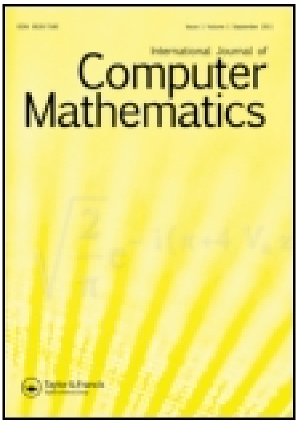

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The construction of mutually independent Hamiltonian cycles in bubble-sort graphs

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A Hamiltonian cycle $C = \langle u_1, u_2, \ldots, u_{n(G)}, u_1 \rangle$ with $n(G) =$ number of vertices of G, is a cycle $C(u_1; G)$, where u_1 is the beginning and ending vertex and u_i is the ith vertex in C and $u_i \neq u_j$ for any $i \neq j$, $1 \leq i, j \leq n(G)$. A set of Hamiltonian cycles $\{C_1, C_2, \ldots, C_k\}$ of G is *mutually independent* if any two different Hamiltonian cycles are independent. For a hamiltonian graph G, the *mutually independent Hamiltonianicity number* of G, denoted by $h(G)$, is the maximum integer k such that for any vertex u of G there exist k-mutually independent Hamiltonian cycles of G starting at u. In this paper, we prove that $h(B_n) = n - 1$ if $n \geq 4$, where B_n is the n-dimensional bubble-sort graph.

Keywords: Hamiltonian cycle; bubble-sort networks; interconnection networks; mutually independent Hamiltonian cycles; Cayley graph

2000 AMS Subject Classifications: 05C38; 05C45; 05C75; 05C90; 68M10

1. Introduction

Let H be a group, and let S be a generating set of H with $S^{-1} = S$. The Cayley graph on a group H with generating set S, denoted by Cay$(H; S)$, is the graph with vertex set H, and for two vertices u and v in H, u is adjacent to v if and only if $v = us$ for some $s \in S$. Hamiltonian cycles in Cayley graphs exist naturally in computing and communication [10], in the study of word-hyperbolic groups and automatic groups [6], in changing–ringing [13], in creating Escher-like repeating patterns in hyperbolic plane 1 [4], and in combinatorial designs [4]. It is conjectured that every connected Cayley graph with more than three vertices is Hamiltonian [3]. Up to now, this conjecture is unsolved. Yet, some Cayley graphs have many more Hamiltonian cycles than we expected. In this paper, we introduce and study the concept of mutually independent Hamiltonian (MIH) cycles in Cayley graphs.

For graph definitions and notations we follow [2]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. We use $n(G)$ to denote $|V|$. Let S be a nonempty subset of $V(G)$. The subgraph

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induced by S is the subgraph of G with its vertex set S and with its edge set consisting of all edges of G joining any two vertices in S. We use $G - S$ to denote the subgraph of G induced by $V - S$. Two vertices u and v are *adjacent* if (u, v) is an edge of G. The set of *neighbours* of u, denoted by $N_G(u)$, is $\{v \mid (u, v) \in E\}$. The *degree* of a vertex u of G, $\deg_G(u)$, is the number of edges incident with u. The *minimum degree* of G, $\delta(G)$, is $\min\{\deg_G(x) \mid x \in V\}$. A graph G is *k-regular* if $\deg_G(u) = k$ for every vertex u in G. A *path* between vertices v_0 and v_k is a sequence of vertices represented by $\langle v_0, v_1, \ldots, v_k\rangle$ with no repeated vertex and (v_i, v_{i+1}) is an edge of G for every i, $0 \le i \le k-1$. We use $Q(i)$ to denote the ith vertex v_i of $Q = \langle v_1, v_2, \ldots, v_k\rangle$. We also write the path $\langle v_0, v_1, \ldots, v_k\rangle$ as $\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k\rangle$, where Q is a path form v_i to v_j. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last. A *Hamiltonian cycle* of G is a cycle that traverses every vertex of G. A graph is *Hamiltonian* if it has a Hamiltonian cycle. A graph $G = (B \cup W, E)$ is *bipartite* with bipartition B and W if $V(G) = B \cup W$, $B \cap W = \emptyset$, and $E(G)$ is a subset of $\{(u, v) \mid u \in B \text{ and } v \in W\}$. Let G be a bipartite graph with bipartition B and W. We say that a Hamiltonian bipartite graph is *Hamiltonian laceable* if there is a Hamiltonian path between any pair of vertices $\{x, y\}$, where x in B and y in W. Let $a, b, m \in \mathbb{Z}$ with $m > 0$. Then a is said to be *congruent* to b *modulo* m, denoted $a \equiv b \mod m$, if $m|(a-b)$.

A Hamiltonian cycle $C(u_1; G)$ of a Hamiltonian graph G is described as $C(u_1; G) = \langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ to emphasize the order of vertices in C. Thus, u_1 is the beginning vertex and u_i is the ith vertex in C. Two Hamiltonian cycles of G beginning at the vertex x, $C_1 = C(u_1; G) = \langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ and $C_2 = C(v_1; G) = \langle v_1, v_2, \ldots, v_{n(G)}, v_1\rangle$, are *independent* if $x = u_1 = v_1$ and $u_i \neq v_i$ for every i, $2 \le i \le n(G)$. Let G be a Hamiltonian graph. A set of Hamiltonian cycles $\{C_1, C_2, \ldots, C_k\}$ of G is *mutually independent* if any two different Hamiltonian cycles are independent. The *mutually independent Hamiltonianicity number* of a Hamiltonian graph G, called the MIH *number* of G and denoted by $h(G)$, is the maximum integer k such that for any vertex u of G there exist k-mutually independent Hamiltonian cycles of G starting at u. Obviously, $h(G) \le \delta(G)$ for a Hamiltonian graph G. The concept of MIH cycles can be applied in many different areas. Interested readers can refer to [7, 9, 11, 12] for a more detailed introduction.

In this paper, we study MIH cycles of n-dimensional bubble-sort graph B_n. In the following section, we give some basic properties for the n-dimensional bubble-sort graph. In Section 3, we construct MIH cycles in B_n and compute $h(B_n)$, the MIH number of B_n.

2. The bubble-sort graphs

We set $\langle n\rangle = \{1, 2, \ldots, n\}$ if n is a positive integer and we set $\langle 0\rangle$ being the empty set. The *n-dimensional bubble-sort graph*, B_n, is the graph with vertex set $V(B_n) = \{u_1, \ldots, u_n \mid u_i \in \langle n\rangle$ and $u_i \neq u_j$ for $i \neq j\}$. The adjacency is defined as follows: $u_1, \ldots, u_{i-1}, u_i, \ldots, u_n$ is adjacent to $v_1, \ldots, v_{i-1}, v_i, \ldots, v_n$ through an edge of dimension i with $2 \le i \le n$ if $v_j = u_j$ for every $j \in \langle n\rangle - \{i-1, i\}$, $v_{i-1} = u_i$, and $v_i = u_{i-1}$, *i.e.*, swap u_{i-1} and u_i. The bubble-sort graphs B_2, B_3, and B_4 are illustrated in Figure 1. It is known that the connectivity of B_n is $(n-1)$. We use boldface to denote vertices in B_n. Hence, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ denote a sequence of vertices in B_n.

By definition, B_n is an $(n-1)$-regular graph with $n!$ vertices. We use $\mathbf{e}$ to denote the vertex $1, 2, \ldots, n$. It is known that B_n is a bipartite graph with one partite set containing those vertices corresponding to odd permutations and the other containing those vertices corresponding to even permutations. We use white vertices to represent those even permutation vertices and use black vertices to represent those odd permutation vertices. Let $\mathbf{u} = u_1, u_2, \ldots, u_n$ be an arbitrary vertex of the bubble-sort graph B_n. We say that u_i is the ith coordinate of $\mathbf{u}$, $(\mathbf{u})_i$, for $1 \le i \le n$. For

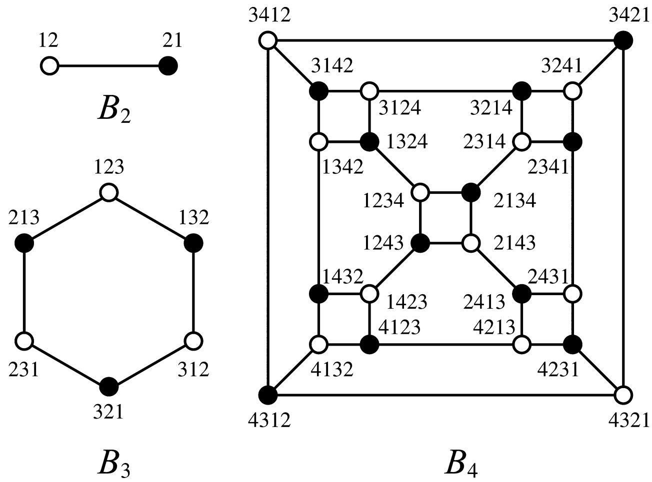

Figure 1. The graphs B_2, B_3, and B_4.

$1 \leq i \leq n$, let $B_{n-1}^{\{i\}}$ be the subgraph of B_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Then B_n can be decomposed into n subgraphs $B_{n-1}^{\{i\}}$, $1 \leq i \leq n$, and each $B_{n-1}^{\{i\}}$ is isomorphic to B_{n-1}. Thus, the bubble-sort graph can also be constructed recursively. Let I be any subset of $\langle n \rangle$. We use B_{n-1}^{I} to denote the subgraph of B_n induced by $\bigcup_{i \in I} V(B_{n-1}^{\{i\}})$. For any two distinct elements i and j in $\langle n \rangle$, we use $E_{n-1}^{i,j}$ to denote the set of edges between $B_{n-1}^{\{i\}}$ and $B_{n-1}^{\{j\}}$. By the definition of B_n, there is exactly one neighbour $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-dimensional edge with $2 \leq i \leq n$. For this reason, we use $(\mathbf{u})^i$ to denote the unique i-neighbour of $\mathbf{u}$. We have $((\mathbf{u})^i)^i = \mathbf{u}$ and $(\mathbf{u})^n \in B_{n-1}^{\{(\mathbf{u})_{n-1}\}}$.

LEMMA 1 *Let i and j be any two distinct elements in $\langle n \rangle$ with $n \geq 3$. Then $|E_{n-1}^{i,j}| = (n-2)!$. Moreover, there are $(n-2)!/2$ edges joining black vertices of $B_{n-1}^{\{i\}}$ to white vertices of $B_{n-1}^{\{j\}}$.*

THEOREM 1 (See [8]) *The bubble-sort graph B_n is Hamiltonian laceable if and only if $n \neq 3$.*

THEOREM 2 (See [1]) *Let $\mathbf{x}$ be a black vertex in B_n with $n \geq 4$. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two distinct white vertices in B_n. There is a Hamiltonian path of $B_n - \{\mathbf{x}\}$ joining $\mathbf{u}$ to $\mathbf{v}$.*

LEMMA 2 *Let $I = \{a_1, a_2, \ldots, a_r\}$ be a subset of $\langle n \rangle$ for some $r \in \langle n \rangle$ with $n \geq 5$. Assume that $\mathbf{u}$ is a white vertex in $B_{n-1}^{\{a_1\}}$ and $\mathbf{v}$ is a black vertex in $B_{n-1}^{\{a_r\}}$. Then there is a Hamiltonian path $\langle \mathbf{u} = \mathbf{x}_1, H_1, \mathbf{y}_1, \mathbf{x}_2, H_2, \mathbf{y}_2, \ldots, \mathbf{x}_r, H_r, \mathbf{y}_r = \mathbf{v} \rangle$ of B_{n-1}^{I} joining $\mathbf{u}$ to $\mathbf{v}$ such that $\mathbf{x}_1 = \mathbf{u}$, $\mathbf{y}_r = \mathbf{v}$, and H_i is a Hamiltonian path of $B_{n-1}^{\{a_i\}}$ joining $\mathbf{x}_i$ to $\mathbf{y}_i$ for every i, $1 \leq i \leq r$.*

Proof We set $\mathbf{x}_1 = \mathbf{u}$ and $\mathbf{y}_r = \mathbf{v}$. By Theorem 1, this lemma holds for $r = 1$. Suppose that $r \geq 2$. By Lemma 1, there are $(n-2)!/2 \geq 3$ edges joining black vertices of $B_{n-1}^{\{a_i\}}$ to white vertices of $B_{n-1}^{\{a_{i+1}\}}$ for every $i \in \langle r-1 \rangle$. We can choose an edge $(\mathbf{y_i}, \mathbf{x_{i+1}}) \in E_{n-1}^{a_i, a_{i+1}}$ with $\mathbf{y}_i$ being a black vertex and $\mathbf{x}_{i+1}$ being a white vertex for every $i \in \langle r-1 \rangle$. By Theorem 1, there is a Hamiltonian path H_i of $B_{n-1}^{\{a_i\}}$ joining $\mathbf{x}_i$ to $\mathbf{y}_i$ for every $i \in \langle r \rangle$. Then the path $\langle \mathbf{u} = \mathbf{x}_1, H_1, \mathbf{y}_1, \mathbf{x}_2, H_2, \mathbf{y}_2, \ldots, \mathbf{x}_r, H_r, \mathbf{y}_r = \mathbf{v} \rangle$ is the desired path. ■

LEMMA 3 *Let $B_{n-1}^{\{a\}}$ and $B_{n-1}^{\{b\}}$ be two distinct subgraphs of B_n with $n \geq 5$. Let* **s** *be a black vertex in $B_{n-1}^{\{a\}}$, let* **t** *be a white vertex in $B_{n-1}^{\{b\}}$, let* **u** *be a white vertex in $B_{n-1}^{\{a\}}$, and let* **v** *be a black vertex in $B_{n-1}^{\{b\}}$. Then there is a Hamiltonian path of $B_{n-1}^{\{a,b\}} - \{\mathbf{s}, \mathbf{t}\}$ joining* **u** *to* **v**.

Proof Let **x** be a white vertex in $B_{n-1}^{\{a\}} - \{\mathbf{u}\}$ with $(\mathbf{x})^n$ being a black vertex in $B_{n-1}^{\{b\}} - \{\mathbf{v}\}$. By Theorem 2, there are Hamiltonian paths: Q_1 of $B_{n-1}^{\{a\}} - \{\mathbf{s}\}$ joining **u** to **x**; Q_2 of $B_{n-1}^{\{b\}} - \{\mathbf{t}\}$ joining $(\mathbf{x})^n$ to **v**. Then $\langle \mathbf{u}, Q_1, \mathbf{x}, (\mathbf{x})^n, Q_2, \mathbf{v} \rangle$ is the Hamiltonian path of $B_{n-1}^{\{a,b\}} - \{\mathbf{s}, \mathbf{t}\}$ joining **u** to **v**. ■

LEMMA 4 *For $n \geq 5$, let* **u** *be a black vertex in $B_{n-1}^{\{n\}}$ and let* **v** *be a white vertex in $B_{n-1}^{\{1\}}$. Then there is a Hamiltonian path of $B_n - \{\mathbf{e}, (\mathbf{e})^n\}$ joining* **u** *to* **v**.

Proof Let **y** be a white vertex in $B_{n-1}^{\{n-1\}}$ with $(\mathbf{y})_{n-1} = n - 2$. By Lemma 3, there is a Hamiltonian path Q_1 of $B_{n-1}^{\{n-1,n\}} - \{\mathbf{e}, (\mathbf{e})^n\}$ joining **u** to **y**. By Lemma 2, there is a Hamiltonian path Q_2 of $\bigcup_{i=1}^{n-2} B_{n-1}^{\{i\}}$ joining the black vertex $(\mathbf{y})^n$ to **v**. Then $\langle \mathbf{u}, Q_1, \mathbf{y}, (\mathbf{y})^n, Q_2, \mathbf{v} \rangle$ is a Hamiltonian path of $B_n - \{\mathbf{e}, (\mathbf{e})^n\}$ joining **u** to **v**. ■

3. The MIH property of B_n

For every i in $\langle n-1 \rangle$ with $n \geq 5$, we set $\mathbf{z}_0^i = (\mathbf{e})^{i+1}$ and we set $\mathbf{z}_j^i = (\mathbf{z}_{j-1}^i)^{i+j+1}$ for any j in $\langle n-i-1 \rangle$. Let A_5^5 be the empty set, let $A_5^i = \{\mathbf{z}_j^i \mid j \in \langle 4-i \rangle \cup \{0\}\}$ for any i in $\langle 4 \rangle$, and let $A_n^i = \{\mathbf{z}_j^i \mid j \in \langle n-i-1 \rangle \cup \{0\}\}$ for any i in $\langle n-1 \rangle$ for $n \geq 6$. We set $X_n^1 = A_n^1 \cup A_n^2 \cup \{\mathbf{e}\}$, $X_n^2 = A_n^2 \cup A_n^4 \cup \{\mathbf{e}\}$, $X_n^3 = A_n^3 \cup A_n^4 \cup \{\mathbf{e}\}$, and $X_n^4 = A_n^3 \cup A_n^5 \cup \{\mathbf{e}\}$ for any $n \geq 5$. We set $Y_n^i = A_n^i \cup A_n^{i+1} \cup \{\mathbf{e}\}$ for $n \geq 6$ and for $3 \leq i \leq n-2$.

LEMMA 5 *There is a Hamiltonian path of $B_5 - X_5^1$ joining a vertex* **u** *with $(\mathbf{u})_5 = 5$ to a vertex* **v** *with $(\mathbf{v})_5 = 1$ such that the colour of* **u** *and the colour of* **v** *are distinct.*

Proof We set $Q_1 = \langle$12435, 21435, 24135, 24315, 42315, 42135, 41235, 14235, 14325, 41325, 43125, 34125, 31425, 31245, 32145, 32415, 34215, 43215$\rangle$. Note that Q_1 is a Hamiltonian path of $B_4^{\{5\}} - (X_5^1 - \{\mathbf{z}_3^1 = 23451, \mathbf{z}_2^2 = 13452\})$ joining the black vertex 12435 to the white vertex 43215.

Case 1 Suppose that **u** is a black vertex and **v** is a white vertex. We set $\mathbf{u} = 12435$ and $\mathbf{x} = 43215$. Let **w** be a black vertex in $B_4^{\{1\}}$ with $(\mathbf{w})_4 = 2$. By Theorem 2, there is a Hamiltonian path Q_2 of $B_4^{\{1\}} - \{\mathbf{z}_3^1\}$ joining the black vertex $(\mathbf{x})^5$ to **w**. Let **y** be any black vertex in $B_4^{\{1\}}$ with $(\mathbf{y})_4 = 4$. Without loss of generality, we write $Q_2 = \langle (\mathbf{x})^5, R_1, \mathbf{y}, \mathbf{m}, R_2, \mathbf{w} \rangle$. By Theorem 2, there is a Hamiltonian path Q_3 of $B_4^{\{2\}} - \{\mathbf{z}_2^2\}$ joining a white vertex **s** with $(\mathbf{s})_4 = 3$ to $(\mathbf{w})^5$. By Lemma 2, there is a Hamiltonian path Q_4 of $B_4^{\{3,4\}}$ joining the white vertex $(\mathbf{y})^5$ to the black vertex $(\mathbf{s})^5$. We let $\mathbf{v} = \mathbf{m}$. Then $\langle \mathbf{u}, Q_1, \mathbf{x}, (\mathbf{x})^5, R_1, \mathbf{y}, (\mathbf{y})^5, Q_4, (\mathbf{s})^5, \mathbf{s}, Q_3, (\mathbf{w})^5, \mathbf{w}, R_2^{-1}, \mathbf{m} = \mathbf{v} \rangle$ is the desired path.

Case 2 Suppose that **u** is a white vertex and **v** is a black vertex. We set $\mathbf{u} = 43215$ and $\mathbf{x} = 12435$. Let **v** be a black vertex in $B_4^{\{1\}}$, and let **s** be a white vertex in $B_4^{\{2\}}$ with $(\mathbf{s})_4 = 4$. By Lemma 3, there is a Hamiltonian path Q_2 of $B_4^{\{1,2\}} - \{\mathbf{z}_3^1, \mathbf{z}_2^2\}$ joining **s** to **v**. By Lemma 2, there is a Hamiltonian path Q_3 of $B_4^{\{3,4\}}$ joining the white vertex $(\mathbf{x})^5$ to the black vertex $(\mathbf{s})^5$. Then $\langle \mathbf{u}, Q_1^{-1}, \mathbf{x}, (\mathbf{x})^5, Q_3, (\mathbf{s})^5, \mathbf{s}, Q_2, \mathbf{v} \rangle$ is the desired path. ■

LEMMA 6 *For $n \geq 5$, there is a Hamiltonian path of $B_n - X_n^1$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_n = n$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_n = 1$ such that the colour of $\mathbf{u}$ and the colour of $\mathbf{v}$ are distinct.*

Proof We prove this statement by induction on n. By Lemma 5, this statement holds for $n = 5$. We suppose that this statement holds for $n - 1$ with $n \geq 6$. We have the following cases.

Case 1 Suppose that $\mathbf{u}$ is a black vertex and $\mathbf{v}$ is a white vertex.

Case 1.1 Suppose that n is even. Thus, $\mathbf{z}_{n-2}^1$ is a black vertex in $B_{n-1}^{\{1\}}$ and $\mathbf{z}_{n-3}^2$ is a white vertex in $B_{n-1}^{\{2\}}$. By induction, there is a Hamiltonian path Q_1 of $B_{n-1}^{\{n\}} - (X_n^1 - \{\mathbf{z}_{n-2}^1, \mathbf{z}_{n-3}^2\})$ joining $\mathbf{u}$ to a white vertex $\mathbf{q}$ with $(\mathbf{q})_{n-1} = 1$. Obviously, $(\mathbf{q})^n$ is the black vertex in $B_{n-1}^{\{1\}}$. Let $\mathbf{s}$ and $\mathbf{w}$ be two white vertices in $B_{n-1}^{\{1\}}$ with $(\mathbf{s})_{n-1} = n - 1$ and $(\mathbf{w})_{n-1} = 2$. By Theorem 2, there is a Hamiltonian path Q_2 of $B_{n-1}^{\{1\}} - \{\mathbf{z}_{n-2}^1\}$ joining $\mathbf{w}$ to $\mathbf{s}$. Without loss of generality, we write $Q_2 = \langle \mathbf{w}, R_1, \mathbf{m}, (\mathbf{q})^n, R_2, \mathbf{s}\rangle$. Let $\mathbf{t}$ be a black vertex in $B_{n-1}^{\{2\}}$ with $(\mathbf{t})_{n-1} = 3$. By Theorem 2, there is a Hamiltonian path Q_3 of $B_{n-1}^{\{2\}} - \{\mathbf{z}_{n-3}^2\}$ joining $\mathbf{t}$ to the black vertex $(\mathbf{w})^n$. By Lemma 2, there is a Hamiltonian path Q_4 of $\bigcup_{i=3}^{n-1} B_{n-1}^{\{i\}}$ joining the black vertex $(\mathbf{s})^n$ in $B_{n-1}^{\{n-1\}}$ to the white vertex $(\mathbf{t})^n$ in $B_{n-1}^{\{3\}}$. We set $\mathbf{v} = \mathbf{m}$. Then $\langle \mathbf{u}, Q_1, \mathbf{q}, (\mathbf{q})^n, R_2, \mathbf{s}, (\mathbf{s})^n, Q_4, (\mathbf{t})^n, \mathbf{t}, Q_3, (\mathbf{w})^n, \mathbf{w}, R_1, \mathbf{m} = \mathbf{v}\rangle$ is the desired path.

Case 1.2 Suppose that n is odd. Thus, $\mathbf{z}_{n-2}^1$ is a white vertex in $B_{n-1}^{\{1\}}$ and $\mathbf{z}_{n-3}^2$ is a black vertex in $B_{n-1}^{\{2\}}$. The proof of this case is similar to Case 1.1.

Case 2 Suppose that $\mathbf{u}$ is a white vertex and $\mathbf{v}$ is a black vertex.

Case 2.1 Suppose that n is even. Thus, $\mathbf{z}_{n-2}^1$ is a black vertex in $B_{n-1}^{\{1\}}$ and $\mathbf{z}_{n-3}^2$ is a white vertex in $B_{n-1}^{\{2\}}$. By induction, there is a hamiltonian path Q_1 of $B_{n-1}^{\{n\}} - (X_n^1 - \{\mathbf{z}_{n-2}^1, \mathbf{z}_{n-3}^2\})$ joining $\mathbf{u}$ to a black vertex $\mathbf{p}$ with $(\mathbf{p})_{n-1} = n - 1$. Let $\mathbf{s}$ and $\mathbf{t}$ be any two white vertices in $B_{n-1}^{\{1\}}$ with $(\mathbf{s})_{n-1} = 2$ and $(\mathbf{t})_{n-1} = 2$. By Theorem 2, there is a Hamiltonian path Q_2 of $B_{n-1}^{\{1\}} - \{\mathbf{z}_{n-2}^1\}$ joining $\mathbf{s}$ to $\mathbf{t}$. Let $\mathbf{y}$ be any white vertex in $B_{n-1}^{\{1\}} - \{\mathbf{s}, \mathbf{t}\}$ with $(\mathbf{y})_{n-1} = 3$. Without loss of generality, we write $Q_2 = \langle \mathbf{s}, R_1, \mathbf{y}, \mathbf{m}, R_2, \mathbf{t}\rangle$. By Theorem 2, there is a Hamiltonian path Q_3 of $B_{n-1}^{\{2\}} - \{\mathbf{z}_{n-3}^2\}$ joining the black vertex $(\mathbf{s})^n$ to the black vertex $(\mathbf{t})^n$. By Lemma 2, there is a Hamiltonian path Q_4 of $\bigcup_{i=3}^{n-1} B_{n-1}^{\{i\}}$ joining the white vertex $(\mathbf{p})^n$ to the black vertex $(\mathbf{y})^n$. Let $\mathbf{v} = \mathbf{m}$. Then $\langle \mathbf{u}, Q_1, \mathbf{p}, (\mathbf{p})^n, Q_4, (\mathbf{y})^n, \mathbf{y}, R_1^{-1}, \mathbf{s}, (\mathbf{s})^n, Q_3, (\mathbf{t})^n, \mathbf{t}, R_2^{-1}, \mathbf{m} = \mathbf{v}\rangle$ is the desired path.

Case 2.2 Suppose that n is odd. Thus, $\mathbf{z}_{n-2}^1$ is a white vertex in $B_{n-1}^{\{1\}}$ and $\mathbf{z}_{n-3}^2$ is a black vertex in $B_{n-1}^{\{2\}}$. Note that n is odd. The proof of this case is similar to Case 2.1. ■

LEMMA 7 *There is a Hamiltonian path of $B_5 - X_5^2$ joining a white vertex $\mathbf{u}$ with $(\mathbf{u})_5 = 5$ to a white vertex $\mathbf{v}$ with $(\mathbf{v})_5 = 1$.*

Proof We set $Q_1 = \langle 24135, 42135, 41235, 41325, 14325, 14235, 12435, 21435, 21345, 23145, 32145, 31245, 31425, 34125, 43125, 43215, 42315, 24315, 23415, 32415, 34215\rangle$ being a hamiltonian path of $B_4^{\{5\}} - (X_5^2 - \{\mathbf{z}_2^2 = 13452, \mathbf{z}_0^4 = 12354\})$ joining the black vertex $\mathbf{p} = 24135$ to the black vertex $\mathbf{q} = 34215$. Let $\mathbf{r}$ be any black vertex in Q_1 with $(\mathbf{r})_4 = 4$. Obviously, $(\mathbf{q})^5$ is a white vertex in $B_4^{\{1\}}$. Without loss of generality, we rewrite $Q_1 = \langle \mathbf{p}, R_1, \mathbf{m}, \mathbf{r}, R_2, \mathbf{q}\rangle$. Let $\mathbf{s}$ be a white vertex in $B_4^{\{4\}}$ with $(\mathbf{s})_4 = 1$, and let $\mathbf{w}$ be a black vertex in $B_4^{\{3\}}$ with $(\mathbf{w})_4 = 2$. By Theorem 1, there is a Hamiltonian path Q_2 of $B_4^{\{3\}}$ joining the white vertex $(\mathbf{p})^5$ to $\mathbf{w}$. By Lemma 3, there is a

Hamiltonian path Q_3 of $B_4^{\{1,2\}} - \{\mathbf{z}_2^2, (\mathbf{q})^5\}$ joining the white vertex $(\mathbf{w})^5$ to the black vertex $(\mathbf{s})^5$. By Theorem 2, there is Hamiltonian path Q_4 of $B_4^{\{4\}} - \{\mathbf{z}_0^4\}$ joining $\mathbf{s}$ to $(\mathbf{r})^5$. We set $\mathbf{v} = (\mathbf{q})^5$ and $\mathbf{u} = \mathbf{m}$. Then $\langle \mathbf{u} = \mathbf{m}, R_1^{-1}, \mathbf{p}, (\mathbf{p})^5, Q_2, \mathbf{w}, (\mathbf{w})^5, Q_3, (\mathbf{s})^5, \mathbf{s}, Q_4, (\mathbf{r})^5, \mathbf{r}, R_2, \mathbf{q}, (\mathbf{q})^5 = \mathbf{v}\rangle$ is the desired path. ■

LEMMA 8 *For $n \geq 5$, there is a Hamiltonian path of $B_n - X_n^2$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_n = n$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_n = 1$, where both $\mathbf{u}$ and $\mathbf{v}$ are black vertices if n is even, and both $\mathbf{u}$ and $\mathbf{v}$ are white vertices if n is odd.*

Proof We prove this statement by induction on n. By Lemma 7, this statement holds on $n = 5$. We suppose that this statement holds on $n - 1$ with $n \geq 6$.

Case 1 Suppose that n is even. It is easy to know that $\mathbf{z}_{n-3}^2$ and $\mathbf{z}_{n-5}^4$ are two white vertices. By induction, there is a Hamiltonian path Q_1 of $B_{n-1}^{\{n\}} - (X_n^2 - \{\mathbf{z}_{n-3}^2, \mathbf{z}_{n-5}^4\})$ joining a white vertex $\mathbf{p}$ with $(\mathbf{p})_{n-1} = n - 1$ to a white vertex $\mathbf{q}$ with $(\mathbf{q})_{n-1} = 1$. Obviously, $(\mathbf{q})^n$ is a black vertex in $B_{n-1}^{\{1\}}$. Let $\mathbf{t}$ be a white vertex in Q_1 with $(\mathbf{t})_{n-1} = 2$. We rewrite $Q_1 = \langle \mathbf{p}, R_1, \mathbf{m}, \mathbf{t}, R_2, \mathbf{q}\rangle$. Let $\mathbf{s}$ be a black vertex in $B_{n-1}^{\{2\}}$ with $(\mathbf{s})_{n-1} = 1$, and let $\mathbf{w}$ be a black vertex in $B_{n-1}^{\{4\}}$ with $(\mathbf{w})_{n-1} = 3$. By Lemma 2, there is a Hamiltonian path Q_2 of $(\cup_{i=5}^{n-1} B_{n-1}^{\{i\}} \bigcup B_{n-1}^{\{3\}})$ joining the black vertex $(\mathbf{p})^n$ to the white vertex $(\mathbf{w})^n$. By Lemma 3, there is a Hamiltonian path Q_3 of $B_{n-1}^{\{1,4\}} - \{\mathbf{z}_{n-5}^4, (\mathbf{q})^n\}$ joining $\mathbf{w}$ to white vertex $(\mathbf{s})^n$. By Theorem 2, there is a Hamiltonian path Q_4 of $B_{n-1}^{\{2\}} - \{\mathbf{z}_{n-3}^2\}$ joining $\mathbf{s}$ to the black $(\mathbf{t})^n$. We set $\mathbf{v} = (\mathbf{q})^n$ and $\mathbf{u} = \mathbf{m}$. Then $\langle \mathbf{u} = \mathbf{m}, R_1^{-1}, \mathbf{p}, (\mathbf{p})^n, Q_2, (\mathbf{w})^n, \mathbf{w}, Q_3, (\mathbf{s})^n, \mathbf{s}, Q_4, (\mathbf{t})^n, \mathbf{t}, R_2, \mathbf{q}, (\mathbf{q})^n = \mathbf{v}\rangle$ is the desired path.

Case 2 Suppose that n is odd. It is easy to know that $\mathbf{z}_{n-3}^2$ and $\mathbf{z}_{n-5}^4$ are two black vertices. The proof of this case is similar to Case 1.

This completes the proof. ■

LEMMA 9 *There is a Hamiltonian path of $B_5 - X_5^3$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_5 = 5$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_5 = 1$ such that the colour of $\mathbf{u}$ and the colour of $\mathbf{v}$ are distinct.*

Proof We set $Q_1 = \langle$21435, 21345, 23145, 23415, 32415, 32145, 31245, 13245, 13425, 31425, 34125, 43125, 41325, 14325, 14235, 41235, 42135, 24135, 24315, 42314, 43215, 34215$\rangle$. Obviously, Q_1 is a Hamiltonian path of $B_4^{\{5\}} - (X_5^3 - \{\mathbf{z}_1^3 = 12453, \mathbf{z}_0^4 = 12354)\}$ joining the white vertex 21435 to the black vertex 34215.

Case 1 Suppose $\mathbf{u}$ is a white vertex and $\mathbf{v}$ is a black vertex. We set $\mathbf{u} = 21435$ and $\mathbf{x} = 34215$. Obviously, $(\mathbf{x})^5$ is the white vertex in $B_4^{\{1\}}$. Let $\mathbf{w}$ be a black vertex in $B_4^{\{1\}}$ with $(\mathbf{w})_4 = 2$. Note that $(\mathbf{w})^5$ is the white vertex in $B_4^{\{2\}}$. By Theorem 1, there is a Hamiltonian path Q_2 of $B_4^{\{1\}}$ joining $(\mathbf{x})^5$ to $\mathbf{w}$. Let $\mathbf{y}$ be any white vertex in $B_4^{\{1\}}$ with $(\mathbf{y})_4 = 3$. Without loss of generality, we write $Q_2 = \langle (\mathbf{x})^5, R_1, \mathbf{y}, \mathbf{m}, R_2, \mathbf{w}\rangle$. Let $\mathbf{t}$ be a white vertex in $B_4^{\{4\}}$ with $(\mathbf{t})_4 = 2$. By Lemma 3, there is a Hamiltonian path Q_3 of $B_4^{\{3,4\}} - \{\mathbf{z}_1^3, \mathbf{z}_0^4\}$ joining the black vertex $(\mathbf{y})^5$ to $\mathbf{t}$. By Theorem 1, there is a Hamiltonian path Q_4 of $B_4^{\{2\}}$ joining the black vertex $(\mathbf{t})^5$ to $(\mathbf{w})^5$. Let $\mathbf{v} = \mathbf{m}$. Then $\langle \mathbf{u}, Q_1, \mathbf{x}, (\mathbf{x})^5, R_1, \mathbf{y}, (\mathbf{y})^5, Q_3, \mathbf{t}, (\mathbf{t})^5, Q_4, (\mathbf{w})^5, \mathbf{w}, R_2^{-1}, \mathbf{m} = \mathbf{v}\rangle$ is the desired path.

Case 2 Suppose $\mathbf{u}$ is a black vertex and $\mathbf{v}$ is a white vertex. We set $\mathbf{u} = 34215$ and $\mathbf{x} = 21435$. Let $\mathbf{y}$ be a white vertex in $B_4^{\{4\}}$ with $(\mathbf{y})_4 = 2$. By Lemma 3, there is a Hamiltonian path Q_2 of $B_4^{\{3,4\}} - \{\mathbf{z}_1^3, \mathbf{z}_0^4\}$ joining the black vertex $(\mathbf{x})^5$ to $\mathbf{y}$. Let $\mathbf{v}$ be any white vertex in $B_4^{\{1\}}$. By Lemma 2, there is a Hamiltonian path Q_3 of $B_4^{\{1,2\}}$ joining the black vertex $(\mathbf{y})^5$ to the white vertex $\mathbf{v}$. Then $\langle \mathbf{u}, Q_1^{-1}, \mathbf{x}, (\mathbf{x})^5, Q_2, \mathbf{y}, (\mathbf{y})^5, Q_3, \mathbf{v}\rangle$ is the desired path. ■

LEMMA 10 *For $n \geq 5$, there is a hamiltonian path of $B_n - X_n^3$ joining a vertex* **u** *with $(\mathbf{u})_n = n$ to a vertex* **v** *with $(\mathbf{v})_n = 1$, where the colour of* **u** *and the colour of* **v** *are distinct.*

Proof We prove this statement by induction on n. By Lemma 9, this statement holds for $n = 5$. We suppose that this statement holds for $n - 1$ with $n - 1 \geq 5$.

Case 1 Suppose that **u** is a white vertex and **v** is a black vertex.

Case 1.1 Suppose that n is even. Thus, $\mathbf{z}_{n-4}^3$ is a black vertex in $B_{n-1}^{\{3\}}$ and $\mathbf{z}_{n-5}^4$ is a white vertex in $B_{n-1}^{\{4\}}$. By induction, there is a Hamiltonian path Q_1 of $B_{n-1}^{\{n\}} - (X_n^3 - \{\mathbf{z}_{n-4}^3, \mathbf{z}_{n-5}^4\})$ joining a white vertex **u** to a black vertex **q** with $(\mathbf{q})_{n-1} = 1$. Obviously, $(\mathbf{q})^n$ is a white vertex in $B_{n-1}^{\{1\}}$. Let **w** be a black vertex in $B_{n-1}^{\{1\}}$ with $(\mathbf{w})_{n-1} = 2$. By Theorem 1, there is a Hamiltonian path Q_2 of $B_{n-1}^{\{1\}}$ joining $(\mathbf{q})^n$ to **w**. Let **y** be any white vertex in $B_{n-1}^{\{1\}}$ with $(\mathbf{y})_{n-1} = 4$. Without loss of generality, we rewrite $Q_2 = \langle (\mathbf{q})^n, R_1, \mathbf{y}, \mathbf{m}, R_2, \mathbf{w}\rangle$. Let **t** be a white vertex in $B_{n-1}^{\{3\}}$ with $(\mathbf{t})_{n-1} = 5$. By Lemma 3, there is a Hamiltonian path Q_3 of $B_{n-1}^{\{3,4\}} - \{\mathbf{z}_{n-4}^3, \mathbf{z}_{n-5}^4\}$ joining the black vertex $(\mathbf{y})^n$ to **t**. By Lemma 2, there is a Hamiltonian path Q_4 of $(\bigcup_{i=5}^{n-1} B_{n-1}^{\{i\}} \bigcup B_{n-1}^{\{2\}})$ joining the black vertex $(\mathbf{t})^n$ in $B_{n-1}^{\{5\}}$ to the white vertex $(\mathbf{w})^n$ in $B_{n-1}^{\{2\}}$. We let $\mathbf{v} = \mathbf{m}$. Then $\langle \mathbf{u}, Q_1, \mathbf{q}, (\mathbf{q})^n, R_1, \mathbf{y}, (\mathbf{y})^n, Q_3, \mathbf{t}, (\mathbf{t})^n, Q_4, (\mathbf{w})^n, \mathbf{w}, R_2^{-1}, \mathbf{m} = \mathbf{v}\rangle$ is the desired path.

Case 1.2 Suppose that n is odd. Thus, $\mathbf{z}_{n-4}^3$ is a white vertex in $B_{n-1}^{\{3\}}$ and $\mathbf{z}_{n-5}^4$ is a black vertex in $B_{n-1}^{\{4\}}$. The proof of this case is similar to Case 1.1.

Case 2 Suppose that **u** is a black vertex and **v** is a white vertex.

Case 2.1 Suppose that n is even. Thus, $\mathbf{z}_{n-4}^3$ is a black vertex in $B_{n-1}^{\{3\}}$ and $\mathbf{z}_{n-5}^4$ is a white vertex in $B_{n-1}^{\{4\}}$. By induction, there is a Hamiltonian path Q_1 of $B_{n-1}^{\{n\}} - (X_n^3 - \{\mathbf{z}_{n-4}^3, \mathbf{z}_{n-5}^4\})$ joining a black vertex **u** to a white vertex **q** with $(\mathbf{q})_{n-1} = n - 1$. Obviously, $(\mathbf{q})^n$ is a black vertex in $B_{n-1}^{\{n-1\}}$. Let **y** be a white vertex in $B_{n-1}^{\{2\}}$ with $(\mathbf{y})_{n-1} = 4$, and let **s** be a white vertex in $B_{n-1}^{\{3\}}$ with $(\mathbf{s})_{n-1} = 1$. By Lemma 2, there is a Hamiltonian path Q_2 of $(\bigcup_{i=5}^{n-1} B_{n-1}^{\{i\}} \cup B_{n-1}^{\{2\}})$ joining $(\mathbf{q})^n$ to **y**. By Lemma 3, there is a Hamiltonian path Q_3 of $B_{n-1}^{\{3,4\}} - \{\mathbf{z}_{n-4}^3, \mathbf{z}_{n-5}^4\}$ joining the black vertex $(\mathbf{y})^n$ to **s**. Let **v** be any white vertex in $B_{n-1}^{\{1\}}$. By Theorem 1, there is a Hamiltonian path Q_4 of $B_{n-1}^{\{1\}}$ joining the black vertex $(\mathbf{s})^n$ to **v**. Then $\langle \mathbf{u}, Q_1, \mathbf{q}, (\mathbf{q})^n, Q_2, \mathbf{y}, (\mathbf{y})^n, Q_3, \mathbf{s}, (\mathbf{s})^n, Q_4, \mathbf{v}\rangle$ is the desired path.

Case 2.2 Suppose that n is odd. Thus, $\mathbf{z}_{n-4}^3$ is a white vertex in $B_{n-1}^{\{3\}}$ and $\mathbf{z}_{n-5}^4$ is a black vertex in $B_{n-1}^{\{4\}}$. The proof of this case is similar to Case 2.1. ■

LEMMA 11 *There is a Hamiltonian path of $B_5 - X_5^4$ joining a black vertex* **u** *with $(\mathbf{u})_5 = 5$ to a black vertex* **v** *with $(\mathbf{v})_5 = 1$.*

Proof We set $Q_1 = \langle 34215, 43215, 42315, 24315, 24135, 42135, 41235, 14235, 14325, 41325, 43125, 34125, 31425, 13425, 13245, 31245, 32145, 32415, 23415, 23145, 21345, 21435\rangle$. Obviously, Q_1 is a Hamiltonian path of $B_4^{\{5\}} - (X_5^4 - \{\mathbf{z}_1^3 = 12453\})$ joining the black vertex 34215 to the white vertex 21435. Let $\mathbf{u} = 34215$ and $\mathbf{x} = 21435$. Let $\mathbf{y}$ be a black vertex in $B_4^{\{3\}}$ with $(\mathbf{y})_4 = 4$. By Theorem 2, there is a Hamiltonian path Q_2 of $B_4^{\{3\}} - \{\mathbf{z}_1^3\}$ joining the black vertex $(\mathbf{x})^5$ to $\mathbf{y}$. Let $\mathbf{v}$ be a black vertex in $B_4^{\{1\}}$. By Lemma 2, there is a hamiltonian path Q_3 of $B_4^{\{1,2,4\}}$ joining the white vertex $(\mathbf{y})^5$ to $\mathbf{v}$. Then $\langle \mathbf{u}, Q_1, \mathbf{x}, (\mathbf{x})^5, Q_2, \mathbf{y}, (\mathbf{y})^5, Q_3, \mathbf{v}\rangle$ is the desired path. ■

LEMMA 12 *For $n \geq 5$, there is a Hamiltonian path of $B_n - X_n^4$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_n = n$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_n = 1$, where both $\mathbf{u}$ and $\mathbf{v}$ are white vertices if n is even and both $\mathbf{u}$ and $\mathbf{v}$ are black vertices if n is odd.*

Proof We prove this statement by induction for n. By Lemma 11, this statement holds for $n = 5$. We suppose that this statement holds on $n-1$ with $n \geq 6$.

Case 1 Suppose that n is even. It is easy to know that $\mathbf{z}_{n-6}^5$ and $\mathbf{z}_{n-4}^3$ are two black vertices. By induction, there is a Hamiltonian path Q_1 of $B_{n-1}^{\{n\}} - (X_n^4 - \{\mathbf{z}_{n-4}^3, \mathbf{z}_{n-6}^5\})$ joining a black vertex $\mathbf{p}$ with $(\mathbf{p})_{n-1} = n-1$ to a black vertex $\mathbf{q}$ with $(\mathbf{q})_{n-1} = 1$. Let $\mathbf{t}$ be a black vertex in Q_1 with $(\mathbf{t})_{n-1} = 3$. We rewrite $Q_1 = \langle \mathbf{p}, R_1, \mathbf{m}, \mathbf{t}, R_2, \mathbf{q}\rangle$. Let $\mathbf{s}$ be a white vertex in $B_{n-1}^{\{3\}}$ with $(\mathbf{s})_{n-1} = 1$, and let $\mathbf{w}$ be a white vertex in $B_{n-1}^{\{5\}}$ with $(\mathbf{w})_{n-1} = 4$. By Lemma 2, there is a Hamiltonian path Q_2 of $(\bigcup_{i=6}^{n-1} B_{n-1}^{\{i\}} \cup B_{n-1}^{\{2,4\}})$ joining the white vertex $(\mathbf{p})^n$ to the black vertex $(\mathbf{w})^n$. By Lemma 3, there is a Hamiltonian path Q_3 of $B_{n-1}^{\{1,5\}} - \{\mathbf{z}_{n-6}^5, (\mathbf{q})^n\}$ joining $\mathbf{w}$ to the black vertex $(\mathbf{s})^n$. By Theorem 2, there is a Hamiltonian path Q_4 of $B_{n-1}^{\{2\}} - \{\mathbf{z}_{n-4}^3\}$ joining $\mathbf{s}$ to the white vertex $(\mathbf{t})^n$. We set $\mathbf{v} = (\mathbf{q})^n$ and $\mathbf{u} = \mathbf{m}$. Then $\langle \mathbf{u} = \mathbf{m}, R_1^{-1}, \mathbf{p}, (\mathbf{p})^n, Q_2, (\mathbf{w})^n, \mathbf{w}, Q_3, (\mathbf{s})^n, \mathbf{s}, Q_4, (\mathbf{t})^n, \mathbf{t}, R_2, \mathbf{q}, (\mathbf{q})^n = \mathbf{v}\rangle$ is the desired path.

Case 2 Suppose that n is odd. It is easy to know that $\mathbf{z}_{n-4}^3$ and $\mathbf{z}_{n-6}^5$ are two white vertices. The proof of this case is similar to Case 1. ■

LEMMA 13 *For $n \geq 6$, there is a Hamiltonian path of $B_n - Y_n^{n-2}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_n = n$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_n = 1$, where the colour of $\mathbf{u}$ and the colour of $\mathbf{v}$ are distinct.*

Proof We know that $Y_n^{n-2} = A_n^{n-2} \cup A_n^{n-1} \cup \{\mathbf{e}\}$, where $A_n^{n-2} = \{(\mathbf{e})^{n-1}, ((\mathbf{e})^{n-1})^n\}$ and $A_n^{n-1} = \{(\mathbf{e})^n\}$. By Lemma 4, there is a Hamiltonian path Q_1 of $B_{n-1}^{\{n\}} - \{(\mathbf{e})^{n-1}, \mathbf{e}\}$ joining a black vertex $\mathbf{p}$ with $(\mathbf{p})_{n-1} = n-1$ to a white vertex $\mathbf{q}$ with $(\mathbf{q})_{n-1} = 1$.

Case 1 Suppose that $\mathbf{u}$ is a black vertex and $\mathbf{v}$ is a white vertex. Let $\mathbf{y}$ be a black vertex in $B_{n-1}^{\{1\}}$ with $(\mathbf{y})_{n-1} = n-1$, and let $\mathbf{s}$ be a white vertex in $B_{n-1}^{\{1\}}$ with $(\mathbf{s})_{n-1} = 2$. By Theorem 1, there is a Hamiltonian path Q_2 of $B_{n-1}^{\{1\}}$ joining the black vertex $(\mathbf{q})^n$ to $\mathbf{s}$. Without loss of generality, we write $Q_2 = \langle (\mathbf{q})^n, R_1, \mathbf{y}, \mathbf{m}, R_2, \mathbf{s}\rangle$. Let $\mathbf{w}$ be a black vertex in $B_{n-1}^{\{n-2\}}$ with $(\mathbf{w})_{n-1} = n-3$. By Lemma 3, there is a Hamiltonian path Q_3 of $B_{n-1}^{\{n-2,n-1\}} - \{((\mathbf{e})^{n-1})^n, (\mathbf{e})^n\}$ joining the white vertex $(\mathbf{y})^n$ to $\mathbf{w}$. By Lemma 2, there is a Hamiltonian path Q_4 of $\bigcup_{i=2}^{n-3} B_{n-1}^{\{i\}}$ joining the white vertex $(\mathbf{w})^n$ to the black vertex $(\mathbf{s})^n$. We set $\mathbf{u} = \mathbf{p}$ and $\mathbf{v} = \mathbf{m}$. Then $\langle \mathbf{u} = \mathbf{p}, Q_1, \mathbf{q}, (\mathbf{q})^n, R_1, \mathbf{y}, (\mathbf{y})^n, Q_3, \mathbf{w}, (\mathbf{w})^n, Q_4, (\mathbf{s})^n, \mathbf{s}, R_2^{-1}, \mathbf{m} = \mathbf{v}\rangle$ is the desired path.

Case 2 Suppose that $\mathbf{u}$ is a white vertex and $\mathbf{v}$ is a black vertex. Let $\mathbf{w}$ be a black vertex in $B_{n-1}^{\{n-2\}}$ with $(\mathbf{w})_{n-1} = n-3$. By Lemma 3, there is a Hamiltonian path Q_2 of $B_{n-1}^{\{n-2,n-1\}} - \{((\mathbf{e})^{n-1})^n, (\mathbf{e})^n\}$ joining the white vertex $(\mathbf{p})^n$ to $\mathbf{w}$. Let $\mathbf{v}$ be a black vertex in $B_{n-1}^{\{1\}}$. By Lemma 2, there is a Hamiltonian path Q_3 of $\bigcup_{i=1}^{n-3} B_{n-1}^{\{i\}}$ joining the white vertex $(\mathbf{w})^n$ to $\mathbf{v}$. We set $\mathbf{u} = \mathbf{q}$. Then $\langle \mathbf{u} = \mathbf{q}, Q_1^{-1}, \mathbf{p}, (\mathbf{p})^n, Q_2, \mathbf{w}, (\mathbf{w})^n, Q_3, \mathbf{v}\rangle$ is the desired path. ■

LEMMA 14 *Let $n \geq 6$. For every $3 \leq i \leq n-2$, there is a Hamiltonian path of $B_n - Y_n^i$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_n = n$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_n = 1$, where the colour of $\mathbf{u}$ and the colour of $\mathbf{v}$ are distinct.*

Proof We prove this statement by induction on n. We have $Y_6^3 = X_6^3$. By Lemmas 10 and 13, the statement holds on $n = 6$. We suppose that this statement holds on $n-1$ with $n \geq 7$.

By induction, there is a Hamiltonian path Q_1 of $B_{n-1}^{\{n\}} - (Y_n^i - \{\mathbf{z}_{n-i-1}^i, \mathbf{z}_{n-i-2}^{i+1}\})$ joining a white vertex $\mathbf{p}$ with $(\mathbf{p})_{n-1} = n-1$ to a black vertex $\mathbf{q}$ with $(\mathbf{q})_{n-1} = 1$.

Case 1 Suppose that $\mathbf{u}$ is a white vertex and $\mathbf{v}$ is a black vertex.

Case 1.1 Suppose that $\mathbf{z}_{n-i-1}^i$ is a white vertex in $B_{n-1}^{\{i\}}$ and $\mathbf{z}_{n-i-2}^{i+1}$ is a black vertex in $B_{n-1}^{\{i+1\}}$. Let $\mathbf{y}$ be a white vertex in $B_{n-1}^{\{1\}}$ with $(\mathbf{y})_{n-1} = i+2$ and $\mathbf{x}$ be a black vertex in $B_{n-1}^{\{1\}}$ with $(\mathbf{x})_{n-1} = i+1$. By Theorem 1, there is a Hamiltonian path Q_2 of $B_{n-1}^{\{1\}}$ joining the white vertex $(\mathbf{q})^n$ to $\mathbf{x}$. Without loss of generality, we rewrite $Q_2 = \langle (\mathbf{q})^n, R_1, \mathbf{y}, \mathbf{m}, R_2, \mathbf{x}\rangle$. Let $\mathbf{w}$ be a white vertex in $B_{n-1}^{\{i-1\}}$ with $(\mathbf{w})_{n-1} = i$. By Lemma 2, there is a Hamiltonian path Q_3 of B_{n-1}^I with $I = \langle n-1\rangle - \{1, i, i+1\}$ joining the black vertex $(\mathbf{y})^n$ to $\mathbf{w}$. By Lemma 3, there is a Hamiltonian path Q_4 of $B_{n-1}^{\{i,i+1\}} - \{\mathbf{z}_{n-i-1}^i, \mathbf{z}_{n-i-2}^{i+1}\}$ joining the black vertex $(\mathbf{w})^n$ to the white vertex $(\mathbf{x})^n$. We set $\mathbf{u} = \mathbf{p}$ and $\mathbf{v} = \mathbf{m}$. Then $\langle \mathbf{u} = \mathbf{p}, Q_1, \mathbf{q}, (\mathbf{q})^n, R_1, \mathbf{y}, (\mathbf{y})^n, Q_3, \mathbf{w}, (\mathbf{w})^n, Q_4, (\mathbf{x})^n, \mathbf{x}, R_2^{-1}, \mathbf{m} = \mathbf{v}\rangle$ is the desired path.

Case 1.2 Suppose that $\mathbf{z}_{n-i-1}^i$ is a black vertex in $B_{n-1}^{\{i\}}$ and $\mathbf{z}_{n-i-2}^{i+1}$ is a white vertex in $B_{n-1}^{\{i+1\}}$. The proof of this case is similarly to Case 1.1.

Case 2 Suppose that $\mathbf{u}$ is a black vertex and $\mathbf{v}$ is a white vertex.

Case 2.1 Suppose that $\mathbf{z}_{n-i-1}^i$ is a white vertex in $B_{n-1}^{\{i\}}$ and $\mathbf{z}_{n-i-2}^{i+1}$ is a black vertex in $B_{n-1}^{\{i+1\}}$. Let $\mathbf{w}$ be a white vertex in $B_{n-1}^{\{n-1\}}$ with $(\mathbf{w})_{n-1} = i$. By Theorem 1, there is a Hamiltonian path Q_2 of $B_{n-1}^{\{n-1\}}$ joining the black vertex $(\mathbf{p})^n$ to $\mathbf{w}$. Let $\mathbf{y}$ be a white vertex in $B_{n-1}^{\{i+1\}}$ with $(\mathbf{y})_{n-1} = i+2$. By Lemma 3, there is a Hamiltonian path Q_3 of $B_{n-1}^{\{i,i+1\}} - \{\mathbf{z}_{n-i-1}^i, \mathbf{z}_{n-i-2}^{i+1}\}$ joining the black vertex $(\mathbf{w})^n$ to the white vertex $\mathbf{y}$. Let v be any white vertex in $B_{n-1}^{\{1\}}$. By Lemma 2, there is a hamiltonian path Q_4 of B_{n-1}^I with $I = \langle n-2\rangle - \{i, i+1\}$ joining the black vertex $(\mathbf{y})^n$ to $\mathbf{v}$. We set $\mathbf{u} = \mathbf{q}$. Then $\langle \mathbf{u} = \mathbf{q}, Q_1^{-1}, \mathbf{p}, (\mathbf{p})^n, Q_2, \mathbf{w}, (\mathbf{w})^n, Q_3, \mathbf{y}, (\mathbf{y})^n, Q_4, \mathbf{v}\rangle$ is the desired path.

Case 2.2 Suppose that $\mathbf{z}_{n-i-1}^i$ is a white vertex in $B_{n-1}^{\{i\}}$ and $\mathbf{z}_{n-i-2}^{i+1}$ is a black vertex in $B_{n-1}^{\{i+1\}}$. The proof of this case is similarly to Case 2.1. Thus, this lemma is proved. ■

THEOREM 3 *For the bubble-sort graph B_5 with $\mathbf{e}$ the vertex denoting identity permutation, there exist four MIH cycles starting at vertex $\mathbf{e}$.*

We give the proof of Theorem 3 in Appendix 1.

Now, we can find the MIH of the bubble-sort graph B_n.

THEOREM 4 *Let $n \geq 6$. We have $h(B_n) \geq n-1$.*

Proof Since B_n is vertex transitive, we show that there are $(n-1)$-mutually independent Hamiltonian cycles of B_n form $\mathbf{e}$. Suppose that $n \geq 6$. Let $\mathbf{v}_1^1, \mathbf{v}_1^2, \ldots, \mathbf{v}_1^n$ be the vertices of $B_{n-1}^{\{1\}}$, $B_{n-1}^{\{2\}}, \ldots, B_{n-1}^{\{n\}}$ with $(\mathbf{v}_1^2)_{n-1} = 4$, $(\mathbf{v}_1^3)_{n-1} = 5$, $(\mathbf{v}_1^4)_{n-1} = 3$, $(\mathbf{v}_1^n)_{n-1} = 1$ and $(\mathbf{v}_1^j)_{n-1} = j+1$

for $5 \leq j \leq n-1$, respectively. By Theorem 1, there are Hamiltonian paths: H_1^2 of $B_{n-1}^{\{2\}}$ joining $\mathbf{z}_{n-3}^2$ to $\mathbf{v}_1^2$; H_1^4 of $B_{n-1}^{\{4\}}$ joining $(\mathbf{v}_1^2)^n$ to $\mathbf{v}_1^4$, H_1^3 of $B_{n-1}^{\{3\}}$ joining $(\mathbf{v}_1^4)^n$ to $\mathbf{v}_1^3$ and H_1^5 of $B_{n-1}^{\{5\}}$ joining $(\mathbf{v}_1^3)^n$ to $\mathbf{v}_1^5$. By Theorem 1, there is a Hamiltonian path H_1^i of $B_{n-1}^{\{i\}}$ joining $(\mathbf{v}_1^{i-1})^n$ to $\mathbf{v}_1^i$ for $6 \leq i \leq n-1$. By Lemma 6, there is a Hamiltonian path H_1^n of $B_{n-1}^{\{n\}} - X_{n-1}^1$ joining $(\mathbf{v}_1^{n-1})^n$ to $\mathbf{v}_1^n$. By Theorem 1, there is a hamiltonian path H_1^1 of $B_{n-1}^{\{1\}}$ joining $(\mathbf{v}_1^n)^n$ to $\mathbf{z}_{n-2}^1$. We set $C_1 = \langle \mathbf{e}, \mathbf{z}_0^2, \ldots, \mathbf{z}_{n-3}^2, H_1^2, \mathbf{v}_1^2, (\mathbf{v}_1^2)^n, H_1^4, \mathbf{v}_1^4, (\mathbf{v}_1^4)^n, H_1^3, \mathbf{v}_1^3, (\mathbf{v}_1^3)^n, H_1^5, \mathbf{v}_1^5, (\mathbf{v}_1^5)^n, H_1^6, \mathbf{v}_1^6, \ldots, (\mathbf{v}_1^{n-2})^n, H_1^{n-1}, \mathbf{v}_1^{n-1}, (\mathbf{v}_1^{n-1})^n, H_1^n, \mathbf{v}_1^n, (\mathbf{v}_1^n)^n, H_1^1, \mathbf{z}_{n-2}^1, \mathbf{z}_{n-3}^1, \ldots, \mathbf{z}_0^1, \mathbf{e} \rangle$ being a Hamiltonian cycle of B_n form $\mathbf{e}$.

Let $\mathbf{v}_2^1, \mathbf{v}_2^2, \ldots, \mathbf{v}_2^n$ be the vertices of $B_{n-1}^{\{1\}}, B_{n-1}^{\{2\}}, \ldots, B_{n-1}^{\{n\}}$ with $(\mathbf{v}_2^2)_{n-1} = 4$, $(\mathbf{v}_2^3)_{n-1} = 5$, $(\mathbf{v}_2^4)_{n-1} = 3$, $(\mathbf{v}_2^n)_{n-1} = 1$, and $(\mathbf{v}_2^j)_{n-1} = j+1$ for $5 \leq j \leq n-1$, respectively. By Theorem 1, there are Hamiltonian paths H_2^4 of $B_{n-1}^{\{4\}}$ joining $\mathbf{z}_{n-5}^4$ to $\mathbf{v}_2^4$; H_2^3 of $B_{n-1}^{\{3\}}$ joining $(\mathbf{v}_2^4)^n$ to $\mathbf{v}_2^3$; and H_2^5 of $B_{n-1}^{\{5\}}$ joining $(\mathbf{v}_2^3)^n$ to $\mathbf{v}_2^5$. By Theorem 1, there is a Hamiltonian path H_2^i of $B_{n-1}^{\{i\}}$ joining $(\mathbf{v}_2^{i-1})^n$ to $\mathbf{v}_2^i$ for $6 \leq i \leq n-1$. By Lemma 8, there is a Hamiltonian path H_2^n of $B_{n-1}^{\{n\}} - X_{n-1}^2$ joining $(\mathbf{v}_2^{n-1})^n$ to $\mathbf{v}_2^n$. By Theorem 1, there are Hamiltonian paths: H_2^1 of $B_{n-1}^{\{1\}}$ joining $(\mathbf{v}_2^n)^n$ to $\mathbf{v}_2^1$ and H_2^2 of $B_{n-1}^{\{2\}}$ joining $(\mathbf{v}_2^1)^n$ to $\mathbf{z}_{n-3}^2$. We set $C_2 = \langle \mathbf{e}, \mathbf{z}_0^4, \ldots, \mathbf{z}_{n-5}^4, H_2^4, \mathbf{v}_2^4, (\mathbf{v}_2^4)^n, H_2^3, \mathbf{v}_2^3, (\mathbf{v}_2^3)^n, H_2^5, \mathbf{v}_2^5, (\mathbf{v}_2^5)^n, H_2^6, \mathbf{v}_2^6, \ldots, (\mathbf{v}_2^{n-2})^n, H_2^{n-1}, \mathbf{v}_2^{n-1}, (\mathbf{v}_2^{n-1})^n, H_2^n, \mathbf{v}_2^n, (\mathbf{v}_2^n)^n, H_2^1, \mathbf{v}_2^1, (\mathbf{v}_2^1)^n, H_2^2, \mathbf{z}_{n-3}^2, \mathbf{z}_{n-4}^2, \ldots, \mathbf{z}_0^2, \mathbf{e} \rangle$ being a Hamiltonian cycle of B_n form $\mathbf{e}$. Let $l = (n-1)(n-1)! - (n-2) + 1$. The lth vertex of C_1 is $(\mathbf{v}_1^n)^n$, which is in $B_{n-1}^{\{1\}}$, and the lth vertex of C_2 is $\mathbf{v}_2^1$, also in $B_{n-1}^{\{1\}}$. Obviously, $((\mathbf{v}_1^n)^n)_{n-1} = n$ and $(\mathbf{v}_2^1)_{n-1} = 2$, then $(\mathbf{v}_1^n)^n \neq \mathbf{v}_2^1$.

Let $\mathbf{v}_3^1, \mathbf{v}_3^2, \ldots, \mathbf{v}_3^n$ be the vertices of $B_{n-1}^{\{1\}}, B_{n-1}^{\{2\}}, \ldots, B_{n-1}^{\{n\}}$ with $(\mathbf{v}_3^2)_{n-1} = 4$, $(\mathbf{v}_3^3)_{n-1} = 5$, $(\mathbf{v}_3^4)_{n-1} = 3$, $(\mathbf{v}_3^n)_{n-1} = 1$, and $(\mathbf{v}_3^j)_{n-1} = j+1$ for $5 \leq j \leq n-1$, respectively. By Theorem 1, there is a Hamiltonian path H_3^3 of $B_{n-1}^{\{3\}}$ joining $\mathbf{z}_{n-4}^3$ to $\mathbf{v}_3^3$ and a Hamiltonian path H_3^5 of $B_{n-1}^{\{5\}}$ joining $(\mathbf{v}_3^3)^n$ to $\mathbf{v}_3^5$. By Theorem 1, there is a Hamiltonian path H_3^i of $B_{n-1}^{\{i\}}$ joining $(\mathbf{v}_3^{i-1})^n$ to $\mathbf{v}_3^i$ for $6 \leq i \leq n-1$. By Lemma 10, there is a Hamiltonian path H_3^n of $B_{n-1}^{\{n\}} - X_{n-1}^3$ joining $(\mathbf{v}_3^{n-1})^n$ to $\mathbf{v}_3^n$. Let $\mathbf{v}_3^1$ be a vertex in $B_{n-1}^{\{1\}}$ with $(\mathbf{v}_3^1)_{n-1} = 2$ such that the vertex $\mathbf{v}_3^1 \notin N((\mathbf{v}_2^n)^n)$ and there exists a vertex $\mathbf{s} \in N(\mathbf{v}_3^1)$ with $\mathbf{s} \neq (\mathbf{v}_2^n)^n$. By Theorem 2, there is a Hamiltonian path H_3^1 of $B_{n-1}^{\{1\}} - \mathbf{v}_3^1$ joining the vertex $(\mathbf{v}_3^n)^n$ to $\mathbf{s}$. Let $\mathbf{v}_3^2$ be a vertex in $B_{n-1}^{\{2\}}$ with $(\mathbf{v}_3^2)_{n-1} = 4$ such that the vertex $\mathbf{v}_3^2 \notin N((\mathbf{v}_2^1)^n)$ and there exists a vertex $\mathbf{t} \in N(\mathbf{v}_3^2)$ with $\mathbf{t} \neq (\mathbf{v}_2^1)^n$. By Theorem 2, there is a Hamiltonian path H_3^2 of $B_{n-1}^{\{2\}}$ joining the vertex $(\mathbf{v}_3^1)^n$ to $\mathbf{t}$. By Theorem 1, there are Hamiltonian paths: H_3^1 of $B_{n-1}^{\{1\}}$ joining $(\mathbf{v}_3^n)^n$ to $\mathbf{v}_3^1$; H_3^2 of $B_{n-1}^{\{2\}}$ joining $(\mathbf{v}_3^1)^n$ to $\mathbf{v}_3^2$; and H_3^4 of $B_{n-1}^{\{4\}}$ joining $(\mathbf{v}_3^2)^n$ to $\mathbf{z}_{n-5}^4$. We set $C_3 = \langle \mathbf{e}, \mathbf{z}_0^3, \ldots, \mathbf{z}_{n-4}^3, H_3^3, \mathbf{v}_3^3, (\mathbf{v}_3^3)^n, H_3^5, \mathbf{v}_3^5, (\mathbf{v}_3^5)^n, H_3^6, \mathbf{v}_3^6, \ldots, (\mathbf{v}_3^{n-2})^n, H_3^{n-1}, \mathbf{v}_3^{n-1}, (\mathbf{v}_3^{n-1})^n, H_3^n, \mathbf{v}_3^n, (\mathbf{v}_3^n)^n, H_3^1, \mathbf{s}, \mathbf{v}_3^1, (\mathbf{v}_3^1)^n, H_3^2, \mathbf{t}, \mathbf{v}_3^2, (\mathbf{v}_3^2)^n, H_3^4, \mathbf{z}_{n-5}^4, \mathbf{z}_{n-6}^4, \ldots, \mathbf{z}_0^4, \mathbf{e} \rangle$ being a Hamiltonian cycle of B_n form $\mathbf{e}$. Let $l = m(n-1)! + (n-4) + 1$ for $1 \leq m \leq n-4$. The lth vertices are coincided in the same subgraph between C_2 and C_3. Obviously, the $(n-1)$-th position of the l-th vertices in C_2 and C_3 are different.

Let $\mathbf{v}_4^1, \mathbf{v}_4^2, \ldots, \mathbf{v}_4^n$ be the vertices of $B_{n-1}^{\{1\}}, B_{n-1}^{\{2\}}, \ldots, B_{n-1}^{\{n\}}$ with $(\mathbf{v}_4^2)_{n-1} = 4$, $(\mathbf{v}_4^3)_{n-1} = 5$, $(\mathbf{v}_4^4)_{n-1} = 3$, $(\mathbf{v}_4^n)_{n-1} = 1$, and $(\mathbf{v}_4^j)_{n-1} = j+1$ for $5 \leq j \leq n-1$, respectively. By Theorem 1, there is a Hamiltonian path H_4^5 of $B_{n-1}^{\{5\}}$ joining $\mathbf{z}_{n-6}^5$ to $\mathbf{v}_4^5$. By Theorem 1, there is a Hamiltonian path H_4^i of $B_{n-1}^{\{i\}}$ joining $(\mathbf{v}_4^{i-1})^n$ to $\mathbf{v}_4^i$ for $6 \leq i \leq n-1$. By Lemma 12, there is a Hamiltonian path H_4^n of $B_{n-1}^{\{n\}} - X_{n-1}^4$ joining $(\mathbf{v}_4^{n-1})^n$ to $\mathbf{v}_4^n$. By Theorem 1, there are Hamiltonian paths: H_4^1 of $B_{n-1}^{\{1\}}$ joining $(\mathbf{v}_4^n)^n$ to $\mathbf{v}_4^1$; H_4^2 of $B_{n-1}^{\{2\}}$ joining $(\mathbf{v}_4^1)^n$ to $\mathbf{v}_4^2$; H_4^4 of $B_{n-1}^{\{4\}}$ joining $(\mathbf{v}_4^2)^n$ to $\mathbf{v}_4^4$; H_4^3 of $B_{n-1}^{\{3\}}$ joining $(\mathbf{v}_4^4)^n$ to $\mathbf{z}_{n-4}^3$. We set $C_4 = \langle \mathbf{e}, \mathbf{z}_0^5, \ldots, \mathbf{z}_{n-6}^5, H_4^5, \mathbf{v}_4^5, (\mathbf{v}_4^5)^n, H_4^6, \mathbf{v}_4^6, \ldots, (\mathbf{v}_4^{n-2})^n, H_4^{n-1}, \mathbf{v}_4^{n-1}, (\mathbf{v}_4^{n-1})^n, H_4^n, \mathbf{v}_4^n, (\mathbf{v}_4^n)^n,$

H_4^1, $\mathbf{v}_4^1$, $(\mathbf{v}_4^1)^n$, H_4^2, $(\mathbf{v}_4^2)^n$, H_4^4, $\mathbf{v}_4^4$, $(\mathbf{v}_4^4)^n$, H_4^3, $\mathbf{z}_{n-4}^3$, $\mathbf{z}_{n-5}^3, \ldots, \mathbf{z}_0^3$, $\mathbf{e}\rangle$ being a Hamiltonian cycle of B_n form $\mathbf{e}$.

Case 3.1 Suppose that $n = 6$. Let $\mathbf{v}_5^1, \mathbf{v}_5^2, \mathbf{v}_5^3, \mathbf{v}_5^4$ and $\mathbf{v}_5^6$ be the vertices of $B_5^{\{1\}}$, $B_5^{\{2\}}$, $B_5^{\{3\}}$, $B_5^{\{4\}}$, and $B_5^{\{6\}}$ with $(\mathbf{v}_5^1)_5 = 2$, $(\mathbf{v}_5^2)_5 = 4$, $(\mathbf{v}_5^4)_5 = 3$, $(\mathbf{v}_5^3)_5 = 5$, and $(\mathbf{v}_5^6)_5 = 1$, respectively. By Theorem 1, there is a Hamiltonian path H_5^6 of $B_5^{\{6\}}$ joining $\mathbf{e}$ to $\mathbf{v}_5^6$. Let $\mathbf{v}_5^1, \mathbf{v}_5^2, \mathbf{v}_5^4$, and $\mathbf{v}_5^3$ be the vertices in $B_{n-1}^{\{1\}}$, $B_{n-1}^{\{2\}}$, $B_{n-1}^{\{4\}}$, and $B_{n-1}^{\{3\}}$, with $(\mathbf{v}_5^1)^{n-1} = 2$, $(\mathbf{v}_5^2)^{n-1} = 4$, $(\mathbf{v}_5^4)^{n-1} = 3$, and $(\mathbf{v}_5^3)^{n-1} = 5$, such that $\mathbf{v}_5^1 \notin N((\mathbf{v}_4^6)^n)$, $\mathbf{v}_5^2 \notin N((\mathbf{v}_4^1)^n)$, $\mathbf{v}_5^4 \notin N((\mathbf{v}_4^2)^n)$, and $\mathbf{v}_5^3 \notin N((\mathbf{v}_4^4)^n)$, respectively. And there exist $\mathbf{s}_5^1) \in N(\mathbf{v}_5^1)$, $\mathbf{s}_5^2 \in N(\mathbf{v}_5^2)$, $\mathbf{s}_5^4 \in N(\mathbf{v}_5^4)$, and $\mathbf{s}_5^3 \in N(\mathbf{v}_5^3)$. By Theorem 2, there are Hamiltonian paths: H_5^1 of $B_5^{\{1\}} - \mathbf{v}_5^1$ joining $(\mathbf{v}_5^n)^n$ to $\mathbf{s}_5^1$; H_5^2 of $B_5^{\{2\}} - \mathbf{v}_5^2$ joining $(\mathbf{v}_5^1)^n$ to $\mathbf{s}_5^2$; H_5^4 of $B_5^{\{4\}} - \mathbf{v}_5^4$ joining $(\mathbf{v}_5^2)^n$ to $\mathbf{s}_5^4$; and H_5^3 of $B_5^{\{3\}} - \mathbf{v}_5^3$ joining $(\mathbf{v}_5^4)^n$ to $\mathbf{s}_5^3$. By Theorem 1, there is a Hamiltonian path H_5^5 of $B_{n-1}^{\{5\}}$ joining $(\mathbf{v}_5^3)^n$ to $\mathbf{z}_0^5$. We set $C_5 = \langle \mathbf{e}$, H_5^6, $\mathbf{v}_5^6$, $(\mathbf{v}_5^6)^6$, H_5^1, $\mathbf{s}_5^1$, $\mathbf{v}_5^1$, $(\mathbf{v}_5^1)^6$, H_5^2, $\mathbf{s}_5^2$, $\mathbf{v}_5^2$, $(\mathbf{v}_5^2)^6$, H_5^4, $\mathbf{s}_5^4$, $\mathbf{v}_5^4$, $(\mathbf{v}_5^4)^6$, H_5^3, $\mathbf{s}_5^3$, $\mathbf{v}_5^3$, $(\mathbf{v}_5^3)^6$, H_5^5, $\mathbf{z}_0^5$, $\mathbf{e}\rangle$ being a Hamiltonian cycle of B_n form $\mathbf{e}$.

Then $\{C_1, C_2, \ldots, C_5\}$ forms a set of five-mutually independent Hamiltonian cycles.

Case 3.2 Suppose that $n > 6$. Let $\mathbf{v}_5^1, \mathbf{v}_5^2, \ldots, \mathbf{v}_5^n$ be the vertices of $B_{n-1}^{\{1\}}, B_{n-1}^{\{2\}}, \ldots, B_{n-1}^{\{n\}}$ with $(\mathbf{v}_5^2)_{n-1} = 4$, $(\mathbf{v}_5^3)_{n-1} = 5$, $(\mathbf{v}_5^4)_{n-1} = 3$, $(\mathbf{v}_5^n)_{n-1} = 1$, and $(\mathbf{v}_5^j)_{n-1} = j + 1$ for $5 \le j \le n - 1$, respectively. By Theorem 1, there is a Hamiltonian path H_5^6 of $B_{n-1}^{\{6\}}$ joining $\mathbf{z}_{n-7}^6$ to $\mathbf{v}_5^6$. By Theorem 1, there is a Hamiltonian path H_5^i of $B_{n-1}^{\{i\}}$ joining $(\mathbf{v}_5^{i-1})^n$ to $\mathbf{v}_5^i$ for $6 \le i \le n - 1$. By Lemma 14, there is a Hamiltonian path H_5^n of $B_{n-1}^{\{n\}} - Y_{n-1}^5$ joining $(\mathbf{v}_5^{n-1})^n$ to $\mathbf{v}_5^n$. Let $\mathbf{v}_5^1, \mathbf{v}_5^2, \mathbf{v}_5^4$, and $\mathbf{v}_5^3$ be the vertices in $B_{n-1}^{\{1\}}$, $B_{n-1}^{\{2\}}$, $B_{n-1}^{\{4\}}$, and $B_{n-1}^{\{3\}}$, with $(\mathbf{v}_5^1)^{n-1} = 2$, $(\mathbf{v}_5^2)^{n-1} = 4$, $(\mathbf{v}_5^4)^{n-1} = 3$, and $(\mathbf{v}_5^3)^{n-1} = 5$, such that $\mathbf{v}_5^1 \notin N((\mathbf{v}_4^6)^n)$, $\mathbf{v}_5^2 \notin N((\mathbf{v}_4^1)^n)$, $\mathbf{v}_5^4 \notin N((\mathbf{v}_4^2)^n)$, and $\mathbf{v}_5^3 \notin N((\mathbf{v}_4^4)^n)$, respectively. And there exist $\mathbf{s}_5^1 \in N(\mathbf{v}_5^1)$, $\mathbf{s}_5^2 \in N(\mathbf{v}_5^2)$, $\mathbf{s}_5^4 \in N(\mathbf{v}_5^4)$, and $\mathbf{s}_5^3 \in N(\mathbf{v}_5^3)$. By Theorem 2, there are Hamiltonian paths: H_5^1 of $B_{n-1}^{\{1\}} - \mathbf{v}_5^1$ joining $(\mathbf{v}_5^n)^n$ to $\mathbf{s}_5^1$; H_5^2 of $B_{n-1}^{\{2\}} - \mathbf{v}_5^2$ joining $(\mathbf{v}_5^1)^n$ to $\mathbf{s}_5^2$; H_5^4 of $B_{n-1}^{\{4\}} - \mathbf{v}_5^4$ joining $(\mathbf{v}_5^2)^n$ to $\mathbf{s}_5^4$; and H_5^3 of $B_{n-1}^{\{3\}} - \mathbf{v}_5^3$ joining $(\mathbf{v}_5^4)^n$ to $\mathbf{s}_5^3$. By Theorem 1, there is a Hamiltonian path H_5^5 of $B_{n-1}^{\{5\}}$ joining $(\mathbf{v}_5^3)^n$ to $\mathbf{z}_{n-6}^5$. We set $C_5 = \langle \mathbf{e}$, $\mathbf{z}_0^6$, $\mathbf{z}_1^6, \ldots, \mathbf{z}_{n-7^6}$, H_5^6, $\mathbf{v}_5^6$, $(\mathbf{v}_5^6)^n$, H_5^7, $\mathbf{v}_5^7, \ldots, (\mathbf{v}_5^{n-1})^n$, H_5^n, $\mathbf{v}_5^n$, $(\mathbf{v}_5^n)^n$, H_5^1, $\mathbf{s}_5^1$, $\mathbf{v}_5^1$, $(\mathbf{v}_5^1)^n$, H_5^2, $\mathbf{s}_5^2$, $\mathbf{v}_5^2$, $(\mathbf{v}_5^2)^n$, H_5^4, $\mathbf{s}_5^4$, $\mathbf{v}_5^4$, $(\mathbf{v}_5^4)^n$, H_5^3, $\mathbf{s}_5^3$, $\mathbf{v}_5^3$, $(\mathbf{v}_5^3)^n$, H_5^5, $\mathbf{z}_{n-6}^5$, $\mathbf{z}_{n-7}^5, \ldots, \mathbf{z}_0^5$, $\mathbf{e}\rangle$ is a Hamiltonian cycle of B_n form $\mathbf{e}$.

Assume that $6 \le i \le n - 2$. Let $\mathbf{v}_i^1$, $\mathbf{v}_i^2, \ldots, \mathbf{v}_i^n$ be the vertices of $B_{n-1}^{\{1\}}, B_{n-1}^{\{2\}}, \ldots, B_{n-1}^{\{n\}}$ with $(\mathbf{v}_1^2)_{n-1} = 4$, $(\mathbf{v}_i^3)_{n-1} = 5$, $(\mathbf{v}_i^4)_{n-1} = 3$, $(\mathbf{v}_i^n)_{n-1} = 1$, and $(\mathbf{v}_i^j)_{n-1} = j + 1$ for $5 \le j \le n - 1$, respectively. By Theorem 1, there is a Hamiltonian path H_i^{i+1} of $B_{n-1}^{\{i+1\}}$ joining $\mathbf{z}_{n-i-2}^{i+1}$ to $\mathbf{v}_i^{i+1}$. By Theorem 1, there is a Hamiltonian path H_i^i of $B_{n-1}^{\{i\}}$ joining $(\mathbf{v}_i^{i-1})^n$ to $\mathbf{z}_{n-i-1}^i$. By Theorem 1, there is a Hamiltonian path H_i^j of $B_{n-1}^{\{j\}}$ joining $(\mathbf{v}_i^{j-1})^n$ to $\mathbf{v}_i^j$ for $6 \le j \le n - 1$ and $j \notin \{i, i + 1\}$. By Lemma 14, there is a Hamiltonian path H_i^n of $B_{n-1}^{\{n\}} - Y_{n-1}^i$ joining $(\mathbf{v}_i^{n-1})^n$ to $\mathbf{v}_i^n$. By Theorem 1, there are Hamiltonian paths: H_i^1 of $B_{n-1}^{\{1\}}$ joining $(\mathbf{v}_i^n)^n$ to $\mathbf{v}_5^1$; H_i^2 of $B_{n-1}^{\{2\}}$ joining $(\mathbf{v}_i^1)^n$ to $\mathbf{v}_5^2$; H_i^4 of $B_{n-1}^{\{4\}}$ joining $(\mathbf{v}_i^2)^n$ to $\mathbf{v}_5^4$; and H_i^3 of $B_{n-1}^{\{3\}}$ joining $(\mathbf{v}_i^4)^n$ to $\mathbf{v}_5^3$. We set $C_i = \langle \mathbf{e}$, $\mathbf{z}_0^{i+1}$, $\mathbf{z}_1^{i+1}, \ldots, \mathbf{z}_{n-i-2}^{i+1}$, H_i^{i+1}, $\mathbf{v}_i^{i+1}$, $(\mathbf{v}_i^{i+1})^n$, H_i^{i+2}, $\mathbf{v}_i^{i+2}, \ldots, (\mathbf{v}_i^{n-1})^n$, H_i^n, $\mathbf{v}_i^n$, $(\mathbf{v}_i^n)^n$, H_i^1, $\mathbf{v}_i^1$, $(\mathbf{v}_i^1)^n$, H_i^2, $\mathbf{v}_i^2$, $(\mathbf{v}_i^2)^n$, H_i^4, $\mathbf{v}_i^4$, $(\mathbf{v}_i^4)^n$, H_i^3, $\mathbf{v}_i^3$, $(\mathbf{v}_i^3)^n$, H_i^5, $\mathbf{v}_i^5, \ldots, (\mathbf{v}_i^{i-1})^n$, H_i^i, $\mathbf{z}_{n-i-1}^i$, $\mathbf{z}_{n-i-2}^i, \ldots, \mathbf{z}_0^i$, $\mathbf{e}\rangle$ being a Hamiltonian cycle of B_n form $\mathbf{e}$.

Let $\mathbf{v}_{n-1}^1$, $\mathbf{v}_{n-1}^2, \ldots, \mathbf{v}_{n-1}^n$ be the vertices of $B_{n-1}^{\{1\}}$, $B_{n-1}^{\{2\}}, \ldots, B_{n-1}^{\{n\}}$ with $(\mathbf{v}_{n-1}^2)_{n-1} = 4$, $(\mathbf{v}_{n-1}^3)_{n-1} = 5$, $(\mathbf{v}_{n-1}^4)_{n-1} = 3$, $(\mathbf{v}_{n-1}^n)_{n-1} = 1$, and $(\mathbf{v}_{n-1}^j)_{n-1} = j + 1$ for $5 \le j \le n - 1$, respectively. By Theorem 1, there is a Hamiltonian path H_{n-1}^n of $B_{n-1}^{\{n\}}$ joining $\mathbf{e}$ to $\mathbf{v}_{n-1}^n$. Again, there is a Hamiltonian path H_{n-1}^i of $B_{n-1}^{\{i\}}$ joining $\mathbf{v}_{n-1}^{i-1}$ to $\mathbf{v}_{n-1}^i$ for $6 \ge i \ge n - 2$. Moreover, there is a Hamiltonian path H_{n-1}^{n-1} of $B_{n-1}^{\{n-1\}}$ joining $(\mathbf{v}_{n-1}^{n-2})^n$ to $\mathbf{z}_0^{n-1}$. By Theorem 1, there are Hamiltonian paths: H_{n-1}^1 of $B_{n-1}^{\{1\}}$ joining $(\mathbf{v}_{n-1}^{n})^n$ to $\mathbf{v}_{n-1}^1$; H_{n-1}^2 of $B_{n-1}^{\{2\}}$ joining $(\mathbf{v}_{n-1}^{1})^n$ to $\mathbf{v}_{n-1}^2$; H_{n-1}^4 of $B_{n-1}^{\{4\}}$ joining $(\mathbf{v}_{n-1}^{2})^n$ to $\mathbf{v}_{n-1}^4$; H_{n-1}^3 of $B_{n-1}^{\{3\}}$ joining $(\mathbf{v}_{n-1}^{4})^n$ to $\mathbf{v}_{n-1}^3$. We set $C_{n-1} = \langle \mathbf{e}, H_{n-1}^n, \mathbf{v}_{n-1}^n, (\mathbf{v}_{n-1}^n)^n, H_{n-1}^1, \mathbf{v}_{n-1}^1, (\mathbf{v}_{n-1}^1)^n, H_{n-1}^2, \mathbf{v}_{n-1}^2, (\mathbf{v}_{n-1}^2)^n, H_{n-1}^4, \mathbf{v}_{n-1}^4, (\mathbf{v}_{n-1}^4)^n, H_{n-1}^3, \mathbf{v}_{n-1}^3, (\mathbf{v}_{n-1}^3)^n, H_{n-1}^5, \mathbf{v}_{n-1}^5, \ldots, (\mathbf{v}_{n-1}^{n-2})^n, H_{n-1}^{n-1}, \mathbf{z}_0^{n-1}, \mathbf{e}\rangle$ being a Hamiltonian cycle of B_n form $\mathbf{e}$.

Then $\{C_1, C_2, \ldots, C_{n-1}\}$ is a set of $(n-1)$-mutually independent Hamiltonian cycles for B_n from $\mathbf{e}$. ■

COROLLARY 1 *For $n \geq 4$, we have $h(B_n) = n - 1$. Moreover, $h(B_3) = 1$.*

Proof Since $\delta(B_n) = n - 1$, $h(B_n) \leq n - 1$. Since B_3 is a cycle with six vertices, it is easy to check that $h(B_3) = 1$. To show $h(B_n) = n - 1$ for $n \geq 4$, we need to construct $(n-1)$-mutually independent Hamiltonian cycles of B_n from every vertex $\mathbf{u}$. Since B_n is vertex transitive, we show that there are $(n-1)$-mutually independent Hamiltonian cycles of B_n from $\mathbf{e}$.

Case 1 Suppose that $n = 4$. We set

$$C_1 = \langle 1234, 2134, 2143, 2413, 2431, 2341, 2314, 3214, 3241, 3421, 4321, 4231,$$
$$4213, 4123, 4132, 4312, 3412, 3142, 3124, 1324, 1342, 1432, 1423, 1243, 1234\rangle,$$
$$C_2 = \langle 1234, 1243, 1423, 1432, 4132, 4123, 4213, 4231, 4321, 4312, 3412, 3421,$$
$$3241, 2341, 2431, 2413, 2143, 2134, 2314, 3214, 3124, 3142, 1342, 1323, 1234\rangle, \text{ and}$$
$$C_3 = \langle 1234, 1324, 3124, 3142, 1342, 1432, 1423, 1243, 2143, 2413, 4213, 4123,$$
$$4132, 4312, 3412, 3421, 4321, 4231, 2431, 2341, 3241, 3214, 2314, 2134, 1234\rangle.$$

Then $\{C_1, C_2, C_3\}$ is a set of three-mutually independent Hamiltonian cycles for B_4 from $\mathbf{e}$.

Case 2 Suppose that $n \geq 5$. By Theorems 3 and 4, there is a set of $(n-1)$-mutually independent Hamiltonian cycles on B_n from $\mathbf{e}$.

Summarily, Case 1 and Case 2, we have $h(B_n) = n - 1$ for $n \geq 4$. ■

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Appendix 1. The proof of Theorem 3

Proof of Theorem 3 Since B_5 is vertex transitive, we show that there are four-mutually independent Hamiltonian cycles of B_5 from $\mathbf{e}$.

Obviously, $(\mathbf{z}_1^2)^5 = 13452$ is a black vertex in $B_4^{\{2\}}$. Let $\mathbf{v}_1^2$ be a white vertex in $B_4^{\{2\}}$ with $(\mathbf{v}_1^2)_4 = 4$. By Theorem 1, there is a Hamiltonian path H_1^2 of $B_4^{\{2\}}$ joining $(\mathbf{z}_1^2)^5$ to $\mathbf{v}_1^2$. Let $\mathbf{v}_1^4$ be a white vertex in $B_4^{\{4\}}$ with $(\mathbf{v}_1^4)_4 = 3$. By Theorem 1, there is a Hamiltonian path H_1^4 of $B_4^{\{4\}}$ joining the black vertex $(\mathbf{v}_1^2)^5$ to $\mathbf{v}_1^4$. Let $\mathbf{v}_1^3 = 12453$ be a white vertex in $B_4^{\{3\}}$, then we have $(\mathbf{v}_1^3)^5 = 12435$ is a black vertex in $B_4^{\{5\}}$. By Theorem 1, there is a Hamiltonian path H_1^3 of $B_4^{\{3\}}$ joining the black vertex $(\mathbf{v}_1^4)^5$ to $\mathbf{v}_1^3$. Let H_1^5 be the Hamiltonian path Q_1 of Lemma 5 joining $(\mathbf{v}_1^3)^5$ to the white vertex $43215 = \mathbf{v}_1^5$. Let $\mathbf{v}_1^1$ be a white vertex in $B_4^{\{1\}}$ with $\mathbf{v}_1^1 = (\mathbf{z}_2^1)^5 = 23451$. By Theorem 1, there is a Hamiltonian path H_1^1 of $B_4^{\{1\}}$ joining the black vertex $(\mathbf{v}_1^5)^5$ to $\mathbf{v}_1^1$. Then $C_1 = \langle \mathbf{e}, \mathbf{z}_0^2 = 13245, \mathbf{z}_1^2 = 13425, (\mathbf{z}_1^2)^5, H_1^2, \mathbf{v}_1^2, (\mathbf{v}_1^2)^5, H_1^4, \mathbf{v}_1^4, (\mathbf{v}_1^4)^5, H_1^3, \mathbf{v}_1^3, (\mathbf{v}_1^3)^5, H_1^5, \mathbf{v}_1^5, (\mathbf{v}_1^5)^5, H_1^1, \mathbf{v}_1^1, \mathbf{z}_2^1 = 23415, \mathbf{z}_1^1 = 23145, \mathbf{z}_0^1 = 21345, \mathbf{e}\rangle$ is the desired cycle.

Obviously, $(\mathbf{e})^5$ is a black vertex in $B_4^{\{4\}}$. Let $\mathbf{v}_2^4$ be a white vertex in $B_4^{\{4\}}$ with $(\mathbf{v}_2^4)_4 = 3$. By Theorem 1, there is a Hamiltonian path H_2^4 of $B_4^{\{4\}}$ joining $(\mathbf{e})^5$ to $\mathbf{v}_2^4$. Let $\mathbf{v}_2^3 = 24153$ that is a white vertex in $B_4^{\{3\}}$. Then we have $(\mathbf{v}_2^3)^5 = 24135$ is a black vertex in $B_4^{\{5\}}$. By Theorem 1, there is a Hamiltonian path H_2^3 of $B_4^{\{3\}}$ joining the black vertex $(\mathbf{v}_2^4)^5$ to $\mathbf{v}_2^3$. Let H_2^5 be the Hamiltonian path Q_1 of Lemma 7 joining $(\mathbf{v}_2^3)^5$ to the black vertex $34215 = \mathbf{v}_2^5$. Let $\mathbf{v}_2^1$ be a black vertex in $B_4^{\{1\}}$ with $(\mathbf{v}_2^1)_4 = 2$. By Theorem 1, there is a Hamiltonian path H_2^1 of $B_4^{\{1\}}$ joining the white vertex $(\mathbf{v}_2^5)^5$ to $\mathbf{v}_2^1$. Let $\mathbf{v}_2^2$ be a black vertex in $B_4^{\{2\}}$ with $\mathbf{v}_2^2 = (\mathbf{z}_1^2)^5 = 13452$. By Theorem 1, there is a Hamiltonian path H_2^2 of $B_4^{\{2\}}$ joining the white vertex $(\mathbf{v}_2^1)^5$ to $\mathbf{v}_2^2$. We set $C_2 = \langle \mathbf{e}, (\mathbf{e})^5, H_2^4, \mathbf{v}_2^4, (\mathbf{v}_2^4)^5, H_2^3, \mathbf{v}_2^3, (\mathbf{v}_2^3)^5, H_2^5, \mathbf{v}_2^5, (\mathbf{v}_2^5)^5, H_2^1, \mathbf{v}_2^1, (\mathbf{v}_2^1)^5, H_2^2, \mathbf{v}_2^2, \mathbf{z}_1^2 = 13425, \mathbf{z}_0^2 = 13245, \mathbf{e}\rangle$. The 94th vertex of C_1 is $(\mathbf{v}_1^5)^5$ in $B_4^{\{1\}}$, and the 94th vertex of C_2 is $\mathbf{v}_2^1$ also in $B_4^{\{1\}}$. Obviously, $((\mathbf{v}_1^5)^5)_4 = 5$ and $(\mathbf{v}_2^1)_4 = 2$, then $(\mathbf{v}_1^5)^5 \neq \mathbf{v}_2^1$. Therefore, C_2 is the desired cycle.

Obviously, $\mathbf{z}_0^3 = 12435$ is a black vertex, and $(\mathbf{z}_0^3)^5$ is a white vertex in $B_4^{\{3\}}$. Let $\mathbf{v}_3^3 = 21453$ that is a black vertex in $B_4^{\{3\}}$, and $(\mathbf{v}_3^3)^5$ is the white vertex in $B_4^{\{5\}}$. By Theorem 1, there is a Hamiltonian path H_3^3 of $B_4^{\{3\}}$ joining $(\mathbf{z}_0^3)^5$ to $\mathbf{v}_3^3$. Let H_3^5 be the Hamiltonian path Q_1 of Lemma 9 joining $(\mathbf{v}_3^3)^5$ to the black vertex $34215 = \mathbf{v}_3^5$. Let $\mathbf{v}_3^1$ be a black vertex in $B_4^{\{1\}}$ with $(\mathbf{v}_3^1)_4 = 2$ such that the vertex $\mathbf{v}_3^1 \notin N((\mathbf{v}_2^5)^5)$ and there exists a white vertex $\mathbf{s} \in N(\mathbf{v}_3^1)$ and $\mathbf{s} \neq (\mathbf{v}_2^5)^5$. By Theorem 2, there is a Hamiltonian path H_3^1 of $B_4^{\{1\}} - \mathbf{v}_3^1$ joining the white vertex $(\mathbf{v}_3^5)^5$ to $\mathbf{s}$. Let $\mathbf{v}_3^2$ be a black vertex in $B_4^{\{2\}}$ with $(\mathbf{v}_3^2)_4 = 4$ such that the vertex $\mathbf{v}_3^2 \notin N((\mathbf{v}_2^1)^5)$ and there exists a white vertex $\mathbf{t} \in N(\mathbf{v}_3^2)$ and $\mathbf{t} \neq (\mathbf{v}_2^1)^5$. By Theorem 1, there is a Hamiltonian path H_3^2 of $B_4^{\{2\}} - \{\mathbf{v}_3^2\}$ joining the white vertex $(\mathbf{v}_3^1)^5$ to $\mathbf{t}$. Obviously, $(\mathbf{e})^5$ is a black vertex in $B_4^{\{4\}}$. By Theorem 1, there is a Hamiltonian path H_3^4 of $B_4^{\{4\}}$ joining the white vertex $(\mathbf{v}_3^2)^5$ to $(\mathbf{e})^5$. We set $C_3 = \langle \mathbf{e}, \mathbf{z}_0^3, (\mathbf{z}_0^3)^5, H_3^3, \mathbf{v}_3^3, (\mathbf{v}_3^3)^5, H_3^5, \mathbf{v}_3^5, (\mathbf{v}_3^5)^5, H_3^1, \mathbf{s}, \mathbf{v}_3^1, (\mathbf{v}_3^1)^5, H_3^2, \mathbf{t}, \mathbf{v}_3^2, (\mathbf{v}_3^2)^5, H_3^4, (\mathbf{e})^5, \mathbf{e}\rangle$. The 26th vertex of C_2 is $(\mathbf{v}_2^4)^5$ in $B_4^{\{3\}}$, and the 26th vertex of C_3 is $\mathbf{v}_3^3$ also in $B_4^{\{3\}}$. Obviously, $((\mathbf{v}_2^4)^5)_4 = 4$ and $(\mathbf{v}_3^3)_4 = 5$, then $(\mathbf{v}_2^4)^5 \neq \mathbf{v}_3^3$. Therefore, C_3 is the desired cycle.

We set $H_4^5 = \langle 21435, 24135, 24315, 23415, 23145, 32145, 31245, 13245, 13425, 31425, 34125, 43125, 41325, 14325, 14235, 41235, 42135, 42315, 43215, 34215, 32415 = \mathbf{v}_4^5\rangle$. Obviously, $\mathbf{v}_4^5$ is the white vertex in $B_4^{\{5\}}$, and $(\mathbf{v}_4^5)^5$ is a black vertex in $B_4^{\{1\}}$. Let $\mathbf{v}_4^1$, $\mathbf{v}_4^2$, and $\mathbf{v}_4^4$ be the white vertices in $B_4^{\{1\}}$, $B_4^{\{2\}}$, and $B_4^{\{4\}}$, with $(\mathbf{v}_4^1)_4 = 2$, $(\mathbf{v}_4^2)_4 = 4$, and $(\mathbf{v}_4^4)_4 = 3$, respectively. It is easy to know that $(\mathbf{v}_4^1)^5$, $(\mathbf{v}_4^2)^5$, and $(\mathbf{v}_4^4)^5$ are the black vertices in $B_4^{\{2\}}$, $B_4^{\{4\}}$, and $B_4^{\{3\}}$, respectively. By Theorem 1, there are the Hamiltonian paths: H_4^1 of $B_4^{\{1\}}$, H_4^2 of $B_4^{\{2\}}$, and H_4^4 of $B_4^{\{4\}}$ joining $(\mathbf{v}_4^5)^5$ to $\mathbf{v}_4^1$, $(\mathbf{v}_4^1)^5$ to $\mathbf{v}_4^2$ and $(\mathbf{v}_4^2)^5$ to $\mathbf{v}_4^4$, respectively. We know that $\mathbf{z}_0^3 = 12435$ is a black vertex and $(\mathbf{z}_0^3)^5$ is a white vertex in $B_4^{\{3\}}$. By Theorem 1, there is a Hamiltonian path H_4^3 of $B_4^{\{3\}}$ joining $(\mathbf{v}_4^4)^5$ to $(\mathbf{z}_0^3)^5$. We set $C_4 = \langle \mathbf{e}, \mathbf{z}_0^1, (\mathbf{z}_0^1)^4, H_4^5, \mathbf{v}_4^5, (\mathbf{v}_4^5)^5, H_4^1, \mathbf{v}_4^1, (\mathbf{v}_4^1)^5, H_4^2, \mathbf{v}_4^2, (\mathbf{v}_4^2)^5, H_4^4, \mathbf{v}_4^4, (\mathbf{v}_4^4)^5, H_4^3, (\mathbf{z}_0^3)^5, \mathbf{z}_0^3, \mathbf{e}\rangle$ is the desired cycle (see Figure A1). ■

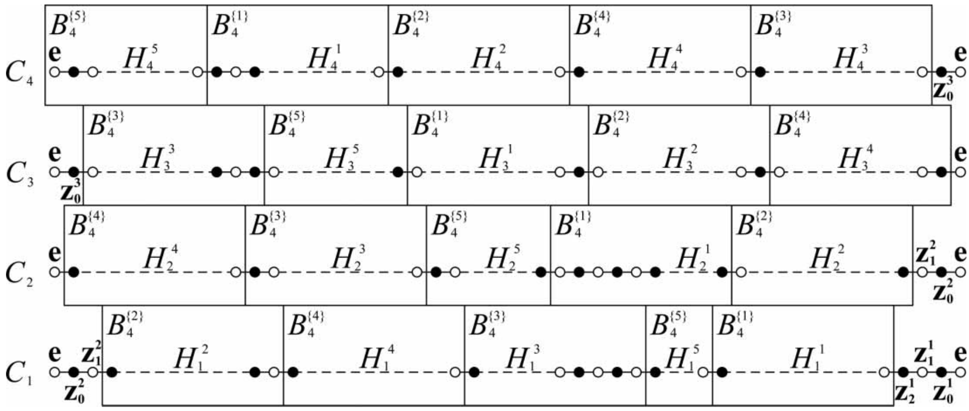

Figure A1. The mutually independent Hamiltonian cycles of B_5.