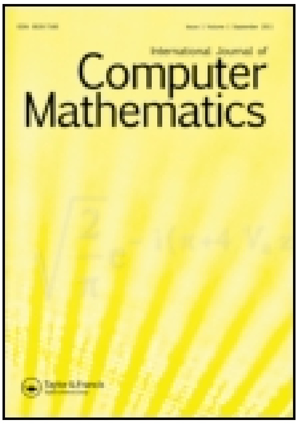

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The honeycomb torus HT(m) is an attractive architecture for distributed processing applications. For analysing its performance, a symmetric generalized honeycomb torus, GHT$(m, n, n/2)$, with $m \geq 2$ and even $n \geq 4$, where $m + n/2$ is even, which is a 3-regular, Hamiltonian bipartite graph, is operated as a platform for combinatorial studies. More specifically, GHT$(m, n, n/2)$ includes GHT$(m, 6m, 3m)$, the isomorphism of the honeycomb torus HT(m). It has been proven that any GHT$(m, n, n/2) - e$ is Hamiltonian for any edge $e \in E(\text{GHT}(m, n, n/2))$. Moreover, any GHT$(m, n, n/2) - F$ is Hamiltonian for any $F = \{u, v\}$ with $u \in B$ and $v \in W$, where B and W are the bipartition of $V(\text{GHT}(m, n, n/2))$ if and only if $n \geq 6$ or $m = 2, n \geq 4$.

Keywords: fault-tolerance; generalized honeycomb torus; graph embedding; Hamiltonian cycle; interconnection networks

2000 AMS Subject Classifications: 05C60; 68M10; 68R10; 68M15; 94C15

1. Introduction

Network topology is a crucial factor in interconnection networks, because it determines the performance of the network [8]. For example, the number of links connected to network nodes can be as small as possible [11], and the network configuration can still fit the geometry of potential applications [9]. In addition, fault-tolerance is desired for offering more systematic reliability.

Honeycomb torus, denoted by HT(m) for any integer $m \geq 1$, is recognized as an attractive, symmetric, alternative architecture to existing torus interconnection networks in parallel and distributed applications [2,6,7,11]. In 1997, Stojmenovic [11] proposed several variations of honeycomb tori. Cho and Hsu [4] then proved that all honeycomb torus networks can be characterized in a unified way, the *generalized honeycomb torus*, denoted by GHT(m, n, s), where $m \geq 2$ is an integer, $n \geq 4$ is an even integer, s is an integer with $0 \leq s < n$, and $m + s$ is even. HT(m),

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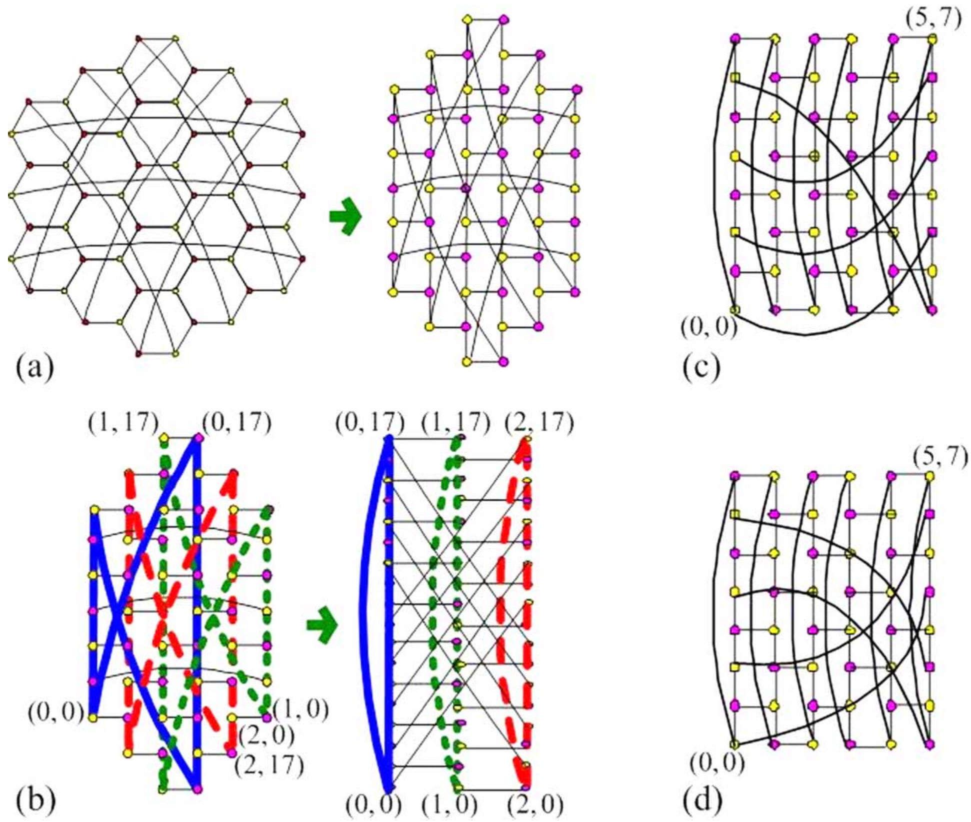

Figure 1. Developing configurations of generalized honeycomb tori. (a) Honeycomb torus HT(3) and its brick presentation [10], (b) presenting HT(3) as GHT(3, 18, 9) [4], (c) GHT(6, 8, 2), and (d) GHT(6, 8, 4).

$m \geq 2$, is isomorphic to GHT$(m, 6m, 3m)$ (Figures 1(a) and (b)). Two examples of GHT(m, n, s) are shown in Figures 1(c) and (d). Ring embedding is one of the most important subjects in interconnection networks [3,6,7,11,12]. Megson *et al*. studied fault-tolerant ring embedding in HT(m) with two adjacent faulty nodes [7]. In this paper, finding a fault-free ring in GHT$(m, n, n/2)$ is intended on the failure of any pair of bipartite nodes and the failure of any one edge. Obviously, GHT$(m, n, n/2)$ serves as a larger platform that contains HT(m) as a subset, and this paper deals with more arbitrary faulty conditions than [7]. Moreover, GHT$(m, n, n/2)$ can offer the property of symmetry, which can help GHT$(m, n, n/2)$ be prototyped for applications. Many studies have been conducted recently on generalized honeycomb tori [9,10,13,14].

This paper is organized in the following way. In Section 2, we define some graphing terms that are used in the paper and give a formal definition of the generalized honeycomb torus. In Section 3, we present a recursive property of ring embedding in GHT$(m, n, 2/n)$. In Section 4, we discuss the ring-embedding properties of GHT$(m, n, n/2) - F$, where F consists of a pair of nodes from opposite sets of the partition of $V($GHT$(m, n, n/2))$. In Section 5, we prove the ring-embedding property of HT$(m, n, n/2) - e$ for any $e \in E($GHT$(m, n, n/2))$.

2. Preliminaries

Computer and communication networks are usually represented by graphs, where nodes represent processors and edges represent links between processors. In this paper, a network is represented as an undirected graph. For graph definition and notation, we follow [1]. $G = (V, E)$ is a *graph*, with V being a finite set and E being a subset of $\{(u, v)|(u, v)$ is an unordered pair of $V\}$. We say that V is the *node* set, and E is the *edge* set of G. Two nodes, u and v, are *adjacent* if $(u, v) \in E$. A *path* is a sequence of nodes such that two consecutive nodes are adjacent. A path is delimited by $<x_0, x_1, x_2, \ldots, x_{n-1}>$. We use P^{-1} to denote the reverse path $<x_{n-1}, \ldots, x_2, x_1, x_0>$ if P is the path $<x_0, x_1, x_2, \ldots, x_{n-1}>$. A path is called a *Hamiltonian path* if its nodes are distinct and span V. A *cycle* is a path of at least three nodes, such that the first node is the same as the last node. A cycle is called a *Hamiltonian cycle* if its nodes are distinct, except for the first node and the last node, and if they span V. A graph is called *Hamiltonian* if it has a Hamiltonian cycle. A graph $G = (V, E)$ is 1-*edge Hamiltonian* if $G - e$ is Hamiltonian for any $e \in E$. A Hamiltonian bipartite graph G is 1_p-*Hamiltonian* if $G - F$ remains Hamiltonian for any $F = \{u, v\}$ with $u \in B$ and $v \in W$, where B and W are the bipartition of G [3,5,7]. The following definition and notions can be found in [4]. For any two positive integers r and d, we use $[r]_d$ to denote $r \pmod d$. Let m, n, and s be positive integers with $m \geq 2$, $n \geq 4$, and both n and $m + s$ are even. The *generalized honeycomb torus* GHT(m, n, s) is the graph with the node set $\{(i, j)|0 \leq i < m, 0 \leq j < n\}$, such that (i, j) and (k, l) with $i \leq k$ are adjacent if they satisfy one of the following conditions:

$$(k, l) = (i, [j \pm 1]_n), \tag{1}$$

$$0 \leq i \leq m - 2, i + j \text{ is odd, and } (k, l) = (i + 1, j), \tag{2}$$

$$i = 0, j \text{ is even, and } (k, l) = (m - 1, [j + s]_n). \tag{3}$$

From condition (1) above, m rings of n nodes can be inherently formed in GHT(m, n, s). The requirement that n be even is essential for establishing bipartite rings, and then the configuration regularly composed of rings can give ideas for applying ring embedding to the study of network performance related to scalability. Moreover, such a performance makes GHT(m, n, s) node-transitive. We can set a node partition W as $\{(i, j)|(i, j) \in V(\text{GHT}(m, n, s))$, and $i + j$ is even$\}$ and another partition set B as $\{(i, j)|(i, j) \in V(\text{GHT}(m, n, s))$, and $i + j$ is odd$\}$. From condition (2), and on the ring at column i, there are edges extending towards the right (to column $i + 1$), where j is odd with i being even, or where j is even with i being odd. Based on condition (3), there are wrap-around edges connecting the first column (column 0) and the last column (column $m - 1$; yet, such wrap-around edges can be transitively assigned to connect any two adjacent columns); however, on establishing bipartite networks, $(m - 1) + (j + s)$ has to be odd, because $0 + j$ is even. In other words, we can find that $m + s$ being even is a reasonable prerequirement.

All GHT(m, n, s) nodes can have exactly three adjacent nodes, which belong to another partition. Consequently, any generalized honeycomb torus is a 3-regular bipartite graph. We are interested in a special type of generalized honeycomb torus, GHT$(m, n, n/2)$. We will prove that any GHT$(m, n, n/2)$ is 1-*edge Hamiltonian* for $n \geq 4$. Moreover, GHT$(m, n, n/2)$ is 1_p-*Hamiltonian* if and only if $n \geq 6$ or $m = 2$, $n \geq 4$. To discuss the 1_p-*Hamiltonian* property of GHT$(m, n, n/2)$, let $F = \{u, v\}$ with $u \in B$ and $v \in W$. We may assume that $(0, 0) \in F$ because GHT$(m, n, n/2)$ is node-transitive. For this reason, we use $\mathbf{F}$ to denote $\{F|\{(0, 0), (x, y)\}|(x, y) \in B\}$. Hence, $F \in \mathbf{F}$. We use (x, y) to denote the unique element in $F - \{(0, 0)\}$. By assumption, $x + y$ is odd. We define the vertically extensive path patterns $I_l(i, j)$ and $I_l^{-1}(i, j)$ for $0 \leq l < m$ and $i \leq j$ as follows:

$$I_l(i, j) = (l, i), (l, i + 1), (l, i + 2), \ldots, (l, j - 1), (l, j),$$

$$I_l^{-1}(i, j) = (l, j), (l, j - 1), (l, j - 2), \ldots, (l, i + 1), (l, i).$$

In Section 4, vertically recursive path patterns are similarly generalized as $Q_l^t(i, j)$, where Q is the name of the path pattern; t is the recursive time equal to $(j - i)/2$; and i and j are row indices of the starting and ending nodes of this path. The Q pattern is typically repeated every two rows, and t can be zero if $i = j$ and omitted if $t = 1$: i.e., $Q_l(i, i + 2)$. Moreover, for sketching required paths, some path patterns are specifically defined as S_k, where k is a sequentially assigned number.

3. A recursive property – extending the column dimension

In this section, assume that $0 \le i < m$, and $G = \text{GHT}(m, n, n/2)$. S denotes a subset of $V(G) \cup E(G)$. Let $A_i = \{((i, j), (i + 1, j)) | 0 \le j < n\}$ for $0 \le i \le m - 2$ and $A_{m-1} = \{((0, [j - n/2]_n), (m - 1, j)) | 0 \le j < n\}$. We define a function, f_i, that maps S from G into $\text{GHT}(m + 2, n, n/2)$ in the following way:

(1) If $(k, l) \in S \cap V(G)$, where $0 \le k \le m - 1$ and $0 \le l \le n - 1$, then

$$f_i((k, l)) = \begin{cases} (k, l) & \text{if } k \le i, \\ (k + 2, l) & \text{if } k > i. \end{cases}$$

(2) If $((k, l), (k', l')) \in S \cap E(G)$, where $k \le k'$, then

$$f_i((k, l), (k', l')) = \begin{cases} (f_i(k, l), f_i(k', l')) & \text{if } \{k, k'\} \ne \{i, [i + 1]_m\}, \\ ((i + 2, l), (i + 3, l)) & \text{if } \{k, k'\} = \{i, i + 1\} \text{ for } 0 \le i \le m - 2, \\ ((0, [l' - n/2]_n), (m + 1, l')) & \text{if } \{k, k'\} = \{0, m - 1\} \text{ for } i = m - 1. \end{cases}$$

$f_i(S)$ is a one-to-one mapping on both nodes and edges. Mapping a Hamiltonian cycle from G into $\text{GHT}(m + 2, n, n/2)$ requires other operations to accommodate extra nodes and edges as a result of extension in the column dimension. Let H be a Hamiltonian cycle of $G - F'$. Although F' theoretically may be any subset of $V(G) \cup E(G)$, in this section, F' specifically denotes a set that includes one faulty edge or a set of a pair of faulty bipartite nodes. For example, a Hamiltonian cycle H of $\text{GHT}(6, 8, 4) - \{(0, 0), (5, 0)\}$ is shown in Figure 2(a). If $H \cap A_i \ne \emptyset$, a Hamiltonian cycle $K_i(H)$ of $\text{GHT}(m + 2, n, n/2) - f_i(F')$ can be constructed as follows.

Step 1. Let $(i, k_0), (i, k_1), \ldots, (i, k_{t-1})$ be the nodes of H on column i. An example is given in Figure 2(a). If $i = 5$, then $t = 1$ and $k_0 = 2$. If $i = 1$, then $t = 3$ and $k_0 = 0, k_1 = 2, k_2 = 6$.
Step 2. Let $\underline{H}$ be the image of H under f_i. For example, in the left half of Figure 2(b), the solid line shows $\underline{H} = f_5(H)$; in the left half of Figure 2(c), the solid line shows $\underline{H} = f_1(H)$.
Step 3. For $0 \le j < t$, we set Z_j (see the dashed line in Figures 2(b) and (c)) as the path:

$$(i, k_j), ([i + 1]_{m+2}, k_j), I_{[i+1]m+2}(k_j, [k_{[j+1]t} - 1]_n), ([i + 1]_{m+2}, [k_{[j+1]t} - 1]_n),$$
$$([i + 2]_{m+2}, [k_{[j+1]t} - 1]_n), I^{-1}_{[i+2]m+2}([k_{[j+1]t} - 1]_n, k_j), ([i + 2]_{m+2}, k_j).$$

It is easy to see that $f_i(H)$ together with the edges of Z_j, with $0 \le j < t$, forms a Hamiltonian cycle of $\text{GHT}(m + 2, n, n/2) - f_i(F')$ (see the right half of Figures 2(b) and (c)). We denote this cycle as $K_i(H)$. Then, we can have the following lemmas.

LEMMA 3.1 *Assume that $0 \le i < m$. Let H be a Hamiltonian cycle of $G - F'$, such that $H \cap A_i \ne \emptyset$. Then, $K_i(H)$ is a Hamiltonian cycle of $\text{GHT}(m + 2, n, n/2) - f_i(F')$. Moreover, $K_i(H)$ contains some edges joining column t to column $[t + 1]_{m+2}$ for any t in $\{i, [i + 1]_{m+2}, [i + 2]_{m+2}\}$.*

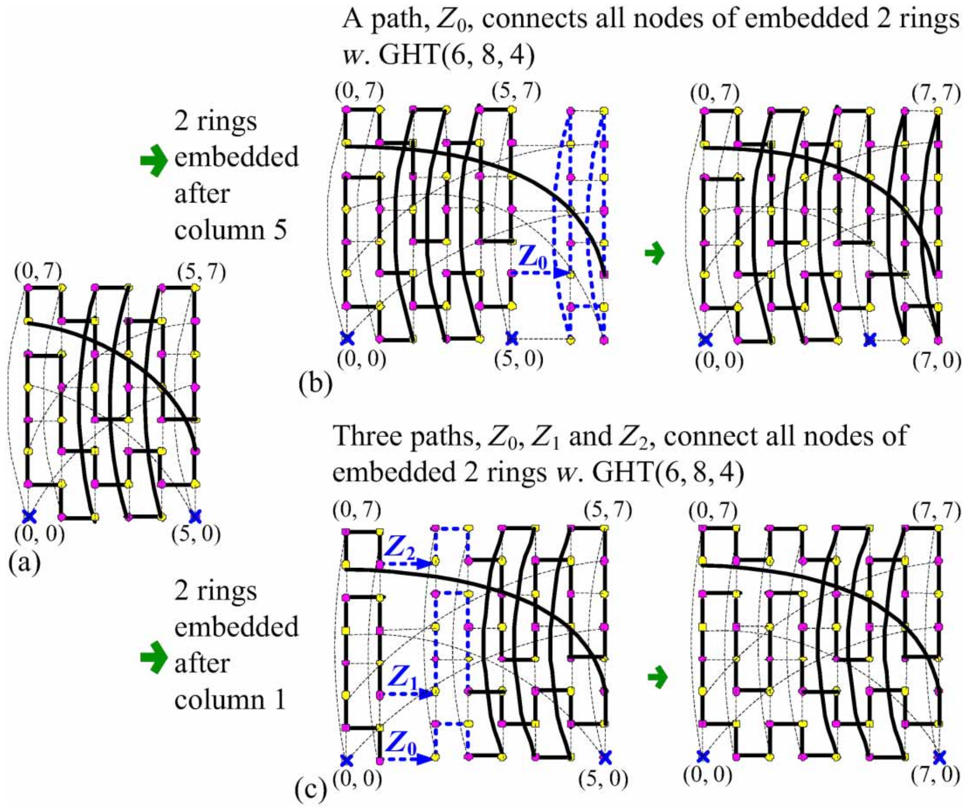

Figure 2. (a) A Hamiltonian cycle H in $\text{GHT}(6, 8, 4) - \{(0, 0), (5, 0)\}$; (b) $K_5(H)$, a Hamiltonian cycle in $\text{GHT}(8, 8, 4) - \{(0, 0), (5, 0)\}$; and (c) $K_1(H)$, a Hamiltonian cycle in $\text{GHT}(8, 8, 4) - \{(0, 0), (7, 0)\}$.

Lemma 3.2 (1) *Suppose that H is a Hamiltonian cycle of* $\text{GHT}(2, n, n/2) - F'$, *such that H contains some edges in* $\{((0, j), (1, j)) | j \text{ is odd}\}$. *Then* $K_0(H)$ *is a Hamiltonian cycle of* $\text{GHT}(4, n, n/2) - f_0(F')$. *Moreover,* $K_0(H)$ *contains some edges joining column t to column* $t + 1$ *for any t in* $\{0, 1, 2\}$. (2) *Suppose that H is a Hamiltonian cycle of* $\text{GHT}(2, n, n/2) - F'$, *such that H contains some edges in* $\{((0, j), (1, [j + n/2]_n)) | j \text{ is even}\}$. *Then* $K_1(H)$ *is a Hamiltonian cycle of* $\text{GHT}(4, n, n/2) - f_1(F')$. *Moreover,* $K_1(H)$ *contains some edges joining column t to column* $[t + 1]_4$ *for any t in* $\{1, 2, 3\}$.

Lemma 3.3 (1) *Suppose that H is a Hamiltonian cycle of* $\text{GHT}(3, n, n/2) - F'$, *such that H contains some edges in* $\{((0, j), (1, j)) | j \text{ is odd}\}$. *Then* $K_0(H)$ *is a Hamiltonian cycle of* $\text{GHT}(5, n, n/2) - f_0(F')$. *Moreover,* $K_0(H)$ *contains some edges joining column t to column* $t + 1$ *for any t in* $\{0, 1, 2\}$. (2) *Suppose that H is a Hamiltonian cycle of* $\text{GHT}(3, n, n/2) - F'$, *such that H contains some edges in* $\{((1, j), (2, j)) | j \text{ is even}\}$. *Then* $K_1(H)$ *is a Hamiltonian cycle of* $\text{GHT}(5, n, n/2) - f_1(F')$. *Moreover,* $K_1(H)$ *contains some edges joining column t to column* $t + 1$ *for any t in* $\{1, 2, 3\}$. (3) *Suppose that H is a Hamiltonian cycle of* $\text{GHT}(3, n, n/2) - F'$, *such that H contains some edges in* $\{((0, j), (2, [j + n/2]_n)) | j \text{ is even}\}$. *Then* $K_2(H)$ *is a Hamiltonian cycle of* $\text{GHT}(5, n, n/2) - f_2(F')$. *Moreover,* $K_2(H)$ *contains some edges joining column t to column* $[t + 1]_5$ *for any t in* $\{2, 3, 4\}$.

We say a Hamiltonian cycle H of $\text{GHT}(2, n, n/2) - F'$ is *regular* if H contains some edges in $\{((0, j), (1, j)) | j \text{ is odd}\}$ and some edges in $\{((0, j), (1, [j + n/2]_n)) | j \text{ is even}\}$. In general, a Hamiltonian cycle H of $G - F'$ is *regular* if $H \cap A_i \neq \emptyset$ for $0 \leq i < m$ and $m \geq 2$. Then the following lemma is derived from Lemmas 3.1–3.3.

LEMMA 3.4 *Suppose that H is a regular Hamiltonian cycle for $G - F'$. Then $K_i(H)$ is a regular Hamiltonian cycle of* $\mathrm{GHT}(m+2, n, n/2) - f_i(F')$ *for every* $0 \leq i < m$.

4. The 1_p-Hamiltonian property of GHT$(m, n, n/2)$

Throughout this section, let $k = n/2$. We first prove that $\mathrm{GHT}(m, n, k)$ is 1_p-*Hamiltonian* for $m = 2, 3, 4$, and then prove the 1_p-*Hamiltonian* property of general m using mathematical induction.

THEOREM 4.1 *Let* $n \equiv 0 (\mathrm{mod}\ 4)$. $\mathrm{GHT}(2, n, k)$ *is* 1_p-*Hamiltonian for* $n \geq 4$.

Proof Let $G = \mathrm{GHT}(2, n, k)$ and $F \in \mathbf{F}(\mathrm{GHT}(2, n, k))$. Using the symmetry of G, it suffices to show that $G - F$ is Hamiltonian, where $F = \{(0,0), (x,y) | x \in \{0,1\}, 0 \leq y \leq k; x + y$ is odd$\}$. We define some path patterns as follows.

$$S_1 = (0,1), (0,2), (1,2+k), (1,3+k), (1,4+k), (0,4),$$
$$S_2 = (0,4), (0,3), (1,3), (1,2), (0,2+k), (0,3+k), (0,4+k), (1,4),$$
$$S_3 = (1,n-3), (1,n-2), (0,k-2), (0,k-1), (0,k), (0,k+1), (1,k+1), (1,k), (1,k-1), (1,k-2), (0,n-2), (0,n-1), (1,n-1), (1,0), (1,1), (0,1),$$
$$S_4 = (1,y-1), (1,y), (1,y+1), (1,y+2), (0,y+2), (0,y+1),$$
$$S_5 = (1,k+y+s), (1,k+y+s-1), (0,k+y+s-1), (0,k+y+s), (0,k+y+s+1), (1,k+y+s+1), (s = 1 \text{ if } y \text{ is odd, otherwise } s = 0),$$
$$S_6 = (1,0), (1,1), (0,1), (0,2), (1,2+k), (1,1+k), (1,k), (1,k-1), (0,k-1), (0,k), (0,k+1), (0,k+2), (1,2),$$
$$A_1(i,i+2) = (1,i), (1,i+1), (0,i+1-k), (0,i+2-k), (1,i+2-k), (1,i+1-k), (0,i+1), (0,i+2), (1,i+2),$$
$$B_1(i,i+2) = (1,i), (1,i+1), (0,i+1), (0,i+2), (1,i+2+k), (1,i+1+k), (0,i+1+k), (0,i+2+k), (1,i+2),$$
$$C_1(i,i+2) = (1,i), (1,i+1), (0,i+1-k), (0,i-k), (1,i-k), (1,i-k+1), (0,i+1), (0,i+2), (1,i+2),$$
$$D_1(i,i+2) = (1,i), (1,i+1), (0,i+1), (0,i+2), (1,i+2+k), (1,i+3+k), (0,i+3+k), (0,i+2+k), (1,i+2),$$

The corresponding Hamiltonian cycles in $G - F$ are constructed below in Cases 1–3.

Case 1 $n = 4$.

(x, y)	the Hamiltonian cycle C in $G - \{(0,0), (x,y)\}$
$(0,1)$	$(0,2), (0,3), (1,3), (1,2), (1,1), (1,0), (0,2)$ (Figure 3(z))
$*(1,0)$	$(0,1), (0,2), (0,3), (1,3), (1,2), (1,1), (0,1)$ (Figure 3(f))
$*(1,2)$	$(0,1), (0,2), (0,3), (1,3), (1,0), (1,1), (0,1)$ (Figure 3(1))

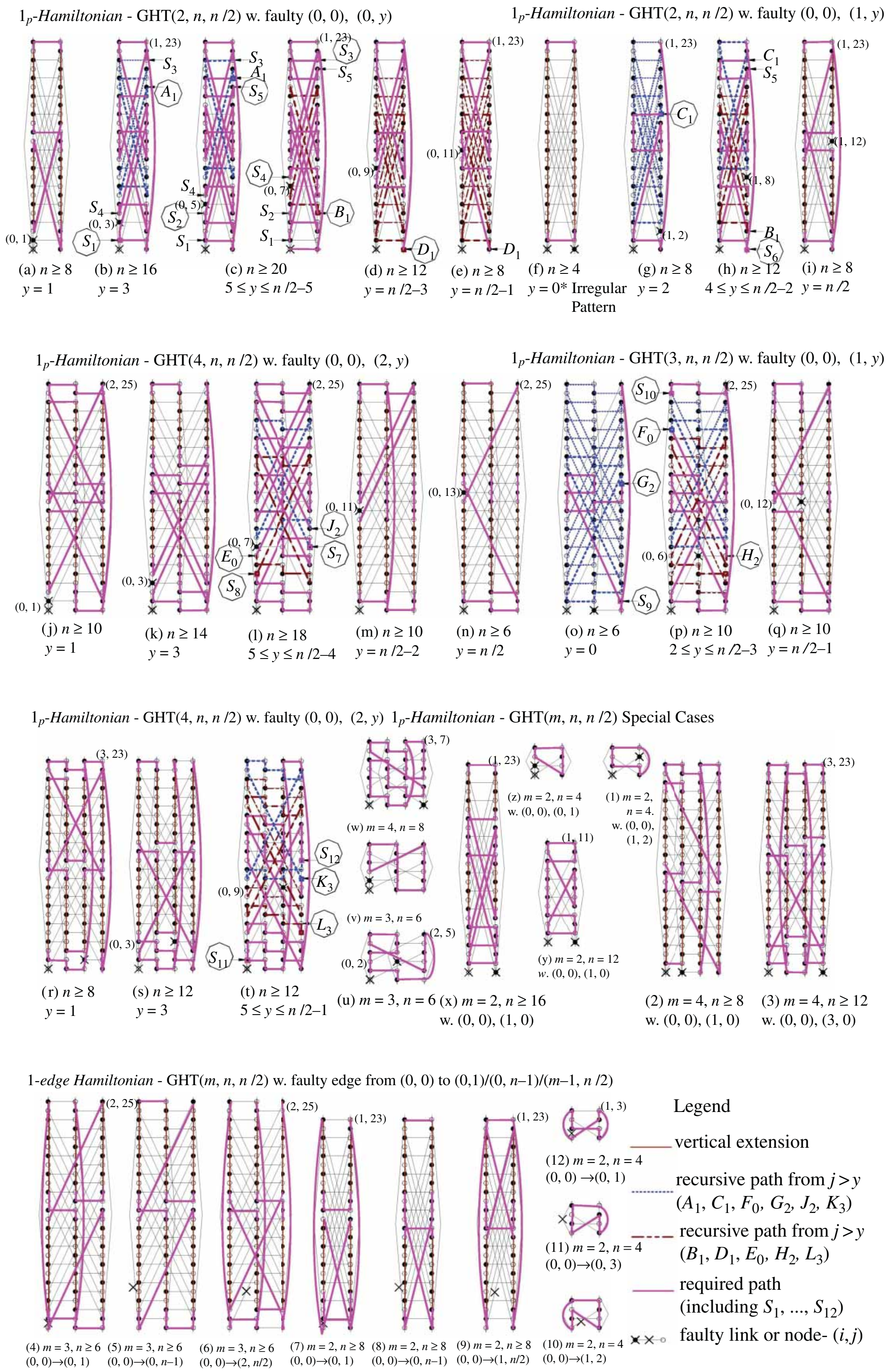

Figure 3. Basic cases for proving GHT$(m, n, n/2)$ 1_p-*Hamiltonian* and 1-*edge Hamiltonian*.

Case 2 $n \geq 8$. $x = 0$, y is odd.

y	the Hamiltonian cycle C in $G - \{(0,0), (0,y)\}$
$y = 1$ ($n \geq 8$)	$(1,0)$, $I_1(1,k+1)$, $(1,k+1)$, $(0,k+1)$, $I_0(k+1,n-1)$, $(0,n-1)$, $(1,n-1)$, $I_1^{-1}(n-1,k+2)$, $(1,k+2)$, $(0,2)$, $I_0(2,k)$, $(0,k)$, $(1,0)$ (Figure 3(a))
$y = 3$ ($n \geq 16$)	$(0,1)$, S_1, $(0,4)$, S_4^{-1}, $(1,2)$, $(0,2+k)$, $(0,3+k)$, $(0,4+k)$, $(0,5+k)$, $(1,5+k)$, $A_1^{(k-8)/2}(5+k,n-3)$, $(1,n-3)$, S_3, $(0,1)$ (Figure 3(b))
$5 \leq y \leq k-5$ ($n \geq 20$)	$(0,1)$, S_1, $(0,4)$, S_2, $(1,4)$, $B_1^{(y-5)/2}(4,y-1)$, $(1,y-1)$, S_4, $(0,y+1)$, $(1,y+1+k)$, S_5, $(1,y+2+k)$, $A_1^{(k-y-5)/2}(y+2+k,n-3)$, $(1,n-3)$, S_3, $(0,1)$ (Figure 3(c))
$y = k-3$ ($n \geq 12$)	$(1,0)$, $D_1^{(k-6)/2}(0,k-6)$, $(1,k-6)$, $(1,k-5)$, $(0,k-5)$, $(0,k-4)$, $(1,n-4)$, $I_1(n-4,n-2)$, $(1,n-2)$, $(0,k-2)$, $I_0(k-2,k+1)$, $(0,k+1)$, $(1,k+1)$, $I_1^{-1}(k+1,k-4)$, $(1,k-4)$, $(0,n-4)$, $I_0(n-4,n-1)$, $(0,n-1)$, $(1,n-1)$, $(1,0)$ (Figure 3(d))
$y = k-1$ ($n \geq 8$)	$(1,0)$, $D_1^{(k-4)/2}(0,k-4)$, $(1,k-4)$, $(1,k-3)$, $(0,k-3)$, $(0,k-2)$, $(1,n-2)$, $(1,n-1)$, $(0,n-1)$, $(0,n-2)$, $(1,k-2)$, $I_1(k-2,k+1)$, $(1,k+1)$, $(0,k+1)$, $(0,k)$, $(1,0)$) (Figure 3(e))

Case 3 $n \geq 8$, $x = 1$, y is even.

y	the Hamiltonian cycle C in $G - \{(0,0), (1,y)\}$
$*y = 0$ ($n \geq 8$)	$(0,1)$, $I_0(1,n-1)$, $(0,n-1)$, $(1,n-1)$, $I_1^{-1}(n-1,1)$, $(1,1)$, $(0,1)$ (Figure 3(f))
$y = 0$ ($n = 12$)	$(0,1)$, $(0,2)$, $(1,8)$, $I_1^{-1}(8,4)$, $(1,4)$, $(0,10)$, $(0,11)$, $(1,11)$, $I_1^{-1}(11,9)$, $(1,9)$, $(0,9)$, $I_0^{-1}(9,3)$, $(0,3)$, $(1,3)$, $(1,2)$, $(1,1)$, $(0,1)$ (Figure 3(y))
$y = 0$ ($n \geq 16$)	$(0,1)$, S_1, $(0,4)$, $(0,3)$, $(1,3)$, $(1,4)$, $(1,5)$, $(0,5)$, $I_0(5,k+1)$, $(0,k+1)$, $(1,k+1)$, $I_1^{-1}(k+1,6)$, $(1,6)$, $(0,k+6)$, $I_0(k+6,n-1)$, $(0,n-1)$, $(1,n-1)$, $I_1^{-1}(n-1,k+5)$, $(1,k+5)$, $(0,k+5)$, $I_0^{-1}(k+5,k+2)$, $(0,k+2)$, $(1,2)$, $(1,1)$, $(0,1)$ (Figure 3(x))
$y = 2$ ($n \geq 8$)	$(1,0)$, $(1,1)$, $(0,1)$, $(0,2)$, $(1,k+2)$, $I_1^{-1}(2+k,k-1)$, $(1,k-1)$, $(0,k-1)$, $I_0(k-1,k+3)$, $(0,k+3)$, $(1,k+3)$, $C_1^{(k-4)/2}(k+3,n-1)$, $(1,n-1)$, $(1,0)$ (Figure 3(g))
$4 \leq y \leq k-2$ ($n \geq 12$)	$(1,0)$, S_6, $(1,2)$, $B_1^{(y-4)/2}(2,y-2)$, $(1,y-2)$, $(1,y-1)$, $(0,y-1)$, $(0,y)$, $(1,y+k)$, S_5, $(1,y+k+1)$, $C_1^{(k-y-2)/2}(y+k+1,n-1)$, $(1,n-1)$, $(1,0)$ (Figure 3(h))
$y = k$ ($n \geq 8$)	$(1,0)$, $(1,1)$, $(0,1)$, $I_0(1,k-2)$, $(0,k-2)$, $(1,n-2)$, $I_1^{-1}(n-2,k+1)$, $(1,k+1)$, $(0,k+1)$, $(0,k)$, $(0,k-1)$, $(1,k-1)$, $I_1^{-1}(k-1,2)$, $(1,2)$, $(0,k+2)$, $I_0(k+2,n-1)$, $(0,n-1)$, $(1,n-1)$, $(1,0)$ (Figure 3(i))

Obviously, these Hamiltonian cycles above in $G - F$ are regular except in the cases marked with an asterisk (*). More precisely, for $n \geq 4$ and for any $F \in \mathbf{F}(\mathrm{GHT}(2,n,k))$, $\mathrm{GHT}(2,n,k) - F$ has a regular Hamiltonian cycle except for GHT(2,4,2)–{(0,0),(1,0)}, GHT(2,4,2)–{(0,0),(1,2)}, and GHT(2,8,4)–{(0,0),(1,0)}. ■

THEOREM 4.2 *Let* $n \equiv 2 \pmod 4$. GHT$(3, n, k)$ *is* 1_p*-Hamiltonian for* $n \geq 6$. *Moreover, there is a regular Hamiltonian cycle in* GHT$(3, n, k) - F$ *for any* $F \in \boldsymbol{F}$(GHT$(3, n, k)$).

Proof Let $G =$ GHT$(3, n, k)$ and $F \in \mathbf{F}$(GHT$(3, n, k)$). Using the symmetry of G, it suffices to show that $G - F$ is Hamiltonian, where $F = \{(0, 0), (x, y) | x \in \{0, 1\}, 0 \leq y \leq k; x + y$ is odd$\}$. We define some path patterns as follows.

$$
\begin{aligned}
S_7 = {} & (2, y), (2, y+1), (1, y+1), (1, y), (1, y-1), (2, y-1), (2, y-2),\\
& (0, y-2+k), (0, y-1+k), (1, y-1+k), (1, y-2+k),\\
& (2, y-2+k), (2, y-1+k), (0, y-1),\\
S_8 = {} & (0, 4), (0, 3), (1, 3), (1, 2), (2, 2), (2, 1), (2, 0), (1, 0), (1, 1), (0, 1), (0, 2),\\
& (2, k+2), I_2^{-1}(k+2, k-2), (2, k-2), (0, n-2), (0, n-1), (1, n-1),\\
& (1, n-2), (2, n-2), (2, n-1), (0, k-1), I_0(k-1, k+2), (0, k+2),\\
& (1, k+2), I_1^{-1}(k+2, k-2), (1, k-2), (0, k-2), (0, k-3), (2, n-3),\\
& (2, n-4), (1, n-4), (1, n-3), (0, n-3), (0, n-4), (2, k-4),\\
S_9 = {} & (2, 1), (0, k+1), (0, k+2), (1, k+2), I_1^{-1}(k+2, k), (1, k), (0, k), I_0^{-1}\\
& (k, k-2), (0, k-2), (1, k-2), (1, k-1), (2, k-1),\\
& I_2(k-1, k+1), (2, k+1),\\
S_{10} = {} & (0, n-2), (0, n-1), (1, n-1), (1, n-2), (2, n-2), (2, n-1), (2, 0),\\
& (1, 0), (1, 1), (0, 1), (0, 2), (2, k+2), (2, k+1), S_9^{-1}, (2, 1), (2, 2),\\
E_0(i, i+2) = {} & (0, i), (2, k+i), (2, k+i-1), (1, k+i-1), (1, k+i), (0, k+i),\\
& (0, k+i-1), (2, i-1), (2, i), (1, i), (1, i+1), (0, i+1), (0, i+2),\\
F_0(i, i+2) = {} & (0, i), (0, i+1), (1, i+1), (1, i), (2, i), (2, i+1), (0, i-k+1), (0, i-k),\\
& (1, i-k), (1, i-k+1), (2, i-k+1), (2, i-k+2), (0, i+2),\\
G_2(i, i+2) = {} & (2, i), (2, i+1), (0, i-k+1), (0, i-k), (1, i-k), (1, i-k+1),\\
& (2, i-k+1), (2, i-k+2), (0, i+2), (0, i+3), (1, i+3),\\
& (1, i+2), (2, i+2),\\
H_2(i, i+2) = {} & (2, i), (1, i), (1, i+1), (0, i+1), (0, i+2), (2, k+i+2), (2, k+i+1),\\
& (1, k+i+1), (1, k+i+2), (0, k+i+2), (0, k+i+1),\\
& (2, i+1), (2, i+2),\\
J_2(i, i+2) = {} & (2, i), (0, i+k), (0, k+i+1), (1, k+i+1), (1, k+i), (2, k+i),\\
& (2, k+i+1), (0, i+1), (0, i+2), (1, i+2), (1, i+3),\\
& (2, i+3), (2, i+2).
\end{aligned}
$$

Case 1 $x = 0$, y is odd.

y	the Hamiltonian cycle C in $G - \{(0,0),(0,y)\}$
$y = 1 (n = 6)$	$(0,2), I_0(2,5), (0,5), (1,5), I_1^{-1}(5,0), (1,0), (2,0), I_2(0,5), (2,5), (0,2)$ (Figure 3(v))
$y = 1 (n \geq 10)$	$(0,2), I_0(2,k-2), (0,k-2), (1,k-2), I_1^{-1}(k-2,0), (1,0), (2,0), (2,n-1), (0,k-1), (0,k), (1,k), (1,k-1), (2,k-1), (2,k), (2,k+1), (1,k+1), I_1(k+1,n-3), (1,n-3), (0,n-3), I_0^{-1}(n-3,k+1), (0,k+1), (2,1), I_2(1,k-2), (2,k-2), (0,n-2), (0,n-1), (1,n-1), (1,n-2), (2,n-2), I_2^{-1}(n-2,k+2), (2,k+2), (0,2)$ (Figure 3(j))
$y = 3 (n \geq 14)$	$(0,1), (0,2), (2,k+2), (2,k+3), (1,k+3), I_1(k+3,n-1), (1,n-1), (0,n-1), I_0^{-1}(n-1,k+3), (0,k+3), (2,3), I_2(3,k+1), (2,k+1), (1,k+1), (1,k+2), (0,k+2), (0,k+1), (2,1), (2,2), (1,2), I_1(2,k), (1,k), (0,k), I_0^{-1}(k,4), (0,4), (2,k+4), I_2(k+4,n-1), (2,n-1), (2,0), (1,0), (1,1), (0,1)$ (Figure 3(k))
$5 \leq y \leq k-4$ $(n \geq 18)$	$(0,4), S_8, (2,k-4), J_2^{-1(k-y-4)/2}(k-4,y), (2,y), S_7, (0,y-1), E_0^{-1(y-5)/2}(y-1,4), (0,4)$ (Figure 3(l))
$y = k-2$ $(n \geq 10)$	$(0,1), I_0(1,k-3), (0,k-3), (2,n-3), I_2^{-1}(n-3,0), (2,0), (1,0), (1,n-1), (0,n-1), I_0^{-1}(n-1,k-1), (0,k-1), (2,n-1), (2,n-2), (1,n-2), I_1^{-1}(n-2,1), (1,1), (0,1)$ (Figure 3(m))
$y = k (n \geq 6)$	$(0,1), I_0(1,k-1), (0,k-1), (2,n-1), I_2^{-1}(n-1,2), (2,2), (1,2), I_1(2,n-1), (1,n-1), (0,n-1), I_0^{-1}(n-1,k+1), (0,k+1), (2,1), (2,0), (1,0), (1,1), (0,1)$ (Figure 3(n))

Case 2 $x = 1$, y is even.

y	the Hamiltonian cycle C in $G - \{(0,0),(1,y)\}$
$y = 0$ $(n \geq 6)$	$(2,1), S_9, (2,k+1), G_2^{(k-3)/2}(k+1,n-2), (2,n-2), (2,n-1), (2,0)$, (2,1) (Figure 3(o))
$y = 2$ $(n = 6)$	$(0,1), (0,2), (0,3), (1,3), (1,4), (1,5), (0,5), (0,4), (2,1), I_2(1,5), (2,5), (2,0), (1,0), (1,1), (0,1)$ (Figure 3(u))
$2 \leq y \leq k-3$ $(n \geq 10)$	$(2,2), H_2^{(y-2)/2}(2,y), (2,y), (2,y+1), (0,k+y+1), F_0^{(k-y-3)/2}(y+k+1,n-2), (0,n-2), S_{10}, (2,2)$ (Figure 3(p))
$y = k-1$ $(n \geq 10)$	$(0,1), I_0(1,k-2), (0,k-2), (1,k-2), I_1^{-1}(k-2,2), (1,2), (2,2), I_2(2,n-1), (2,n-1), (0,k-1), (0,k), (1,k), I_1(k,n-1), (1,n-1), (0,n-1), I_0^{-1}(n-1,k+1), (0,k+1), (2,1), (2,0), (1,0), (1,1), (0,1)$ (Figure 3(q))

It is easy to see that these Hamiltonian cycles above of $G - F$ are regular. ■

We would like to apply Theorem 4.1 and Lemma 3.2 to prove that GHT$(4,n,k)$ is 1_p-*Hamiltonian* for $n \geq 8$ and that a regular Hamiltonian cycle exists when GHT$(4,n,k)$ contains a pair of faulty nodes. However, with Theorem 4.1, no regular Hamiltonian cycle exists in GHT$(2,8,4) - \{(0,0),(1,0)\}$. Thus, we need the following lemma.

LEMMA 4.3 *Let* $n \equiv 0 (\mathrm{mod}\, 4)$ *and* $n \geq 8$. $\mathrm{GHT}(4, n, k) - F$ *has a regular Hamiltonian cycle when* $F \in \{\{(0, 0), (1, 0)\}, \{(0, 0), (3, 0)\}\}$.

Proof Let $F = \{(0, 0), (x, 0)\} | x = 1, 3\}$. The corresponding regular Hamiltonian cycles are listed in the following table.

x	the Hamiltonian cycle C in $G - F$
$x = 1$ $(n \geq 8)$	$(0, 1), I_0(1, k-1), (0, k-1), (1, k-1), I_1(k-1, n-3), (1, n-3), (0, n-3), I_0^{-1}(n-3, k), (0, k), (3, 0), I_3(0, k-3), (3, k-3), (2, k-3), I_2^{-1}(k-3, 0), (2, 0), (2, n-1), (3, n-1), I_3^{-1}(n-1, k-2), (3, k-2), (0, n-2), (0, n-1), (1, n-1), (1, n-2), (2, n-2), I_2^{-1}(n-2, k-2), (2, k-2), (1, k-2), I_1^{-1}(k-2, 1), (1, 1), (0, 1)$ (Figure 3(2))
$x = 3$ $(n = 8)$	$(0, 1), I_0(1, 5), (0, 5), (1, 5), I_1^{-1}(5, 2), (1, 2), (2, 2), (2, 1), (3, 1), (3, 2), (0, 6), (0, 7), (1, 7), (1, 6), (2, 6), I_2^{-1}(6, 3), (2, 3), (3, 3), I_3(3, 7), (3, 7), (2, 7), (2, 0), (1, 0), (1, 1), (0, 1)$(Figure 3(w))
$x = 3$ $(n \geq 12)$	$(0, 1), (1, 1), (1, 2), (1, 3), (0, 3), I_0(3, k+1), (0, k+1), (1, k+1), I_1^{-1}(k+1, 4), (1, 4), (2, 4), I_2(4, k+2), (2, k+2), (1, k+2), I_1(k+2, n-2), (1, n-2), (2, n-2), I_2^{-1}(n-2, k+3), (2, k+3), (3, k+3), I_3(k+3, n-1), (3, n-1), (2, n-1), (2, 0), (1, 0), (1, n-1), (0, n-1), I_0^{-1}(n-1, k+2), (0, k+2), (3, 2), (3, 1), (2, 1), (2, 2), (2, 3), (3, 3), I_3(3, k+2), (3, k+2), (0, 2), (0, 1)$ (Figure 3(3))

■

THEOREM 4.4 *Let* $n \equiv 0 (\mathrm{mod}\ 4)$. $\mathrm{GHT}(4, n, k)$ *is* 1_p*-Hamiltonian for* $n \geq 8$. *Moreover, there is a regular Hamiltonian cycle in* $\mathrm{GHT}(4, n, k) - F$ *for any* $F \in \boldsymbol{F}(\mathrm{GHT}(4, n, k))$.

Proof Let $G = \mathrm{GHT}(4, n, k)$. With Theorem 4.1 and Lemmas 3.2 and 4.3, $G - \{(0, 0), (x, y)\}$ is Hamiltonian, and the corresponding Hamiltonian cycles are regular when $x \in \{0, 1, 3\}$. Thus, using the symmetry of G, we need to show only that $G - F$ is Hamiltonian, where $F = \{(0, 0), (2, y) | 0 \leq y \leq k;\ y \text{ is odd}\}$. We will now define some path patterns.

$$\begin{aligned}
S_{11} = {} & (0, 1), (1, 1), (1, 0), (2, 0), (2, n-1), (3, n-1), (3, 0), (0, k), I_0(k, k+3), \\
& (0, k+3), (1, k+3), I_1^{-1}(k+3), (1, k), (2, k), I_2(k, k+4), (2, k+4), \\
& (1, k+4), (1, k+5), (0, k+5), (0, k+4), (3, 4), \\
S_{12} = {} & (3, k), I_3(k, k+4), (3, k+4), (0, 4), (0, 5), (1, 5), (1, 4), (2, 4), (2, 3), (3, 3), \\
& (3, 2), (3, 1), (2, 1), (2, 2), (1, 2), (1, 3), (0, 3), (0, 2), (0, 1), \\
L_3(i, i+2) = {} & (3, i), (3, i+1), (2, i+1), (2, i+2), (1, i+2), (1, i+3), (0, i+3), \\
& (0, i+2), (3, k+i+2), (3, k+i+1), (2, k+i+1), (2, k+i+2), \\
& (1, k+i+2), (1, k+i+3), (0, k+i+3), (0, k+i+2), (3, i+2), \\
K_3(i, i+2) = {} & (3, i), (0, k+i), (0, k+i+1), (1, k+i+1), (1, k+i), (2, k+i), \\
& (2, k+i-1), (3, k+i-1), (3, k+i), (0, i), (0, i+1), (1, i+1), (1, i), \\
& (2, i), (2, i+1), (3, i+1), (3, i+2).
\end{aligned}$$

y	the Hamiltonian cycle C in $G - \{(0,0),(2,y)\}$
$y = 1$ ($n \geq 8$)	$(0,1)$, $I_0(1,k-2)$, $(0,k-2)$, $(3,n-2)$, $I_3^{-1}(n-2,k-1)$, $(3,k-1)$, $(2,k-1)$, $I_2^{-1}(k-1,2)$, $(2,2)$, $(1,2)$, $I_1(2,k-1)$, $(1,k-1)$, $(0,k-1)$, $I_0(k-1,n-3)$, $(0,n-3)$, $(1,n-3)$, $I_1^{-1}(n-3,k)$, $(1,k)$, $(2,k)$, $I_2(k,n-2)$, $(2,n-2)$, $(1,n-2)$, $(1,n-1)$, $(0,n-1)$, $(0,n-2)$, $(3,k-2)$, $I_3^{-1}(k-2,0)$, $(3,0)$, $(3,n-1)$, $(2,n-1)$, $(2,0)$, $(1,0)$, $(1,1)$, $(0,1)$ (Figure 3(r))
$y = 3$ ($n \geq 8$)	$(0,1)$, $(0,2)$, $(3,k+2)$, $I_3(k+2,n-1)$, $(3,n-1)$, $(3,0)$, $(3,1)$, $(2,1)$, $(2,2)$, $(1,2)$, $(1,3)$, $(0,3)$, $I_0(3,k+1)$, $(0,k+1)$, $(1,k+1)$, $I_1^{-1}(k+1,4)$, $(1,4)$, $(2,4)$, $I_2(4,k+1)$, $(2,k+1)$, $(3,k+1)$, $I_3^{-1}(k+1,2)$, $(3,2)$, $(0,k+2)$, $I_0(k+2,n-1)$, $(0,n-1)$, $(1,n-1)$, $I_1^{-1}(n-1,k+2)$, $(1,k+2)$, $(2,k+2)$, $I_2(k+2,n-1)$, $(2,n-1)$, $(2,0)$, $(1,0)$, $(1,1)$, $(0,1)$ (Figure 3(s))
$5 \leq y \leq k-1$ ($n \geq 12$)	$(0,1)$, S_{11}, $(3,4)$, $L_3^{(y-5)/2}(4,y-1)$, $(3,y-1)$, $(3,y)$, $(3,y+1)$, $K_3^{(k-y-1)/2}(y+1,k)$, $(3,k)$, S_{12}, $(0,1)$ (Figure 3(t))

It is obvious that the Hamiltonian cycles of $G - F$ shown above are regular. ■

In general, after establishing the start regularity pattern with 1_p-*Hamiltonian* property, extending the 1_p-*Hamiltonian* property for a larger m can be proven by the available condition; i.e., the regularity (for recursively embedding a pair of rings between and before/after the pair of faulty nodes). Theorem 4.1 has proven the related performance for $m = 2$, and $n \geq 4$. However, some patterns listed in this theorem are not regular and cannot be used (for $n = 4$) or used directly for proving 1_p-*Hamiltonian* when $m > 2$; therefore, Lemma 4.3 has to be discussed.

Theorem 4.2 proves the related performance for $m = 3$, $n \geq 6$, and all patterns shown in this theorem are regular. Therefore, extending the performance for $m > 3$ being odd can be proven through recursive ring embedding that is started from $m = 3$. Moreover, an extension of such 1_p-*Hamiltonian* proof for $m > 2$ being even needs to consider the regularity pattern for a pair of faulty nodes separated by odd columns; hence, Theorem 4.4 has been proven.

THEOREM 4.5 GHT(m,n,k) *is* 1_p-*Hamiltonian if and only if either* $n \geq 6$ *or* $m = 2$, $n \geq 4$.

Proof By Theorem 4.1, GHT(2, 4, 2) is 1_p-*Hamiltonian*; we first prove that GHT(m,4,2) is not 1_p-*Hamiltonian* for any $m \neq 2$. If m is odd, by definition, GHT(m,4,2) cannot exist because m+2 cannot be even. Hence, we show only that GHT(m,4,2) is not 1_p-*Hamiltonian* for any $m \geq 4$, m being even. Suppose that $F = \{(0,0),(1,2)\}$. Obviously, $\deg_{\mathrm{GHT}(m,4,2)-F}(v) = 2$ where $v \in \{(0,1),(0,3),(1,1),(1,3)\}$. Therefore, any Hamiltonian cycle of GHT$(m,4,2) - F$ must include the following edge set: $\{((0,1),(0,2)), ((0,2),(0,3)), ((0,3),(1,3)), ((1,3),(1,0)), ((1,0),(1,1)), ((1,1),(0,1))\}$; however, this edge set induces a cycle of length 6. Thus, GHT$(m,4,2)$ is not 1_p-*Hamiltonian* if $m \neq 2$.

Then we prove that GHT(m,n,k), $n \geq 6$, is 1_p-*Hamiltonian* by induction with m. By Theorem 4.1, 4.2, and 4.4 and Lemma 4.3, we know that GHT(m,n,k) is 1_p-*Hamiltonian* for $n \geq 6$, $m = 2, 3, 4$. Using node-transitiveness, we let $F \in \mathbf{F}(\mathrm{GHT}(m,n,k))$ and (x,y) be the only element in $F - \{(0,0)\}$. Let m' be an integer with the same parity (even or odd) of m and $3 \leq m' < m$. Now we consider the case where $m \geq 5$. Assume that GHT(m',n,k) is 1_p-*Hamiltonian* and a regular Hamiltonian cycle exists in GHT$(m',n,k) - F$. Suppose that $x < m-2$. By induction, a regular Hamiltonian cycle H of GHT$(m-2,n,k) - F$ exists. By Lemma 3.4, $K_{m'-1}(H)$ is a regular Hamiltonian cycle of GHT$(m,n,k) - F$. Suppose that $x \geq m-2$. By induction, there exists a regular Hamiltonian cycle H of GHT$(m-2, n, k) - \{(0,0), (x-2, y)\}$. By Lemma 3.4, $K_0(H)$ is a regular Hamiltonian cycle of GHT$(m, n, k) - F$. Hence, the theorem holds for $n \geq 6$. The theorem is proven. ■

5. The 1-edge Hamiltonian property of GHT$(m, n, n/2)$

Throughout this section, let $k = n/2$. We first prove that GHT(m, n, k) is 1-*edge Hamiltonian* for $m \in \{2, 3\}$ and then prove the 1-*edge Hamiltonian* property of general m using mathematical induction.

THEOREM 5.1 *Let* $n \equiv 0 \pmod 4$. GHT$(2, n, k)$ *is* 1-*edge Hamiltonian for* $n \geq 4$. *Moreover, there is a regular Hamiltonian cycle in* GHT$(2, n, k) - e$ *for any* $e \in E($GHT$(2, n, k))$.

Proof Let $G =$ GHT$(2, n, k)$. Using the symmetry of G, we need to show only that $G - e$ has a regular Hamiltonian cycle for $e \in \{((0,0),(0,1)), ((0,0),(0,n-1)), ((0,1),(1,1)), ((0,0),(1,k))\}$. There are two cases.

Case 1 $n = 4$. The corresponding Hamiltonian cycles are as follows:

e	the Hamiltonian cycle C in $G - e$
$((0,0),(0,1))$	$(0,0), (0,3), (0,2), (0,1), (1,1), (1,0), (1,3), (1,2), (0,0)$ (Figure 3(12))
$((0,0),(0,n-1))$ or $((0,1),(1,1))$	$(0,0), I_0(0,3), (0,3), (1,3), (1,0), (1,1), (1,2), (0,0)$ (Figure 3(11))
$((0,0),(1,k))$	$(0,0), (0,1), (0,2), (1,0), I_1(0,3), (1,3), (0,3), (0,0)$ (Figure 3(10))

Case 2 $n \geq 8$. The corresponding Hamiltonian cycles are as follows:

e	the Hamiltonian cycle C in $G - e$
((0,0),(0,1))	$(0,0), (1,k), I_1^{-1}(k,1), (1,1), (0,1), I_0(1,k), (0,k), (1,0), (1,n-1), I_1^{-1}(n-1,k+1), (1,k+1), (0,k+1), I_0(k+1,n-1), (0,n-1), (0,0)$ (Figure 3(7))
((0,0),(0,$n-1$)) or ((0,1),(1,1))	$(0,0), (1,k), I_1(k,n-1), (1,n-1), (0,n-1), I_0^{-1}(n-1,k), (0,k), (1,0), I_1(0,k-1), (1,k-1), (0,k-1), I_0^{-1}(k-1,0), (0,0)$ (Figure 3(8))
$((0,0),(1,k))$	$(0,0), I_0(0,k-2), (0,k-2), (1,n-2), I_1^{-1}(n-2,k-1), (1,k-1), (0,k-1), I_0(k-1,n-2), (0,n-2), (1,k-2), I_1^{-1}(k-2,0), (1,0), (1,n-1), (0,n-1), (0,0)$ (Figure 3(9))

Obviously, the Hamiltonian cycles above are regular. The theorem is proven. ■

THEOREM 5.2 *Let* $n \equiv 2 \pmod 4$. GHT$(3, n, k)$ *is* 1-*edge Hamiltonian for* $n \geq 4$. *Moreover, there is a regular Hamiltonian cycle in* GHT$(3, n, k) - e$ *for any* $e \in E($GHT$(3, n, k))$.

Proof Let $G =$ GHT$(3, n, k)$. Using the symmetry of G, we need to show only that $G - e$ has a regular Hamiltonian cycle for $e \in \{((0,0),(0,1)), ((0,0),(0,n-1)), ((0,1),(1,1)), ((0,0),$

$(2, k))\}$. The corresponding Hamiltonian cycles are as follows:

e	the Hamiltonian cycle C in $G - e$
$((0,0),(0,1))$	$(0,0),(0,n-1),I_0^{-1}(n-1,k),(0,k),(1,k),I_1^{-1}(k,1),(1,1),(0,1),$ $I_0(1,k-1),(0,k-1),(2,n-1),I_2^{-1}(n-1,k+1),(2,k+1),$ $(1,k+1),I_1(k+1,n-1),(1,n-1),(1,0),(2,0),I_2(0,k),(2,k),$ $(0,0)$ (Figure 3(4))
$((0,0),(0,n-1))$ or $((0,1),(1,1))$	$(0,0),I_0(0,k-1),(0,k-1),(2,n-1),I_2^{-1}(n-1,k+1),(2,k+1),$ $(1,k+1),I_1(k+1,n-1),(1,n-1),(0,n-1),I_0^{-1}(n-1,k),(0,k),$ $(1,k),I_1^{-1}(k,0),(1,0),(2,0),I_2(0,k),(2,k),(0,0)$ (Figure 3(5))
$((0,0),(1,k))$	$(0,0),(0,n-1),I_0^{-1}(n-1,k+1),(0,k+1),(2,1),(2,0),(2,n-1),$ $I_2^{-1}(n-1,k+2),(2,k+2),(0,2),I_0(2,k),(0,k),(1,k),I_1^{-1}(k,2),$ $(1,2),(2,2),I_2(2,k+1),(2,k+1),(1,k+1),I_1(k+1,n-1),$ $(1,n-1),(1,0),(1,1),(0,1),(0,0)$ (Figure 3(6))

It is obvious that the Hamiltonian cycles of $G - F$ shown above are regular. The theorem is proven. ■

In general, after establishing the start regularity pattern with 1-*edge Hamiltonian* property, extension of the 1-*edge Hamiltonian* property for a larger m can be proven by the available condition, i.e., the regularity (for recursively embedding a pair of rings before/after the faulty edges). Theorem 5.1 has proven 1-*edge Hamiltonian* performance for $m = 2$, and $n \geq 4$, and all patterns shown in this theorem are regular. Theorem 5.2 has proven the related performance for $m = 3$, $n \geq 6$ (the smallest n can only be 6 with m being odd because $m + k$ should be even), and all patterns shown in this theorem are regular.

THEOREM 5.3 GHT(m, n, k) *is* 1-*edge Hamiltonian for all* $n \geq 4$.

Proof With Theorems 5.1 and 5.2, we know that GHT(m, n, k) is 1-*edge Hamiltonian* and that a regular Hamiltonian cycle exists in GHT$(m, n, k) - e$, where $e \in E$(GHT(m, n, k), for $m \in \{2, 3\}$. Recursively using Lemma 3.4, GHT(m, n, k) is 1-*edge Hamiltonian* for any $n \geq 4$. ■

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