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On the largest eigenvalues of bipartite graphs which are nearly complete

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ABSTRACT

For positive integers p, q, r, s and t satisfying $rt \leqslant p$ and $st \leqslant q$, let $G(p, q; r, s; t)$ be the bipartite graph with partite sets $\{u_1, \ldots, u_p\}$ and $\{v_1, \ldots, v_q\}$ such that u_i and v_j are not adjacent if and only if there exists a positive integer k with $1 \leqslant k \leqslant t$ such that $(k-1)r + 1 \leqslant i \leqslant kr$ and $(k-1)s + 1 \leqslant j \leqslant ks$. In this paper we study the largest eigenvalues of bipartite graphs which are nearly complete. We first compute the largest eigenvalue (and all other eigenvalues) of $G(p, q; r, s; t)$, and then list nearly complete bipartite graphs according to the magnitudes of their largest eigenvalues. These results give an affirmative answer to [1, Conjecture 1.2] when the number of edges of a bipartite graph with partite sets U and V, having $|U| = p$ and $|V| = q$ for $p \leqslant q$, is $pq - 2$. Furthermore, we refine [1, Conjecture 1.2] for the case when the number of edges is at least $pq - p + 1$.

1. Introduction and preliminary

Let G be a (simple) graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{v_i, v_j | v_i$ and v_j are adjacent$\}$. If H is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G. If $H \neq G$, then H is a *proper* subgraph. If H is a proper subgraph of G, and there is no proper subgraph H' of G such that H is a proper subgraph of H', then H is a *maximal proper* subgraph of G.

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The *adjacency matrix of* G on n vertices is the n by n matrix $A(G)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i, v_j \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

Note that $A(G)$ is symmetric. Hence, all the eigenvalues of $A(G)$ are real. The eigenvalues of $A(G)$ are called *eigenvalues of the graph* G. We can list the eigenvalues of graph G in non-increasing order

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G).$$

By [4, Theorem 8.4.5], we have the following result.

Proposition 1. *If H is a subgraph of G, then*

$$\lambda_1(H) \leqslant \lambda_1(G).$$

A graph G is *bipartite* if its vertex set can be partitioned into two parts U and V, so called *partite sets*, such that every edge has one end in U and the other in V. Let G be a bipartite graph with partite sets $U = \{u_1, \ldots, u_p\}$ and $V = \{v_1, \ldots, v_q\}$, and A be the $(p+q)$ by $(p+q)$ adjacency matrix of G of the form

$$\begin{bmatrix} O & B \\ B^T & O \end{bmatrix}, \tag{1}$$

where $B = [b_{ij}]$ is the p by q matrix such that

$$b_{ij} = \begin{cases} 1, & \text{if } u_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

If B is the p by q matrix with each entry equal to 1, then the bipartite graph G is a *complete* bipartite graph, and denoted by $K_{p,q}$. The following two results describe spectral properties of bipartite graphs (Theorem 2; see [8, Theorem 8.6.9]) and the matrix product of the form BB^T (Proposition 3; see [4]).

Theorem 2. *Let G be a bipartite graph, and A be its adjacency matrix. Then the eigenvalues of A are symmetric about the origin, i.e., if $\lambda \in \sigma(A)$, then $-\lambda \in \sigma(A)$. Moreover, $\mu(\geqslant 0)$ is an eigenvalue of A^2 if and only if $\pm\sqrt{\mu}$ are eigenvalues of A.*

For each $\mathbf{x} \in \mathbb{R}^n$, if an n by n real symmetric matrix M satisfies $\mathbf{x}^T M \mathbf{x} \geqslant 0$, then M is a *positive semidefinite* matrix. It is well known that each eigenvalue of a positive semidefinite matrix is a nonnegative real number (see [4]).

Proposition 3. *Let B be a p by q matrix. Then*

(a) *The rank of B is equal to that of the p by p matrix BB^T.*
(b) *The matrix BB^T of order p is positive semidefinite.*
(c) *The number of nonzero eigenvalues of BB^T is equal to the rank of BB^T.*
(d) *If $p \leqslant q$, then the q by q matrix B^TB has the same eigenvalues as BB^T of order p, counting multiplicity, together with additional $(q-p)$ eigenvalues equal to 0.*

For positive integers p, q, r, s and t satisfying $rt \leqslant p$ and $st \leqslant q$, we define $G(p, q; r, s; t)$ to be the bipartite graph with partite sets $\{u_1, \ldots, u_p\}$ and $\{v_1, \ldots, v_q\}$ such that u_i and v_j are not adjacent if and only if there exists a positive integer k with $1 \leqslant k \leqslant t$ such that $(k-1)r + 1 \leqslant i \leqslant kr$ and $(k-1)s + 1 \leqslant j \leqslant ks$. In Fig. 1 there is an edge between u_i and v_j if and only if they are not connected by a dotted line.

In [2,3] the largest and the second largest eigenvalues of certain types of trees are obtained. This motivates our study of the largest eigenvalues of bipartite graphs which are close to a complete bipartite graph. In this paper we find all the eigenvalues of the bipartite graph $G(p, q; r, s; t)$ by computing eigenvalues of the square of the adjacency matrix of $G(p, q; r, s; t)$, and list its eigenvalues according to their

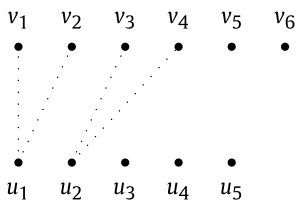

Fig. 1. $G(5,6;\ 1,2;\ 2)$.

magnitudes. Using this result, we list nearly complete bipartite graphs according to the magnitudes of their largest eigenvalues. These results give an affirmative answer to [1, Conjecture 1.2] when the number of edges of a bipartite graph with partite sets U and V, having $|U| = p$ and $|V| = q$ for $p \leqslant q$, is $pq - 2$. Furthermore, by Theorem 7, we refine [1, Conjecture 1.2] when the number of edges is at least $pq - p + 1$ (see Conjecture 11).

2. Eigenvalues of $G(p, q;\ r, s;\ t)$

Let A be the adjacency matrix of $G(p, q;\ r, s;\ t)$ of the form (1). If we can compute eigenvalues of A^2, then, by Theorem 2, we can find the eigenvalues of A. Note that

$$A^2 = \begin{bmatrix} O & B \\ B^T & O \end{bmatrix}\begin{bmatrix} O & B \\ B^T & O \end{bmatrix} = \begin{bmatrix} BB^T & O \\ O & B^TB \end{bmatrix}.$$

By Proposition 3(d), it suffices to compute eigenvalues of BB^T.

Let $(a)_{r\times s}$ be the r by s matrix each of whose entries is a. When $r = s$, we use $(a)_r$ to denote $(a)_{r\times r}$. Then

$$B = \left[\begin{array}{cccc|c} (0)_{r\times s} & (1)_{r\times s} & \cdots & (1)_{r\times s} & \\ (1)_{r\times s} & (0)_{r\times s} & \ddots & \vdots & \\ \vdots & \ddots & \ddots & (1)_{r\times s} & (1)_{rt\times(q-st)} \\ (1)_{r\times s} & \cdots & (1)_{r\times s} & (0)_{r\times s} & \\ \hline \multicolumn{4}{c|}{(1)_{(p-rt)\times st}} & (1)_{(p-rt)\times(q-st)} \end{array}\right],$$

and

$$BB^T = \left[\begin{array}{cccc|c} (q-s)_r & (q-2s)_r & \cdots & (q-2s)_r & \\ (q-2s)_r & (q-s)_r & \ddots & \vdots & \\ \vdots & \ddots & \ddots & (q-2s)_r & (q-s)_{rt\times(p-rt)} \\ (q-2s)_r & \cdots & (q-2s)_r & (q-s)_r & \\ \hline \multicolumn{4}{c|}{(q-s)_{(p-rt)\times rt}} & (q)_{(p-rt)} \end{array}\right].$$

Assume that $p > rt$ and $q > st$. Then it can be shown that, by elementary row and column operations, B has the same rank as that of the $(t+1)$ by $(t+1)$ matrix

$$C = \left[\begin{array}{cccc|c} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots \\ 1 & \cdots & 1 & 0 & 1 \\ \hline 1 & \cdots & & 1 & 1 \end{array}\right]. \tag{2}$$

If $t = 1$, then

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

which is nonsingular. For $t \geqslant 2$, note that the $(1, 1)$-block of C is $(1)_t - I_t$, which is nonsingular. Let R_i be the ith row of the matrix (2). By using the row operation $(t-1)R_{t+1}$ and then $R_{t+1} - R_i$ for each $i = 1, 2, \ldots, t$, we can obtain the following matrix whose rank is equal to that of C:

$$\left[\begin{array}{cccc|c} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots \\ 1 & \cdots & 1 & 0 & 1 \\ \hline 0 & \cdots & & 0 & -1 \end{array}\right].$$

This implies that the rank of C (and hence BB^T) is $t + 1$.

If $p = rt$ (resp. $q = st$), then the matrix C is obtained by deleting the last row (resp. the last column) from the matrix in the form (2). By Proposition 3(a) and (c), we get the following result.

Proposition 4. *If $p > rt$ and $q > st$, then BB^T has $t + 1$ nonzero eigenvalues, and if $p = rt$ or $q = st$, then BB^T has t nonzero eigenvalues.*

The p by p matrix BB^T can be rewritten as follows:

$$BB^T = (q - 2s)(1)_p + sM, \tag{3}$$

where

$$M = \left[\begin{array}{ccc|c} (1)_r & & O & \\ & \ddots & & (1)_{rt\times(p-rt)} \\ O & & (1)_r & \\ \hline & (1)_{(p-rt)\times rt} & & (2)_{p-rt} \end{array}\right]_{p\times p}. \tag{4}$$

In the following we find some eigenvalues and their corresponding eigenvectors of M and use them to find the eigenvalues of BB^T. The vector $\mathbf{e}_i$ denotes the vector with exactly one nonzero entry that is 1 and located in the ith position. For a square matrix M of order p and a p by 1 vector $\mathbf{x}$, if $M\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is a *nullvector* of M.

Proposition 5. *For $p > rt$ and $q > st$, let BB^T be of the form (3), and M be the p by p matrix of the form (4). Then the following hold:*

(a) *The $p - (t + 1)$ nonzero vectors in the set*

$$\{\mathbf{e}_{rk+1} - \mathbf{e}_{rk+j} | k = 0, 1, \ldots, t-1 \text{ and } j = 2, \ldots, r\} \cup \{\mathbf{e}_{rt+1} - \mathbf{e}_j | j = rt + 2, \ldots, p\} \tag{5}$$

are linearly independent nullvectors of M.

(b) *The $p - (t + 1)$ linearly independent vectors in (5) are nonzero nullvectors of BB^T.*

(c) *For $t \geqslant 2$, the scalar r is an eigenvalue of M with multiplicity $t - 1$, and the $t - 1$ nonzero vectors in the set*

$$\left\{ \mathbf{x}_k \,\middle|\, \mathbf{x}_k = \sum_{i=1}^{r} \mathbf{e}_i - \sum_{j=1}^{r} \mathbf{e}_{rk+j}, k = 1, \ldots, t-1 \right\} \tag{6}$$

are linearly independent eigenvectors corresponding to the eigenvalue r.

(d) *When $t \geqslant 2$, the $(t - 1)$ nonzero vectors in (6) are linearly independent eigenvectors of BB^T corresponding to the eigenvalue rs.*

Proof. A direct computation proves (a) and (c). Since the nonzero vectors in (5) and (6) are nullvectors of the matrix $(1)_p$ in (3), (b) and (d) follow. □

By Proposition 4, when $p > rt$ and $q > st$, there are $t + 1$ nonzero eigenvalues of BB^T. Hence, by Proposition 5, there are two more nonzero eigenvalues of BB^T to be computed. Let W_1 and W_2 be subspaces of $\mathbb{R}^p$. We say that W_1 and W_2 are *perpendicular* (denoted by $W_1 \perp W_2$) provided that for any vectors $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$, $\mathbf{w}_1$ is perpendicular to $\mathbf{w}_2$, i.e., $\mathbf{w}_1^T\mathbf{w}_2 = 0$. For a subspace W of $\mathbb{R}^p$, we define $W^\perp$ as follows:

$$W^\perp = \{\mathbf{z} | \mathbf{z}^T\mathbf{w} = 0 \text{ for each } \mathbf{w} \in W\}.$$

Let S be a p by p matrix. We say that W is *invariant under* S if $S\mathbf{w} \in W$ for each $\mathbf{w} \in W$. The following result can be found in [6, Theorems 4.2 and 4.3].

Theorem 6. *Let S be a p by p real symmetric matrix. Then the following hold:*

(a) *If E_1 and E_2 are the eigenspaces of S corresponding to distinct eigenvalues, then*
$$E_1 \perp E_2.$$
(b) *If W is a subspace of $\mathbb{R}^p$ which is invariant under S, then $W^\perp$ is also invariant under S.*

Let W be the vector space spanned by the vectors in (5) and (6). It can be verified that $W^\perp$ is spanned by $\mathbf{z}_1$ and $\mathbf{z}_2$, i.e.,

$$W^\perp = \langle \mathbf{z}_1, \mathbf{z}_2 \rangle,$$

where

$$\mathbf{z}_1 = \begin{bmatrix} (1)_{rt\times 1} \\ (0)_{(p-rt)\times 1} \end{bmatrix} \quad \text{and} \quad \mathbf{z}_2 = \begin{bmatrix} (0)_{rt\times 1} \\ (1)_{(p-rt)\times 1} \end{bmatrix}.$$

Moreover, by Theorem 6, the eigenvectors of BB^T corresponding to the remaining two nonzero eigenvalues are in $W^\perp$, and $(BB^T)\mathbf{z} \in W^\perp$ for every $\mathbf{z} \in W^\perp$. Let $\mathcal{Z} = \{\mathbf{z}_1, \mathbf{z}_2\}$. Then the eigenvalues of the 2 by 2 $\mathcal{Z}$-*matrix* for BB^T (see [7, p. 329]) are the remaining two eigenvalues of BB^T (see [5, Proposition 1.5.4]). To find the $\mathcal{Z}$-matrix for BB^T, we compute $(BB^T)\mathbf{z}_i$ for each $i = 1, 2$:

$$(BB^T)\mathbf{z}_1 = [r(q-s) + r(t-1)(q-2s)]\mathbf{z}_1 + [rt(q-s)]\mathbf{z}_2,$$
$$(BB^T)\mathbf{z}_2 = [(p-rt)(q-s)]\mathbf{z}_1 + [q(p-rt)]\mathbf{z}_2.$$

Hence, the 2 by 2 $\mathcal{Z}$-matrix for BB^T is

$$\begin{bmatrix} r(q-s) + r(t-1)(q-2s) & (p-rt)(q-s) \\ rt(q-s) & q(p-rt) \end{bmatrix}.$$

By computing the eigenvalues of the $\mathcal{Z}$-matrix for BB^T, we get the eigenvalues of BB^T:

$$\frac{pq - 2rst + rs \pm \sqrt{(pq - 2rst + rs)^2 - 4rs(p-rt)(q-st)}}{2}. \tag{7}$$

By Theorem 2, Propositions 3, 4 and 5 along with the eigenvalues of BB^T in (7), we have found all the eigenvalues of $G(p, q; r, s; t)$.

Theorem 7. *Let A be the adjacency matrix of the bipartite graph $G(p, q; r, s; t)$. Then the following hold:*

(a) *For $p > rt$, $q > st$ and $t \geqslant 2$, the eigenvalues of A are*

$$\lambda_1 = \sqrt{\frac{pq-2rst+rs+\sqrt{(pq-2rst+rs)^2-4rs(p-rt)(q-st)}}{2}} \geqslant \lambda_2 = \sqrt{rs}$$

$$\lambda_3 = \sqrt{\frac{pq-2rst+rs-\sqrt{(pq-2rst+rs)^2-4rs(p-rt)(q-st)}}{2}} \geqslant \lambda_4 = 0$$

$$\geqslant -\lambda_3 \geqslant -\lambda_2 \geqslant -\lambda_1.$$

Moreover, the multiplicities of $\pm\lambda_1$ and $\pm\lambda_3$ are 1, the multiplicities of $\pm\lambda_2$ are $t-1$, and the multiplicity of $\lambda_4 = 0$ is $(p+q)-2(t+1)$.

(b) *If $p = rt$ or $q = st$ with $t \geqslant 2$, then the eigenvalues of A are*

$$\lambda_1 = \sqrt{pq-2rst+rs} \geqslant \lambda_2 = \sqrt{rs} \geqslant \lambda_3 = 0 \geqslant -\lambda_2 \geqslant -\lambda_1.$$

(c) *If $t = 1$, then the eigenvalues of A are*

$$\lambda_1 = \sqrt{\frac{pq-rs+\sqrt{(pq-rs)^2-4rs(p-r)(q-s)}}{2}}$$

$$\geqslant \lambda_2 = \sqrt{\frac{pq-rs-\sqrt{(pq-rs)^2-4rs(p-r)(q-s)}}{2}}$$

$$\geqslant 0 \geqslant -\lambda_2 \geqslant -\lambda_1.$$

Proof. We here show that for nonnegative λ_1, λ_2 and λ_3 in (a), $\lambda_1^2 \geqslant \lambda_2^2 \geqslant \lambda_3^2$ when $p \geqslant rt$, $q \geqslant st$ and $t \geqslant 2$. Then the orders of eigenvalues according to their magnitudes in (a), (b) and (c) follow.

We first show that $\lambda_1^2 \geqslant \lambda_2^2$. Since $p \geqslant rt$ and $q \geqslant rt$, it follows that

$$\begin{aligned} pq-2rst+rs &\geqslant rst^2-2rst+rs \\ &= rs(t^2-2t+1) \\ &= rs(t-1)^2. \end{aligned}$$

Hence, for $t \geqslant 3$, $\lambda_1^2 \geqslant \lambda_2^2$. Let $t = 2$. Consider

$$\frac{pq-3rs+\sqrt{(pq-3rst)^2-4rs(p-2r)(q-2s)}}{2} - rs. \tag{8}$$

If $pq \geqslant 5rs$, then (8) is nonnegative and hence $\lambda_1^2 \geqslant \lambda_2^2$.

Suppose that $pq < 5rs$, i.e., $5rs - pq > 0$. We show that

$$(pq-3rs)^2 - 4rs(p-2r)(q-2s) - (5rs-pq)^2 \tag{9}$$

is nonnegative. By a simple calculation, it can be shown that (9) is equal to

$$8rs^2p + 8r^2sq - 32r^2s^2.$$

Since $p \geqslant 2r$ and $q \geqslant 2s$, we have

$$8rs^2p + 8r^2sq - 32r^2s^2 \geqslant 16r^2s^2 + 16r^2s^2 - 32r^2s^2 = 0.$$

Hence, $\lambda_1^2 \geqslant \lambda_2^2$.

Next, we show that $\lambda_2^2 \geqslant \lambda_3^2$ when $p \geqslant rt, q \geqslant st$ and $t \geqslant 2$. The difference $\lambda_2^2 - \lambda_3^2$ is equal to

$$\frac{rs + 2rst - pq + \sqrt{(pq - 2rst + rs)^2 - 4rs(p - rt)(q - st)}}{2}.$$

If $rs + 2rst - pq \geqslant 0$, $\lambda_2^2 \geqslant \lambda_3^2$.

Suppose that $rs + 2rst - pq < 0$, i.e., $pq - 2rst - rs > 0$. Then, in order to show $\lambda_2^2 \geqslant \lambda_3^2$, it suffices to show that

$$(pq - 2rst + rs)^2 - 4rs(p - rt)(q - st) - (pq - 2rst - rs)^2 \tag{10}$$

is nonnegative. By a simple calculation, it can be shown that (10) is equal to

$$4rs^2tp + 4r^2stq - 4r^2s^2t^2 - 8r^2s^2t.$$

Since $p \geqslant rt, q \geqslant st$ and $t \geqslant 2$, we have

$$\begin{aligned} 4rs^2tp + 4r^2stq - 4r^2s^2t^2 - 8r^2s^2t &\geqslant 4r^2s^2t^2 + 4r^2s^2t^2 - 4r^2s^2t^2 - 8r^2s^2t \\ &= 4r^2s^2t(t-2) \\ &\geqslant 0. \end{aligned}$$

Hence, the result follows. □

3. List of nearly complete bipartite graphs

We now list bipartite graphs, missing at most two edges from a complete bipartite graph, according to the magnitudes of their largest eigenvalues λ_1. We denote by $G^{(i)}$ the bipartite graph with the ith largest λ_1 among all bipartite graphs with $2n$ vertices.

Theorem 8. *For* $n \geqslant 3$, $G^{(1)} = K_{n,n}$, $G^{(2)} = K_{n-1,n+1}$, $G^{(3)} = G(n, n; 1, 1; 1)$, $G^{(4)} = G(n-1, n+1; 1, 1; 1)$, $G^{(5)} = G(n, n; 2, 1; 1)$, $G^{(6)} = G(n, n; 1, 1; 2)$, $G^{(7)} = K_{n-2,n+2}$, $G^{(8)} = G(n-1, n+1; 2, 1; 1)$, $G^{(9)} = G(n-1, n+1; 1, 2; 1)$ *and* $G^{(10)} = G(n-1, n+1; 1, 1; 2)$.

Proof. Let H be a subgraph of $K_{p,q}$ with $1 \leqslant p \leqslant q$ and $p + q = 2n$. Note that

$$\lambda_1(K_{p,q}) = \sqrt{pq} \tag{11}$$

and $\sqrt{pq} \leqslant n$. Furthermore, by Proposition 1, $\lambda_1(H) \leqslant \sqrt{pq}$. Hence,

$$G^{(1)} = K_{n,n}.$$

By Proposition 1 and the fact that $\sqrt{pq}$ with $p + q = 2n$ is increasing as the value of p grows from 1 to n, it is sufficient to consider $G(n, n; 1, 1; 1)$ and $K_{n-1,n+1}$ for $G^{(2)}$. By Theorem 7 and (11), we have $\lambda_1(G^{(2)}) = \max\left\{\frac{1}{\sqrt{2}}[(n^2-1) + (n^4 - 6n^2 + 8n - 3)^{1/2}]^{1/2}, (n^2-1)^{1/2}\right\}$. Note that $\frac{1}{\sqrt{2}}[(n^2-1) + (n^4 - 6n^2 + 8n - 3)^{1/2}]^{1/2} \leqslant \frac{1}{\sqrt{2}}[(n^2-1) + (n^4 - 6n^2 + 8n - 3 + 4n^2 - 8n + 4)^{1/2}]^{1/2} \leqslant (n^2 - 1)^{1/2}$. Hence,

$$G^{(2)} = K_{n-1,n+1}.$$

Similarly, in order to find $G^{(3)}$, it suffices to consider $G(n, n; 1, 1; 1)$, $G(n-1, n+1; 1, 1; 1)$ and $K_{n-2,n+2}$. We compute the largest largest eigenvalues of these three bipartite graphs by Theorem 7 and (11), and then use the facts, $n^4 - 6n^2 + 8n - 3 + 4n^2 - 8n + 4 \geqslant n^4 - 8n^2 + 8n + 4$ for $n \geqslant 3$ and $(n-4)^{1/2} = \frac{1}{\sqrt{2}}[(n^2-1) + (n^4 - 14n^2 + 49)^{1/2}]^{1/2}$ in order to compare the largest eigenvalues. This gives

$$G^{(3)} = G(n, n; 1, 1; 1).$$

Next, we consider the maximal proper subgraphs of $G^{(1)}, G^{(2)}, G^{(3)}$, and the complete bipartite graph $K_{n-2,n+2}$ for $G^{(4)}$, using Theorem 7 and (11). By repeating this process, considering the maximal subgraphs of $G^{(1)}, \ldots, G^{(i)}$, and a complete bipartite graph $K_{p,q}$ with $p + q = 2n$ (which was not considered in the previous steps), we can get $G^{(i+1)}$ for $i = 3, \ldots, 9$. □

Example 9. The following is the list of bipartite graphs with 40 vertices according to the magnitudes of the largest eigenvalues:

	Graph	λ_1
1.	$K_{20,20}$	20
2.	$K_{19,20}$	19.97498436
3.	$G(20, 20;\ 1, 1;\ 1)$	19.95227248
4.	$G(19, 21;\ 1, 1;\ 1)$	19.92720282
5.	$G(20, 20;\ 2, 1;\ 1)$	19.90663008
6.	$G(20, 20;\ 1, 1;\ 2)$	19.90432602
7.	$K_{18,20}$	19.89974874
8.	$G(19, 21;\ 2, 1;\ 1)$	19.88164226
9.	$G(19, 21;\ 1, 2;\ 1)$	19.88138664
10.	$G(19, 21;\ 1, 1;\ 2)$	19.87920160

The computation of λ_1 can be done by the open source mathematical software SAGE (see http://www.sagemath.org). The following is a SAGE code for $\lambda_1(G(20, 20;\ 2, 1;\ 1))$:

```
p = 20; q = 20; r = 2; s = 1; t = 1
def a(i,j):
  for k in [1..t]:
    if (i in [r * (k − 1) + 1..r * k]) and (j in [s * (k − 1) + 1..s * k]):
      return 0
  else:
    return 1

B = matrix([[a(i,j) for j in [1..q]] for i in [1..p]])
E = (B * B.transpose()).eigenvalues()
print sqrt(E[p − 1])
```

From Theorem 8 it follows that

$$\lambda_1(G(p, q;\ 2, 1;\ 1)) \geqslant \lambda_1(G(p, q;\ 1, 2;\ 1)) \geqslant \lambda_1(G(p, q;\ 1, 1;\ 2))$$

for $(p, q) \in \{(n, n), (n - 1, n + 1)\}$. This can be generalized to the case for any positive integers p, q with $2 \leqslant p \leqslant q$.

Proposition 10. *Let p and q be positive integers. If* $2 \leqslant p \leqslant q$, *then*

$$\lambda_1(G(p, q;\ 2, 1;\ 1)) \geqslant \lambda_1(G(p, q;\ 1, 2;\ 1)) \geqslant \lambda_1(G(p, q;\ 1, 1;\ 2)).$$

Proof. This can be shown by first computing the eigenvalues, using Theorem 7, and then using a direct comparison. □

Proposition 10 gives an affirmative answer to [1, Conjecture 1.2] when the number of edges of a bipartite graph with partite sets U and V, having $|U| = p$ and $|V| = q$ for $p \leqslant q$, is $pq - 2$. By Theorem 7, we can refine [1, Conjecture 1.2] when the number of edges is at least $pq - p + 1$.

Conjecture 11. *For positive integers p, q and k satisfying $p \leqslant q$ and $k < p$, let G be a bipartite graph with partite sets U and V, having $|U| = p$ and $|V| = q$, and $|E(G)| = pq - k$. Then*

$$\lambda_1(G) \leqslant \lambda_1(G(p, q; k, 1; 1)) = \sqrt{\frac{pq - k + \sqrt{p^2q^2 - 6pqk + 4pk + 4qk^2 - 3k^2}}{2}}.$$

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