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Measuring production yield for processes with multiple characteristics

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Numerous capability indices have been proposed to measure the performance of processes with multiple characteristics. The index S_{pk}^T provides an exact measure on the production yield of multinormal processes in which the characteristics are mutually independent. In this paper, we thoroughly investigate the relationship between process parameters and the sampling distribution of S_{pk}^T. Our investigation shows that for a fixed S_{pk}^T, the variance of sample estimator of S_{pk}^T is restricted in an interval. For reliability consideration, the maximal variance is used in the estimation and testing of the production yield to ensure the level of confidence. Also, information about sample sizes required for specified precision of estimation and for convergence is determined. At last, we implement a trivariate process with data collected from a plastics manufacturing industrial to demonstrate the practicability of the proposed method in measuring the production yield.

Keywords: capability indices; manufacturing processes; production yield; reliability

1. Introduction

Numerous capability indices have been proposed to measure the production yield, which is an important concern for manufacturing factories. The production yield is defined as the percentage of processed product units passing the inspection. For a measured characteristic X, the production yield can be defined as follows, where $F(x)$ is the cumulative distribution function of X, USL and LSL are the upper and lower specification limits of the product characteristic, respectively:

$$\begin{aligned} Yield &= \int_{LSL}^{USL} \mathrm{d}F(x) \\ &= \Pr(LSL \le X \le USL). \end{aligned}$$

For normal processes with a single characteristic, the production yield can be measured by some well-known process capability indices. For example, Boyles (1991) found that for an on-centre process, $Yield = 2\Phi(3C_p) - 1$, and for any process, $2\Phi(3C_{pk}) - 1 \le Yield < \Phi(3C_{pk})$. Ruczinski (1996) obtained a lower bound on the production yield as $Yield \ge 2\Phi(3C_{pm}) - 1$. Furthermore, Boyles (1994) proposed the yield index S_{pk},

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which provides an exact measure on the production yield, $Yield = 2\Phi(3S_{pk}) - 1$. These indices have been explicitly defined as follows:

$$C_p = \frac{USL - LSL}{6\sigma},$$

$$C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}},$$

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - \mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu - LSL}{\sigma}\right)\right\},$$

where μ is the process mean, σ is the process standard deviation, T is the target value, $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution $N(0, 1)$, and $\Phi^{-1}(\cdot)$ is the inverse of $\Phi(\cdot)$.

The above indices have been designed for measuring the capability of processes with a single characteristic. However, commonly the manufactured product involves more than one characteristic. That is, manufactured items require values of several different characteristics for adequate description of their quality. The assessed quality of a product depends on the combined effects of those characteristics, rather than on their individual values. Normally each of those characteristics must satisfy certain specifications, and the corresponding production yield can be expressed as:

$$Yield = \Pr(LSL_1 \le X_1 \le USL_1 \cap \cdots \cap LSL_v \le X_v \le USL_v),$$

where $X_1, \ldots, X_v$ represents v characteristics of the investigated product.

To measure the performance of processes with multiple characteristics, various capability indices are proposed for assessing process capability. For example, Pearn *et al.* (1992) proposed the indices ${}_vC_{pm}$, ${}_vC_p$ which require the assumption of multivariate normality and thus the tolerance region is ellipsoidal. Taam *et al.* (1993) proposed an index MC_{pm} defined as a ratio of a modified tolerance region and a scaled 99.73% process region. Chen (1994) proposed the MC_p index given by a ratio of the width of the tolerance interval centred at the target value and the width of the interval that the probability of a process falling within this interval is $1 - \alpha$. The MC_p index can be easily interpreted for the univariate normal cases. Shahriari *et al.* (1995) proposed a multivariate capability vector denoted as $[C_{pM}, PV, LI]$ representing, respectively, a ratio of regions (similar to MC_p and MC_{pm}), locations of centres, and whether or not the modified process region is contained within the tolerance region. Wang and Du (2000) presented a comparison of three methods proposed by Taam *et al.* (1993), Chen (1994), and Shahriari *et al.* (1995). They applied the principal component analysis (PCA) to these process capability indices to handle normal and non-normal data. The capability indices proposed by Pearn *et al.* (1992), Taam *et al.* (1993) and Shahriari *et al.* (1995) require the assumption of multivariate normality, while those proposed by Chen (1994) and Wang and Du (2000) make no assumptions.

However, the issues between the production yield of processes with multiple characteristics and the capability indices have received little attention. In this paper, we focus on a more newly proposed index S_{pk}^T, which can provide an exact measure on the

production yield for the multinormal processes in which the characteristics are mutually independent. The sampling distribution of S_{pk}^T is investigated based on various combinations of the production parameters. A statistical method for estimating the true production yield is presented by the lower confidence bound of S_{pk}^T. Hypothesis testing on the production yield is also conducted. For the engineers' usefulness, critical values of the hypothesis testing for some commonly used capability requirements are tabulated. Finally, a real world example is presented to demonstrate the applicability of the proposed approach.

2. Production yield of multivariate processes

2.1 *General consideration*

Let $\underset{\sim}{\mathrm{X}} = (X_1, \ldots, X_v)^T$ be the vector of the v interested variables of a specific product which is used for statistical process control. Assume that $\underset{\sim}{\mathrm{X}}$ has a multinormal distribution, $N_v(\underset{\sim}{\mathrm{X}}|\mu, \Sigma)$, where μ is the mean vector and Σ represents the covariance matrix of $\underset{\sim}{\mathrm{X}}$. Under the assumptions mentioned, the production yield can be represented as:

$$Yield = \int_{LSL_v}^{USL_v} \cdots \int_{LSL_1}^{USL_1} N_v(\underset{\sim}{\mathrm{X}}|\mu, \Sigma)\, \mathrm{d}X_1 \cdots \mathrm{d}X_v,$$

where LSL_i and USL_i, $i = 1, \ldots, v$, are the v lower and upper specification limits, respectively. If the v variables are mutually independent, the production yield can be alternatively represented as:

$$Yield = \prod_{i=1}^{v} \left[\Phi\left(\frac{USL_i - \mu_i}{\sigma_i}\right) - \Phi\left(\frac{LSL_i - \mu_i}{\sigma_i}\right) \right],$$

where μ_i and σ_i are the mean and standard deviation of the ith characteristic X_i, respectively.

2.2 *Yield index* S_{pk}^T

Chen *et al.* (2003) proposed the following capability index, referred to as S_{pk}^T:

$$S_{pk}^T = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{i=1}^{v}(2\Phi(3S_{pki}) - 1) + 1\right]\right\},$$

where S_{pki} denotes the S_{pk} value of the ith characteristic for $i = 1, 2, \ldots, v$, and v is the number of characteristics. The index S_{pk}^T may be viewed as a generalisation of the yield index S_{pk} proposed by Boyles (1994) for processes with a single characteristic. For multinormal processes in which the characteristics are mutually independent, there is a one-to-one correspondence between the index S_{pk}^T and the overall production yield, $Yield = 2\Phi(3S_{pk}^T) - 1$. For example, if $S_{pk}^T = 1.50$, then the overall production yield is roughly 99.9993%, or equally, the fraction of defectives is roughly 7 parts per million (ppm). Table 1 displays various commonly used capability requirements and the corresponding production yields as well as non-conformities in ppm.

Table 1. Various S_{pk}^T values and the corresponding production yields as well as non-conformities in parts per million (ppm).

S_{pk}^T	*Yield*	ppm
1.00	0.997300204	2699.796
1.05	0.998367295	1632.705
1.10	0.999033152	966.848
1.15	0.999439413	560.587
1.20	0.999681783	318.217
1.25	0.999823165	176.835
1.30	0.999903807	96.193
1.35	0.999948782	51.218
1.40	0.999973309	26.691
1.45	0.999986386	13.614
1.50	0.999993205	6.795
1.55	0.999996681	3.319
1.60	0.999998413	1.587
1.65	0.999999258	0.742
1.70	0.999999660	0.340
1.75	0.999999848	0.152
1.80	0.999999933	0.067
1.85	0.999999971	0.029
1.90	0.999999988	0.012
1.95	0.999999995	0.005
2.00	0.999999998	0.002

2.3 *Sampling distribution of* S_{pk}^T

The natural estimator of S_{pk}^T is defined as:

$$\hat{S}_{pk}^T = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{i=1}^{v}(2\Phi(3\hat{S}_{pki}) - 1) + 1\right]\right\},$$

where $\hat{S}_{pki}$ denotes the estimator of S_{pki}. For univariate normal processes, the index S_{pk} provides an exact measure on the production yield. Lee *et al.* (2002) derived the sampling distribution of the univariate capability index S_{pk}, which is an asymptotic normal distribution with mean S_{pk} and variance $(a^2 + b^2)/36n\phi^2(3S_{pk})$, where ϕ is the probability density function (PDF) of the standard normal distribution $N(0, 1)$.

$$a = \frac{1}{\sqrt{2}}\left\{\frac{1 - C_{dr}}{C_{dp}}\phi\left(\frac{1 - C_{dr}}{C_{dp}}\right) + \frac{1 + C_{dr}}{C_{dp}}\phi\left(\frac{1 + C_{dr}}{C_{dp}}\right)\right\},$$

$$b = \phi\left(\frac{1 - C_{dr}}{C_{dp}}\right) - \phi\left(\frac{1 + C_{dr}}{C_{dp}}\right),$$

$$C_{dr} = \frac{\mu - \mathrm{m}}{\mathrm{d}},$$

$$C_{dp} = \frac{\sigma}{\mathrm{d}},$$

$$\mathrm{m} = (USL + LSL)/2,$$

$$\mathrm{d} = (USL - LSL)/2.$$

Inheriting the single variate result of Lee *et al.* (2002) and applying the Taylor expansion of *v*-variate, Pearn *et al.* (2006) showed that the sampling distribution of S_{pk}^T is an asymptotic normal distribution with mean S_{pk}^T and variance:

$$\frac{1}{36n\phi^2(3S_{pk}^T)}\sum_{j=1}^{v}\left\{\left(a_j^2+b_j^2\right)\left[\frac{\prod_{i=1}^{v}(2\Phi(3S_{pki})-1)^2}{(2\Phi(3S_{pkj})-1)^2}\right]\right\},$$

where a_j, b_j, S_{pkj} are the corresponding parameters of the *j*th characteristic. That is,

$$\hat{S}_{pk}^T \sim N\left(S_{pk}^T, \frac{1}{36n\phi^2(3S_{pk}^T)}\sum_{j=1}^{v}\left\{\left(a_j^2+b_j^2\right)\left[\frac{\prod_{i=1}^{v}(2\Phi(3S_{pki})-1)^2}{(2\Phi(3S_{pkj})-1)^2}\right]\right\}\right).$$

It should be noted that the production yield index S_{pk}^T provides an exact yield measure based on the assumption of multinormal processes with independent characteristics. With dependent characteristics, our proposed procedure will use the principal component analysis (PCA) to project the original set of dependent variables Y into a set of independent ones X, which is used to estimate the index value of S_{pk}^T. Furthermore, if the original characteristics is not multinormal but any particular distribution, we will assume that Y is some transformation $\mathrm{Y} = h(\mathrm{W})$, where W is the original non-normal variables, so that $\tilde{\mathrm{Y}}$ is normally distributed. In practice, some Box-and-Cox type transformations can help improve the normality of the data. In the rest of the article, to avoid further notation we will assume that the original observations follow a multinormal distribution with *v* independent characteristics, that is $\underset{\sim}{\mathrm{X}} \sim N_v(\underset{\sim}{X}|\mu, \Sigma)$.

3. Estimation of the production yield

3.1 *Lower confidence bound*

In practice, to estimate the true production yield in a conservative way, the engineers will take the lower confidence bound as the true yield estimation. Since the sampling distribution of S_{pk}^T is presented in $3v+1$ parameters, S_{pkj}, a_j, b_j, $j=1,\ldots,v$ and S_{pk}^T, the lower confidence bound is also a function of those parameters. Thus, we have to consider the effect of all the parameters in the calculation of lower confidence bound to ensure that the lower bounds obtained are reliable. The word 'reliable' here means that the probability that the obtained lower bound (subject to the sample estimate) is smaller than the actual capability index S_{pk}^T is greater than the desired confidence level. In the following, we present how to obtain the reliable lower bound in the two- and three-dimensional processes. We also execute cases in higher (greater than three) dimensions. To save the capacity of the article, the high dimensional cases are dropped. Fortunately, in the high dimensional cases, the effects of all the parameters in the calculation of lower confidence bound come out in a similar pattern as in the two- or three-dimensional cases.

3.1.1 *Two-dimensional processes*

For processes with two interested characteristics, i.e., $v=2$, we perform extensive calculations to obtain the variance of $\hat{S}_{pk}^T$, $Var(\hat{S}_{pk}^T)$, which directly affects the value of the lower confidence bound. We note that for an identical S_{pk}^T, there are numerous combinations of S_{pk1} and S_{pk2}, and when one of them becomes larger, the other one will converge to S_{pk}^T. Figure 1 shows the relationship between S_{pk1} and S_{pk2} under various S_{pk}^T.

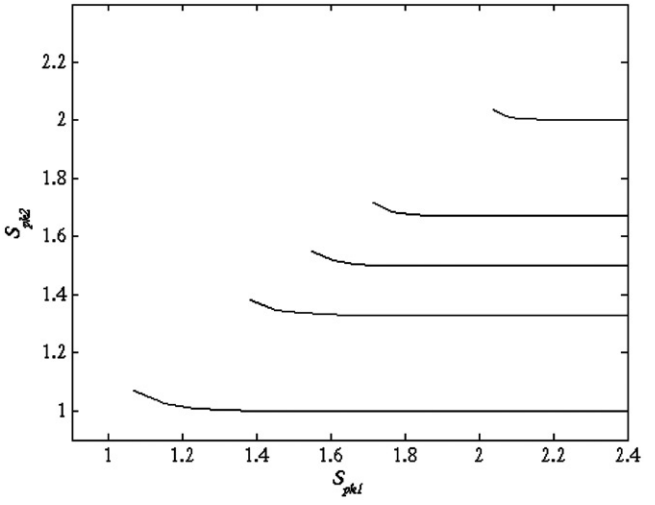

Figure 1. Contour of combinations of S_{pk1} and S_{pk2} for $S_{pk}^T = 1.00, 1.33, 1.50, 1.67, 2.00$ (from bottom to top).

Furthermore, for a fixed S_{pkj}, there are also numerous combinations of a_j and b_j. Table 2 displays a few combinations of the parameters S_{pk1}, S_{pk2}, a_1, a_2, b_1, and b_2 under $S_{pk}^T = 1$, and the corresponding variance of $\hat{S}_{pk}^T$.

Our extensive calculation results show that: (i) the variance of $\hat{S}_{pk}^T$ obtains its minimum at $S_{pk1} = S_{pk2}$, and maximum at $S_{pk1} \geq 2.5$ and $S_{pk2} = S_{pk}^T$ (or $S_{pk1} = S_{pk}^T$ and $S_{pk2} \geq 2.5$); and (ii) for fixed S_{pk1} and S_{pk2}, the variance of $\hat{S}_{pk}^T$ reaches its maximum at $b_1 = b_2 = 0$, i.e., the mean vector is on-centre, $\mu = \mathrm{m}$. Figure 2 shows the $nVar(\hat{S}_{pk}^T)$ of two-dimensional processes for $S_{pk}^T = 1.00, 1.33, 1.50, 1.67, 2.00$ with combinations of S_{pk1} and S_{pk2} as in Figure 1. It is clear that the $nVar(\hat{S}_{pk}^T)$ reaches its maximum when $S_{pk1} \geq 2.5$ and $S_{pk2} = S_{pk}^T$. Thus, in the calculations of lower confidence bound of S_{pk}^T for the two-dimensional processes, we will set $S_{pk1} = 2.5$, $S_{pk2} = S_{pk}^T$ (or $S_{pk1} = S_{pk}^T$ and $S_{pk2} = 2.5$), $a_1 = \sqrt{2}(3S_{pk1})\phi(3S_{pk1})$, $a_2 = \sqrt{2}(3S_{pk2})\phi(3S_{pk2})$, and $b_1 = b_2 = 0$ to make a reliable yield estimation.

3.1.2 *Three- and high-dimensional processes*

For three-dimensional processes, we perform similar calculations as in the two-dimensional cases. Figure 3 shows the combinations of S_{pk1}, S_{pk2} and S_{pk3} of three-dimensional processes for $S_{pk}^T = 1.00$. Note that the plots in Figure 3 are identical, just with opposite angles. The top point in Figure 3 represents the combination of (S_{pk1}, S_{pk2}, S_{pk3}) = (1.1066, 1.1066, 1.1066) for $S_{pk}^T = 1.00$, which can be shown with the smallest corresponding variance (as in Figure 5). Furthermore, we can see that S_{pk3} converges to S_{pk}^T as S_{pk1} and S_{pk2} become larger. In such a condition, the variance of $\hat{S}_{pk}^T$ can be shown to be the largest (see Figure 5). Similar to Figure 3, Figure 4 presents the combinations of $\hat{S}_{pk}^T$ for three-dimensional processes for $S_{pk}^T = 1.00, 1.33, 1.50, 1.67$ and 2.00. It can be seen that the patterns of combinations of S_{pk1}, S_{pk2} and S_{pk3} for a fixed S_{pk}^T are the same in three-dimensional cases. There is a peak point with $S_{pk1} = S_{pk2} = S_{pk3}$, and a convergence with $S_{pk3} = S_{pk}^T$ and $S_{pk1} = S_{pk2} \geq 2.5$.

Table 2. Variance of $\hat{S}_{pk}^{T}$ versus combinations of the parameters under $S_{pk}^{T}=1$.

S_{pk1}	S_{pk2}	a_1	a_2	b_1	b_2	$nVar(\hat{S}_{pk}^{T})$
1.06832	1.06832	0.01064	0.01064	**0.00000**	**0.00000**	**0.319084**
1.06832	1.06832	0.01063	0.01063	0.00015	0.00015	0.319084
1.06832	1.06832	0.01063	0.01063	0.00031	0.00031	0.319084
1.06832	1.06832	0.01063	0.01063	0.00047	0.00047	0.319084
1.06832	1.06832	0.01062	0.01062	0.00064	0.00064	0.319082
1.06832	1.06832	0.01060	0.01060	0.00082	0.00082	0.319077
1.06832	1.06832	0.01059	0.01059	0.00100	0.00100	0.319069
1.06832	1.06832	0.01057	0.01057	0.00118	0.00118	0.319054
1.06832	1.06832	0.01055	0.01055	0.00137	0.00137	0.319028
1.06832	1.06832	0.01052	0.01052	0.00157	0.00157	0.318987
1.66832	1.00002	1.02652E-05	0.01880	**0.00000**	**0.00000**	**0.499832**
1.66832	1.00002	1.02641E-05	0.01880	1.48866E-07	0.00027	0.499832
1.66832	1.00002	1.02607E-05	0.01879	3.05375E-07	0.00055	0.499832
1.66832	1.00002	1.02544E-05	0.01878	4.69921E-07	0.00083	0.499831
1.66832	1.00002	1.02449E-05	0.01877	6.42929E-07	0.00113	0.499829
1.66832	1.00002	1.02316E-05	0.01874	8.24847E-07	0.00143	0.499823
1.66832	1.00002	1.02138E-05	0.01872	1.01616E-06	0.00175	0.499813
1.66832	1.00002	1.01907E-05	0.01868	1.21739E-06	0.00207	0.499794
1.66832	1.00002	1.01612E-05	0.01864	1.42911E-06	0.00240	0.499763
1.66832	1.00002	1.01239E-05	0.01860	1.65195E-06	0.00275	0.499712
2.26832	1.00000	3.37837E-10	0.01880	**0.00000**	**0.00000**	**0.500000**
2.26832	1.00000	3.37801E-10	0.01880	4.9441E-12	0.00027	0.500000
2.26832	1.00000	3.37682E-10	0.01879	1.02364E-11	0.00055	0.500000
2.26832	1.00000	3.37459E-10	0.01878	1.59017E-11	0.00083	0.499999
2.26832	1.00000	3.37109E-10	0.01877	2.19671E-11	0.00113	0.499997
2.26832	1.00000	3.36602E-10	0.01875	2.84607E-11	0.00143	0.499991
2.26832	1.00000	3.35893E-10	0.01872	3.54154E-11	0.00175	0.499981
2.26832	1.00000	3.34914E-10	0.01869	4.28673E-11	0.00207	0.499962
2.26832	1.00000	3.33572E-10	0.01865	5.08564E-11	0.00240	0.499931
2.26832	1.00000	3.31662E-10	0.01860	5.94371E-11	0.00275	0.499880

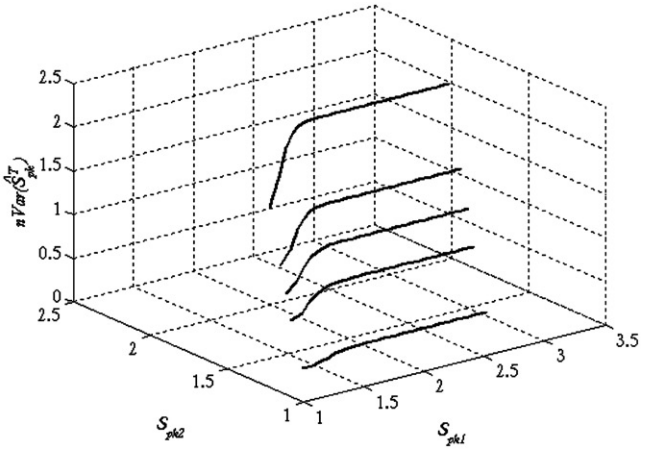

Figure 2. Variance plots of two-dimensional processes for $S_{pk}^{T}=1.00$, 1.33, 1.50, 1.67, 2.00 (from bottom to top).

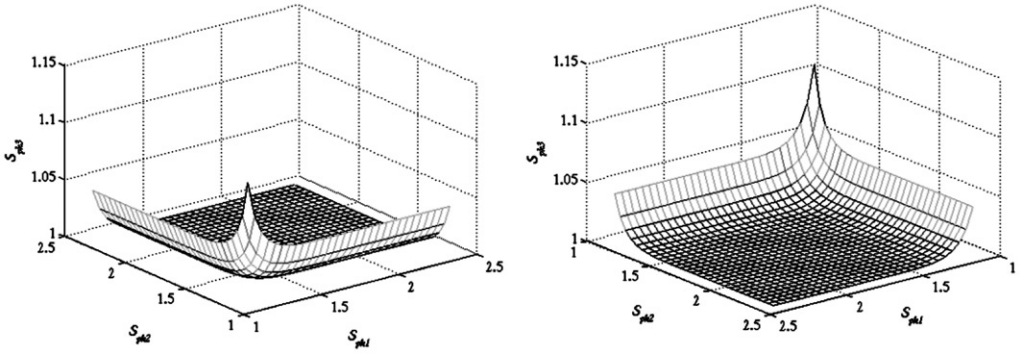

Figure 3. Surface of combinations of S_{pk1}, S_{pk2} and S_{pk3} for $S_{pk}^T = 1.00$.

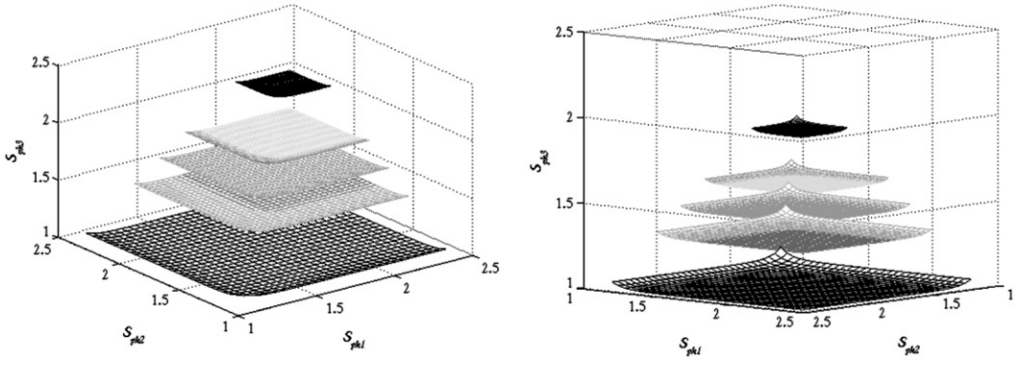

Figure 4. Surfaces of combinations of S_{pk1}, S_{pk2} and S_{pk3} for $S_{pk}^T = 1.00$, 1.33, 1.50, 1.67, 2.00 (from bottom to top).

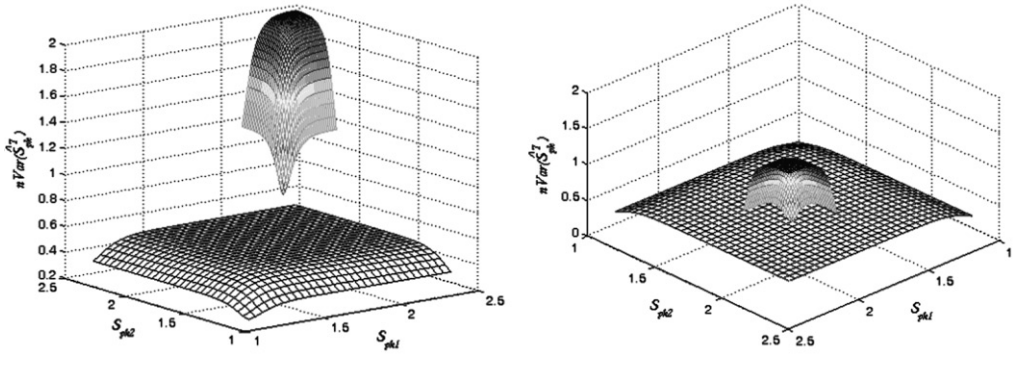

Figure 5. Variance plots of three-dimensional processes for $S_{pk}^T = 1.0$ and 2.0 (bottom to top).

Table 3 displays a few combinations of the parameters S_{pk1}, S_{pk2}, S_{pk3}, a_1, a_2, a_3, b_1, b_2, and b_3 under $S_{pk}^T = 1$ as well as the corresponding variance of $\hat{S}_{pk}^T$. Similar to the two-dimensional cases, the calculation results show that: (i) the variance of $\hat{S}_{pk}^T$ obtains its minimum at $S_{pk1} = S_{pk2} = S_{pk3}$, and maximum at $S_{pki} = S_{pk}^T$ and $S_{pkj} \geq 2.5$, where $j \neq i$; and

Table 3. Combinations of the parameters and the corresponding $nVar(\hat{S}_{pk}^T)$ for $S_{pk}^T = 1$.

S_{pk1}	S_{pk2}	S_{pk3}	a_1	a_2	a_3	b_1	b_2	b_3	$nVar(\hat{S}_{pk}^T)$
1.10661	1.10661	1.10661	0.00757	0.00757	0.00757	**0.00000**	**0.00000**	**0.00000**	**0.242496**
1.10661	1.10661	1.10661	0.00757	0.00757	0.00757	0.00011	0.00011	0.00011	0.242496
1.10661	1.10661	1.10661	0.00757	0.00757	0.00757	0.00022	0.00022	0.00022	0.242496
1.10661	1.10661	1.10661	0.00757	0.00757	0.00757	0.00034	0.00034	0.00034	0.242495
1.10661	1.10661	1.10661	0.00756	0.00756	0.00756	0.00046	0.00046	0.00046	0.242494
1.10661	1.10661	1.10661	0.00755	0.00755	0.00755	0.00058	0.00058	0.00058	0.242490
1.10661	1.10661	1.10661	0.00754	0.00754	0.00754	0.00071	0.00071	0.00071	0.242483
1.10661	1.10661	1.10661	0.00753	0.00753	0.00753	0.00084	0.00084	0.00084	0.242470
1.10661	1.10661	1.10661	0.00751	0.00751	0.00751	0.00098	0.00098	0.00098	0.242448
1.10661	1.10661	1.10661	0.00749	0.00749	0.00749	0.00112	0.00112	0.00112	0.242413
2.10661	1.10661	1.04043	0.00000	0.00757	0.01350	**0.00000**	**0.00000**	**0.00000**	**0.337989**
2.10661	1.10661	1.04043	0.00000	0.00757	0.01350	0.00000	0.00011	0.00019	0.337989
2.10661	1.10661	1.04043	0.00000	0.00757	0.01349	0.00000	0.00022	0.00039	0.337988
2.10661	1.10661	1.04043	0.00000	0.00757	0.01348	0.00000	0.00034	0.00060	0.337988
2.10661	1.10661	1.04043	0.00000	0.00756	0.01347	0.00000	0.00046	0.00081	0.337986
2.10661	1.10661	1.04043	0.00000	0.00755	0.01346	0.00000	0.00058	0.00103	0.337982
2.10661	1.10661	1.04043	0.00000	0.00754	0.01344	0.00000	0.00071	0.00126	0.337973
2.10661	1.10661	1.04043	0.00000	0.00753	0.01341	0.00000	0.00084	0.00149	0.337958
2.10661	1.10661	1.04043	0.00000	0.00751	0.01338	0.00000	0.00098	0.00174	0.337932
2.10661	1.10661	1.04043	0.00000	0.00749	0.01335	0.00000	0.00112	0.00198	0.337890
2.10661	2.10661	1.00000	0.00000	0.00000	0.01880	**0.00000**	**0.00000**	**0.00000**	**0.500000**
2.10661	2.10661	1.00000	0.00000	0.00000	0.01880	0.00000	0.00000	0.00027	0.500000
2.10661	2.10661	1.00000	0.00000	0.00000	0.01879	0.00000	0.00000	0.00055	0.500000
2.10661	2.10661	1.00000	0.00000	0.00000	0.01878	0.00000	0.00000	0.00083	0.499999
2.10661	2.10661	1.00000	0.00000	0.00000	0.01877	0.00000	0.00000	0.00113	0.499997
2.10661	2.10661	1.00000	0.00000	0.00000	0.01875	0.00000	0.00000	0.00143	0.499991
2.10661	2.10661	1.00000	0.00000	0.00000	0.01872	0.00000	0.00000	0.00175	0.499981
2.10661	2.10661	1.00000	0.00000	0.00000	0.01869	0.00000	0.00000	0.00207	0.499962
2.10661	2.10661	1.00000	0.00000	0.00000	0.01865	0.00000	0.00000	0.00240	0.499930
2.10661	2.10661	1.00000	0.00000	0.00000	0.01860	0.00000	0.00000	0.00275	0.499880

(ii) for fixed S_{pk1}, S_{pk2}, and S_{pk3}, the variance of $\hat{S}_{pk}^T$ reaches its maximum at $b_1 = b_2 = b_3 = 0$, that is, the mean vector is on-centre.

Figure 5 shows the corresponding variance of $\hat{S}_{pk}^T$ with parameters S_{pk1}, S_{pk2}, S_{pk3} identical to those as in Figure 4 for $S_{pk}^T = 1.00$ and 2.00. Again, the plots in Figure 5 are the same plot with opposite angles. It can be seen that the larger the S_{pk3}, the smaller the $nVar(\hat{S}_{pk}^T)$, and the larger the S_{pk}^T, the greater the variation of the $nVar(\hat{S}_{pk}^T)$. We should remark that to make a reliable decision, the maximal $nVar(\hat{S}_{pk}^T)$ for a fixed S_{pk}^T is the only concern, and the $nVar(\hat{S}_{pk}^T)$ reaches its maximum at $S_{pk1} \geq 2.5$, $S_{pk2} \geq 2.5$ and $S_{pk3} = S_{pk}^T$.

For processes with high (greater than three) dimensions, the relationship between variance of $\hat{S}_{pk}^T$ and the $3v+1$ parameters is in a similar pattern as in the two- or three-dimensional processes. Figure 6 shows the plots for four-dimensional cases. Figures 6(a)–6(b) present the combinations of S_{pk2}, S_{pk3} and S_{pk4} for $S_{pk1} = 2.5$ and $S_{pk}^T = 1.0$, 1.5, 2.0 from bottom to top. Figures 6(c)–6(d) present the $nVar(\hat{S}_{pk}^T)$ versus S_{pk2}, S_{pk3} with parameter combinations as in Figures 6(a)–6(b). Again, Figures 6(a) and 6(b) are the same plots with opposite angle, and so are Figures 6(c) and 6(d).

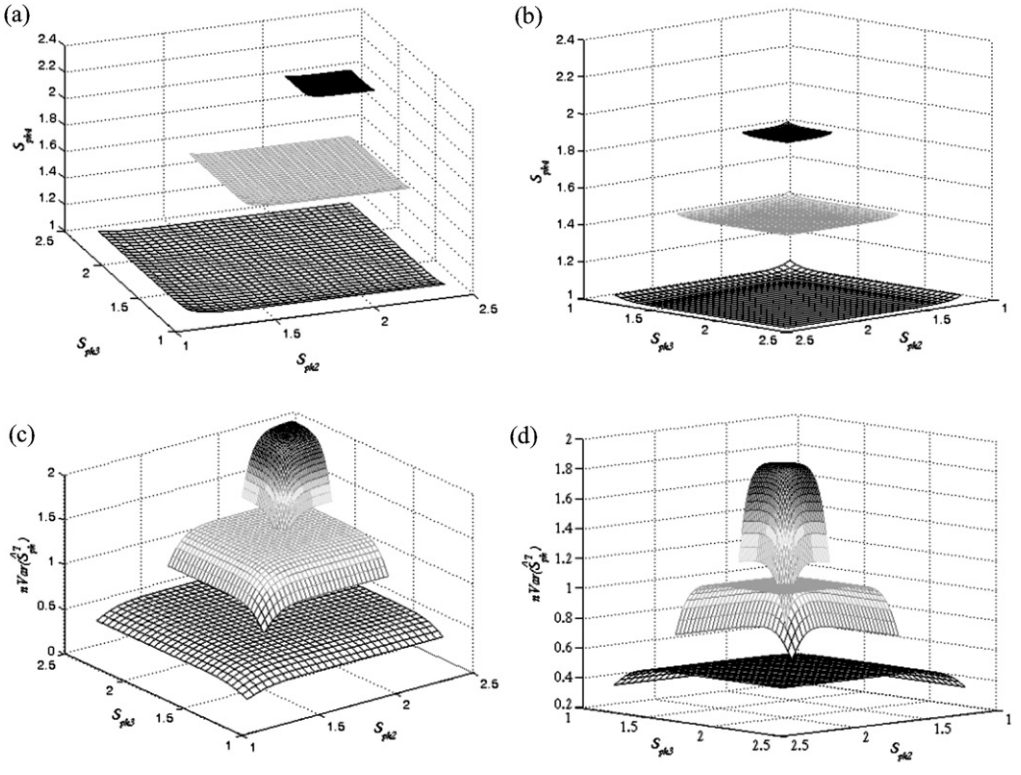

Figure 6. Plots of four-dimensional processes with $S_{pk1}=2.5$ and $S_{pk}^{T}=1.0$, 1.5, 2.0 (from bottom to top): (a)–(b) surfaces of combinations of S_{pk2}, S_{pk3} and S_{pk4}; (c) –(d) $nVar(\hat{S}_{pk}^{T})$ versus S_{pk2} and S_{pk3} with parameter combinations as in (a) and (b).

In brief: (i) for an identical S_{pk}^{T}, variance of $\hat{S}_{pk}^{T}$ is maximal at $S_{pki}=S_{pk}^{T}$ and $S_{pkj}\geqq 2.5$, where $j\neq i$, (also variance of $\hat{S}_{pk}^{T}$ is minimal while all v S_{pkj} are equal); and (ii) for fixed S_{pkj}, where $j=1,\ldots,v$, the variance of $\hat{S}_{pk}^{T}$ reaches its maximum at $b_j=0$, i.e., the mean vector is on-centre. Hence, in the calculation of lower confidence bound of S_{pk}^{T}, we will set $S_{pki}=S_{pk}^{T}$ and $S_{pkj}=2.5$, for all $j\neq i$, $a_j=\sqrt{2}(3S_{pkj})\phi(3S_{pkj})$, and $b_j=0$ for all $j=1,\ldots,v$. In this way, the level of confidence can be ensured, and the decisions (lower confidence bounds) made based on such an approach are indeed more reliable.

We note that with the above parameter setting, the sampling distribution of S_{pk}^{T} can be rewritten in a shorter and simpler form, that is:

$$\hat{S}_{pk}^{T} \sim N\left(S_{pk}^{T}, \frac{\left(S_{pk}^{T}\right)^{2}}{2n}\right).$$

Thus, given an estimate $\hat{S}_{pk}^{T}$, a sample size n and a confidence level $1-\alpha$, the lower confidence bound of S_{pk}^{T} (denoted as $S_{pk}^{T(L)}$) can be obtained:

$$S_{pk}^{T(L)}=\frac{\hat{S}_{pk}^{T}}{1+Z_{\alpha}/\sqrt{2n}}$$

Table 4. Lower confidence bound of S_{pk}^T for various $\hat{S}_{pk}^T$, $n = 5(5)100$, and $\alpha = 0.05$.

	$\hat{S}_{pk}^T$										
n	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
5	0.6578	0.7236	0.7893	0.8551	0.9209	0.9867	1.0525	1.1183	1.1840	1.2498	1.3156
10	0.7311	0.8042	0.8773	0.9504	1.0235	1.0966	1.1697	1.2428	1.3159	1.3890	1.4622
15	0.7690	0.8459	0.9228	0.9997	1.0766	1.1535	1.2304	1.3073	1.3842	1.4611	1.5380
20	0.7936	0.8729	0.9523	1.0316	1.1110	1.1904	1.2697	1.3491	1.4284	1.5078	1.5872
25	0.8112	0.8924	0.9735	1.0546	1.1357	1.2169	1.2980	1.3791	1.4603	1.5414	1.6225
30	0.8248	0.9073	0.9898	1.0722	1.1547	1.2372	1.3197	1.4022	1.4847	1.5672	1.6496
35	0.8357	0.9192	1.0028	1.0864	1.1699	1.2535	1.3371	1.4206	1.5042	1.5878	1.6714
40	0.8446	0.9291	1.0135	1.0980	1.1825	1.2669	1.3514	1.4359	1.5203	1.6048	1.6893
45	0.8522	0.9374	1.0226	1.1079	1.1931	1.2783	1.3635	1.4488	1.5340	1.6192	1.7044
50	0.8587	0.9446	1.0304	1.1163	1.2022	1.2881	1.3739	1.4598	1.5457	1.6316	1.7174
55	0.8644	0.9508	1.0373	1.1237	1.2102	1.2966	1.3830	1.4695	1.5559	1.6424	1.7288
60	0.8694	0.9563	1.0433	1.1302	1.2172	1.3041	1.3911	1.4780	1.5650	1.6519	1.7388
65	0.8739	0.9613	1.0487	1.1361	1.2234	1.3108	1.3982	1.4856	1.5730	1.6604	1.7478
70	0.8779	0.9657	1.0535	1.1413	1.2291	1.3169	1.4047	1.4925	1.5803	1.6681	1.7559
75	0.8815	0.9697	1.0579	1.1460	1.2342	1.3223	1.4105	1.4987	1.5868	1.6750	1.7631
80	0.8849	0.9734	1.0619	1.1504	1.2388	1.3273	1.4158	1.5043	1.5928	1.6813	1.7698
85	0.8879	0.9767	1.0655	1.1543	1.2431	1.3319	1.4207	1.5095	1.5983	1.6871	1.7759
90	0.8907	0.9798	1.0689	1.1580	1.2471	1.3361	1.4252	1.5143	1.6034	1.6924	1.7815
95	0.8933	0.9827	1.0720	1.1614	1.2507	1.3400	1.4294	1.5187	1.6081	1.6974	1.7867
100	0.8958	0.9853	1.0749	1.1645	1.2541	1.3437	1.4332	1.5228	1.6124	1.7020	1.7916

where Z_α is the upper 100α percentile of the standard normal distribution. Table 4 lists the lower confidence bounds of S_{pk}^T for the estimates $\hat{S}_{pk}^T = 1.0(0.1)2.0$, $n = 5(5)100$, and the confidence level $1 - \alpha = 95\%$.

We remark that the lower confidence bound obtained according to our methodology is identical for processes of various dimensions, since the reliable lower confidence bound (based on the maximum $Var(\hat{S}_{pk}^T)$ for a fixed S_{pk}^T) is a function of $\hat{S}_{pk}^T$ and n only.

3.2 *Sample size determination*

The sample size determination is important, as it directly relates to the cost of the data collection plan. Applying the parameter setting as in computing the lower confidence bound, the sample size required for a given estimation precision R can be expressed as:

$$n \geq \frac{1}{2}\left(\frac{Z_\alpha}{\frac{1}{R} - 1}\right)^2,$$

where $R = S_{pk}^{T(L)}/\hat{S}_{pk}^T$. Note that no matter what value of $\hat{S}_{pk}^T$ is, the ratio of the lower bound $S_{pk}^{T(L)}$ to the estimate (value of $\hat{S}_{pk}^T$) is identical, because of the parameters setting in our methodology. Thus, given a desired estimation precision R, a significance level α and the parameters setting as previously, the required sample size n can be obtained. Table 5 shows the sample size n required for desired estimation precisions $R = 0.75(0.01)0.95$ and significant levels $\alpha = 0.1$, 0.05, 0.025, 0.01. For example, if the desired estimation precision

Table 5. Sample size required for specific estimation precision.

R	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$
0.75	8	13	18	25
0.76	9	14	20	28
0.77	10	16	22	31
0.78	11	18	25	35
0.79	12	20	28	39
0.80	14	22	31	44
0.81	15	25	35	50
0.82	18	29	40	57
0.83	20	33	46	65
0.84	23	38	53	75
0.85	27	44	62	87
0.86	31	52	73	103
0.87	37	61	87	122
0.88	45	73	104	146
0.89	54	89	126	178
0.90	67	110	156	220
0.91	84	139	197	277
0.92	109	179	255	358
0.93	145	239	340	478
0.94	202	333	472	665
0.95	297	489	694	977

R is set to 0.80, then with $\alpha = 0.05$ the sample size required is 22. Thus, if a sample estimate $\hat{S}_{pk}^{T} = 1.5$ is obtained, then we can conclude that the actual value of S_{pk}^{T} would be no less than $1.5 \times 0.80 = 1.2$, or equally, be 95% confident that the production yield would be greater 0.999681783.

We further consider how large a sample size, n, should be collected to ensure that the sample estimator $\hat{S}_{pk}^{T}$ is close to the actual capability performance S_{pk}^{T} within a designated accuracy ε. The word 'close' here means that the occurring probability is greater than a desired level $1 - \alpha$, say 0.95. That is:

$$
\begin{aligned}
&\Pr\left\{\left|\hat{S}_{pk}^{T} - S_{pk}^{T}\right| \le \varepsilon\right\} \ge 1 - \alpha \\
&\Rightarrow \Pr\left\{\frac{\hat{S}_{pk}^{T} - S_{pk}^{T}}{\sqrt{Var\left(\hat{S}_{pk}^{T}\right)}} \le \frac{\varepsilon}{\sqrt{Var\left(\hat{S}_{pk}^{T}\right)}}\right\} \ge 1 - \frac{\alpha}{2} \\
&\Rightarrow \frac{\varepsilon}{\sqrt{Var\left(\hat{S}_{pk}^{T}\right)}} \ge \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \\
&\Rightarrow Var\left(\hat{S}_{pk}^{T}\right) \le \frac{\varepsilon^2}{\left[\Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\right]^2} \\
&\Rightarrow n \ge \frac{(S_{pk}^{T})^2}{2} \times \frac{\left[\Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\right]^2}{\varepsilon^2}.
\end{aligned}
$$

Table 6. Sample sizes required to converge.

		Designated accuracy, ε									
S_{pk}^T	α	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
1.00	0.01	33175	8294	3687	2074	1327	922	678	519	410	332
	0.025	25120	6280	2792	1570	1005	698	513	393	311	252
	0.05	19208	4802	2135	1201	769	534	392	301	238	193
1.33	0.01	58683	14671	6521	3668	2348	1631	1198	917	725	587
	0.025	44434	11109	4938	2778	1778	1235	907	695	549	445
	0.05	33976	8494	3776	2124	1360	944	694	531	420	340
1.50	0.01	74643	18661	8294	4666	2986	2074	1524	1167	922	747
	0.025	56519	14130	6280	3533	2261	1570	1154	884	698	566
	0.05	43217	10805	4802	2702	1729	1201	882	676	534	433
1.67	0.01	92521	23131	10281	5783	3701	2571	1889	1446	1143	926
	0.025	70056	17514	7784	4379	2803	1946	1430	1095	865	701
	0.05	53568	13392	5952	3348	2143	1488	1094	837	662	536
2.00	0.01	132698	33175	14745	8294	5308	3687	2709	2074	1639	1327
	0.025	100478	25120	11165	6280	4020	2792	2051	1570	1241	1005
	0.05	76830	19208	8537	4802	3074	2135	1568	1201	949	769

Table 6 displays the required sample sizes n for $\hat{S}_{pk}^T$ to converge to the actual S_{pk}^T within a designated accuracy ε, with $\varepsilon = 0.01(0.01)0.10$. For example, for $S_{pk}^T = 1.0$ and $\alpha = 0.05$, a sample size of $n \geq 193$ ensures that the difference between the sample estimator $\hat{S}_{pk}^T$ and the actual performance S_{pk}^T is smaller than 0.1. This convergence investigated is not for practical purpose, but to illustrate the behaviour and the rate of convergence for the normal approximation.

4. Testing on the production yield

From a customer's view point, to determine whether or not a production process is capable is the main work. To deal with this, we consider the following hypothesis testing:

H_0: $Yield \leq C$ (process is incapable),
H_1: $Yield > C$ (process is capable), where C is a designated constant.

Since the production yield for a process with multiple characteristics has a one-to-one relationship with the yield index S_{pk}^T, $Yield = 2\Phi(3S_{pk}^T) - 1$, testing the above hypotheses is equal to testing:

H_0: $S_{pk}^T \leq S$,
H_1: $S_{pk}^T > S$, where $S = \frac{1}{3}\Phi^{-1}[\frac{1}{2}(C+1)]$.

Based on the sampling distribution of S_{pk}^T and a desired confidence level $1-\alpha$, the decision rule for this hypothesis testing should be to reject H_0 if the testing statistic $\hat{S}_{pk}^T > c_0$, where c_0 is the critical value that satisfies:

$$\Pr\left\{\hat{S}_{pk}^T \geq c_0 | H_0 : S_{pk}^T \leq S\right\} \leq \alpha.$$

Note that the smaller the value of S_{pk}^T, the larger the probability of $\hat{S}_{pk}^T > c_0$. Thus, the above probability should be calculated with H_0: $S_{pk}^T = S$. Also, since $\hat{S}_{pk}^T$ is asymptotically normal distributed with mean S_{pk}^T and for a given fixed S_{pk}^T the largest $Var(\hat{S}_{pk}^T)$, $(S_{pk}^T)^2/2n$, is identical, the critical value c_0 is also a function of S_{pk}^T only, which can be expressed as:

$$\begin{aligned} c_0 &= S_{pk}^T + Z_\alpha \sqrt{Var(\hat{S}_{pk}^T)} \\ &= S + Z_\alpha \frac{S}{\sqrt{2n}}, \end{aligned}$$

where Z_α is the upper 100α percentile of the standard normal distribution. Table 7 shows the critical values for testing the production yield for a process with multiple characteristics by the yield index S_{pk}^T, covering the most commonly used performance requirements $S_{pk}^T = 1.00$, 1.33, 1.50, 1.67 and 2.00.

5. A real world application

To illustrate how to apply the proposed method to estimate and test production yield of a multivariate process, we implement a real-world application presented in the article of Wang and Chen (1998). The data collected from a plastics manufacturing industrial has three interesting characteristics: depth (D), length (L), and width (W). Specification limits and target value (*LSL*, *T*, *USL*) for D, L, W are set to (2.1, 2.2, 2.3), (304.5, 304.8, 305.1), and (304.5, 304.8, 305.1), respectively. The sample mean vector and sample covariance matrix calculated from 50 collected observations were:

$$\hat{\mu} = \begin{pmatrix} 2.16 \\ 304.72 \\ 304.77 \end{pmatrix} \text{ and } \hat{\Sigma} = \begin{pmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0071 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{pmatrix}.$$

By performing the principal component analysis (PCA), the eigenvectors u_1, u_2, u_3 and eigenvalues λ_1, λ_2, λ_3 of the sample covariance matrix can be obtained as in Table 8. In the article of Wang and Chen (1998), the principal components (PCs) were calculated to estimate the multivariate capability indices MC_p, MC_{pk}, MC_{pm}, and MC_{pmk}. However, none of the indices can provide an adequate connection to the production yield. Next, we would employ the proposed methodology by the yield index S_{pk}^T to the estimation and testing on the production yield.

Let U be the matrix of (u_1, u_2, u_3), $Y = (D, L, W)^T$, and $X = U^T Y = (X_1, X_2, X_3)^T$. Then, according to PCA, X_i is the ith principal component with variance $\tilde{\lambda}_i$, and all the X_i's are independent. Based on such a transformation, the specification limits and target value (*LSL*, *T*, *USL*) for X_1, X_2, X_3 become (368.14092, 368.55476, 368.9686), (−216.82815, −216.6969, −216.56565), and (55.00047, 55.0394, 55.07833), respectively. Consequently, the sample mean vector of $\underset{\sim}{X}$ becomes:

$$\begin{aligned} \bar{X} &= \begin{pmatrix} 0.5222 & 0.5824 & 0.6230 \\ 0.8385 & -0.2172 & -0.4998 \\ -0.1558 & 0.7834 & -0.6017 \end{pmatrix} \begin{pmatrix} 2.16 \\ 304.72 \\ 304.77 \end{pmatrix} \\ &= \begin{pmatrix} 368.46859 \\ -216.69807 \\ 55.001011 \end{pmatrix}. \end{aligned}$$

Table 7. Critical values for testing production yield with $\alpha = 0.05, 0.025, 0.01$.

	S_{pk}^T														
	1.00			1.33			1.50			1.67			2.00		
n	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
5	1.5202	1.6198	1.7357	2.0218	2.1543	2.3084	2.2802	2.4297	2.6035	2.5387	2.7051	3.0403	3.0403	3.2396	3.4713
10	1.3678	1.4383	1.5202	1.8192	1.9129	2.0219	2.0517	2.1574	2.2803	2.2842	2.4019	2.7356	2.7356	2.8765	3.0404
15	1.3003	1.3578	1.4247	1.7294	1.8059	1.8949	1.9505	2.0368	2.1371	2.1715	2.2676	2.6006	2.6006	2.7157	2.8495
20	1.2601	1.3099	1.3678	1.6759	1.7422	1.8192	1.8901	1.9649	2.0517	2.1043	2.1875	2.5202	2.5202	2.6198	2.7357
25	1.2326	1.2772	1.3290	1.6394	1.6987	1.7676	1.8489	1.9158	1.9935	2.0585	2.1329	2.4652	2.4652	2.5544	2.6580
30	1.2124	1.2530	1.3003	1.6124	1.6665	1.7294	1.8185	1.8796	1.9505	2.0246	2.0926	2.4247	2.4247	2.5061	2.6007
35	1.1966	1.2343	1.2781	1.5915	1.6416	1.6998	1.7949	1.8514	1.9171	1.9983	2.0612	2.3932	2.3932	2.4685	2.5561
40	1.1839	1.2191	1.2601	1.5746	1.6214	1.6759	1.7759	1.8287	1.8901	1.9771	2.0360	2.3678	2.3678	2.4383	2.5202
45	1.1734	1.2066	1.2452	1.5606	1.6048	1.6561	1.7601	1.8099	1.8678	1.9596	2.0150	2.3468	2.3468	2.4132	2.4904
50	1.1645	1.1960	1.2326	1.5488	1.5907	1.6394	1.7467	1.7940	1.8490	1.9447	1.9973	2.3290	2.3290	2.3920	2.4653
55	1.1568	1.1869	1.2218	1.5386	1.5785	1.6250	1.7353	1.7803	1.8327	1.9319	1.9821	2.3137	2.3137	2.3738	2.4436
60	1.1502	1.1789	1.2124	1.5297	1.5680	1.6125	1.7252	1.7684	1.8186	1.9208	1.9688	2.3003	2.3003	2.3578	2.4247
65	1.1443	1.1719	1.2040	1.5219	1.5586	1.6014	1.7164	1.7579	1.8061	1.9109	1.9571	2.2885	2.2885	2.3438	2.4081
70	1.1390	1.1657	1.1966	1.5149	1.5503	1.5915	1.7085	1.7485	1.7949	1.9022	1.9466	2.2780	2.2780	2.3313	2.3932
75	1.1343	1.1600	1.1900	1.5086	1.5428	1.5826	1.7015	1.7401	1.7849	1.8943	1.9373	2.2686	2.2686	2.3201	2.3799
80	1.1300	1.1550	1.1839	1.5030	1.5361	1.5746	1.6951	1.7324	1.7759	1.8872	1.9288	2.2601	2.2601	2.3099	2.3678
85	1.1262	1.1503	1.1784	1.4978	1.5299	1.5673	1.6892	1.7255	1.7676	1.8807	1.9210	2.2523	2.2523	2.3006	2.3569
90	1.1226	1.1461	1.1734	1.4931	1.5243	1.5606	1.6839	1.7191	1.7601	1.8747	1.9140	2.2452	2.2452	2.2922	2.3468
95	1.1193	1.1422	1.1688	1.4887	1.5191	1.5545	1.6790	1.7133	1.7532	1.8693	1.9075	2.2387	2.2387	2.2844	2.3375
100	1.1163	1.1386	1.1645	1.4847	1.5143	1.5488	1.6745	1.7079	1.7468	1.8642	1.9015	2.2326	2.2326	2.2772	2.3290

Table 8. Eigenvectors and eigenvalues of the sample covariance.

	Eigenvector		
Characteristics	u_1	u_2	u_3
D	0.5222	0.8385	−0.1558
L	0.5824	−0.2172	0.7834
W	0.6230	−0.4998	−0.6017
Eigenvalue, λ_i	0.0037	0.0015	0.0006
% explained	63.93	25.11	10.96

Table 9. Correlation coefficients between the characteristics and principal components.

	Principal components		
Characteristics	X_1	X_2	X_3
D	0.6932	0.7087	−0.0833
L	0.4204	−0.0998	0.2277
W	0.8474	−0.4328	−0.3296

The correlation between the ith characteristic and the jth principal component is given by:

$$\rho_{ij} = u_{ij}\sqrt{\frac{\lambda_j}{s_{ii}}},$$

where u_{ij} denotes the coefficient for the ith characteristic in the jth principal component, λ_j denotes the eigenvalue associated with the principal component, and s_{ii} is the variance of the ith characteristic. The correlations between characteristics and principal components are tabulated in Table 9. As we can see, at least one of the absolute correlation coefficients between the first two principal components and the original characteristics is greater than 0.7, which is generally identified as highly correlated. The absolute value of correlations between the third principal component and the three characteristics are all less than 0.33, which indicates that the third principal component does not correlate well with the three original characteristics. Thus, the first two principal components are used to evaluate the capability performance of the three-dimensional process.

Applying the formulae of $\hat{S}_{pki}$ and $\hat{S}^T_{pk}$:

$$\hat{S}_{pki} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - \bar{X}_i}{s_i}\right) + \frac{1}{2}\Phi\left(\frac{\bar{X}_i - LSL}{s_i}\right)\right\}$$

$$\hat{S}^T_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{i=1}^{v}\left(2\Phi(3\hat{S}_{pki}) - 1\right) + 1\right]\right\},$$

we can calculate that:

$$\hat{S}_{pk1} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{368.9686 - 368.46859}{\sqrt{0.0037}}\right) + \frac{1}{2}\Phi\left(\frac{368.46859 - 368.14092}{\sqrt{0.0037}}\right)\right\}$$
$$= 1.8367$$

$$\hat{S}_{pk2} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{-216.56565 + 216.69807}{\sqrt{0.0015}}\right) + \frac{1}{2}\Phi\left(\frac{-216.69807 + 216.82815}{\sqrt{0.0015}}\right)\right\}$$
$$= 1.1291$$
$$\hat{S}^T_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{i=1}^{v}\left(2\Phi(3\hat{S}_{pki}) - 1\right) + 1\right]\right\}$$
$$= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}[(2\Phi(3 \times 1.8367) - 1) \times (2\Phi(3 \times 1.1291) - 1) + 1]\right\}$$
$$= 1.1291.$$

Thus, the 95% lower confidence bound of S^T_{pk} can be calculated as:

$$S^{T(L)}_{pk} = \frac{\hat{S}^T_{pk}}{1 + Z_\alpha/\sqrt{2n}}$$
$$= \frac{1.1291}{1 + 1.645/\sqrt{2 \times 50}}$$
$$= 0.9696.$$

Then, we can conclude that the true value of the yield index S^T_{pk} would be no less than 0.9696, or equally, be 95% confident that the production yield would be greater than 0.9964.

To test the yield performance, H_0: $S^T_{pk} \leq 1.0$ versus H_1: $S^T_{pk} > 1.0$, one could calculate the critical value c_0:

$$c_0 = S + Z_\alpha \frac{S}{\sqrt{2n}}$$
$$= 1.0 + 1.645\frac{1.0}{\sqrt{2 \times 50}}$$
$$= 1.1645.$$

Since the estimated S^T_{pk}, $\hat{S}^T_{pk} = 1.1291$, is smaller than the critical value, $c_0 = 1.1645$, one could conclude that the sample does not provide sufficient evidence to support that the process capability performance S^T_{pk} is larger than 1.0, or equally, be 95% confident that the production yield is greater than 0.9973.

6. Conclusion

Production yield is the most common and standard criterion for evaluating the performance of products manufactured. Numerous multivariate capability indices have been proposed to measure the performance of processes with multiple characteristics. However, few of them can be applied to measure the production yield for processes with multiple characteristics. The capability index S^T_{pk} proposed by Chen *et al.* (2003) provides an exact yield measure for multinormal processes with independent characteristics. With the aids of principal component analysis (PCA) and some normalising methods, e.g., Box-and-Cox transformation, the yield index S^T_{pk} can be applied to measure the production yield of multivariate processes.

Assuring the production yield for processes with multiple characteristics to meet the requirement is important. So, the proposition of a technique of assuring production yield is necessary in this field. In this paper, statistical inferences on the capability index S_{pk}^{T} have been considered. To make reliable decisions, we investigated the effect of all the parameters on the sampling distribution of S_{pk}^{T}, and obtained the lower confidence bounds and critical values which can ensure that the risk of making incorrect decisions will be smaller than the significant level α. A real-world application, with data investigated by Wang and Chen (1998), was executed to illustrate the applicability of the proposed approach. The proposed methodology can be used to make a reliable decision to determine whether the production meets the yield requirement, and bridges the gap between the theoretical development and factory application.

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