda}}(x_0)|^2 dy \leqslant \tilde{\theta}^{2(j-1)\mu}\tilde{J}^2_{\varepsilon,\lambda} \\ \displaystyle⨍_{B(x_0,\tilde{\theta}^{j-1})\cap\Omega_m^\lambda} \varepsilon^2|u_{\varepsilon,\lambda} - U_{b_{\varepsilon,\lambda}}(x_0)|^2 dy \leqslant \tilde{\theta}^{2(j-1)\mu}\tilde{J}^2_{\varepsilon,\lambda} \end{cases}$$

This implies

$$\begin{cases} \displaystyle⨍_{B(x,\tilde{\theta}^j/2)\cap\Omega} |\Pi_\lambda U_{\varepsilon,\lambda} - (\Pi_\lambda U_{\varepsilon,\lambda})_{x,\tilde{\theta}^j/2}|^2 dy \leqslant c\tilde{\theta}^{2j\mu}\tilde{J}^2_{\varepsilon,\lambda} \\ \displaystyle⨍_{B(x,\tilde{\theta}^j/2)\cap\Omega_m^\lambda} \varepsilon^2|u_{\varepsilon,\lambda} - (\Pi_\lambda U_{\varepsilon,\lambda})_{x,\tilde{\theta}^j/2}|^2 dy \leqslant c\tilde{\theta}^{2j\mu}\tilde{J}^2_{\varepsilon,\lambda} \end{cases} \tag{48}$$

where c is independent of $\varepsilon, \lambda, \omega$. Clearly, (48) also holds for any radius $r \geqslant \tilde{\theta}^j/2$. We shift the coordinate such that the origin is at x. For radius $r < \tilde{\theta}^j/2$, we define

$$\begin{cases} \hat{U}_\varepsilon(y) \equiv \hat{J}_{\varepsilon,\lambda}^{-1}\left|\dfrac{\tilde{\theta}^j}{2}\right|^{-\mu}\left(U_{\varepsilon,\lambda}\left(\left|\dfrac{\tilde{\theta}^j}{2}\right|y\right) - (\Pi_\lambda U_{\varepsilon,\lambda})_{x,|\tilde{\theta}^j/2|}\right) \\ \hat{Q}_\varepsilon(y) \equiv \hat{J}_{\varepsilon,\lambda}^{-1}\left|\dfrac{\tilde{\theta}^j}{2}\right|^{1-\mu} Q_{\varepsilon,\lambda}\left(\left|\dfrac{\tilde{\theta}^j}{2}\right|y\right) & \text{in } B(x,1)\cap\Omega_f^\lambda/\left|\dfrac{\tilde{\theta}^j}{2}\right| \\ \hat{F}_\varepsilon(y) \equiv \hat{J}_{\varepsilon,\lambda}^{-1}\left|\dfrac{\tilde{\theta}^j}{2}\right|^{2-\mu}\left(F_{\varepsilon,\lambda}\left(\left|\dfrac{\tilde{\theta}^j}{2}\right|y\right) - \left|\dfrac{\varepsilon}{\lambda}\right|^2\omega(\Pi_\lambda U_{\varepsilon,\lambda})_{x,|\tilde{\theta}^j/2|}\right) \end{cases}$$

$$\begin{cases} \hat{u}_\varepsilon(y) \equiv \hat{J}_{\varepsilon,\lambda}^{-1}\left|\dfrac{\tilde{\theta}^j}{2}\right|^{-\mu}\left(u_{\varepsilon,\lambda}\left(\left|\dfrac{\tilde{\theta}^j}{2}\right|y\right) - (\Pi_\lambda U_{\varepsilon,\lambda})_{x,|\tilde{\theta}^j/2|}\right) \\ \hat{q}_\varepsilon(y) \equiv \hat{J}_{\varepsilon,\lambda}^{-1}\left|\dfrac{\tilde{\theta}^j}{2}\right|^{1-\mu} q_{\varepsilon,\lambda}\left(\left|\dfrac{\tilde{\theta}^j}{2}\right|y\right) & \text{in } B(x,1)\cap\Omega_m^\lambda/\left|\dfrac{\tilde{\theta}^j}{2}\right| \\ \hat{f}_\varepsilon(y) \equiv \hat{J}_{\varepsilon,\lambda}^{-1}\left|\dfrac{\tilde{\theta}^j}{2}\right|^{2-\mu}\left(f_{\varepsilon,\lambda}\left(\left|\dfrac{\tilde{\theta}^j}{2}\right|y\right) - \dfrac{|\frac{\varepsilon}{\lambda}|^2\omega}{\varepsilon}(\Pi_\lambda U_{\varepsilon,\lambda})_{x,|\tilde{\theta}^j/2|}\right) \end{cases}$$

Then these functions satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\lambda/|\tilde\theta^j/2|}(\nabla\hat U_\varepsilon+\hat Q_\varepsilon))+\left|\frac{\tilde\theta^j}{2}\right|^2\left|\frac{\varepsilon}{\lambda}\right|^2\omega\hat U_\varepsilon=\hat F_\varepsilon & \text{in } B(x,1)\cap\Omega_f^\lambda/\left|\frac{\tilde\theta^j}{2}\right| \\ -\varepsilon\nabla\cdot(\mathbf{k}_{\lambda/|\tilde\theta^j/2|}(\varepsilon\nabla\hat u_\varepsilon+\hat q_\varepsilon))+\left|\frac{\tilde\theta^j}{2}\right|^2\left|\frac{\varepsilon}{\lambda}\right|^2\omega\hat u_\varepsilon=\varepsilon\hat f_\varepsilon & \text{in } B(x,1)\cap\Omega_m^\lambda/\left|\frac{\tilde\theta^j}{2}\right| \\ \mathbf{K}_{\lambda/|\tilde\theta^j/2|}(\nabla\hat U_\varepsilon+\hat Q_\varepsilon)\cdot\vec{\mathbf{n}}=\varepsilon\mathbf{k}_{\lambda/|\tilde\theta^j/2|}(\varepsilon\nabla\hat u_\varepsilon+\hat q_\varepsilon)\cdot\vec{\mathbf{n}} & \text{on } B(x,1)\cap\partial\Omega_m^\lambda/\left|\frac{\tilde\theta^j}{2}\right| \\ \hat U_\varepsilon=\hat u_\varepsilon & \text{on } B(x,1)\cap\partial\Omega_m^\lambda/\left|\frac{\tilde\theta^j}{2}\right| \end{cases} \tag{49}$$

where $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\Omega_m^\lambda/|\tilde\theta^j/2|$. Equation (48) implies

$$\begin{aligned} &\|\hat U_\varepsilon\mathscr{X}_{\Omega_f^\lambda/|\tilde\theta^j/2|}+\varepsilon\hat u_\varepsilon\mathscr{X}_{\Omega_m^\lambda/|\tilde\theta^j/2|}\|_{L^2(B(x,1))}+\|\frac{\hat Q_\varepsilon}{\tilde\varepsilon_0}\mathscr{X}_{\Omega_f^\lambda/|\tilde\theta^j/2|}+\hat q_\varepsilon\mathscr{X}_{\Omega_m^\lambda/|\tilde\theta^j/2|}\|_{L^{3+\delta}(B(x,1))} \\ &+\lambda 2/\tilde\theta^j\|\mathscr{X}_{\Omega_f^\lambda/|\tilde\theta^j/2|}\nabla\cdot\hat Q_\varepsilon+\varepsilon\mathscr{X}_{\Omega_m^\lambda/|\tilde\theta^j/2|}\nabla\cdot\hat q_\varepsilon\|_{L^{3+\delta}(B(x,1))} \\ &+\|\frac{\hat F_\varepsilon}{\tilde\varepsilon_0}\|_{L^{3+\delta}(B(x,1)\cap\Omega_f^\lambda/|\tilde\theta^j/2|)}+\|\frac{\lambda 2}{\tilde\varepsilon_0\tilde\theta^j}\hat f_\varepsilon\|_{L^{3+\delta}(B(x,1)\cap\Omega_m^\lambda/|\tilde\theta^j/2|)}\leqslant c \end{aligned}$$

where c is independent of $\varepsilon,\lambda,\omega$. Apply Lemma 4.3 to (49) to obtain

$$\begin{cases} \displaystyle\fint_{B(x,r)\cap\Omega}|\Pi_\lambda U_{\varepsilon,\lambda}-(\Pi_\lambda U_{\varepsilon,\lambda})_{x,r}|^2\mathrm{d}y\leqslant cr^{2\mu}\hat J_{\varepsilon,\lambda}^2 \\ \displaystyle\fint_{B(x,r)\cap\Omega_m^\lambda}\varepsilon^2|u_{\varepsilon,\lambda}-(\Pi_\lambda U_{\varepsilon,\lambda})_{x,r}|^2\mathrm{d}y\leqslant cr^{2\mu}\hat J_{\varepsilon,\lambda}^2 \end{cases} \quad \text{for } r<\tilde\theta^j/2 \tag{50}$$

Equations (48), (50), and Lemma 3.1 imply the estimate (47) for case (1).

For case (2), we observe, by Lemma 4.5,

$$\begin{cases} \displaystyle\fint_{B(x_0,r)\cap\Omega}|\Pi_\lambda U_{\varepsilon,\lambda}-U_{b_{\varepsilon,\lambda}}(x_0)|^2\mathrm{d}y\leqslant cr^{2\mu}\tilde J_{\varepsilon,\lambda}^2 \\ \displaystyle\fint_{B(x_0,r)\cap\Omega_m^\lambda}\varepsilon^2|u_{\varepsilon,\lambda}-U_{b_{\varepsilon,\lambda}}(x_0)|^2\mathrm{d}y\leqslant cr^{2\mu}\tilde J_{\varepsilon,\lambda}^2 \end{cases} \quad \text{for } r\geqslant\lambda/\tilde\varepsilon_0$$

where the constant c is independent of $\varepsilon,\lambda,\omega$. This implies

$$\begin{cases} \displaystyle\fint_{B(x,r/2)\cap\Omega}|\Pi_\lambda U_{\varepsilon,\lambda}-(\Pi_\lambda U_{\varepsilon,\lambda})_{x,r/2}|^2\mathrm{d}y\leqslant cr^{2\mu}\tilde J_{\varepsilon,\lambda}^2 \\ \displaystyle\fint_{B(x,r/2)\cap\Omega_m^\lambda}\varepsilon^2|u_{\varepsilon,\lambda}-(\Pi_\lambda U_{\varepsilon,\lambda})_{x,r/2}|^2\mathrm{d}y\leqslant cr^{2\mu}\tilde J_{\varepsilon,\lambda}^2 \end{cases} \quad \text{for } r\geqslant\lambda/\tilde\varepsilon_0 \tag{51}$$

We shift the coordinate so that the origin is at x. Take $r=2\lambda/\tilde\varepsilon_0$ in (51) and define

$$\begin{cases} \hat U_\varepsilon(y)\equiv\hat J_{\varepsilon,\lambda}^{-1}\lambda^{-\mu}(U_{\varepsilon,\lambda}(\lambda y)-(\Pi_\lambda U_{\varepsilon,\lambda})_{x,\lambda/\tilde\varepsilon_0}) \\ \hat Q_\varepsilon(y)\equiv\hat J_{\varepsilon,\lambda}^{-1}\lambda^{1-\mu}Q_{\varepsilon,\lambda}(\lambda y) \\ \hat F_\varepsilon(y)\equiv\hat J_{\varepsilon,\lambda}^{-1}\lambda^{2-\mu}\left(F_{\varepsilon,\lambda}(\lambda y)-\left|\frac{\varepsilon}{\lambda}\right|^2\omega(\Pi_\lambda U_{\varepsilon,\lambda})_{x,\lambda/\tilde\varepsilon_0}\right) \end{cases} \quad \text{in } B\left(x,\frac{1}{\tilde\varepsilon_0}\right)\cap\Omega_f^\lambda/\lambda$$

$$\begin{cases} \hat{u}_\varepsilon(y) \equiv \hat{J}^{-1}_{\varepsilon,\lambda} \lambda^{-\mu}(u_{\varepsilon,\lambda}(\lambda y) - (\Pi_\lambda U_{\varepsilon,\lambda})_{x,\lambda/\tilde{\varepsilon}_0}) \\ \hat{q}_\varepsilon(y) \equiv \hat{J}^{-1}_{\varepsilon,\lambda} \lambda^{1-\mu} q_{\varepsilon,\lambda}(\lambda y) \\ \hat{f}_\varepsilon(y) \equiv \hat{J}^{-1}_{\varepsilon,\lambda} \lambda^{2-\mu}\left(f_{\varepsilon,\lambda}(\lambda y) - \frac{|\frac{\varepsilon}{\lambda}|^2 \omega}{\varepsilon} (\Pi_\lambda U_{\varepsilon,\lambda})_{x,\lambda/\tilde{\varepsilon}_0}\right) \end{cases} \quad \text{in } B\left(x, \frac{1}{\tilde{\varepsilon}_0}\right) \cap \Omega^\lambda_m/\lambda$$

$$\hat{U}_{b_\varepsilon} \equiv \hat{J}^{-1}_{\varepsilon,\lambda} \lambda^{-\mu}(U_{b_{\varepsilon,\lambda}}(\lambda y) - (\Pi_\lambda U_{\varepsilon,\lambda})_{x,\lambda/\tilde{\varepsilon}_0}) \quad \text{in } B(x, \frac{1}{\tilde{\varepsilon}_0}) \cap \Omega/\lambda$$

Then these functions satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\lambda/\lambda}(\nabla \hat{U}_\varepsilon + \hat{Q}_\varepsilon)) + \varepsilon^2 \omega \hat{U}_\varepsilon = \hat{F}_\varepsilon & \text{in } B\left(x, \frac{1}{\tilde{\varepsilon}_0}\right) \cap \Omega^\lambda_f/\lambda \\ -\varepsilon\nabla\cdot(\mathbf{k}_{\lambda/\lambda}(\varepsilon\nabla \hat{u}_\varepsilon + \hat{q}_\varepsilon)) + \varepsilon^2 \omega \hat{u}_\varepsilon = \varepsilon \hat{f}_\varepsilon & \text{in } B\left(x, \frac{1}{\tilde{\varepsilon}_0}\right) \cap \Omega^\lambda_m/\lambda \\ \mathbf{K}_{\lambda/\lambda}(\nabla \hat{U}_\varepsilon + \hat{Q}_\varepsilon)\cdot\vec{\mathbf{n}} = \varepsilon \mathbf{k}_{\lambda/\lambda}(\varepsilon\nabla \hat{u}_\varepsilon + \hat{q}_\varepsilon)\cdot\vec{\mathbf{n}} & \text{on} B\left(x, \frac{1}{\tilde{\varepsilon}_0}\right) \cap \partial\Omega^\lambda_m/\lambda \\ \hat{U}_\varepsilon = \hat{u}_\varepsilon & \text{on } B\left(x, \frac{1}{\tilde{\varepsilon}_0}\right) \cap \partial\Omega^\lambda_m/\lambda \\ \hat{U}_\varepsilon = \hat{U}_{b_\varepsilon} & \text{on } B\left(x, \frac{1}{\tilde{\varepsilon}_0}\right) \cap \partial\Omega/\lambda \end{cases}$$

where $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\Omega^\lambda_m/\lambda$, and

$$\begin{aligned} &\|\hat{U}_\varepsilon \mathscr{X}_{\Omega^\lambda_f/\lambda} + \varepsilon \hat{u}_\varepsilon \mathscr{X}_{\Omega^\lambda_m/\lambda}\|_{L^2(B(x,1/\tilde{\varepsilon}_0))} + \|\hat{Q}_\varepsilon \mathscr{X}_{\Omega^\lambda_f/\lambda} + \hat{q}_\varepsilon \mathscr{X}_{\Omega^\lambda_m/\lambda}\|_{L^{3+\delta}(B(x,1/\tilde{\varepsilon}_0))} \\ &+\|\nabla\cdot\hat{Q}_\varepsilon \mathscr{X}_{\Omega^\lambda_f/\lambda} + \varepsilon\nabla\cdot\hat{q}_\varepsilon \mathscr{X}_{\Omega^\lambda_m/\lambda}\|_{L^{3+\delta}(B(x,1/\tilde{\varepsilon}_0))} \\ &+\|\hat{F}_\varepsilon \mathscr{X}_{\Omega^\lambda_f/\lambda} + \hat{f}_\varepsilon \mathscr{X}_{\Omega^\lambda_m/\lambda}\|_{L^{3+\delta}(B(x,1/\tilde{\varepsilon}_0))} + [\hat{U}_{b_\varepsilon}]_{C^{0,1}(B(x,1/\tilde{\varepsilon}_0)\cap\Omega/\lambda)} \leqslant c \end{aligned}$$

where c is independent of $\varepsilon, \lambda, \omega$. Lemma 8.29 [9] and Lemma 3.4 imply $[\hat{U}_\varepsilon]_{C^{0,\mu}(B(x,1/2\tilde{\varepsilon}_0)\cap\Omega^\lambda_f/\lambda)} \leqslant c$. Combining with (51) and Lemma 3.1, we obtain the local Hölder estimate (47) for case (2). □

By energy method, partition of unity, Lemma 4.3, and Lemma 4.6, we conclude:

Corollary 4.1
Under A1–A2, the solution of

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\lambda(\nabla U_{\varepsilon,\lambda} + Q_{\varepsilon,\lambda})) + \left|\frac{\varepsilon}{\lambda}\right|^2 \omega U_{\varepsilon,\lambda} = F_{\varepsilon,\lambda} & \text{in } \Omega^\lambda_f \\ -\varepsilon\nabla\cdot(\mathbf{k}_\lambda(\varepsilon\nabla u_{\varepsilon,\lambda} + q_{\varepsilon,\lambda})) + \left|\frac{\varepsilon}{\lambda}\right|^2 \omega u_{\varepsilon,\lambda} = \varepsilon f_{\varepsilon,\lambda} & \text{in } \Omega^\lambda_m \\ \mathbf{K}_\lambda(\nabla U_{\varepsilon,\lambda} + Q_{\varepsilon,\lambda})\cdot\vec{\mathbf{n}}^\lambda = \varepsilon \mathbf{k}_\lambda(\varepsilon\nabla u_{\varepsilon,\lambda} + q_{\varepsilon,\lambda})\cdot\vec{\mathbf{n}}^\lambda & \text{on } \partial\Omega^\lambda_m \\ U_{\varepsilon,\lambda} = u_{\varepsilon,\lambda} & \text{on } \partial\Omega^\lambda_m \\ U_{\varepsilon,\lambda} = U_{b_{\varepsilon,\lambda}} & \text{on } \partial\Omega \end{cases}$$

satisfies

$$[U_{\varepsilon,\lambda}]_{C^{0,\mu}(\Omega^\lambda_f)} \leqslant c(\|Q_{\varepsilon,\lambda}, \lambda\nabla\cdot Q_{\varepsilon,\lambda}, F_{\varepsilon,\lambda}\|_{L^{3+\delta}(\Omega^\lambda_f)} + \|q_{\varepsilon,\lambda}, \lambda\varepsilon\nabla\cdot q_{\varepsilon,\lambda}, \lambda f_{\varepsilon,\lambda}\|_{L^{3+\delta}(\Omega^\lambda_m)} + [U_{b_{\varepsilon,\lambda}}]_{C^{0,1}(\Omega)})$$

where $\mu \equiv 1 - 3/(3+\delta), \delta > 0, \omega \geqslant 0, \lambda \geqslant \varepsilon > 0$, and c is independent of ε, λ.

Theorem 2.1 is a direct consequence of Corollary 4.1.

5. Uniform Hölder gradient estimate

In this section, we prove Theorem 2.2. The Hölder gradient estimate in the interior region is derived in Section 5.1, and the estimate around the boundary region is in Section 5.2. Define the left and the right limits, ξ_- and ξ_+, on ∂Y_m as $\xi_-(x) \equiv \lim_{\substack{x' \to 0 \\ x+x' \in Y_m}} \xi(x+x')$

and $\xi_+(x) \equiv \lim_{\substack{x' \to 0 \\ x+x' \in Y_f}} \xi(x+x')$ for $x \in \partial Y_m$. For each $i=1,2,3$, we define a periodic function $\mathbb{X}_\varepsilon^{*,i}$ in R^3 with period Y as the solution of the following problem: In each cell Y, function $\mathbb{X}_\varepsilon^{*,i}$ satisfies

$$\begin{cases} -\nabla \cdot (\mathbf{K}\nabla(\mathbb{X}_\varepsilon^{*,i}+y_i))=0 & \text{in } Y_f \\ -\varepsilon^2\nabla \cdot (\mathbf{k}\nabla(\mathbb{X}_\varepsilon^{*,i}+y_i))=0 & \text{in } Y_m \\ \mathbf{K}\nabla(\mathbb{X}_\varepsilon^{*,i}+y_i)\cdot \vec{\mathbf{n}}_y=\varepsilon^2\mathbf{k}\nabla(\mathbb{X}_\varepsilon^{*,i}+y_i)\cdot \vec{\mathbf{n}}_y & \text{on } \partial Y_m \\ \mathbb{X}_{\varepsilon,+}^{*,i}=\mathbb{X}_{\varepsilon,-}^{*,i} & \text{on } \partial Y_m \\ \int_Y \mathbb{X}_\varepsilon^{*,i} dy=0 & \end{cases} \tag{52}$$

where $\vec{\mathbf{n}}_y$ denotes the unit outward normal vector on ∂Y_m. Clearly, $\mathbb{X}_\varepsilon^{*,i}$ is solvable [2]. We define $\mathbb{X}_{\varepsilon,v}^{(i)}(x) \equiv v\mathbb{X}_\varepsilon^{*,i}(x/v)$ in Ω, $\mathbb{X} \equiv (\mathbb{X}_\varepsilon^{*,i})$, and $\mathbb{X}_{\varepsilon,v} \equiv (\mathbb{X}_{\varepsilon,v}^{(i)})$.

5.1. Interior gradient estimate

We assume $\overline{B(0,1)} \subset \Omega$.

Lemma 5.1

For any $\delta>0$, there are constants $\theta \in (0,1)$ (depending on $\delta, \mathbf{K}$) and $\varepsilon_0 \in (0,1)$ (depending on $\theta, \delta, \mathbf{k}$) such that if $U_{\varepsilon,v}, u_{\varepsilon,v}, F_{\varepsilon,v}, f_{\varepsilon,v}$ satisfy

$$\begin{cases} -\nabla \cdot (\mathbf{K}_v\nabla U_{\varepsilon,v})=F_{\varepsilon,v} & \text{in } B(0,1)\cap\Omega_f^v \\ -\varepsilon^2\nabla \cdot (\mathbf{k}_v\nabla u_{\varepsilon,v})=\varepsilon f_{\varepsilon,v} & \text{in } B(0,1)\cap\Omega_m^v \\ \mathbf{K}_v\nabla U_{\varepsilon,v}\cdot\vec{\mathbf{n}}^v=\varepsilon^2\mathbf{k}_v\nabla u_{\varepsilon,v}\cdot\vec{\mathbf{n}}^v & \text{on } B(0,1)\cap\partial\Omega_m^v \\ U_{\varepsilon,v}=u_{\varepsilon,v} & \text{on } B(0,1)\cap\partial\Omega_m^v \end{cases} \tag{53}$$

and if

$$\max\{\|U_{\varepsilon,v}\|_{L^\infty(B(0,1)\cap\Omega_f^v)}, \varepsilon\|u_{\varepsilon,v}\|_{L^2(B(0,1)\cap\Omega_m^v)}, \varepsilon_0^{-1}\|F_{\varepsilon,v}\mathscr{X}_{\Omega_f^v}+vf_{\varepsilon,v}\mathscr{X}_{\Omega_m^v}\|_{L^{3+\delta}(B(0,1))}\}\leqslant 1 \tag{54}$$

then, for any $\varepsilon\leqslant v\leqslant\varepsilon_0$ and $\mu=1-3/(3+\delta)$

$$\begin{cases} \sup_{B(0,\theta)} |\Pi_v U_{\varepsilon,v}(x)-\Pi_v U_{\varepsilon,v}(0)-(x+\Pi_v\mathbb{X}_{\varepsilon,v}(x))\mathbf{b}_{\varepsilon,v}|\leqslant\theta^{1+\mu/2} \\ \displaystyle\int_{B(0,\theta)\cap\Omega_m^v} \varepsilon^2|u_{\varepsilon,v}(x)-\Pi_v U_{\varepsilon,v}(0)-(x+\mathbb{X}_{\varepsilon,v}(x))\mathbf{b}_{\varepsilon,v}|^2 dx\leqslant\theta^{2(1+\mu/2)} \end{cases} \tag{55}$$

where $\mathbf{b}_{\varepsilon,v} \equiv (\check{\mathbf{K}}^{-1}/|B(0,\theta)|)\int_{B(0,\theta)\cap\Omega_f^v}\mathbf{K}_v\nabla U_{\varepsilon,v}dx$ and $\check{\mathbf{K}}$ is the one in (2).

Proof

If U is a solution of $\mathscr{L}U\equiv-\nabla\cdot(\check{\mathbf{K}}\nabla U)=0$ in $B(0,1)$, then

$$\|U\|_{C^{2,\beta}(B(0,1/2))}\leqslant c\|U\|_{L^\infty(B(0,1))}$$

where $\beta\in(0,1)$ and c depends on $\check{\mathbf{K}}$. If μ' satisfies $\mu<\mu'<1$, then by [9]

$$\sup_{|x|<\theta}|U(x)-U(0)-x(\nabla U)_{0,\theta}|\leqslant\theta^{1+\mu'/2}\|U\|_{L^\infty(B(0,1))} \tag{56}$$

for θ (depending on $\delta, \check{\mathbf{K}}$) sufficiently small. Fix a value θ and we prove $(55)_1$ by contradiction. If not, there is a sequence $\{U_{\varepsilon,v}, u_{\varepsilon,v}, F_{\varepsilon,v}, f_{\varepsilon,v}\}$ satisfying (53) and

$$\begin{cases} \max\{\|U_{\varepsilon,v}\|_{L^\infty(B(0,1)\cap\Omega_f^v)}, \varepsilon\|u_{\varepsilon,v}\|_{L^2(B(0,1)\cap\Omega_m^v)}\}\leqslant 1 \\ \lim_{\varepsilon\leqslant v\to 0}\|F_{\varepsilon,v}\|_{L^{3+\delta}(B(0,1)\cap\Omega_f^v)}+\|vf_{\varepsilon,v}\|_{L^{3+\delta}(B(0,1)\cap\Omega_m^v)}=0 \\ \sup_{B(0,\theta)}|\Pi_v U_{\varepsilon,v}(x)-\Pi_v U_{\varepsilon,v}(0)-(x+\Pi_v\mathbb{X}_{\varepsilon,v}(x))\mathbf{b}_{\varepsilon,v}|>\theta^{1+\mu/2} \end{cases} \tag{57}$$

After extraction of a subsequence (same notation for subsequence), we have, by Lemma 4.3

$$\begin{cases} \Pi_\nu U_{\varepsilon,\nu} \to U & \text{in } L^\infty(B(0,\theta)) \text{ strongly} \\ \mathbf{K}_\nu \mathscr{X}_{\Omega_f^\nu} \nabla U_{\varepsilon,\nu} \to \check{\mathbf{K}} \nabla U & \text{in } L^2(B(0,\theta)) \text{ weakly} \quad \text{as } \varepsilon \leqslant \nu \to 0 \\ F_{\varepsilon,\nu} \mathscr{X}_{\Omega_f^\nu}, \nu f_{\varepsilon,\nu} \mathscr{X}_{\Omega_m^\nu} \to 0 & \text{in } L^2(B(0,1)) \text{ strongly} \end{cases} \tag{58}$$

Note U is a solution of $\mathscr{L}U = 0$. Together with (56), $(57)_3$, and (58), we get contradiction. Thus $(55)_1$ holds. Define

$$\begin{cases} \hat{U}_\varepsilon(x) \equiv \theta^{-(1+\mu/2)} (\Pi_\nu U_{\varepsilon,\nu}(x) - \Pi_\nu U_{\varepsilon,\nu}(0) - (x + \Pi_\nu \mathbb{X}_{\varepsilon,\nu}(x)) \mathbf{b}_{\varepsilon,\nu}) \\ \hat{u}_\varepsilon(x) \equiv \theta^{-(1+\mu/2)} (u_{\varepsilon,\nu}(x) - \Pi_\nu U_{\varepsilon,\nu}(0) - (x + \mathbb{X}_{\varepsilon,\nu}(x)) \mathbf{b}_{\varepsilon,\nu}) \end{cases} \quad \text{in } \Omega_m^\nu$$

Then $\hat{u}_\varepsilon(x)$ satisfies, in $\nu(Y_m + j) \subset B(0,\theta) \cap \Omega_m^\nu$ for some $j \in Z^3$

$$\begin{cases} -\varepsilon^2 \nabla \cdot (\mathbf{k}_\nu \nabla \hat{u}_\varepsilon) = \theta^{-(1+\mu/2)} \varepsilon f_{\varepsilon,\nu} & \text{in } \nu(Y_m + j) \\ \hat{u}_\varepsilon = \hat{U}_\varepsilon & \text{on } \nu(\partial Y_m + j) \end{cases}$$

An analogous argument as that in Lemma 4.1 implies $\varepsilon \|\hat{u}_\varepsilon\|_{L^2(B(0,\theta)\cap\Omega_m^\nu)} \leqslant 1$. Thus $(55)_2$ holds in Ω_m^ν. □

Lemma 5.2
For any $\delta > 0$, there are constants $\theta \in (0,1)$ (depending on $\delta, \mathbf{K}$) and $\varepsilon_0 \in (0,1)$ (depending on $\theta, \delta, \mathbf{k}$) such that if

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\lambda \nabla U_{\varepsilon,\lambda}) = F_{\varepsilon,\lambda} & \text{in } B(0,1) \cap \Omega_f^\lambda \\ -\varepsilon^2 \nabla \cdot (\mathbf{k}_\lambda \nabla u_{\varepsilon,\lambda}) = \varepsilon f_{\varepsilon,\lambda} & \text{in } B(0,1) \cap \Omega_m^\lambda \\ \mathbf{K}_\lambda \nabla U_{\varepsilon,\lambda} \cdot \vec{\mathbf{n}}^\lambda = \varepsilon^2 \mathbf{k}_\nu \nabla u_{\varepsilon,\lambda} \cdot \vec{\mathbf{n}}^\lambda & \text{on } B(0,1) \cap \partial\Omega_m^\lambda \\ U_{\varepsilon,\lambda} = u_{\varepsilon,\lambda} & \text{on } B(0,1) \cap \partial\Omega_m^\lambda \end{cases} \tag{59}$$

then, for any $\varepsilon \leqslant \lambda \leqslant \varepsilon_0$ and k satisfying $\lambda/\theta^k \leqslant \varepsilon_0$, there are constants $\mathbf{a}_k^{\varepsilon,\lambda}$, $\mathbf{b}_k^{\varepsilon,\lambda}$ so that the solutions of (59) satisfy

$$\begin{cases} |\mathbf{a}_k^{\varepsilon,\lambda}| + |\mathbf{b}_k^{\varepsilon,\lambda}| \leqslant c J_{\varepsilon,\lambda} \\ \sup\limits_{B(0,\theta^k)} |\Pi_\lambda U_{\varepsilon,\lambda} - \Pi_\lambda U_{\varepsilon,\lambda}(0) - \lambda \mathbf{a}_k^{\varepsilon,\lambda} - (x + \Pi_\lambda \mathbb{X}_{\varepsilon,\lambda}(x)) \mathbf{b}_k^{\varepsilon,\lambda}| \leqslant \theta^{k(1+\mu/2)} J_{\varepsilon,\lambda} \\ \displaystyle\fint_{B(0,\theta^k)\cap\Omega_m^\lambda} \varepsilon^2 |u_{\varepsilon,\lambda} - \Pi_\lambda U_{\varepsilon,\lambda}(0) - \lambda \mathbf{a}_k^{\varepsilon,\lambda} - (x + \mathbb{X}_{\varepsilon,\lambda}(x)) \mathbf{b}_k^{\varepsilon,\lambda}|^2 \mathrm{d}x \leqslant \theta^{2k(1+\mu/2)} J_{\varepsilon,\lambda}^2 \end{cases} \tag{60}$$

where $\mu \equiv 1 - 3/(3+\delta)$, c is independent of ε, λ, and

$$J_{\varepsilon,\lambda} \equiv \|U_{\varepsilon,\lambda}\|_{L^\infty(B(0,1)\cap\Omega_f^\lambda)} + \varepsilon \|u_{\varepsilon,\lambda}\|_{L^2(B(0,1)\cap\Omega_m^\lambda)} + \|\varepsilon_0^{-1} F_{\varepsilon,\lambda} \mathscr{X}_{\Omega_f^\lambda} + f_{\varepsilon,\lambda} \mathscr{X}_{\Omega_m^\lambda}\|_{L^{3+\delta}(B(0,1))}$$

Proof
This is done by induction. Define

$$\hat{U}_\varepsilon \equiv \frac{U_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}, \quad \hat{u}_\varepsilon \equiv \frac{u_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}, \quad \hat{F}_\varepsilon \equiv \frac{F_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}, \quad \hat{f}_\varepsilon \equiv \frac{f_{\varepsilon,\lambda}}{J_{\varepsilon,\lambda}}$$

Then these functions satisfy (53) and (54) with $\nu = \lambda$. By Lemma 5.1, we obtain (60) for $k = 1$ case, where

$$\mathbf{a}_1^{\varepsilon,\lambda} = 0, \quad \mathbf{b}_1^{\varepsilon,\lambda} = \frac{\check{\mathbf{K}}^{-1}}{|B(0,\theta)|} \int_{B(0,\theta)\cap\Omega_f^\lambda} \mathbf{K}_\lambda \nabla U_{\varepsilon,\lambda} \mathrm{d}x$$

If (60) holds for some k satisfying $\lambda/\theta^k \leqslant \varepsilon_0$, then we define, in $B(0,1) \cap \Omega_f^\lambda/\theta^k$

$$\begin{cases} \hat{U}_\varepsilon(x) \equiv \dfrac{\theta^{-k(1+\mu/2)}}{J_{\varepsilon,\lambda}} (U_{\varepsilon,\lambda}(\theta^k x) - \Pi_\lambda U_{\varepsilon,\lambda}(0) - \lambda \mathbf{a}_k^{\varepsilon,\lambda} - (\theta^k x + \mathbb{X}_{\varepsilon,\lambda}(\theta^k x)) \mathbf{b}_k^{\varepsilon,\lambda}) \\ \hat{F}_\varepsilon(x) \equiv \dfrac{\theta^{k(1-\mu/2)}}{J_{\varepsilon,\lambda}} F_{\varepsilon,\lambda}(\theta^k x) \end{cases}$$

and, in $B(0,1)\cap\Omega_m^\lambda/\theta^k$

$$\begin{cases} \hat{u}_\varepsilon(x)\equiv\dfrac{\theta^{-k(1+\mu/2)}}{J_{\varepsilon,\lambda}}(u_{\varepsilon,\lambda}(\theta^k x)-\Pi_\lambda U_{\varepsilon,\lambda}(0)-\lambda\mathbf{a}_k^{\varepsilon,\lambda}-(\theta^k x+\mathbb{X}_{\varepsilon,\lambda}(\theta^k x))\mathbf{b}_k^{\varepsilon,\lambda}) \\ \hat{f}_\varepsilon(x)\equiv\dfrac{\theta^{k(1-\mu/2)}}{J_{\varepsilon,\lambda}}f_{\varepsilon,\lambda}(\theta^k x) \end{cases}$$

They satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\lambda/\theta^k}\nabla\hat{U}_\varepsilon)=\hat{F}_\varepsilon & \text{in } B(0,1)\cap\Omega_f^\lambda/\theta^k \\ -\varepsilon^2\nabla\cdot(\mathbf{k}_{\lambda/\theta^k}\nabla\hat{u}_\varepsilon)=\varepsilon\hat{f}_\varepsilon & \text{in } B(0,1)\cap\Omega_m^\lambda/\theta^k \\ \mathbf{K}_{\lambda/\theta^k}\nabla\hat{U}_\varepsilon\cdot\vec{\mathbf{n}}=\varepsilon^2\mathbf{k}_{\lambda/\theta^k}\nabla\hat{u}_\varepsilon\cdot\vec{\mathbf{n}} & \text{on } B(0,1)\cap\partial\Omega_m^\lambda/\theta^k \\ \hat{U}_\varepsilon=\hat{u}_\varepsilon & \text{on } B(0,1)\cap\partial\Omega_m^\lambda/\theta^k \end{cases}$$

where $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\Omega_m^\lambda/\theta^k$. By induction,

$$\begin{cases} \max\{\|\hat{U}_\varepsilon\|_{L^\infty(B(0,1)\cap\Omega_f^\lambda/\theta^k)},\varepsilon\|\hat{u}_\varepsilon\|_{L^2(B(0,1)\cap\Omega_m^\lambda/\theta^k)}\}\leqslant 1 \\ \|\varepsilon_0^{-1}\hat{F}_\varepsilon\mathscr{X}_{\Omega_f^\lambda/\theta^k}+\hat{f}_\varepsilon\mathscr{X}_{\Omega_m^\lambda/\theta^k}\|_{L^{3+\delta}(B(0,1))}\leqslant 1 \end{cases} \tag{61}$$

Apply Lemma 5.1 (take $v=\lambda/\theta^k$) to obtain

$$\begin{cases} \sup_{B(0,\theta)}|\Pi_{\lambda/\theta^k}\hat{U}_\varepsilon-\Pi_{\lambda/\theta^k}\hat{U}_\varepsilon(0)-(x+\Pi_{\lambda/\theta^k}\mathbb{X}_{\varepsilon,\lambda/\theta^k}(x))\hat{\mathbf{b}}_k^{\varepsilon,\lambda}|\leqslant\theta^{1+\mu/2} \\ \displaystyle⨍_{B(0,\theta)\cap\Omega_m^\lambda/\theta^k}\varepsilon^2|\hat{u}_\varepsilon-\Pi_{\lambda/\theta^k}\hat{U}_\varepsilon(0)-(x+\mathbb{X}_{\varepsilon,\lambda/\theta^k}(x))\hat{\mathbf{b}}_k^{\varepsilon,\lambda}|^2\mathrm{d}x\leqslant\theta^{2(1+\mu/2)} \end{cases} \tag{62}$$

where

$$\hat{\mathbf{b}}_k^{\varepsilon,\lambda}\equiv\frac{\check{\mathbf{K}}^{-1}}{|B(0,\theta)|}\int_{B(0,\theta)\cap\Omega_f^\lambda/\theta^k}\mathbf{K}_{\lambda/\theta^k}\nabla\hat{U}_\varepsilon\mathrm{d}x$$

By Lemma 2.1, $(62)_1$ can be written as

$$\begin{aligned} \sup_{B(0,\theta)}|&\Pi_\lambda U_{\varepsilon,\lambda}(\theta^k x)-\Pi_\lambda U_{\varepsilon,\lambda}(0)+\lambda\Pi_1\mathbb{X}(0)\mathbf{b}_k^{\varepsilon,\lambda}-(\theta^k x+\Pi_\lambda\mathbb{X}_{\varepsilon,\lambda}(\theta^k x))\mathbf{b}_k^{\varepsilon,\lambda} \\ &-J_{\varepsilon,\lambda}\theta^{k(1+\mu/2)}(x+\theta^{-k}\Pi_\lambda\mathbb{X}_{\varepsilon,\lambda}(\theta^k x))\hat{\mathbf{b}}_k^{\varepsilon,\lambda}|\leqslant J_{\varepsilon,\lambda}\theta^{(k+1)(1+\mu/2)} \end{aligned} \tag{63}$$

Define

$$\mathbf{a}_{k+1}^{\varepsilon,\lambda}\equiv-\Pi_1\mathbb{X}(0)\mathbf{b}_k^{\varepsilon,\lambda}\quad\text{and}\quad\mathbf{b}_{k+1}^{\varepsilon,\lambda}\equiv\mathbf{b}_k^{\varepsilon,\lambda}+J_{\varepsilon,\lambda}\theta^{k\mu/2}\hat{\mathbf{b}}_k^{\varepsilon,\lambda} \tag{64}$$

By (61) and (62), $|\hat{\mathbf{b}}_k^{\varepsilon,\lambda}|$ are bounded uniformly in ε,λ,k. Thus $(60)_1$ holds. Substituting (64) into (63) and making the change of variables $\theta^k x\to x$, we obtain $(60)_2$. $(60)_3$ is derived in a similar way as $(60)_2$. □

Lemma 5.3
Under the assumptions of Lemma 5.2

$$\|\nabla U_{\varepsilon,\lambda}\|_{L^\infty(B(0,1/2)\cap\Omega_f^\lambda)}\leqslant cJ_{\varepsilon,\lambda} \tag{65}$$

where constant c is independent of ε.

Proof
Let $k\in N$ such that $\lambda/\theta^k\leqslant\varepsilon_0<\lambda/\theta^{k+1}$. By Lemma 5.2

$$\begin{cases} \sup_{B(0,\lambda/\varepsilon_0)}|\Pi_\lambda U_{\varepsilon,\lambda}-\Pi_\lambda U_{\varepsilon,\lambda}(0)-\lambda\mathbf{a}_k^{\varepsilon,\lambda}-(x+\Pi_\lambda\mathbb{X}_{\varepsilon,\lambda}(x))\mathbf{b}_k^{\varepsilon,\lambda}|\leqslant c\left|\dfrac{\lambda}{\varepsilon_0}\right|^{1+\mu/2}J_{\varepsilon,\lambda} \\ \displaystyle⨍_{B(0,\lambda/\varepsilon_0)\cap\Omega_m^\lambda}\varepsilon^2|u_{\varepsilon,\lambda}-\Pi_\lambda U_{\varepsilon,\lambda}(0)-\lambda\mathbf{a}_k^{\varepsilon,\lambda}-(x+\mathbb{X}_{\varepsilon,\lambda}(x))\mathbf{b}_k^{\varepsilon,\lambda}|^2\mathrm{d}x\leqslant c|\frac{\lambda}{\varepsilon_0}|^{2(1+\mu/2)}J_{\varepsilon,\lambda}^2 \end{cases} \tag{66}$$

Define, in $B(0,1/\varepsilon_0)\cap\Omega_f^\lambda/\lambda$

$$\begin{cases} \hat{U}_\varepsilon(x)\equiv J_{\varepsilon,\lambda}^{-1}\lambda^{-(1+\mu/2)}(U_{\varepsilon,\lambda}(\lambda x)-\Pi_\lambda U_{\varepsilon,\lambda}(0)-\lambda\mathbf{a}_k^{\varepsilon,\lambda}-(\lambda x+\mathbb{X}_{\varepsilon,\lambda}(\lambda x))\mathbf{b}_k^{\varepsilon,\lambda}) \\ \hat{F}_\varepsilon(x)\equiv J_{\varepsilon,\lambda}^{-1}\lambda^{1-\mu/2}F_{\varepsilon,\lambda}(\lambda x) \end{cases}$$

and, in $B(0,1/\varepsilon_0)\cap\Omega_m^\lambda/\lambda$

$$\begin{cases} \hat{u}_\varepsilon(x)\equiv J_{\varepsilon,\lambda}^{-1}\lambda^{-(1+\mu/2)}(u_{\varepsilon,\lambda}(\lambda x)-\Pi_\lambda U_{\varepsilon,\lambda}(0)-\lambda\mathbf{a}_k^{\varepsilon,\lambda}-(\lambda x+\mathbb{X}_{\varepsilon,\lambda}(\lambda x))\mathbf{b}_k^{\varepsilon,\lambda}) \\ \hat{f}_\varepsilon(x)\equiv J_{\varepsilon,\lambda}^{-1}\lambda^{1-\mu/2}f_{\varepsilon,\lambda}(\lambda x) \end{cases}$$

Then these functions satisfy (53) in domain $B(0,1/\varepsilon_0)\cap(\Omega_f^\lambda\cup\Omega_m^\lambda)/\lambda$ and, by (66)

$$\|\hat{U}_\varepsilon\|_{L^\infty(B(0,1/\varepsilon_0)\cap\Omega_f^\lambda/\lambda)}+\varepsilon\|\hat{u}_\varepsilon\|_{L^2(B(0,1/\varepsilon_0)\cap\Omega_m^\lambda/\lambda)}+\|\hat{F}_\varepsilon\mathscr{X}_{\Omega_f^\lambda/\lambda}+\hat{f}_\varepsilon\mathscr{X}_{\Omega_m^\lambda/\lambda}\|_{L^{3+\delta}(B(0,1/\varepsilon_0))}\leqslant c$$

where c is independent of ε. Lemma 3.3 implies

$$\|\hat{U}_\varepsilon\|_{C^{1,\mu}(B(0,1/2\varepsilon_0)\cap\Omega_f^\lambda/\lambda)}\leqslant c \tag{67}$$

Since $\nabla\hat{U}_\varepsilon(x)=\frac{\nabla U_{\varepsilon,\lambda}(\lambda x)-(I+\nabla\mathbb{X}(x))\mathbf{b}_k^{\varepsilon,\lambda}}{\lambda^{\mu/2}J_{\varepsilon,\lambda}}$, we obtain $|\nabla U_{\varepsilon,\lambda}(\lambda x)|\leqslant cJ_{\varepsilon,\lambda}$ for $x\in B(0,\frac{1}{2\varepsilon_0})\cap\Omega_f^\lambda/\lambda$ by (67) and Lemma 5.2. We conclude (65). □

5.2. Boundary gradient estimate

As in Section 4.2, we assume (37)–(38). Let ρ be a smooth non-negative function satisfying $\rho\in C_0^\infty(R^3)$, $\rho=1$ on $\{|x|<1\}$, and $\rho=0$ on $|x|>\frac{3}{2}$. For each $i=1,2,3$ and $s>0$, we consider the following problem: Find $\mathbb{W}_{\varepsilon,\varepsilon/s}^{(i)}\in H^1(B(0,2)\cap\Omega/s)$ such that

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\varepsilon/s}\nabla(\mathbb{W}_{\varepsilon,\varepsilon/s}^{(i)}(x)+x_i))=0 & \text{in } B(0,2)\cap\Omega_f^\varepsilon/s \\ -\varepsilon^2\nabla\cdot(\mathbf{k}_{\varepsilon/s}\nabla(\mathbb{W}_{\varepsilon,\varepsilon/s}^{(i)}(x)+x_i))=0 & \text{in } B(0,2)\cap\Omega_m^\varepsilon/s \\ \mathbf{K}_{\varepsilon/s}\nabla(\mathbb{W}_{\varepsilon,\varepsilon/s}^{(i)}+x_i)\cdot\vec{\mathbf{n}}=\varepsilon^2\mathbf{k}_{\varepsilon/s}\nabla(\mathbb{W}_{\varepsilon,\varepsilon/s}^{(i)}+x_i)\cdot\vec{\mathbf{n}} & \text{on } B(0,2)\cap\partial\Omega_m^\varepsilon/s \\ \mathbb{W}_{\varepsilon,\varepsilon/s,+}^{(i)}=\mathbb{W}_{\varepsilon,\varepsilon/s,-}^{(i)} & \text{on } B(0,2)\cap\partial\Omega_m^\varepsilon/s \\ \mathbb{W}_{\varepsilon,\varepsilon/s}^{(i)}(x)=(1-\rho(x))\mathbb{X}_{\varepsilon,\varepsilon/s}^{(i)} & \text{on } \begin{cases} B(0,2)\cap\partial\Omega/s \\ \partial B(0,2)\cap\Omega/s \end{cases} \end{cases} \tag{68}$$

where $\mathbb{X}_{\varepsilon,\varepsilon/s}^{(i)}$ is defined at the beginning of Section 5, $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\Omega_m^\varepsilon/s$, and both $\mathbb{W}_{\varepsilon,\varepsilon/s,+}^{(i)}$, $\mathbb{W}_{\varepsilon,\varepsilon/s,-}^{(i)}$ are defined at the beginning of Section 5.

Lemma 5.4
Solution of (68) exists and $\|\mathbb{W}_{\varepsilon,\varepsilon/s}^{(i)}\|_{L^\infty(B(0,2)\cap\Omega/s)}\leqslant c\varepsilon/s$, where c is independent of ε.

Proof
Existence of the solution of (68) is clear. Define $\mathbb{Z}_{\varepsilon,\varepsilon/s}^{(i)}\equiv\mathbb{W}_{\varepsilon,\varepsilon/s}^{(i)}-\mathbb{X}_{\varepsilon,\varepsilon/s}^{(i)}$. Then $\mathbb{Z}_{\varepsilon,\varepsilon/s}^{(i)}$ satisfies

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\varepsilon/s}\nabla\mathbb{Z}_{\varepsilon,\varepsilon/s}^{(i)})=0 & \text{in } B(0,2)\cap\Omega_f^\varepsilon/s \\ -\varepsilon^2\nabla\cdot(\mathbf{k}_{\varepsilon/s}\nabla\mathbb{Z}_{\varepsilon,\varepsilon/s}^{(i)})=0 & \text{in } B(0,2)\cap\Omega_m^\varepsilon/s \\ \mathbf{K}_{\varepsilon/s}\nabla\mathbb{Z}_{\varepsilon,\varepsilon/s}^{(i)}\cdot\vec{\mathbf{n}}=\varepsilon^2\mathbf{k}_{\varepsilon/s}\nabla\mathbb{Z}_{\varepsilon,\varepsilon/s}^{(i)}\cdot\vec{\mathbf{n}} & \text{on } B(0,2)\cap\partial\Omega_m^\varepsilon/s \\ \mathbb{Z}_{\varepsilon,\varepsilon/s,+}^{(i)}=\mathbb{Z}_{\varepsilon,\varepsilon/s,-}^{(i)} & \text{on } B(0,2)\cap\partial\Omega_m^\varepsilon/s \\ \mathbb{Z}_{\varepsilon,\varepsilon/s}^{(i)}=-\mathbb{X}_{\varepsilon,\varepsilon/s}^{(i)}\,\rho(x) & \text{on } \begin{cases} B(0,2)\cap\partial\Omega/s \\ \partial B(0,2)\cap\Omega/s \end{cases} \end{cases} \tag{69}$$

We claim the maximal value of $\mathbb{Z}^{(i)}_{\varepsilon,\varepsilon/s}$ is on the boundary $B(0,2)\cap\partial\Omega/s$ and $\partial B(0,2)\cap\Omega/s$. By Theorem 3.5 [9], the maximal value cannot be in the interior regions of $B(0,2)\cap\Omega^{\varepsilon}_f/s$ and $B(0,2)\cap\Omega^{\varepsilon}_m/s$. Hence, it is on the boundary $B(0,2)\cap\partial\Omega^{\varepsilon}_m/s$ or the boundary $B(0,2)\cap\partial\Omega/s$ and $\partial B(0,2)\cap\Omega/s$. If the maximal value appears at $x_0\in B(0,2)\cap\partial\Omega^{\varepsilon}_m/s$, then $\mathbf{k}_{\varepsilon/s}\nabla\mathbb{Z}^{(i)}_{\varepsilon,\varepsilon/s}\cdot\vec{\mathbf{n}}|_{x=x_0}>0$ and $\mathbf{K}_{\varepsilon/s}\nabla\mathbb{Z}^{(i)}_{\varepsilon,\varepsilon/s}\cdot\vec{\mathbf{n}}|_{x=x_0}<0$ by Lemma 3.4 [9]. But this is inconsistent with $(69)_3$. Thus we know that the maximal value appears on the boundary $B(0,2)\cap\partial\Omega/s$ and $\partial B(0,2)\cap\Omega/s$. Similarly, the minimum value of $\mathbb{Z}^{(i)}_{\varepsilon,\varepsilon/s}$ is also on the boundary $B(0,2)\cap\partial\Omega/s$ and $\partial B(0,2)\cap\Omega/s$. This implies

$$\|\mathbb{W}^{(i)}_{\varepsilon,\varepsilon/s}\|_{L^\infty(B(0,2)\cap\Omega/s)}\leqslant 2\|\mathbb{X}^{(i)}_{\varepsilon,\varepsilon/s}\|_{L^\infty(B(0,2)\cap\Omega/s)}\leqslant c\varepsilon/s$$

□

Lemma 5.5

For any $\delta>0$, there are constants $\tilde{\theta}\in(0,1)$ (depending on $\delta,\alpha,\mathbf{K},\mathbf{k}$) and $\tilde{\varepsilon}_0\in(0,1)$ (depending on $\theta,\delta,\mathbf{k}$) satisfying $\tilde{\theta}<\theta$, $\tilde{\varepsilon}_0<\varepsilon_0$ (θ, ε_0 are those in Lemma 5.1) such that if $U_{\varepsilon,v},u_{\varepsilon,v},F_{\varepsilon,v},f_{\varepsilon,v}$ satisfy

$$\begin{cases}-\nabla\cdot(\mathbf{K}_v\nabla U_{\varepsilon,v})=F_{\varepsilon,v} & \text{in } B(0,1)\cap\Omega^v_f\\ -\varepsilon^2\nabla\cdot(\mathbf{k}_v\nabla u_{\varepsilon,v})=\varepsilon f_{\varepsilon,v} & \text{in } B(0,1)\cap\Omega^v_m\\ \mathbf{K}_v\nabla U_{\varepsilon,v}\cdot\vec{\mathbf{n}}^v=\varepsilon^2\mathbf{k}_v\nabla u_{\varepsilon,v}\cdot\vec{\mathbf{n}}^v & \text{on } B(0,1)\cap\partial\Omega^v_m\\ U_{\varepsilon,v}=u_{\varepsilon,v} & \text{on } B(0,1)\cap\partial\Omega^v_m\\ U_{\varepsilon,v}=U_{b_{\varepsilon,v}} & \text{on } B(0,1)\cap\partial\Omega\end{cases}$$

and if

$$\begin{cases}U_{b_{\varepsilon,v}}(0)=\nabla_T U_{b_{\varepsilon,v}}(0)=0\\ \max\{\|U_{\varepsilon,v}\|_{L^\infty(B(0,1)\cap\Omega^v_f)},\varepsilon\|u_{\varepsilon,v}\|_{L^2(B(0,1)\cap\Omega^v_m)}\\ \quad\|\varepsilon_0^{-1}F_{\varepsilon,v}\mathscr{X}_{\Omega^v_f}+vf_{\varepsilon,v}\mathscr{X}_{\Omega^v_m}\|_{L^{3+\delta}(B(0,1))}+[U_{b_{\varepsilon,v}}]_{C^{1,\alpha}(B(0,1)\cap\Omega)}\}\leqslant 1\end{cases}$$

where $\nabla_T U_{b_{\varepsilon,v}}(0)$ is the tangential derivative of $U_{b_{\varepsilon,v}}$ at 0, then, for $\varepsilon\leqslant v\leqslant\tilde{\varepsilon}_0$, $\tau=\min(\alpha/2,\mu/2)$,

$$\begin{cases}\sup_{B(0,\tilde{\theta})\cap\Omega}|\Pi_v U_{\varepsilon,v}-(x_3+\Pi_v\mathbb{W}^{(3)}_{\varepsilon,v}(x))\mathbf{d}_{\varepsilon,v}|\leqslant\tilde{\theta}^{1+\tau}\\ \displaystyle\fint_{B(0,\tilde{\theta})\cap\Omega^v_m}\varepsilon^2|u_{\varepsilon,v}-(x_3+\mathbb{W}^{(3)}_{\varepsilon,v}(x))\mathbf{d}_{\varepsilon,v}|^2\mathrm{d}x\leqslant\tilde{\theta}^{2(1+\tau)}\end{cases}$$

where $\mathbf{d}_{\varepsilon,v}$ is the third component of

$$\frac{\check{\mathbf{K}}^{-1}}{|B(0,\tilde{\theta})|}\int_{B(0,\tilde{\theta})\cap\Omega^v_f}\mathbf{K}_v\nabla U_{\varepsilon,v}\mathrm{d}x$$

Proof

The proof is similar to that of Lemma 5.1. Let (U,U_b) satisfy

$$\begin{cases}-\nabla\cdot(\check{\mathbf{K}}\nabla U)=0 & \text{in } B(0,1)\cap\Omega\\ U=U_b & \text{on } B(0,1)\cap\partial\Omega\end{cases}$$

and $U_b(0)=\nabla_T U_b(0)=0$. By regularity theory, there exists τ' satisfying $\tau<\tau'<\min(\mu,\alpha)$ and $\tilde{\theta}\in(0,1)$ such that

$$\sup_{B(0,\tilde{\theta})\cap\Omega}|U-x_3(\partial_3 U)_{0,\tilde{\theta}}|\leqslant\tilde{\theta}^{1+\tau'}(\|U\|_{L^\infty(B(0,1)\cap\Omega)}+[U_b]_{C^{1,\alpha}(B(0,1)\cap\Omega)})\tag{70}$$

If we fix $\tilde{\theta}$ so that (70) holds, the conclusion will follow by contradiction provided we prove $\lim_{\varepsilon\to 0}\|\mathbb{W}^{(3)}_{\varepsilon,v}\|_{L^\infty(B(0,2)\cap\Omega)}=0$. But that is the result of Lemma 5.4. Thus we prove this lemma. □

Lemma 5.6
For any $\delta>0$, there are constants $\tilde{\theta}\in(0,1)$ (depending on $\delta,\alpha,\mathbf{K},\mathbf{k}$) and $\tilde{\varepsilon}_0\in(0,1)$ (depending on $\theta,\delta,\mathbf{k}$) satisfying $\tilde{\theta}<\theta$, $\tilde{\varepsilon}_0<\varepsilon_0$ (θ, ε_0 are those in Lemma 5.1) such that if $U_\varepsilon, u_\varepsilon, F_\varepsilon, f_\varepsilon, U_{b_\varepsilon}$ satisfies

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\varepsilon\nabla U_\varepsilon)=F_\varepsilon & \text{in } B(0,1)\cap\Omega_f^\varepsilon \\ -\varepsilon^2\nabla\cdot(\mathbf{k}_\varepsilon\nabla u_\varepsilon)=\varepsilon f_\varepsilon & \text{in } B(0,1)\cap\Omega_m^\varepsilon \\ \mathbf{K}_\varepsilon\nabla U_\varepsilon\cdot\vec{\mathbf{n}}^\varepsilon=\varepsilon^2\mathbf{k}_\varepsilon\nabla u_\varepsilon\cdot\vec{\mathbf{n}}^\varepsilon & \text{on } B(0,1)\cap\partial\Omega_m^\varepsilon \\ U_\varepsilon=u_\varepsilon & \text{on } B(0,1)\cap\partial\Omega_m^\varepsilon \\ U_\varepsilon=U_{b_\varepsilon} & \text{on } B(0,1)\cap\partial\Omega \end{cases} \tag{71}$$

and if $U_{b_\varepsilon}(0)=\nabla_T U_{b_\varepsilon}(0)=0$, then, for all $\varepsilon\leqslant\tilde{\varepsilon}_0$ and k satisfying $\varepsilon/\tilde{\theta}^k\leqslant\tilde{\varepsilon}_0$, there exists constant $\mathbf{d}_k^\varepsilon$ satisfying

$$\begin{cases} |\mathbf{d}_k^\varepsilon|\leqslant c\tilde{J}_\varepsilon \\ \sup\limits_{B(0,\tilde{\theta}^k)\cap\Omega} |\Pi_\varepsilon U_\varepsilon-\sum\limits_{j=0}^{k-1}\tilde{\theta}^{\tau j}(x_3+\tilde{\theta}^j\Pi_{\varepsilon/\tilde{\theta}^j}\mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^j}(\tilde{\theta}^{-j}x))\mathbf{d}_j^\varepsilon|\leqslant\tilde{\theta}^{k(1+\tau)}\tilde{J}_\varepsilon \\ \displaystyle\fint_{B(0,\tilde{\theta}^k)\cap\Omega_m^\varepsilon}\varepsilon^2|u_\varepsilon-\sum_{j=0}^{k-1}\tilde{\theta}^{\tau j}(x_3+\tilde{\theta}^j\mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^j}(\tilde{\theta}^{-j}x))\mathbf{d}_j^\varepsilon|^2\,dx\leqslant\tilde{\theta}^{2k(1+\tau)}\tilde{J}_\varepsilon^2 \end{cases} \tag{72}$$

where τ is same as that in Lemma 5.5 and

$$\tilde{J}_\varepsilon\equiv\|U_\varepsilon\|_{L^\infty(B(0,1)\cap\Omega_f^\varepsilon)}+\varepsilon\|u_\varepsilon\|_{L^2(B(0,1)\cap\Omega_m^\varepsilon)}+[U_{b_\varepsilon}]_{C^{1,\alpha}(B(0,1)\cap\Omega)}$$
$$+\|\tilde{\varepsilon}_0^{-1}F_\varepsilon\mathscr{X}_{\Omega_f^\varepsilon}+f_\varepsilon\mathscr{X}_{\Omega_m^\varepsilon}\|_{L^{3+\delta}(B(0,1))}$$

Proof
This is done by induction on k. When $k=1$, (72) holds by Lemma 5.5 with $v=\varepsilon$. $\mathbf{d}_0^\varepsilon$ is the third component of

$$\frac{\check{\mathbf{K}}^{-1}}{|B(0,\tilde{\theta})|}\int_{B(0,\tilde{\theta})\cap\Omega_f^\varepsilon}\mathbf{K}_\varepsilon\nabla U_\varepsilon\,dx$$

If (72) holds for some k satisfying $\varepsilon/\tilde{\theta}^k\leqslant\tilde{\varepsilon}_0$, then we define, in $B(0,1)\cap\Omega_f^\varepsilon/\tilde{\theta}^k$,

$$\begin{cases} \hat{U}_\varepsilon(x)\equiv\tilde{J}_\varepsilon^{-1}\tilde{\theta}^{-k(1+\tau)}\left(U_\varepsilon(\tilde{\theta}^k x)-\sum\limits_{j=0}^{k-1}\tilde{\theta}^{\tau j}(\tilde{\theta}^k x_3+\tilde{\theta}^j\mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^j}(\tilde{\theta}^{k-j}x))\mathbf{d}_j^\varepsilon\right) \\ \hat{F}_\varepsilon(x)\equiv\tilde{J}_\varepsilon^{-1}\tilde{\theta}^{k(1-\tau)}F_\varepsilon(\tilde{\theta}^k x) \\ \hat{U}_{b_\varepsilon}(x)\equiv\tilde{J}_\varepsilon^{-1}\tilde{\theta}^{-k(1+\tau)}\left(U_{b_\varepsilon}(\tilde{\theta}^k x)-\sum\limits_{j=0}^{k-1}\tilde{\theta}^{\tau j}\tilde{\theta}^k x_3\mathbf{d}_j^\varepsilon\right) \end{cases}$$

and, in $B(0,1)\cap\Omega_m^\varepsilon/\tilde{\theta}^k$,

$$\begin{cases} \hat{u}_\varepsilon(x)\equiv\tilde{J}_\varepsilon^{-1}\tilde{\theta}^{-k(1+\tau)}\left(u_\varepsilon(\tilde{\theta}^k x)-\sum\limits_{j=0}^{k-1}\tilde{\theta}^{\tau j}(\tilde{\theta}^k x_3+\tilde{\theta}^j\mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^j}(\tilde{\theta}^{k-j}x))\mathbf{d}_j^\varepsilon\right) \\ \hat{f}_\varepsilon(x)\equiv\tilde{J}_\varepsilon^{-1}\tilde{\theta}^{k(1-\tau)}f_\varepsilon(\tilde{\theta}^k x) \end{cases}$$

Then those functions satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\varepsilon/\tilde{\theta}^k}\nabla\hat{U}_\varepsilon)=\hat{F}_\varepsilon & \text{in } B(0,1)\cap\Omega_f^\varepsilon/\tilde{\theta}^k \\ -\varepsilon^2\nabla\cdot(\mathbf{k}_{\varepsilon/\tilde{\theta}^k}\nabla\hat{u}_\varepsilon)=\varepsilon\hat{f}_\varepsilon & \text{in } B(0,1)\cap\Omega_m^\varepsilon/\tilde{\theta}^k \\ \mathbf{K}_{\varepsilon/\tilde{\theta}^k}\nabla\hat{U}_\varepsilon\cdot\vec{\mathbf{n}}=\varepsilon^2\mathbf{k}_{\varepsilon/\tilde{\theta}^k}\nabla\hat{u}_\varepsilon\cdot\vec{\mathbf{n}} & \text{on } B(0,1)\cap\partial\Omega_m^\varepsilon/\tilde{\theta}^k \\ \hat{U}_\varepsilon=\hat{u}_\varepsilon & \text{on } B(0,1)\cap\partial\Omega_m^\varepsilon/\tilde{\theta}^k \\ \hat{U}_\varepsilon=\hat{U}_{b_\varepsilon} & \text{on } B(0,1)\cap\partial\Omega/\tilde{\theta}^k \end{cases}$$

where $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\Omega^\varepsilon_m/\tilde{\theta}^k$. Note that by (38), $[\hat{U}_{b_\varepsilon}]_{C^{1,\alpha}(B(0,1)\cap\Omega/\tilde{\theta}^k)}\leqslant c^*$, where c^* depends only on domain. If necessary, one may adjust the $\tilde{J}_\varepsilon$ so that, by induction,

$$\max\{\|\hat{U}_\varepsilon\|_{L^\infty(B(0,1)\cap\Omega^\varepsilon_f/\tilde{\theta}^k)},\varepsilon\|\hat{u}_\varepsilon\|_{L^2(B(0,1)\cap\Omega^\varepsilon_m/\tilde{\theta}^k)},[\hat{U}_{b_\varepsilon}]_{C^{1,\alpha}(B(0,1)\cap\Omega/\tilde{\theta}^k)} + \|\varepsilon_0^{-1}\hat{F}_\varepsilon\mathscr{X}_{\Omega^\varepsilon_f/\tilde{\theta}^k}+\hat{f}_\varepsilon\mathscr{X}_{\Omega^\varepsilon_m/\tilde{\theta}^k}\|_{L^{3+\delta}(B(0,1))}\}\leqslant 1 \tag{73}$$

Apply Lemma 5.5 (in this case $v=\varepsilon/\tilde{\theta}^k$),

$$\begin{cases}\sup\limits_{B(0,\tilde{\theta})\cap\Omega/\tilde{\theta}^k}|\Pi_{\varepsilon/\tilde{\theta}^k}\hat{U}_\varepsilon-(x_3+\Pi_{\varepsilon/\tilde{\theta}^k}\mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^k}(x))\hat{\mathbf{d}}^\varepsilon_k|\leqslant\tilde{\theta}^{1+\tau}\\ \displaystyle\fint_{B(0,\tilde{\theta})\cap\Omega^\varepsilon_m/\tilde{\theta}^k}\varepsilon^2|\hat{u}_\varepsilon-(x_3+\mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^k}(x))\hat{\mathbf{d}}^\varepsilon_k|^2\mathrm{d}x\leqslant\tilde{\theta}^{2(1+\tau)}\end{cases} \tag{74}$$

where $\hat{\mathbf{d}}^\varepsilon_k$ is the third component of

$$\frac{\check{\mathbf{K}}^{-1}}{|B(0,\tilde{\theta})|}\int_{B(0,\tilde{\theta})\cap\Omega^\varepsilon_f/\tilde{\theta}^k}\mathbf{K}_{\varepsilon/\tilde{\theta}^k}\nabla\hat{U}_\varepsilon\mathrm{d}x$$

Equations (73) and (74) imply that $|\hat{\mathbf{d}}^\varepsilon_k|$ are bounded uniformly in ε, k. Rewrite $(74)_1$ in terms of U_ε in $B(0,\tilde{\theta}^{k+1})$ to obtain, by Lemma 2.1,

$$\sup_{B(0,\tilde{\theta}^{k+1})\cap\Omega}\left|\Pi_\varepsilon U_\varepsilon-\sum_{j=0}^{k-1}\tilde{\theta}^{\tau j}(x_3+\tilde{\theta}^j\Pi_{\varepsilon/\tilde{\theta}^j}\mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^j}(\tilde{\theta}^{-j}x))\mathbf{d}^\varepsilon_j-\tilde{\theta}^{k\tau}\tilde{J}_\varepsilon(x_3+\tilde{\theta}^k\Pi_{\varepsilon/\tilde{\theta}^k}\mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^k}(\tilde{\theta}^{-k}x))\hat{\mathbf{d}}^\varepsilon_k\right|\leqslant\tilde{\theta}^{(k+1)(1+\tau)}\tilde{J}_\varepsilon$$

Set $\mathbf{d}^\varepsilon_k\equiv\tilde{J}_\varepsilon\hat{\mathbf{d}}^\varepsilon_k$, we conclude that $(72)_{1,2}$ hold for $k+1$. By a similar argument as above, we obtain $(72)_3$ holds for $k+1$ by $(74)_2$. □

Lemma 5.7
Under the assumptions of Lemma 5.6 except $U_{b_\varepsilon}(0)=\nabla_T U_{b_\varepsilon}(0)=0$,

$$\|\nabla U_\varepsilon\|_{L^\infty(B(0,1/2)\cap\Omega^\varepsilon_f)}\leqslant c\tilde{J}_\varepsilon \tag{75}$$

where constant c is independent of ε.

Proof
Because of (37), we introduce a local coordinate $x=(x',x_3)$ so that

$$B(0,1)\cap\Omega=\{(x',x_3)\in\Omega||x'|^2+|x_3|^2<1,\phi(x')<x_3\}$$

To obtain the Lipschitz estimate (75), it is suffice to show

$$\sup_{(0,x_3)\in B(0,1/2)\cap\Omega^\varepsilon_f}|\nabla U_\varepsilon(0,x_3)|\leqslant c\tilde{J}_\varepsilon \tag{76}$$

The reason is that one can repeat the same argument by varying the origin along the boundary $B(0,1)\cap\partial\Omega$ and by adjusting the constant c to obtain the estimate.

Define $\widehat{\mathbb{W}}_{\varepsilon,\varepsilon/s}=(\mathbb{W}^{(1)}_{\varepsilon,\varepsilon/s},\mathbb{W}^{(2)}_{\varepsilon,\varepsilon/s})\in H^1(B(0,1)\cap\Omega/s,R^2)$ for $s>0$, and

$$\begin{cases}\hat{U}_\varepsilon(x)\equiv U_\varepsilon(x)-U_{b_\varepsilon}(0)-(x'+\widehat{\mathbb{W}}_{\varepsilon,\varepsilon}(x))\nabla_T U_{b_\varepsilon}(0) & \text{in } B(0,1)\cap\Omega^\varepsilon_f\\ \hat{u}_\varepsilon(x)\equiv u_\varepsilon(x)-U_{b_\varepsilon}(0)-(x'+\widehat{\mathbb{W}}_{\varepsilon,\varepsilon}(x))\nabla_T U_{b_\varepsilon}(0) & \text{in } B(0,1)\cap\Omega^\varepsilon_m\\ \hat{U}_{b_\varepsilon}\equiv U_{b_\varepsilon}(x)-U_{b_\varepsilon}(0)-x'\nabla_T U_{b_\varepsilon}(0) & \text{on } B(0,1)\cap\Omega\end{cases} \tag{77}$$

Then $\hat{U}_\varepsilon,\hat{u}_\varepsilon,F_\varepsilon,f_\varepsilon,\hat{U}_{b_\varepsilon}$ satisfy (71) and $\widehat{U}_{b_\varepsilon}(0)=\nabla_T\widehat{U}_{b_\varepsilon}(0)=0$. Hence, the assumptions of Lemma 5.6 hold for $\hat{U}_\varepsilon,\hat{u}_\varepsilon,F_\varepsilon,f_\varepsilon,\hat{U}_{b_\varepsilon}$. Let k satisfy $\varepsilon/\tilde{\theta}^k\leqslant\tilde{\varepsilon}_0<\varepsilon/\tilde{\theta}^{k+1}$. For any $x\equiv(0,x_3)\in B(0,\frac{1}{2})\cap\Omega^\varepsilon_f$, we have either case (1): $\frac{1}{2}\tilde{\theta}^\ell<x_3\leqslant\frac{1}{2}\tilde{\theta}^{\ell-1}$ for $1\leqslant\ell\leqslant k$ or case (2): $0\leqslant x_3\leqslant\frac{1}{2}\tilde{\theta}^k$.

For case (1): By Lemma 5.6, we have

$$\begin{cases} \sup\limits_{B(0,\tilde{\theta}^{\ell-1})\cap\Omega} \left| \Pi_\varepsilon \hat{U}_\varepsilon - \sum\limits_{j=0}^{\ell-2} \tilde{\theta}^{\tau j}(y_3 + \tilde{\theta}^j \Pi_{\varepsilon/\tilde{\theta}^j} \mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^j}(\tilde{\theta}^{-j}y))\mathbf{d}_j^\varepsilon \right| \leqslant \tilde{\theta}^{\ell(1+\tau)}\tilde{J}_\varepsilon \\ \displaystyle\fint_{B(0,\tilde{\theta}^{\ell-1})\cap\Omega_m^\varepsilon} \varepsilon^2 \left| \hat{u}_\varepsilon - \sum\limits_{j=0}^{\ell-2} \tilde{\theta}^{\tau j}(y_3 + \tilde{\theta}^j \mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^j}(\tilde{\theta}^{-j}y))\mathbf{d}_j^\varepsilon \right|^2 \mathrm{d}y \leqslant \tilde{\theta}^{2\ell(1+\tau)}\tilde{J}_\varepsilon^2 \end{cases} \tag{78}$$

Hence, by Lemma 5.4 and (78),

$$\begin{cases} \sup\limits_{B(0,\tilde{\theta}^{\ell-1})\cap\Omega_f^\varepsilon} |\hat{U}_\varepsilon| \leqslant c\tilde{\theta}^{\ell(1+\tau)}\tilde{J}_\varepsilon + c\tilde{J}_\varepsilon(\eta(x)+\varepsilon)\sum\limits_{j=0}^{\ell-2}\tilde{\theta}^{\tau j} \leqslant c\tilde{J}_\varepsilon \eta(x) \\ \displaystyle\fint_{B(0,\tilde{\theta}^{\ell-1})\cap\Omega_m^\varepsilon} \varepsilon^2 |\hat{u}_\varepsilon|^2 \mathrm{d}y \leqslant c\tilde{J}_\varepsilon^2 \eta^2(x) \end{cases} \tag{79}$$

In (79), we use the smoothness of the domain, $\eta(x) \equiv |x-x_0|$, where $x_0 \in \partial\Omega$ so that $|x-x_0| = \min_{y\in\partial\Omega}|x-y|$, and $\varepsilon \leqslant \tilde{\varepsilon}_0 \tilde{\theta}^k \leqslant 2\tilde{\varepsilon}_0 \frac{1}{2}\tilde{\theta}^\ell \leqslant 2\tilde{\varepsilon}_0 x_3 \leqslant c\tilde{\varepsilon}_0 \eta(x)$. By (77) and (79), we conclude that

$$\begin{cases} \sup\limits_{B(x,\eta(x)/2)\cap\Omega_f^\varepsilon} |U_\varepsilon - U_{b_\varepsilon}(0)| \leqslant c\tilde{J}_\varepsilon \eta(x) \\ \displaystyle\fint_{B(x,\eta(x)/2)\cap\Omega_m^\varepsilon} \varepsilon^2 |u_\varepsilon - U_{b_\varepsilon}(0)|^2 \mathrm{d}y \leqslant c\tilde{J}_\varepsilon^2 \eta^2(x) \end{cases} \tag{80}$$

Following the reasoning of case (1) in the proof of Lemma 4.6 as well as employing (80) and Lemma 5.3, we see $\|\nabla U_\varepsilon\|_{L^\infty(B(x,\eta(x)/4)\cap\Omega_f^\varepsilon)} \leqslant c\tilde{J}_\varepsilon$. This proves (76) for case (1).

For case (2): Apply Lemma 5.6 to obtain

$$\begin{cases} \sup\limits_{B(0,\tilde{\theta}^k)\cap\Omega} \left| \Pi_\varepsilon \hat{U}_\varepsilon - \sum\limits_{j=0}^{k-1} \tilde{\theta}^{\tau j}(y_3 + \tilde{\theta}^j \Pi_{\varepsilon/\tilde{\theta}^j} \mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^j}(\tilde{\theta}^{-j}y))\mathbf{d}_j^\varepsilon \right| \leqslant \tilde{J}_\varepsilon \tilde{\theta}^{k(1+\tau)} \\ \displaystyle\fint_{B(0,\tilde{\theta}^k)\cap\Omega_m^\varepsilon} \varepsilon^2 \left| \hat{u}_\varepsilon - \sum\limits_{j=0}^{k-1} \tilde{\theta}^{\tau j}(y_3 + \tilde{\theta}^j \mathbb{W}^{(3)}_{\varepsilon,\varepsilon/\tilde{\theta}^j}(\tilde{\theta}^{-j}y))\mathbf{d}_j^\varepsilon \right|^2 \mathrm{d}y \leqslant \tilde{J}_\varepsilon^2 \tilde{\theta}^{2k(1+\tau)} \end{cases}$$

By (77) and Lemma 5.4,

$$\begin{cases} \sup\limits_{B(0,\tilde{\theta}^k)\cap\Omega_f^\varepsilon} |U_\varepsilon - U_{b_\varepsilon}(0)| \leqslant c\tilde{J}_\varepsilon \varepsilon \\ \displaystyle\fint_{B(0,\tilde{\theta}^k)\cap\Omega_m^\varepsilon} \varepsilon^2 |u_\varepsilon - U_{b_\varepsilon}(0)|^2 \mathrm{d}y \leqslant c\tilde{J}_\varepsilon^2 \varepsilon^2 \end{cases} \tag{81}$$

where c is independent of ε. Define

$$\begin{cases} \hat{U}_\varepsilon(y) \equiv \varepsilon^{-1}\tilde{J}_\varepsilon^{-1}(U_\varepsilon(\varepsilon y) - U_{b_\varepsilon}(0)) & \text{in } \Omega_f^\varepsilon/\varepsilon \\ \hat{F}_\varepsilon(y) \equiv \varepsilon\tilde{J}_\varepsilon^{-1} F_\varepsilon(\varepsilon y) & \text{in } \Omega_f^\varepsilon/\varepsilon \\ \hat{u}_\varepsilon(y) \equiv \varepsilon^{-1}\tilde{J}_\varepsilon^{-1}(u_\varepsilon(\varepsilon y) - U_{b_\varepsilon}(0)) & \text{in } \Omega_m^\varepsilon/\varepsilon \\ \hat{f}_\varepsilon(y) \equiv \varepsilon\tilde{J}_\varepsilon^{-1} f_\varepsilon(\varepsilon y) & \text{in } \Omega_m^\varepsilon/\varepsilon \\ \hat{U}_{b_\varepsilon}(y) \equiv \varepsilon^{-1}\tilde{J}_\varepsilon^{-1}(U_{b_\varepsilon}(\varepsilon y) - U_{b_\varepsilon}(0)) & \text{on } \Omega/\varepsilon \end{cases}$$

By (81)

$$\|\hat{U}_\varepsilon\|_{L^\infty(B(0,1)\cap\Omega_f^\varepsilon/\varepsilon)} + \varepsilon\|\hat{u}_\varepsilon\|_{L^2(B(0,1)\cap\Omega_m^\varepsilon/\varepsilon)} + [\hat{U}_{b_\varepsilon}]_{C^{1,\alpha}(B(0,1)\cap\Omega/\varepsilon)} + \|\hat{F}_\varepsilon \mathscr{X}_{\Omega_f^\varepsilon/\varepsilon} + \hat{f}_\varepsilon \mathscr{X}_{\Omega_m^\varepsilon/\varepsilon}\|_{L^{3+\delta}(B(0,1))} \leqslant c$$

where c is independent of ε. $\hat{U}_\varepsilon, \hat{u}_\varepsilon, \hat{F}_\varepsilon, \hat{f}_\varepsilon, \hat{U}_{b_\varepsilon}$ satisfy (71) in domain $B(0,1)\cap\Omega/\varepsilon$. Lemma 3.4 implies $\|\hat{U}_\varepsilon\|_{C^{1,\mu}(B(0,1)\cap\Omega_f^\varepsilon/\varepsilon)} \leqslant c$. This gives the proof of (76) for case (2). □

Theorem 2.2 is a direct consequence of energy method, partition of unity, Lemma 5.3, and Lemma 5.7.

6. Proof of Theorem 2.3

Let $\mathcal{G}_\varepsilon(x,y)$ be Green's function of

$$\begin{cases} -\nabla_x\cdot(\mathbf{K}_\varepsilon\nabla_x\mathcal{G}_\varepsilon)=\delta(x-y) & \text{in } \Omega_f^\varepsilon \\ -\varepsilon^2\nabla_x\cdot(\mathbf{k}_\varepsilon\nabla_x\mathcal{G}_\varepsilon)=\delta(x-y) & \text{in } \Omega_m^\varepsilon \\ \mathbf{K}_\varepsilon\nabla_x\mathcal{G}_\varepsilon\cdot\vec{\mathbf{n}}^\varepsilon=\varepsilon^2\mathbf{k}_\varepsilon\nabla_x\mathcal{G}_\varepsilon\cdot\vec{\mathbf{n}}^\varepsilon & \text{on } \partial\Omega_m^\varepsilon \\ \mathcal{G}_{\varepsilon,+}=\mathcal{G}_{\varepsilon,-} & \text{on } \partial\Omega_m^\varepsilon \\ \mathcal{G}_\varepsilon=0 & \text{on } \partial\Omega \end{cases} \tag{82}$$

The left and the right limits, $\mathcal{G}_{\varepsilon,-}$ and $\mathcal{G}_{\varepsilon,+}$, on $\partial\Omega_m^\varepsilon$ are defined at the beginning of Section 5.

Lemma 6.1
If $x,y\in\Omega_f^\varepsilon$, $|\mathcal{G}_\varepsilon(x,y)|\leqslant c/|x-y|$, where c is independent of ε.

Proof
If $x,y\in\Omega_f^\varepsilon$, define $|x-y|=r$. Let $F\in C_0^\infty(B(x,r/3))$ and $U_\varepsilon\mathcal{X}_{\Omega_f^\varepsilon}+u_\varepsilon\mathcal{X}_{\Omega_m^\varepsilon}$ be the solution of

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\varepsilon\nabla U_\varepsilon)=F & \text{in } \Omega_f^\varepsilon \\ -\varepsilon^2\nabla\cdot(\mathbf{k}_\varepsilon\nabla u_\varepsilon)=\varepsilon F & \text{in } \Omega_m^\varepsilon \\ \mathbf{K}_\varepsilon\nabla U_\varepsilon\cdot\vec{\mathbf{n}}^\varepsilon=\varepsilon^2\mathbf{k}_\varepsilon\nabla u_\varepsilon\cdot\vec{\mathbf{n}}^\varepsilon & \text{on } \partial\Omega_m^\varepsilon \\ U_\varepsilon=u_\varepsilon & \text{on } \partial\Omega_m^\varepsilon \\ U_\varepsilon=0 & \text{on } \partial\Omega \end{cases}$$

By Green's theorem [9] and Lemmas 4.3 and 4.6

$$\begin{cases} U_\varepsilon(y)=\int_{B(x,r/3)\cap\Omega}(\mathcal{G}_\varepsilon(z,y)F\mathcal{X}_{\Omega_f^\varepsilon}+\varepsilon\mathcal{G}_\varepsilon(z,y)F\mathcal{X}_{\Omega_m^\varepsilon})\mathrm{d}z \\ |U_\varepsilon(y)|\leqslant c\left(⨍_{B(y,r/3)\cap\Omega}U_\varepsilon^2(z)\mathcal{X}_{\Omega_f^\varepsilon}+\varepsilon^2u_\varepsilon^2(z)\mathcal{X}_{\Omega_m^\varepsilon}\mathrm{d}z\right)^{1/2} \end{cases} \quad \text{for } y\in\Omega_f^\varepsilon \tag{83}$$

By (83), Lemma 2.1, and Sobolev inequality [9]

$$\left|\int_{B(x,r/3)\cap\Omega}(\mathcal{G}_\varepsilon(z,y)F\mathcal{X}_{\Omega_f^\varepsilon}+\varepsilon\mathcal{G}_\varepsilon(z,y)F\mathcal{X}_{\Omega_m^\varepsilon})\mathrm{d}z\right|\leqslant c\left|⨍_{B(y,r/3)\cap\Omega}|U_\varepsilon^2(z)\mathcal{X}_{\Omega_f^\varepsilon}+\varepsilon^2u_\varepsilon^2(z)\mathcal{X}_{\Omega_m^\varepsilon}|\mathrm{d}z\right|^{1/2}$$

$$\leqslant c\left|⨍_{B(y,r/3)\cap\Omega}|U_\varepsilon(z)\mathcal{X}_{\Omega_f^\varepsilon}+\varepsilon u_\varepsilon(z)\mathcal{X}_{\Omega_m^\varepsilon}|^6\mathrm{d}z\right|^{1/6}\leqslant\frac{c}{r^{1/2}}(\|\nabla U_\varepsilon\|_{L^2(\Omega_f^\varepsilon)}+\varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega_m^\varepsilon)})\leqslant cr^{1/2}\|F\|_{L^2(\Omega)} \tag{84}$$

Multiply (84) by r^{-3} to obtain

$$\left|⨍_{B(x,r/3)\cap\Omega}\mathcal{G}_\varepsilon(z,y)F\mathcal{X}_{\Omega_f^\varepsilon}+\varepsilon\mathcal{G}_\varepsilon(z,y)F\mathcal{X}_{\Omega_m^\varepsilon}\mathrm{d}z\right|\leqslant\frac{c}{r}\left|⨍_{B(x,r/3)\cap\Omega}F^2(z)\mathrm{d}z\right|^{1/2} \tag{85}$$

Equations (82), (85) and Lemmas 4.3, 4.6 imply, for $x\in\Omega_f^\varepsilon$

$$|\mathcal{G}_\varepsilon(x,y)|\leqslant c\left|⨍_{B(x,r/3)\cap\Omega}\mathcal{G}_\varepsilon^2(z,y)\mathcal{X}_{\Omega_f^\varepsilon}+\varepsilon^2\mathcal{G}_\varepsilon^2(z,y)\mathcal{X}_{\Omega_m^\varepsilon}\mathrm{d}z\right|^{1/2}\leqslant\frac{c}{r}$$

Thus we prove this lemma. □

Lemma 6.2
If $\mathcal{P}_\varepsilon(x,y)\equiv\mathbf{K}_\varepsilon\nabla_y\mathcal{G}(x,y)\cdot\vec{\mathbf{n}}$, there is a constant c independent of ε such that, for all $x\in\Omega_f^\varepsilon, y\in\partial\Omega$,

$$|\mathcal{P}_\varepsilon(x,y)|\leqslant\frac{c\eta(x)}{|x-y|^3}$$

where $\vec{\mathbf{n}}$ is the unit outward normal vector on $\partial\Omega$ and $\eta(x)$ is the distance from x to $\partial\Omega$.

Proof
If $x,y\in\Omega_f^\varepsilon$ and $r=|x-y|$, then $|\mathscr{G}_\varepsilon(x,y)|\leqslant c/r$, where c is independent of ε by Lemma 6.1. We first claim if $x,y\in\Omega_f^\varepsilon$

$$|\mathscr{G}_\varepsilon(x,y)|\leqslant\frac{c\eta(y)}{|x-y|^2} \tag{86}$$

It suffices to consider $\eta(y)\leqslant\frac{1}{3}|x-y|$ case. Let $y^*\in\partial\Omega$ satisfy $\eta(y)=|y-y^*|$. Since $\mathscr{G}_\varepsilon(x,\cdot)$ is an adjoint solution of (82) in $B(y^*,\frac{1}{2}r)\cap\Omega$ and by Lemma 5.7, $|\mathscr{G}_\varepsilon(x,z)|\leqslant\frac{c|z-y^*|}{r}$ for $z\in B(y^*,\frac{1}{3}r)\cap\Omega$. In particular, if we take $z=y$, then (86) holds for $\eta(y)\leqslant\frac{1}{3}|x-y|$.

We next claim if $x,y\in\Omega_f^\varepsilon$

$$|\mathscr{G}_\varepsilon(x,y)|\leqslant\frac{c\eta(x)\eta(y)}{|x-y|^3} \tag{87}$$

It suffices to consider $\eta(x)\leqslant\frac{1}{3}|x-y|$ case. Indeed (87) can be seen by (86) and by tracing the argument for (86). The lemma follows from (87). □

By Theorem 2.1, Theorem 2.3 can be proved by showing that the solutions of

$$\begin{cases}-\nabla\cdot(\mathbf{K}_\varepsilon\nabla U_\varepsilon)=0 & \text{in } \Omega_f^\varepsilon\\ -\varepsilon^2\nabla\cdot(\mathbf{k}_\varepsilon\nabla u_\varepsilon)=0 & \text{in } \Omega_m^\varepsilon\\ \mathbf{K}_\varepsilon\nabla U_\varepsilon\cdot\vec{\mathbf{n}}^\varepsilon=\varepsilon^2\mathbf{k}_\varepsilon\nabla u_\varepsilon\cdot\vec{\mathbf{n}}^\varepsilon & \text{on } \partial\Omega_m^\varepsilon\\ U_\varepsilon=u_\varepsilon & \text{on } \partial\Omega_m^\varepsilon\\ U_\varepsilon=U_{b_\varepsilon} & \text{on } \partial\Omega\end{cases}$$

satisfy $[U_\varepsilon]_{C^{0,\mu}(\Omega_f^\varepsilon)}\leqslant c[U_{b_\varepsilon}]_{C^{0,\mu}(\partial\Omega)}$, where constant c is independent of ε. By Green's theorem

$$U_\varepsilon(y)=\int_{\partial\Omega}\mathscr{P}_\varepsilon(x,y)U_{b_\varepsilon}(x)\mathrm{d}\sigma_x \quad \text{for } y\in\Omega_f^\varepsilon$$

By Lemma 6.2, maximal principle [9], and argument in Section 3, Theorem 2.3 can be proved by a standard procedure (see [18, pp. 101–103] and [8, pp. 841–843]).

7. Proof of Theorem 2.4

If the domain Ω is in $C^{3,\alpha}$ and $F\in W^{2,3+\delta}(\Omega)$, the solution U_0 of (8) satisfies, by Calderon–Zygmund estimate [9], $\|U_0\|_{W^{4,3+\delta}(\Omega)}\leqslant c\,\|F\|_{W^{2,3+\delta}(\Omega)}$, where c is a constant depending on Ω. Define

$$\begin{cases}V_\varepsilon(x)\equiv U_\varepsilon(x)-U_0(x)-\mathbb{X}_{\varepsilon,\varepsilon}(x)\nabla U_0(x)-\mathbb{Y}_\varepsilon(x)\nabla^2U_0(x) & \text{in } \Omega_f^\varepsilon\\ v_\varepsilon(x)\equiv u_\varepsilon(x)-U_0(x)-\mathbb{X}_{\varepsilon,\varepsilon}(x)\nabla U_0(x)-\mathbb{Y}_\varepsilon(x)\nabla^2U_0(x) & \text{in } \Omega_m^\varepsilon\end{cases} \tag{88}$$

where $\mathbb{X}_{\varepsilon,\varepsilon}$ is defined in (52), $\mathbb{Y}_\varepsilon(x)\equiv\varepsilon^2\mathbb{Y}(x/\varepsilon)$, and $\mathbb{Y}$ is a matrix function satisfying

$$\begin{cases}\nabla\cdot(\mathbf{K}\nabla\mathbb{Y})+\nabla(\mathbf{K}\mathbb{X})=\dfrac{\check{\mathbf{K}}}{|Y_f|}-\mathbf{K}(I+\nabla\mathbb{X}) & \text{in } Y_f\\ \varepsilon^2\nabla\cdot(\mathbf{k}\nabla\mathbb{Y})+\varepsilon^2\nabla(\mathbf{k}\mathbb{X})=-\varepsilon^2\mathbf{k}(I+\nabla\mathbb{X}) & \text{in } Y_m\\ \mathbf{K}(\nabla\mathbb{Y}\cdot\vec{\mathbf{n}}_y+\mathbb{X}\vec{\mathbf{n}}_y)=\varepsilon^2\mathbf{k}(\nabla\mathbb{Y}\cdot\vec{\mathbf{n}}_y+\mathbb{X}\vec{\mathbf{n}}_y) & \text{on } \partial Y_m\\ \mathbb{Y}_+=\mathbb{Y}_- & \text{on } \partial Y_m\\ \mathbb{Y} \text{ is } 1-\text{periodic in } y \text{ and } \displaystyle\int_{Y_f}\mathbb{Y}(y)\mathrm{d}y=0 & \end{cases} \tag{89}$$

$\vec{\mathbf{n}}_y$ denotes the unit outward normal vector on ∂Y_m. By energy method, $\mathbb{Y}$ is solvable uniquely. Equations (7)–(8) and (88)–(89) imply that

$$\begin{cases}-\nabla\cdot(\mathbf{K}_\varepsilon(\nabla V_\varepsilon+\mathbb{Y}_\varepsilon\nabla^3U_0))=\mathbf{K}_\varepsilon(\mathbb{X}_{\varepsilon,\varepsilon}\nabla\Delta U_0+\nabla\mathbb{Y}_\varepsilon\nabla^3U_0) & \text{in } \Omega_f^\varepsilon\\ -\varepsilon^2\nabla\cdot(\mathbf{k}_\varepsilon(\nabla v_\varepsilon+\mathbb{Y}_\varepsilon\nabla^3U_0))=\varepsilon^2\mathbf{k}_\varepsilon(\mathbb{X}_{\varepsilon,\varepsilon}\nabla\Delta U_0+\nabla\mathbb{Y}_\varepsilon\nabla^3U_0) & \text{in } \Omega_m^\varepsilon\\ \mathbf{K}_\varepsilon(\nabla V_\varepsilon+\mathbb{Y}_\varepsilon\nabla^3U_0)\cdot\vec{\mathbf{n}}^\varepsilon=\varepsilon^2\mathbf{k}_\varepsilon(\nabla v_\varepsilon+\mathbb{Y}_\varepsilon\nabla^3U_0)\cdot\vec{\mathbf{n}}^\varepsilon & \text{on } \partial\Omega_m^\varepsilon\\ V_\varepsilon=v_\varepsilon & \text{on } \partial\Omega_m^\varepsilon\\ V_\varepsilon=-\mathbb{X}_{\varepsilon,\varepsilon}\nabla U_0-\mathbb{Y}_\varepsilon\nabla^2U_0 & \text{on } \partial\Omega\end{cases}$$

By Theorem 2.3, $\|V_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)}\leqslant c\varepsilon^{1-\mu}\|F\|_{W^{2,3+\delta}(\Omega)}$. Thus we get Theorem 2.4.

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References

1. Allaire G. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis* 1992; **23**:1482–1518.
2. Ladyzhenskaya OA, Uraltseva NN. *Elliptic and Quasilinear Elliptic Equations*. Academic Press: New York, 1968.
3. Huang J, Zou J. Some new a priori estimates for second-order elliptic and parabolic interface problems. *Journal of Differential Equations* 2002; **184**:570–586.
4. Jikov VV, Kozlov SM, Oleinik OA. *Homogenization of Differential Operators and Integral Functions*. Springer: Berlin, 1994.
5. Schweizer B. Uniform estimates in two periodic homogenization problems. *Communications on Pure and Applied Mathematics* 2000; **LIII**:1153–1176.
6. Masmoudi N. Some uniform elliptic estimates in porous media. *Comptes Rendus de l'Academie des Sciences, Paris, Series I* 2004; **339**:849–854.
7. Li YY, Vogelius M. Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. *Archive for Rational Mechanics and Analysis* 2000; **153**:91–151.
8. Avellaneda M, Lin F-H. Compactness methods in the theory of homogenization. *Communications on Pure and Applied Mathematics* 1987; **XI**: 803–847.
9. Gilbarg D, Trudinger NS. *Elliptic Partial Differential Equations of Second Order* (2nd edn). Springer: Berlin, 1983.
10. Acerbi E, Piat V Chiado, Dal Maso G, Percivale D. An extension theorem from connected sets, and homogenization in general periodic domains. *Nonlinear Analysis* 1992; **18**:481–496.
11. Giaquinta M. *Multiple Integrals in the Calculus of Variations*. Annals of Mathematical Studies, vol. 105. Princeton University Press: Princeton, NJ, 1983.
12. Chen G, Zhou J. *Boundary Element Methods*. Academic Press: New York, 1992.
13. Triebel H. *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland: Amsterdam, 1978.
14. Taylor ME. *Pseudodifferential Operators*. Princeton University Press: Princeton, NJ, 1981.
15. Escauriaza L, Fabes EB, Verchota G. On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries. *Proceedings of the American Mathematical Society* 1992; **115**(4):1069–1076.
16. Conway JB. *A Course in Functional Analysis*. Springer: Berlin, 1985.
17. Fujiwara D, Morimoto H. An L_r-theorem of the Helmholtz decomposition of vector fields. *Journal of the Faculty of Science, University of Tokyo, Section IA Mathematics* 1977; **24**(3):685–700.
18. Avellaneda M, Lin F-H. Homogenization of elliptic problems with L^p boundary data. *Applied Mathematics and Optimization* 1987; **15**:93–107.