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Procedure of the convolution method for estimating production yield with sample size information

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The yield index S_{pk} proposed by Boyles (1994. Process capability with asymmetric tolerances. *Communications in Statistics – Simulation and Computation*, 23 (1), 615–643) provides an exact measure on the production yield of normal processes. Lee *et al*. (Lee, J.C., Hung, H.N., Pearn, W.L. and Kueng, T.L., 2002. On the distribution of the estimated process yield index S_{pk}. *Quality and Reliability Engineering International*, 18 (2), 111–116) considered a normal approximation for estimating S_{pk}. In this paper, we consider a convolution approximation for estimating S_{pk}, and compare with the normal approximation. The comparison results show that the convolution method does provide a more accurate estimation to S_{pk} as well as the production yield than the normal approximation. An efficient step-by-step procedure based on the convolution method is developed to illustrate how to estimate the production yield. Also investigated is the accuracy of the convolution method which provides useful information about sample size required for designated power levels, and for convergence.

Keywords: production yield; process capability; quality assurance; critical value; power of test

1. Introduction

Production yield, for a long time, has been a standard criterion used in the manufacturing industry as a common measure on process performance, and defined as the percentage of processed product unit that falls within the manufacturing specification limits. For product units falling out of the manufacturing tolerance, additional cost would be incurred to the factory for scrapping or repairing the product. All passed product units, which incur no additional cost to the factory, are equally accepted by the producer. Numerous process capability indices (PCI) have been proposed to the manufacturing industry, to provide numerical measures on the production yield as well as process performance. Those indices, such as C_p, C_{pk}, C_{pm}, C_{pmk}, and S_{pk}, establish the relationship between the actual process performance and the manufacturing specifications, which

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have been the focus of the recent research in statistical and quality assurance literatures. The explicit forms of the indices are defined as follows:

$$C_p = \frac{USL - LSL}{6\sigma}, \quad C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \quad C_{pmk} = \min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\},$$

and

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - \mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu - LSL}{\sigma}\right)\right\},$$

where USL and LSL are the upper and lower specification limits, respectively, μ is the process mean, σ is the process standard deviation, T is the target value, $\Phi(\cdot)$ is the cumulatively distribution function (CDF) of the standard normal variable, and $\Phi^{-1}(\cdot)$ is the inverse function of $\Phi(\cdot)$.

The index C_p measures the overall process variation relative to the specification tolerance, therefore only reflects the process precision (the product consistency) (see Juran 1974, Kane 1986). Owing to the simplicity of the design, C_p cannot reflect the tendency of process centring. In order to reflect the deviations of process mean from the target value, several indices similar in nature to C_p, such as C_{pk}, C_{pm}, C_{pmk}, have been proposed. Those indices attempt to take into consideration the magnitude of process variance as well as process location. The C_{pk} index was developed because the C_p index can not adequately deal with cases that process mean is not centred. However, a large value of C_{pk} does not really say anything about the location of the mean in the tolerance interval. The C_{pk} index has been regarded as a yield-based index since it provides bounds on production yield for a normally distributed process, $2\Phi(3C_{pk}) - 1 \leq Yield \leq \Phi(3C_{pk})$ (Boyles 1991). The C_p and C_{pk} indices are appropriate measures of progress for quality improvement paradigms in which reduction of variability is the guiding principle and production yield is the primary measure of success.

Taguchi, on the other hand, emphasises the loss in a product's worth, rather than the production yield, when one of its characteristics departs from the target value. Hsiang and Taguchi (1985) introduced the index C_{pm}, which was also proposed independently by Chan *et al.* (1988). The C_{pm} index is related to the idea of squared error loss, $loss(X) = (X - T)^2$, and has been called the Taguchi index. The C_{pm} index incorporates the process variation with respect to the target value with the manufacturing specifications preset in the factory, which reflects the degree of process targeting. Chan *et al.* (1988) also discussed the sampling properties of the natural estimator of C_{pm}. Boyles (1991) provided a definitive analysis of C_{pm} and its usefulness in measuring process targeting. Pearn and Shu (2003) provided explicit formulas with efficient algorithms to obtain the lower confidence bound of C_{pm} using the maximum likelihood estimator (MLE) of C_{pm}. Pearn *et al.* (2004b) developed a two-phase supplier selection procedure based on the C_{pm} index providing useful information about sample size required for a designated selection power.

Pearn *et al.* (1992) proposed a third-generation capability index called C_{pmk}, which is constructed by combining the merits of the three indices C_p, C_{pk}, and C_{pm}. The index C_{pmk} alters the user either the process variance increases or the process mean deviates from its target value. The C_{pmk} index responds to the departure of the process mean from the target value T faster than the other three indices C_p, C_{pk}, and C_{pm}, while it remains sensitive to the changes of process variation. Vännman and Kotz (1995) obtained the distribution of the estimated $C_p(u, v)$ for cases with on-centre target. By taking $u = 1$ and $v = 1$, the distribution of $C_p(1, 1) = C_{pmk}$ is obtained. Chen and Hsu (1995) proposed the asymptotic sampling distribution of C_{pmk}, and showed that the estimated C_{pmk} is consistent, asymptotically unbiased estimator of C_{pmk} and is asymptotically normal while the fourth moment of the characteristic X is finite. Wright (1998) derived an explicit but rather complicated expression of the probability density function (PDF) of the estimated C_{pmk}. Pearn and Lin (2002) alternatively expressed the CDF and PDF of the estimated C_{pmk} in terms of a mixture of the chi-square distribution and normal distribution. The CDF form of the estimated C_{pmk} obtained by Pearn and Lin considerably simplify the complexity for analysing the statistical properties of the estimated C_{pmk}.

Note that the indices C_{pm} and C_{pmk} are defined to emphasise the loss in a product's worth when one of its characteristics departs from the target value T, and the indices C_p and C_{pk} can only provide a lower bound or interval estimation on the production yield. Only the yield index S_{pk} provides an exact measure on the production yield. We remark that the indices presented above are designed to monitor the performance for stable normal or near-normal processes with symmetric tolerances. In practice, the process mean μ and the process variance σ^2 are unknown. To calculate the index value, sample data must be collected, and a great degree of uncertainty may be introduced into the assessments due to the sampling errors. As the use of the capability indices grows more widespread, users are becoming educated and sensitive to the impact of the estimators and their distributions, learning that capability measures must be reported in confidence intervals or via capability testing. Statistical properties of the estimators of those indices under various process conditions have been investigated extensively, including Chan *et al.* (1988), Pearn *et al.* (1992, 2003, 2004a, b), Kotz and Johnson (1993, 2002), Vännman and Kotz (1995), Vännman (1997), Kotz and Lovelace (1998), Chen (2000), Zhang (2001), Lee *et al.* (2002), Xie *et al.* (2002), Spiring *et al.* (2003), Montgomery (2005), Wu (2007).

2. The yield index S_{pk}

Boyles (1994) proposed a yield measurement index, referred to as S_{pk}, based on the production yield of normal processes. The yield index S_{pk}, as defined previously, also can be alternatively expressed as

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1 - C_{dr}}{C_{dp}}\right) + \frac{1}{2}\Phi\left(\frac{1 + C_{dr}}{C_{dp}}\right)\right\},$$

where $C_{dr} = (\mu - m)/d$, $C_{dp} = \sigma/d$, $m = (USL + LSL)/2$ is the midpoint of the specification limits, and $d = (USL - LSL)/2$ is the half length of the specification interval.

As mentioned previously, the index C_{pk} can only provide interval estimation on the production yield. The indices C_{pm} and C_{pmk} are defined by being related to the customer's loss. Only the yield index S_{pk} can provide a one-to-one correspondence to the production yield, which can be expressed as

$$Yield = 2\Phi\,(3S_{pk}) - 1.$$

Table 1. Various S_{pk} values and the corresponding production yields as well as non-conformities in PPM.

S_{pk}	Yield	PPM
1.00	0.997300204	2699.796
1.10	0.999033152	966.848
1.20	0.999681783	318.217
1.30	0.999903807	96.193
1.33	0.999933927	66.073
1.40	0.999973309	26.691
1.50	0.999993205	6.795
1.60	0.999998413	1.587
1.67	0.999999456	0.544
1.70	0.999999660	0.340
1.80	0.999999933	0.067
1.90	0.999999988	0.012
2.00	0.999999998	0.002

Table 1 summarises the corresponding production yields as well as non-conformities in parts per million (PPM) for $S_{pk} = 1.0(0.1)2.0$, including the most commonly used performance requirements: 1.00, 1.33, 1.50, 1.67, and 2.00. For example, if a process has capability index value $S_{pk} = 1.50$, then the yield of the process is 0.999993205 and the corresponding non-conformities is roughly seven parts per million.

Assume that $X_1, \ldots, X_{\text{n}}$ be a random sample of the characteristic from a normal process. The natural estimator of S_{pk} is defined as

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - \bar{X}}{S}\right) + \frac{1}{2}\Phi\left(\frac{\bar{X} - LSL}{S}\right)\right\},$$

and can also be expressed as

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1 - \hat{C}_{dr}}{\hat{C}_{dp}}\right) + \frac{1}{2}\Phi\left(\frac{1 + \hat{C}_{dr}}{\hat{C}_{dp}}\right)\right\},$$

where $\hat{C}_{dr} = (\bar{X} - m)/d$ and $\hat{C}_{dp} = S/d$ are natural estimators of C_{dr} and C_{dp}, respectively, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean, and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is the sample variance. The distribution of the natural estimator of S_{pk} is mathematically intractable as it is a complex function of the statistics $\bar{X}$ and S^2 (or $\hat{C}_{dr}$ and $\hat{C}_{dp}$). However, we can profile the sampling distribution of S_{pk} by using a simulation technique. Figure 1 shows the histograms of $\hat{S}_{pk}$ with simulation parameters $S_{pk} = 1.0$, $\xi = (\mu - m)/\sigma = 0$, and sample size $n = 20$, 30, 50, 80 each with 10,000 simulated $\hat{S}_{pk}$. The histograms reveal that the probability density function (PDF) of $\hat{S}_{pk}$ is nearly bell-shaped, symmetric to the real S_{pk} for large sample sizes, and slightly skewed to the right for small sample sizes.

Many researchers have focused on the sampling distribution of S_{pk}. Lee *et al.* (2002) derived a normal approximated distribution of the estimated S_{pk}. Pearn *et al.* (2004a) investigated the accuracy of the normal approximation computationally, and suggested that a sample size greater than 150 is required for the normal approximation sufficiently accurate. Pearn and Cheng (2007) further derived a normal approximated distribution of

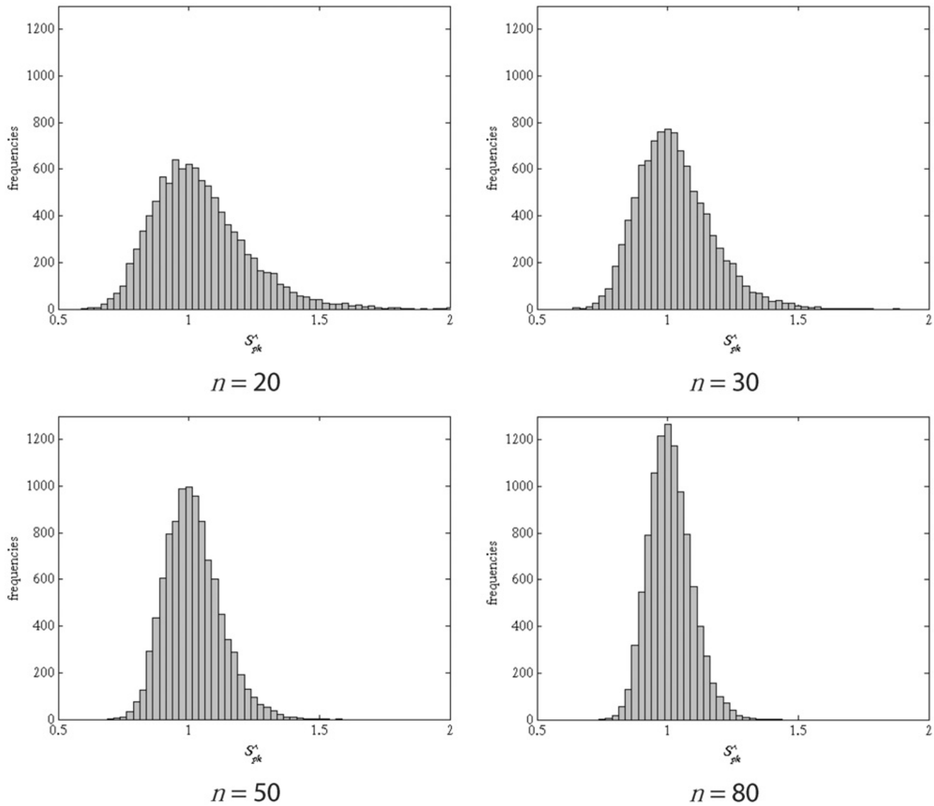

Figure 1. Histograms of $\hat{S}_{pk}$ with simulation parameters $S_{pk} = 1.0$ and $\xi = 0$.

the estimated S_{pk} under multiple samples, and investigated the sample sizes required to converge to S_{pk} within a designated accuracy. Chen (2005) considered that the formula of the normal approximation is messy and cumbersome to deal with. Chen (2005) applied four bootstrap methods to find the lower confidence bounds on S_{pk}, and showed that the standard bootstrap (SB) method significantly outperforms the other three bootstrap methods in coverage fraction. We note, however, the bootstrap re-sampling method results in different solutions each time, while the theoretical sampling distribution approach provides a unique lower bound for the same sample estimates.

The distribution of $\hat{S}_{pk}$ is analytically intractable, but approximate distributions of $\hat{S}_{pk}$ can be obtained. In the following sections, two approximate distributions are considered and compared to the distribution of the estimated S_{pk} obtained via simulations.

3. Normal approximation of $\hat{S}_{pk}$: $\hat{S}'_{pk}$

Lee *et al*. (2002) considered a normal approximation of $\hat{S}_{pk}$, which is denoted $\hat{S}'_{pk}$ in this paper. The normal distribution of $\hat{S}'_{pk}$ is distributed with a mean S_{pk} and a variance $(a^2 + b^2)/[36n\varphi^2(3S_{pk})]$, i.e.

$$\hat{S}'_{pk} \sim N\left(S_{pk}, \frac{a^2 + b^2}{36n\phi^2(3S_{pk})}\right),$$

where

$$a = \frac{1}{\sqrt{2}}\left\{\frac{1-C_{dr}}{C_{dp}}\phi\left(\frac{1-C_{dr}}{C_{dp}}\right) + \frac{1+C_{dr}}{C_{dp}}\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right\},$$

and

$$b = \phi\left(\frac{1-C_{dr}}{C_{dp}}\right) - \phi\left(\frac{1+C_{dr}}{C_{dp}}\right).$$

The normal approximation is useful in statistical inferences for S_{pk}. Consider the following null versus alternative hypotheses:

H_0: $S_{pk} \leq C$, a specified value;
H_1: $S_{pk} > C$.

The decision rule with $1-\alpha$ confidence level should be that to reject the null hypothesis H_0 if the sample statistic $\hat{S}_{pk}$ is equal to or larger than the critical value c_0, where c_0 satisfies the following equation

$$\Pr\left\{\hat{S}'_{pk} \geq c_0 \mid H_0\colon S_{pk} \leq C\right\} \leq \alpha.$$

Lee *et al.* (2002) suggested performing the hypothesis testing with the test statistic

$$T = \left(\hat{S}_{pk} - C\right)\frac{6\sqrt{n}\phi\left(3\hat{S}_{pk}\right)}{\sqrt{\hat{a}^2 + \hat{b}^2}},$$

where $\hat{a}$ and $\hat{b}$ are the natural estimators of a and b, with C_{dr} and C_{dp} replaced by $\hat{C}_{dr}$ and $\hat{C}_{dp}$, respectively. Then, the decision rule becomes that the null hypothesis H_0 would be rejected if $T \geq z_\alpha$, where z_α is the upper $100\alpha\%$ point of the standard normal distribution.

This approach is intuitive and reasonable, but introduces additional sampling errors from estimating a and b (or C_{dr} and C_{dp}) with $\hat{a}$ and $\hat{b}$ (or $\hat{C}_{dr}$ and $\hat{C}_{dp}$). Thus, it would certainly become less reliable. For example, in Table 2 the sample estimate of S_{pk} in Process B is larger than the one in Process A, but contradictorily it turns out a smaller test statistic T in Process B. Table 2 shows a couple of examples for testing H_0: $S_{pk} \leq 1.0$ versus H_1: $S_{pk} > 1.0$ in which the sample estimate of S_{pk} is larger (e.g. Processes B, D, F, and H), but on the contrary, the corresponding test statistic T is smaller.

Table 2. Contradiction between $\hat{S}_{pk}$ and test statistic T in Lee's method.

Process	$\bar{X}$	S	$\hat{C}_{dr}$	$\hat{C}_{dp}$	$\hat{S}_{pk}$	T
A	7.695115	1.365970	0.139023	0.273194	1.114490	0.807547
B	7.674245	1.372115	0.134849	0.274423	1.114555	0.807412
C	7.707630	1.335160	0.141526	0.267032	1.134942	1.207505
D	7.681125	1.342895	0.136225	0.268579	1.135032	1.207252
E	7.683340	1.314965	0.136668	0.262993	1.156439	1.063747
F	7.650165	1.324405	0.130033	0.264881	1.156573	1.063459
G	7.700125	1.219685	0.140025	0.243937	1.234395	1.929673
H	7.680760	1.224995	0.136152	0.244999	1.234452	1.929267

Pearn *et al.* (2004a) showed that for a specific S_{pk} (e.g. $S_{pk} = \mathrm{C}$), the variance of $\hat{S}'_{pk}$ would be the largest with on-centre processes, i.e. with $\xi = (\mu - m)/\sigma = 0$. Consequently, the critical value of testing H_0: $S_{pk} \le \mathrm{C}$ versus H_1: $S_{pk} > \mathrm{C}$ would be the largest, and the test statistic T would be the smallest with $\xi = 0$. Hence, for practical purpose we would obtain the test statistic (or critical value) with $\xi = 0$ without having to further estimate the parameter ξ (or parameters a and b). The test statistic T obtained in this way is increasing in $\hat{S}_{pk}$, and there would be no contradiction. Pearn *et al.* (2004a) listed in the Table III of the published paper the critical values c_0 of the $\hat{S}'_{pk}$ approach which were obtained by the following probability

$$\Pr\left\{\hat{S}'_{pk} \ge c_0 | S_{pk} \le \mathrm{C} \text{ and } \xi = 0\right\} \le \alpha.$$

Lee *et al.* (2002) showed that the normal distribution of $\hat{S}'_{pk}$ can produce an adequate approximation to the actual distribution of $\hat{S}_{pk}$ for a large enough sample size. However, Pearn *et al.* (2004a) noted that the normal approximation would significantly under-calculate the critical values for small sample sizes, and suggested that a sample of size greater than 150 is recommended in real applications, for which the magnitude of under-calculation would be as large as 0.02 at most. Since the critical value of the $\hat{S}'_{pk}$ approach is significantly under-calculated for small sample sizes, it is necessary to do some improvement.

4. Convolution approximation of $\hat{S}_{pk}$: $\hat{S}''_{pk}$

The critical value obtained from the normal approximation is significantly under-calculated for small sample sizes. Thus, we go further to do some improvement by considering a convolution approximation of the estimated S_{pk}. First, we define the two random variables $Z = \sqrt{n}(\bar{X} - \mu)/\sigma$ and $Y = \sqrt{n}(S^2 - \sigma^2)/2\sigma^2$. The two random variables Z and Y are independent since $\bar{X}$ and S^2 are independent variables. It is well-known that the variable Z follows the standard normal distribution $N(0, 1)$ according to the famous Central Limit Theory, and Y can be expressed as a function of a chi-square random variable with $n - 1$ degrees of freedom, i.e.

$$Z \sim N(0, 1), \qquad Y \sim \frac{\sqrt{n}}{2}\left(\frac{\chi^2_{n-1}}{(n-1)} - 1\right).$$

Then, we can rewrite the form of $\hat{S}_{pk}$ as the following analytical expansion:

$$\hat{S}_{pk} = S_{pk} + D_1 Z + D_2 Y + D_3 Z^2 + D_4 ZY + D_5 Y^2 + O_p\left(\frac{1}{n\sqrt{n}}\right),$$

where

$$D_1 = \frac{1}{\sqrt{n}}\left(\frac{-\lambda_0}{6\phi(3S_{pk})}\right), \quad D_2 = \frac{1}{\sqrt{n}}\left(\frac{-\lambda_1}{6\phi(3S_{pk})}\right), \quad D_3 = \frac{1}{n}\left(\frac{S_{pk}\lambda_0^2}{8\left[\phi(3S_{pk})\right]^2} - \frac{\lambda_1}{12\phi(3S_{pk})}\right),$$

$$D_4 = \frac{1}{n}\left(\frac{S_{pk}\lambda_0\lambda_1}{4\left[\phi(3S_{pk})\right]^2} + \frac{\lambda_0 - \lambda_2}{6\phi(3S_{pk})}\right), \quad D_5 = \frac{1}{n}\left(\frac{S_{pk}\lambda_1^2}{8\left[\phi(3S_{pk})\right]^2} + \frac{3\lambda_1 - \lambda_3}{12\phi(3S_{pk})}\right),$$

and

$$\lambda_k = \left(\frac{1-C_{dr}}{C_{dp}}\right)^k \phi\left(\frac{1-C_{dr}}{C_{dp}}\right) + (-1)^{k+1} \times \left(\frac{1+C_{dr}}{C_{dp}}\right)^k \phi\left(\frac{1+C_{dr}}{C_{dp}}\right), \quad k = 0, 1, 2, 3.$$

Let

$$\hat{S}''_{pk} = S_{pk} + D_1 Z + D_2 Y + D_3 Z^2 + D_4 ZY + D_5 Y^2.$$

The cumulative distribution function (CDF) of $\hat{S}''_{pk}$, $F_{\hat{S}''_{pk}}(x)$, then can be derived by the probability

$$F_{\hat{S}''_{pk}}(x) = \Pr\left\{\hat{S}''_{pk} - x \le 0\right\} = \Pr\left\{D_3\left(Z + \frac{D_1 + D_4 Y}{2D_3}\right)^2 - \frac{E_1(Y+E_3)^2}{4D_3} + \frac{\Delta_1(x)}{4D_3E_1} \le 0\right\},$$

where

$$E_1 = D_4^2 - 4D_3D_5, \quad E_2 = D_1D_4 - 2D_2D_3, \quad E_3 = \frac{E_2}{E_1}, \quad E_4 = 4D_3S_{pk} - D_1^2,$$

and

$$\Delta_1(x) = E_2^2 - E_1(4D_3x - E_4).$$

The explicit form of the CDF of $\hat{S}''_{pk}$ is presented in the Appendix. The CDF of $\hat{S}''_{pk}$ consists of eight parts according to the signs of D_3, E_1 and $y_0 + E_3$, where $y_0 = -\frac{\sqrt{n}}{2}$ is the minimal value of the variable Y. Applying the Leibniz's rule for derivatives, we can also obtain the probability density function (PDF) of $\hat{S}''_{pk}$.

Again, we consider the following hypothesis testing

H_0: $S_{pk} \le C$, a specified value;
H_1: $S_{pk} > C$.

It is inevitable to face the same problem or contradiction as in the normal approximation. Thus, we examine the behaviour of the critical values c_0 against the parameter ξ before we do the hypothesis testing for S_{pk}. We perform extensive calculations to obtain the critical values c_0 for $\xi = 0(0.05)3.0$, $n = 20(10)200$, $S_{pk} = 1.0(0.1)2.0$, 1.33, 1.67, and confidence level $1 - \alpha = 0.95$. Figure 2 shows parts of the results for ξ versus the critical values. The parameter values we investigated, $\xi = 0(0.05)3.0$, cover a wide range of applications with process capability $S_{pk} \ge 1.0$. Note that for an on-centre process the yield index $S_{pk} < 1.0$ indicates that six-sigma of the process is larger than the manufacturing specification tolerance, i.e. $6\sigma > USL - LSL$, and such a process is said to be inadequate. The results of our extensive calculations show the following features of the critical values obtained from the convolution approximation.

(i) The critical value obtains its maximum with ξ around 0.5, minimum with $\xi = 0$, and stays at the same value for $\xi \ge 1.0$ in all cases.
(ii) The critical value reaches its maximum with ξ slightly larger than 0.5 for $n \le 50$, and with ξ slightly smaller than 0.5 for $n > 50$.
(iii) The larger the sample size n, the *smaller* the difference between the maximal and minimal critical values.
(iv) The larger the sample size n, the *larger* the difference between the maximal critical value (with ξ around 0.5) and the converged critical value (with $\xi \ge 1.0$).

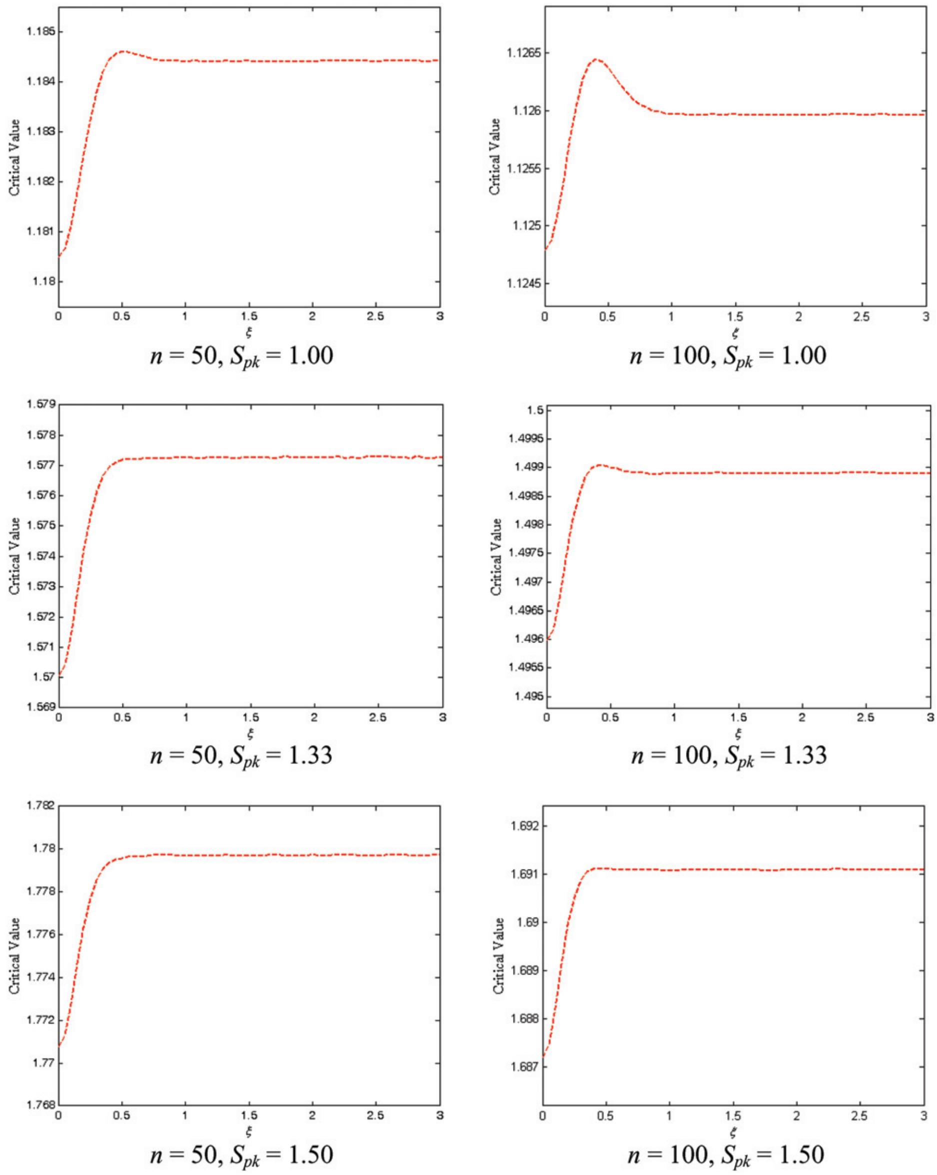

Figure 2. ξ versus critical values for various n and S_{pk}.

(v) The larger the value of S_{pk}, the *larger* the difference between the maximal and minimal critical values.

(vi) The larger the value of S_{pk}, the *smaller* the difference between the maximal critical value (with ξ around 0.5) and the converged one (with $\xi \geq 1.0$).

(vii) The difference between the maximal critical value (with ξ around 0.5) and the converged critical value (with $\xi \geq 1.0$) is always less than 0.0006.

(viii) The critical value is increasing in S_{pk} (the testing parameter), and decreasing in sample size n, which is definite in the statistical inference.

For assurance purposes, we calculate the critical value based on the convolution approximation with $\xi = 0.5$ to obtain the maximal critical value c_0 for testing the hypotheses H_0: $S_{pk} \leq \mathrm{C}$ versus H_1: $S_{pk} > \mathrm{C}$, since the critical value c_0 reaches its maximum with ξ around 0.5 in all cases.

$$\Pr\left\{\hat{S}''_{pk} \geq c_0 | S_{pk} \leq C \text{ and } \xi = 0.5\right\} \leq \alpha.$$

Thus, the level of confidence can be ensured, and the decisions made based on such an approach are indeed more reliable. We note that the above result is impossible to prove mathematically.

5. Comparisons of both approximations

5.1 *Comparison of probability curves*

We perform extensive calculations to draw the PDFs and CDFs of $\hat{S}'_{pk}$ and $\hat{S}''_{pk}$ as well as the density and distribution curves of the estimated S_{pk} via simulation for process parameters $\xi = 0(0.25)1.0$, $S_{pk} = 1.0(0.25)2.0$, and sample size $n = 30$, 50, 80, 100. Each of the density and distribution curves is obtained by 1,000,000 simulated $\hat{S}_{pk}$. Parts of the calculation results are presented in Figure 3. The calculation results also reveal the following general features.

(i) The density curve of $\hat{S}_{pk}$ is nearly bell-shaped, symmetric to the real S_{pk}, and so are the PDFs of $\hat{S}'_{pk}$ and $\hat{S}''_{pk}$.
(ii) The tail probability of $\hat{S}''_{pk}$ is closer to the one of the simulated $\hat{S}_{pk}$ than that of $\hat{S}'_{pk}$.
(iii) The CDFs of $\hat{S}'_{pk}$ and $\hat{S}''_{pk}$ are closer to the distribution curves of $\hat{S}_{pk}$ with a large ξ than those with a small ξ.
(iv) The larger the sample sizes n, the smaller the variance of $\hat{S}_{pk}$, $\hat{S}'_{pk}$, and $\hat{S}''_{pk}$, which is definite for all sample estimators.

The calculation results show that the cumulative distribution functions of $\hat{S}''_{pk}$ are closer to the distribution curves of the simulated $\hat{S}_{pk}$ than CDFs of $\hat{S}'_{pk}$. Though the distribution function of $\hat{S}''_{pk}$ (the convolution method) is more complicated than that of $\hat{S}'_{pk}$ (the normal approximation), it does produce a more accurate approximation to the sampling distribution of S_{pk} than the normal approximation. Besides, by the high development of computer technology, complex functions are no longer a problem for calculation. Making a decision accurately is relatively more important than easy calculation.

5.2 *Comparison of critical values*

Decision rule for testing hypotheses H_0: $S_{pk} \leq \mathrm{C}$ versus H_1: $S_{pk} > \mathrm{C}$ based on the normal and convolution approximations are conducted, respectively. The critical value which is closer to the critical value of the simulated $\hat{S}_{pk}$ is regarded as the more accurate and reliable one. We know that for the same S_{pk}, the maximal variance of $\hat{S}'_{pk}$ occurs at $\xi = 0$, i.e. process mean is on the centre of the specification limits (Pearn *et al.* 2004a). Thus, when testing the hypotheses based on the distribution of $\hat{S}'_{pk}$, we would set $\xi = 0$ to obtain the maximal critical value of the normal approximation. On the other hand, we would set $\xi = 0.5$ while testing the hypotheses based on the convolution method for the same reason.

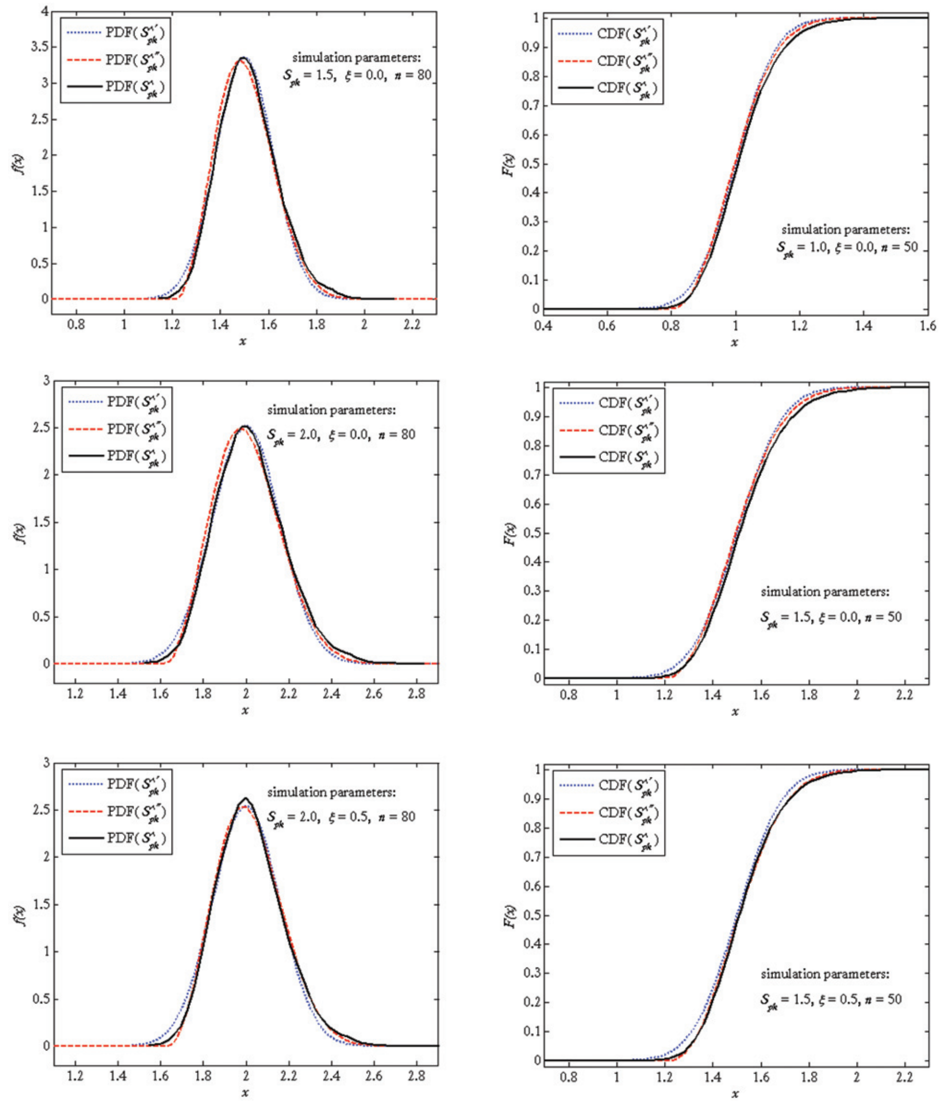

Figure 3. The PDF (l.h.s.) and CDF (r.h.s.) of $\hat{S}'_{pk}$ and $\hat{S}''_{pk}$ as well as the density and distribution curves of $\hat{S}_{pk}$ via simulation.

To compute the critical value of the convolution method, we developed a Matlab program (available on request). The program reads the minimal capability requirement C, the significant level α, and the sample size n, and outputs with the critical value. Table 3 shows the critical values for the normal and convolution approximations as well as the one of $\hat{S}_{pk}$ via simulation, for testing hypotheses H_0: $S_{pk} \leq C$ versus H_1: $S_{pk} > C$ with significant level $\alpha = 0.05$. The critical values of $\hat{S}_{pk}$ (via simulation) and $\hat{S}'_{pk}$ (the normal approximation) are extracts from those presented in the paper of Pearn *et al.* (2004a). We note that the critical values of the convolution approximation are always larger than those of the normal approximation, and are closer to the critical values of the

Table 3. Critical values of the two approximations versus the simulated ones.

S_{pk}	1.00			1.33			1.50			1.67			2.00		
n	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$
20	1.26	1.31	1.37	1.68	1.74	1.82	1.89	1.97	2.05	2.11	2.19	2.30	2.52	2.63	2.74
25	1.23	1.27	1.31	1.64	1.69	1.75	1.85	1.91	1.98	2.06	2.13	2.20	2.47	2.56	2.63
30	1.21	1.25	1.28	1.61	1.66	1.70	1.82	1.87	1.93	2.03	2.09	2.14	2.43	2.50	2.57
35	1.20	1.23	1.25	1.59	1.63	1.67	1.80	1.84	1.89	2.00	2.05	2.10	2.39	2.46	2.51
40	1.18	1.21	1.23	1.58	1.61	1.64	1.78	1.82	1.85	1.98	2.02	2.06	2.37	2.42	2.47
45	1.17	1.20	1.22	1.56	1.59	1.61	1.76	1.80	1.82	1.96	2.00	2.02	2.35	2.40	2.43
50	1.16	1.18	1.20	1.55	1.58	1.60	1.75	1.78	1.80	1.95	1.98	2.01	2.33	2.38	2.40
55	1.16	1.18	1.19	1.54	1.56	1.58	1.74	1.77	1.79	1.93	1.97	1.99	2.31	2.36	2.38
60	1.15	1.17	1.18	1.53	1.55	1.57	1.73	1.75	1.77	1.92	1.95	1.98	2.30	2.34	2.36
65	1.14	1.16	1.17	1.52	1.54	1.56	1.72	1.74	1.76	1.91	1.94	1.96	2.29	2.32	2.34
70	1.14	1.15	1.16	1.52	1.54	1.55	1.71	1.73	1.77	1.90	1.93	1.95	2.28	2.31	2.33
75	1.13	1.15	1.15	1.51	1.53	1.54	1.70	1.72	1.74	1.89	1.92	1.94	2.27	2.30	2.31
80	1.13	1.14	1.15	1.50	1.52	1.53	1.70	1.72	1.73	1.89	1.91	1.93	2.26	2.29	2.31
85	1.13	1.14	1.14	1.50	1.51	1.53	1.69	1.71	1.72	1.88	1.90	1.92	2.25	2.28	2.30
90	1.12	1.13	1.14	1.49	1.51	1.52	1.68	1.70	1.71	1.88	1.90	1.91	2.25	2.27	2.28
95	1.12	1.13	1.14	1.49	1.50	1.51	1.68	1.70	1.71	1.87	1.89	1.90	2.24	2.26	2.27
100	1.12	1.13	1.13	1.49	1.50	1.50	1.67	1.69	1.70	1.86	1.88	1.89	2.23	2.26	2.27
105	1.11	1.12	1.13	1.48	1.49	1.50	1.67	1.69	1.70	1.86	1.88	1.89	2.23	2.25	2.26
110	1.11	1.12	1.13	1.48	1.49	1.50	1.67	1.68	1.69	1.86	1.87	1.89	2.22	2.24	2.25
115	1.11	1.12	1.12	1.47	1.49	1.49	1.66	1.68	1.69	1.85	1.87	1.88	2.22	2.24	2.25
120	1.11	1.11	1.12	1.47	1.48	1.49	1.66	1.67	1.68	1.85	1.86	1.87	2.21	2.23	2.24
125	1.10	1.11	1.12	1.47	1.48	1.49	1.66	1.67	1.68	1.84	1.86	1.86	2.21	2.23	2.24
130	1.10	1.11	1.12	1.47	1.48	1.48	1.65	1.67	1.68	1.84	1.86	1.86	2.20	2.22	2.23
135	1.10	1.11	1.11	1.46	1.47	1.48	1.65	1.66	1.67	1.84	1.85	1.86	2.20	2.22	2.23
140	1.10	1.11	1.11	1.46	1.47	1.48	1.65	1.66	1.67	1.83	1.85	1.86	2.20	2.21	2.22
145	1.10	1.10	1.11	1.46	1.47	1.48	1.65	1.67	1.66	1.83	1.84	1.85	2.19	2.21	2.22
150	1.10	1.10	1.11	1.46	1.47	1.47	1.64	1.65	1.66	1.83	1.84	1.85	2.19	2.21	2.21
155	1.09	1.10	1.10	1.45	1.46	1.47	1.64	1.65	1.66	1.83	1.84	1.84	2.19	2.20	2.21
160	1.09	1.10	1.10	1.45	1.46	1.47	1.64	1.65	1.65	1.82	1.84	1.84	2.19	2.20	2.21
165	1.09	1.10	1.10	1.45	1.46	1.46	1.64	1.65	1.65	1.82	1.83	1.84	2.18	2.20	2.20
170	1.09	1.10	1.10	1.45	1.46	1.46	1.63	1.64	1.65	1.82	1.83	1.84	2.18	2.19	2.20
175	1.09	1.09	1.10	1.45	1.46	1.46	1.63	1.64	1.65	1.82	1.83	1.83	2.18	2.19	2.20
180	1.09	1.09	1.10	1.45	1.45	1.46	1.63	1.64	1.65	1.82	1.83	1.83	2.17	2.19	2.19
185	1.09	1.09	1.09	1.44	1.45	1.46	1.63	1.64	1.64	1.81	1.82	1.83	2.17	2.18	2.19
190	1.08	1.09	1.09	1.44	1.45	1.45	1.63	1.64	1.64	1.81	1.82	1.83	2.17	2.18	2.19
195	1.08	1.09	1.09	1.44	1.45	1.45	1.63	1.63	1.64	1.81	1.82	1.82	2.17	2.18	2.18
200	1.08	1.09	1.09	1.44	1.45	1.45	1.62	1.63	1.64	1.81	1.82	1.82	2.16	2.18	2.18

simulated $\hat{S}_{pk}$, the $100(1-\alpha)$ percentile point of $\hat{S}_{pk}$ under H_0. That is the accuracy of the convolution method is greater than the normal approximation.

Note that the normal approximation significantly under-calculates the critical values for small sample sizes, particularly for $n \le 40$, as the magnitude of the under-calculation exceeds 0.1. The large magnitude of under-calculation results in huge probability of wrongly rejecting H_0: $S_{pk} \le C$ while actually the yield index S_{pk} is smaller than or equal to a specific value C, which incurring the risk of the customers by accepting products with less quality assurance. Therefore, for short run applications, one should avoid using the normal approximation. It is also noted that the magnitude of under-calculation of the normal approximation can be as large as 0.03 for $n = 110$, and 0.02 for $n = 150$. Thus, in real applications a sample of size greater than 150 is recommended for using the normal approximation (Pearn *et al.* 2004a).

The convolution method also under-calculates the critical values, but it always provides a closer estimate to the critical value of $\hat{S}_{pk}$ than the normal approximation. The magnitude of the under-calculation of the convolution approximation is as large as 0.03 for $n = 60$, 0.02 for $n = 70$, and 0.01 for $n = 90$. As we know previously, the magnitude of under-calculation of the normal approximation is as large as 0.03 for $n = 110$, and 0.02 for $n = 150$. That is, if the allowable magnitude of under-calculation is 0.02, a sample size of 70 is enough for using the convolution method, while a sample size of 150 is enough for the normal approximation, which is more than twice sample sizes of the convolution method. Thus, the proposed convolution method does provide a better reliability assurance than the existing normal approximation.

5.3 *Comparison of powers*

The power of test calculates the probability of correctly rejecting the null hypothesis H_0: $S_{pk} \leq C$ while actually $S_{pk} > C$. It is well known that the power of test is the larger the better. As we know previously, the critical value of the normal approximation is highly under-calculated for small sample sizes. Consequently, the power (probability of rejecting H_0) would be highly over-calculated if the under-calculated critical value is used as the testing rule. To compare both approximations on the same basis, we define and calculate the power of both approximations as the probability that the sample estimator is larger than the critical value c_0 of simulated $\hat{S}_{pk}$ as follows:

$$\text{power}(\hat{S}'_{pk}) = \Pr(\hat{S}'_{pk} > c_0 \mid S_{pk} \text{ under } H_1,\ \xi = 0),$$

$$\text{power}(\hat{S}''_{pk}) = \Pr(\hat{S}''_{pk} > c_0 \mid S_{pk} \text{ under } H_1,\ \xi = 0.5).$$

Figure 4 shows the powers of the normal and convolution approximations for testing

(a) H_0: $S_{pk} \leq 1.0$ versus H_1: $S_{pk} > 1.0$ with sample size $n = 30$,
(b) H_0: $S_{pk} \leq 1.0$ versus H_1: $S_{pk} > 1.0$ with sample size $n = 50$,
(c) H_0: $S_{pk} \leq 1.5$ versus H_1: $S_{pk} > 1.5$ with sample size $n = 30$, and
(d) H_0: $S_{pk} \leq 1.5$ versus H_1: $S_{pk} > 1.5$ with sample size $n = 50$ and $\alpha = 0.05$.

Obviously, the power of the convolution method is always greater than the normal approximation, which means the capability of correctly rejecting the bad product lots of the convolution method is stronger than that of the normal approximation.

So far we know that the tail probability and critical value of the convolution method are closer to those of the simulated $\hat{S}_{pk}$ than those of the normal approximation, and the power of the convolution method is always greater than that of the normal approximation. All of above indicate that the convolution method does make a more accurate and reliable approximation to the sampling behaviour of S_{pk} than the normal approximation.

Following, we develop an efficient step-by-step procedure based on the convolution method for testing hypotheses H_0: $S_{pk} \leq C$ versus H_1: $S_{pk} > C$, where C is the minimal capability requirement defined by the customer or product designer. Engineers or practitioners can easily apply the procedure to their in-plant applications to obtain reliable decisions.

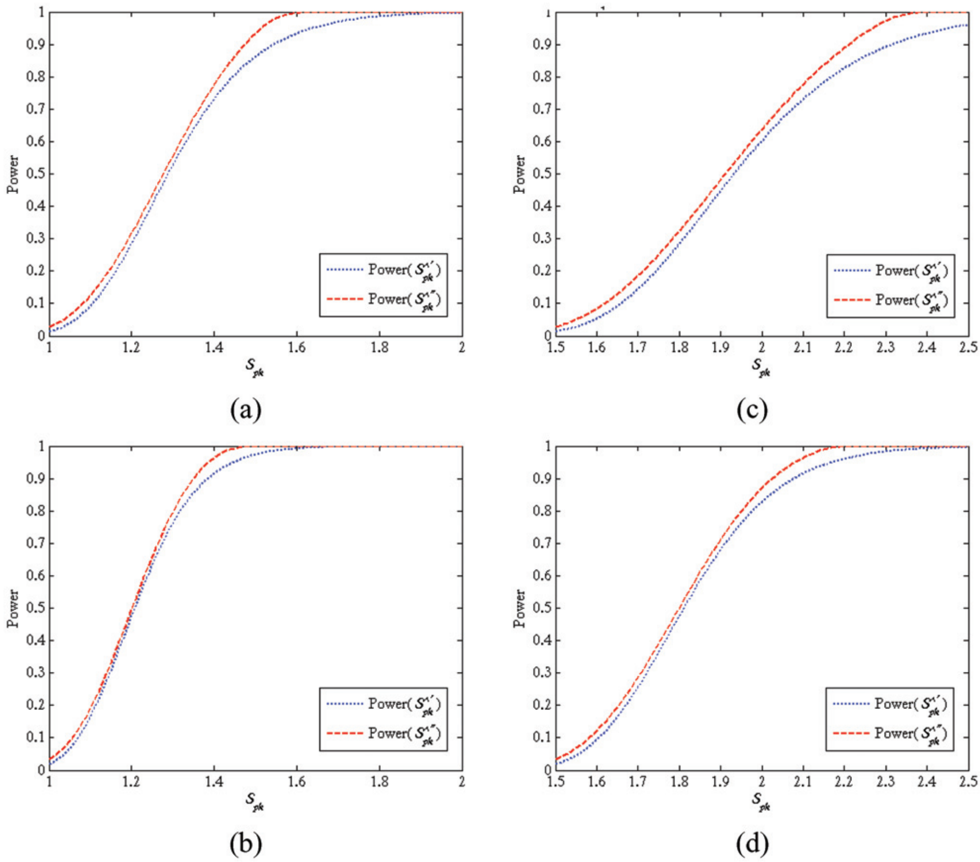

Figure 4. Power curves for testing (a) H_0: $S_{pk} \leq 1.0$ vs H_1: $S_{pk} > 1.0$, $n = 30$; (b) H_0: $S_{pk} \leq 1.0$ vs H_1: $S_{pk} > 1.0$, $n = 50$; (c) H_0: $S_{pk} \leq 1.5$ vs H_1: $S_{pk} > 1.5$, $n = 30$; (d) H_0: $S_{pk} \leq 1.5$ vs H_1: $S_{pk} > 1.5$, $n = 50$.

Procedure for using the convolution method

Step 1: Decide the minimal capability requirement C of S_{pk} (normally set to 1.00, 1.33, 1.50, 1.67 or 2.0), and the significant level α (normally set to 0.10, 0.05, or 0.025).

Step 2: Randomly sample n samples from the products.

Step 3: Calculate the sample estimate of $\hat{S}_{pk}$.

Step 4: Check out Table 3 or run the Matlab program (available on request) for the critical value based on the corresponding capability requirement of S_{pk}, significant level and sample size n.

Step 5: Conclude that the product capability S_{pk} is larger than the minimal capability requirement C, and production yield is larger than $2\Phi(3 \times \text{minimal requirement}) - 1$ with $100(1-\alpha)\%$ confidence level, if the sample estimate $\hat{S}_{pk}$ is larger than or equal to the critical value c_0 of the convolution method. Otherwise, we do not have sufficient information to make such a conclusion.

6. Accuracy analysis

The information of required sample size is important for in-plant applications, as it directly relates to the cost of the data collection plan. Following, we investigate the accuracy of the convolution method which provides useful information about the sample size required for designated power levels and for convergence.

6.1 *Sample size required for designated power*

The decision rule of hypothesis testing depends solely on the significant level α, the maximal probability of Type I error, and ignores the probability of Type II error β. Once the sample size n and α risk are chosen for testing a hypothesis, the power of test $1-\beta$, the probability of correctly rejecting H_0 while H_1 is true, will be fixed. To decrease the β risk and in the meantime maintain the α risk in a small level, the sample size should be increased.

The required sample size of the convolution method can be obtained by a recursive search with the following two constraints:

$$\Pr\left\{\hat{S}''_{pk} \geq c_0 \mid S_{pk} \leq \mathrm{C} \text{ and } \xi = 0.5\right\} \leq \alpha,$$

and

$$\Pr\left\{\hat{S}''_{pk} \geq c_0 \mid S_{pk} > \mathrm{C} \text{ and } \xi = 0.5\right\} \geq 1-\beta,$$

where c_0 is the critical value of the convolution method. Table 4 shows the minimal sample size required for testing H_0: $S_{pk} \leq \mathrm{C}$, $\mathrm{C} = 1.0$, 1.33, 1.50, 1.67, while actually $S_{pk} = \mathrm{C} + h$, $h = 0.15(0.05)0.35$, with designated α levels = 0.1, 0.05, 0.025, and power levels = 0.7, 0.8, 0.9, 0.95.

Note that the sample size required is a function of the α and power levels, the minimal capability requirement C of S_{pk}, and the difference between the actual value of S_{pk} and the minimal requirement C. Table 4 shows that the larger the difference, the smaller the sample size required for fixed α and power levels. For fixed α, minimal requirement C, and actual value of S_{pk}, the sample size increases as the designated power level increases. This phenomenon can be explained easily, since the smaller the difference and the greater the desired power level, the more sample size should be collected to account for the smaller uncertainty in the estimation.

6.2 *Sample size required for convergence*

Table 5 displays the sample sizes required for the convolution approximation to converge to S_{pk} within a designated accuracy $\varepsilon = 0.12(0.01)0.03$.

$$\Pr\left\{\left|\hat{S}''_{pk} - S_{pk}\right| \leq \varepsilon\right\} \geq 1-\alpha$$

For example, for $S_{pk} = 1.33$ with risk $\alpha = 0.025$, a sample size of $n \geq 3831$ ensures that the difference between sample estimate and actual parameter would be no greater than 0.03 with 97.5% confidence. Thus, if $\hat{S}_{pk} = 1.33$, then we may conclude that the actual S_{pk} is greater than 1.3, actually in the interval of $(1.30, 1.36)$, with 97.5% confidence. Note that

Table 4. Sample size required for designated power levels of the convolution method.

		Power						Power			
α	S_{pk}	0.7	0.8	0.9	0.95	α	S_{pk}	0.7	0.8	0.9	0.95
(a) H_0: $S_{pk} \leq 1.0$ vs H_1: $S_{pk} > 1.0$						(b) H_0: $S_{pk} \leq 1.33$ vs H_1: $S_{pk} > 1.33$					
0.10	1.15	83	113	161	207	0.10	1.48	142	194	278	360
	1.20	49	66	212	265		1.53	83	113	161	207
	1.25	33	44	62	78		1.58	55	75	106	135
	1.30	24	32	44	55		1.63	40	53	75	95
	1.35	18	24	33	40		1.68	30	40	56	71
0.05	1.15	120	156	212	265	0.05	1.48	205	267	366	458
	1.20	71	91	124	154		1.53	120	156	212	264
	1.25	47	61	82	101		1.58	79	103	140	173
	1.30	34	44	59	72		1.63	57	74	99	123
	1.35	26	33	44	53		1.68	43	56	75	92
0.025	1.15	158	199	262	321	0.025	1.48	270	340	451	554
	1.20	93	116	153	187		1.53	157	198	262	321
	1.25	62	77	101	125		1.58	104	131	172	210
	1.30	44	56	73	88		1.63	74	94	123	149
	1.35	33	42	55	66		1.68	56	71	93	112
(c) H_0: $S_{pk} \leq 1.5$ vs H_1: $S_{pk} > 1.5$						(d) H_0: $S_{pk} \leq 1.67$ vs H_1: $S_{pk} > 1.67$					
0.10	1.65	179	244	352	454	0.10	1.82	220	301	433	560
	1.70	104	142	203	261		1.87	128	174	250	321
	1.75	69	94	133	170		1.92	84	113	163	210
	1.80	49	67	94	120		1.97	60	82	116	148
	1.85	37	50	71	90		2.02	46	62	87	110
0.05	1.65	259	337	462	579	0.05	1.82	318	415	569	712
	1.70	150	195	267	333		1.87	185	240	328	407
	1.75	99	129	175	218		1.92	122	158	215	268
	1.80	71	92	125	155		1.97	87	113	153	190
	1.85	54	69	94	116		2.02	65	85	115	142
0.025	1.65	340	429	569	700	0.025	1.82	418	528	700	860
	1.70	197	249	329	403		1.87	242	305	404	497
	1.75	130	164	216	264		1.92	159	201	265	325
	1.80	93	117	154	187		1.97	114	143	189	230
	1.85	70	88	116	141		2.02	85	108	142	172

the investigation is not for practical purpose. But, the computations illustrate the rate of convergence for the convolution approximation to converge to actual S_{pk}.

7. Conclusions

Production yield is the most common and standard criteria used in the manufacturing industry for measuring process performance. The yield index S_{pk} provides a one-to-one measure on the yield of normal processes, while no other indices can. The statistical properties of the natural estimator of S_{pk} are mathematically intractable, and the existing approach (the normal approximation) does not provide adequate accuracy, particularly, for small sample sizes. In this paper, we considered the convolution approximation.

Table 5. Sample size required for the convolution approximation to converge.

S_{pk}	α	Designated accuracy, ε									
		0.12	0.11	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03
1.00	0.1	88	106	129	159	203	266	363	523	819	1459
	0.05	127	151	184	228	289	378	518	744	1164	2072
	0.025	167	200	242	299	380	496	676	975	1524	2288
1.33	0.1	159	189	230	284	361	472	644	929	1210	1789
	0.05	226	270	327	405	513	672	875	1319	1543	2706
	0.025	298	355	430	531	673	880	1199	1727	2183	**3831**
1.50	0.1	203	242	294	364	461	603	822	1184	1852	3296
	0.05	289	345	418	517	655	840	1168	1683	2631	4680
	0.025	380	453	549	678	859	1122	1529	2203	3443	5225
1.67	0.1	253	302	366	453	574	750	1022	1473	2303	4097
	0.05	361	430	521	644	811	1066	1452	2092	3271	5818
	0.025	473	564	683	767	1068	1396	1901	2738	4280	7455
2.00	0.1	366	437	529	654	828	1082	1474	2124	3321	5444
	0.05	521	621	752	929	1177	1538	2094	3017	4716	8387
	0.025	683	731	985	1217	1540	2013	2741	3946	6170	10970

The proposed approach, indeed, outperforms the existing method in providing more accurate and reliable estimation for S_{pk} as well as production yield. An efficient step-by-step procedure is developed for using the convolution method to estimate the production yield. The accuracy of the convolution method is also investigated, which provides useful information about the sample size required for designated power levels, and for convergence. The sample size information and the efficient step-by-step procedure are useful to the practitioners for making reliable decisions regarding process performance based on production yield.

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Appendix: CDF of $\hat{S}''_{pk}$

A. Notations

To simplify the derivation, we define the following notations. Let

$$Z = \sqrt{n}(\bar{X} - \mu)/\sigma, \text{ and } Y = \sqrt{n}(S^2 - \sigma^2)/2\sigma^2.$$

Consider the following analytical expansion of $\hat{S}_{pk}$

$$\hat{S}_{pk} = S_{pk} + D_1 Z + D_2 Y + D_3 Z^2 + D_4 ZY + D_5 Y^2 + O_p\left(\frac{1}{n\sqrt{n}}\right),$$

where

$$D_1 = \frac{1}{\sqrt{n}}\left(\frac{-\lambda_0}{6\phi(3S_{pk})}\right), \quad D_2 = \frac{1}{\sqrt{n}}\left(\frac{-\lambda_1}{6\phi(3S_{pk})}\right), \quad D_3 = \frac{1}{n}\left(\frac{S_{pk}\lambda_0^2}{8\left[\phi(3S_{pk})\right]^2} - \frac{\lambda_1}{12\phi(3S_{pk})}\right),$$

$$D_4 = \frac{1}{n}\left(\frac{S_{pk}\lambda_0\lambda_1}{4\left[\phi(3S_{pk})\right]^2} + \frac{\lambda_0 - \lambda_2}{6\phi(3S_{pk})}\right), \quad D_5 = \frac{1}{n}\left(\frac{S_{pk}\lambda_1^2}{8\left[\phi(3S_{pk})\right]^2} + \frac{3\lambda_1 - \lambda_3}{12\phi(3S_{pk})}\right),$$

and

$$\lambda_k = \left(\frac{1-C_{dr}}{C_{dp}}\right)^k \phi\left(\frac{1-C_{dr}}{C_{dp}}\right) + (-1)^{k+1} \times \left(\frac{1+C_{dr}}{C_{dp}}\right)^k \phi\left(\frac{1+C_{dr}}{C_{dp}}\right), \quad k = 0, 1, 2, 3.$$

Let

$$\hat{S}''_{pk} = S_{pk} + D_1 Z + D_2 Y + D_3 Z^2 + D_4 Z Y + D_5 Y^2.$$

The CDF of $\hat{S}''_{pk}$ then can be derived by the probability

$$F_{\hat{S}''_{pk}}(x) = \Pr\left\{\hat{S}''_{pk} - x \le 0\right\} = \Pr\left\{D_3\left(Z + \frac{D_1 + D_4 Y}{2D_3}\right)^2 - \frac{E_1(Y+E_3)^2}{4D_3} + \frac{\Delta_1(x)}{4D_3E_1} \le 0\right\},$$

where

$$E_1 = D_4^2 - 4D_3D_5, \quad E_2 = D_1D_4 - 2D_2D_3, \quad E_3 = \frac{E_2}{E_1}, \quad E_4 = 4D_3S_{pk} - D_1^2,$$

$$\Delta_1(x) = E_2^2 - E_1(4D_3x - E_4), \quad \text{and} \quad \Delta_2(Y;x) = E_1(Y+E_3)^2 - \frac{\Delta_1(x)}{E_1}.$$

Moreover, we define the following notations through the derivation:

$$u_1(x) = -E_3 - \sqrt{\frac{\Delta_1(x)}{E_1^2}}, \quad v_1(x) = -E_3 + \sqrt{\frac{\Delta_1(x)}{E_1^2}},$$

$$u_2(y;x) = -\left(\frac{D_1 + D_4 y}{2D_3}\right) - \sqrt{\frac{\Delta_2(y;x)}{4D_3^2}}, \quad v_2(y;x) = -\left(\frac{D_1 + D_4 y}{2D_3}\right) + \sqrt{\frac{\Delta_2(y;x)}{4D_3^2}},$$

$y_0 = -\sqrt{n}/2$ is the minimum value of the random variable Y, and $\psi(\cdot)$ is the PDF of the random variable Y.

B. CDF of $\hat{S}''_{pk}$

Case 1: For $D_3 < 0$, $E_1 < 0$, $y_0 > -E_3$.

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{v_1(x)}^{\infty} \psi(y)\mathrm{d}y + \int_{y_0}^{v_1(x)} \left\{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\right\}\psi(y)\mathrm{d}y;$$

$$X \ge \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3}, \quad F(x) = 1.$$

Case 2: For $D_3 < 0$, $E_1 < 0$, $y_0 < -E_3$.

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$
$$F(x) = \int_{v_1(x)}^{\infty} \psi(y)\mathrm{d}y + \int_{y_0}^{v_1(x)} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)\mathrm{d}y;$$
$$\frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3} \le x < \frac{E_4 + E_2E_3}{4D_3},$$
$$F(x) = \int_{y_0}^{u_1(x)} \psi(y)\mathrm{d}y + \int_{v_1(x)}^{\infty} \psi(y)\mathrm{d}y + \int_{u_1(x)}^{v_1(x)} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)\mathrm{d}y;$$
$$x \ge \frac{E_4 + E_2E_3}{4D_3}, \quad F(x) = 1.$$

Case 3: For $D_3 < 0$, $E_1 > 0$, $y_0 > -E_3$.

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$
$$F(x) = \int_{y_0}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)\mathrm{d}y;$$
$$x \ge \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$
$$F(x) = \int_{y_0}^{v_1(x)} \psi(y)\mathrm{d}y + \int_{v_1(x)}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)\mathrm{d}y.$$

Case 4: For $D_3 < 0$, $E_1 > 0$, $y_0 < -E_3$.

$$x < \frac{E_4 + E_2E_3}{4D_3},$$
$$F(x) = \int_{y_0}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)\mathrm{d}y;$$
$$\frac{E_4 + E_2E_3}{4D_3} \le x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$
$$F(x) = \int_{u_1(x)}^{v_1(x)} \psi(y)\mathrm{d}y + \int_{y_0}^{u_1(x)} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)\mathrm{d}y$$
$$+ \int_{v_1(x)}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)\mathrm{d}y;$$
$$x \ge \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$
$$F(x) = \int_{y_0}^{v_1(x)} \psi(y)\mathrm{d}y + \int_{v_1(x)}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)\mathrm{d}y.$$

Case 5: For $D_3 > 0,\ E_1 > 0,\ y_0 > -E_3$.

$$\begin{aligned}
x &< \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},\\
F(x) &= \int_{v_1(x)}^{\infty} \left\{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\right\}\psi(y)\mathrm{d}y;\\
x &\geq \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},\\
F(x) &= \int_{y_0}^{\infty} \left\{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\right\}\psi(y)\mathrm{d}y.
\end{aligned}$$

Case 6: For $D_3 > 0,\ E_1 > 0,\ y_0 < -E_3$.

$$\begin{aligned}
x &< \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},\\
F(x) &= \int_{v_1(x)}^{\infty} \left\{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\right\}\psi(y)\mathrm{d}y; \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3} \leq x < \frac{E_4 + E_2E_3}{4D_3},\\
F(x) &= \int_{y_0}^{u_1(x)} \left\{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\right\}\psi(y)\mathrm{d}y + \int_{v_1(x)}^{\infty} \left\{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\right\}\psi(y)\mathrm{d}y;\\
x &\geq \frac{E_4 + E_2E_3}{4D_3},\\
F(x) &= \int_{y_0}^{\infty} \left\{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\right\}\psi(y)\mathrm{d}y.
\end{aligned}$$

Case 7: For $D_3 > 0,\ E_1 < 0,\ y_0 > -E_3$.

$$\begin{aligned}
x &< \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3}, \quad F(x) = 0;\\
x &\geq \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3}, \quad F(x) = \int_{y_0}^{v_1(x)} \left\{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\right\}\psi(y)\mathrm{d}y.
\end{aligned}$$

Case 8: For $D_3 > 0,\ E_1 < 0,\ y_0 < -E_3$.

$$\begin{aligned}
x &< \frac{E_4 + E_2E_3}{4D_3},\ F(x) = 0; \quad \frac{E_4 + E_2E_3}{4D_3} \leq x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},\\
F(x) &= \int_{u_1(x)}^{v_1(x)} \left\{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\right\}\psi(y)\mathrm{d}y;\\
x &\geq \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},\\
F(x) &= \int_{y_0}^{v_1(x)} \left\{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\right\}\psi(y)\mathrm{d}y.
\end{aligned}$$