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D-Stability Bound Analysis for Discrete Multiparameter Singularly Perturbed Systems

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***Abstract*—The D-stability (i.e., the stability in the sense that all the poles of a system are lying inside the disk $D(\alpha, r)$) problem for discrete multiparameter singularly perturbed systems is considered in this brief. A two-stage method is first developed to analyze the stability relationship between the discrete multiparameter singularly perturbed systems and their corresponding reduced systems. An upper bound of the singular perturbation parameters is then derived such that the D-stability of the reduced systems implies that of the original systems, provided that the singular perturbation parameters are small enough to be within this bound. This fact enables us to investigate D-stability of the original systems by establishing that of their corresponding reduced systems.**

***Index Terms*—D-stability, singular perturbation parameters.**

I. Introduction

The problem of pole assignment in linear system theory has been discussed by many authors and solved in many ways. However, locations of poles vary and they cannot be fixed due to parametric uncertainties, e.g., identification errors, ageing of devices, variation of operating points, etc.. Consequently, placing all poles in a desired region rather than choosing an exact assignment may be more satisfactory in practical applications. A well-known desired region for the discrete systems is a disk $D(\alpha, r)$ centered at $(\alpha, 0)$ with radius r, in which $|\alpha| + r < 1$. The assignment of all the poles of a system in the specified disk $D(\alpha, r)$ shown in Fig. 1 is referred to as the D-pole placement problem.

Singularly perturbed systems have been extensively studied in recent years; see Kokotovic *et al.* [1] and the references therein. The work on the discrete singularly perturbed systems with multiple parameters, which dealt with the multitime scales in discrete dynamic systems can be found in Mahmoud [2]. A key to the analysis of singularly perturbed systems lies in the construction of reduced systems. It is noted that the approximation of original systems via the corresponding reduced systems is valid only when the singular perturbation parameters of these systems are sufficiently small. Therefore, it is imperative to find an upper bound of the singular perturbation parameters such that the stability of the original systems can be investigated by establishing that of their corresponding reduced system, provided that the singular perturbation parameters are small enough to be within this bound. The upper bound of the singular perturbation parameter for the asymptotic stability analysis of discrete single-parameter singularly perturbed systems was discussed by Li and Li [3]. However, since there exist multiple parameters in most singularly perturbed dynamic systems, the analysis of the upper bound of the singular perturbation parameter as described in Li and Li [3] is thus impractical for the real control systems. Although the continuous multiparameter singularly perturbed systems have been investigated by many authors, see Khalil and Kokotovic [4] and the references therein, with recent developments in microprocessor technology, however, it becomes even more important to focus the analysis and design of the feedback control systems by using digital equipment. This in turn will promote the study of discrete multiparameter singularly perturbed systems. It is seen that the backward Euler discretization of the continuous multiparameter singularly perturbed systems can provide us with the discrete multiparameter singularly perturbed system (see [5]). On the other hand, due to the presence of parametric uncertainties in practical systems, it is imperative to consider the D-stability problem of the discrete multiparameter singularly perturbed systems. A literature search indicates that the D-stability problem of finding an upper bound of singular perturbation parameters for the discrete singularly perturbed systems with multiple parameters remains unresolved.

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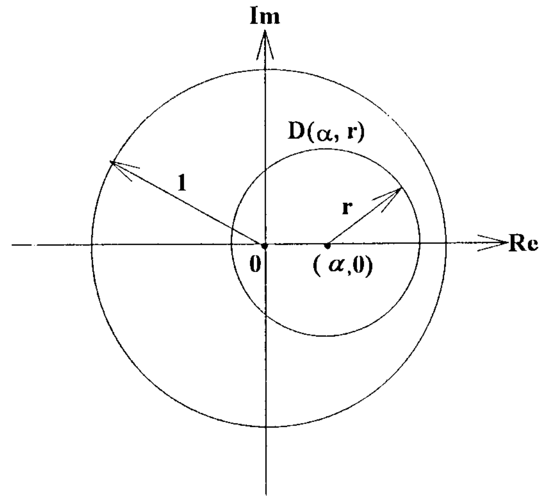

Fig. 1. A specified disk $D(\alpha, r)$.

Hence, in this brief, research on time-scale modeling is extended to include the discrete multiparameter singularly perturbed systems. An algorithm is proposed to find an upper bound of the singular perturbation parameters for the D-stability analysis of the discrete singularly perturbed systems with multiple parameters. If the singular perturbation parameters are small enough to be within this bound, the D-stability of the reduced systems can imply that of the original systems.

II. D-Stability Analysis

Consider the discrete system:

$$x(k+1) = Ax(k) \tag{1}$$

in which $x(k) \in R^n$ and A is a constant matrix with appropriate dimensions.

Definition 1: The system (1) is said to be $D(\alpha, r)$-stable if all the poles of the system (1) are within the specific disk $D(\alpha, r)$ centered at $(\alpha, 0)$ with radius r, in which $|\alpha| + r < 1$ (see Fig. 1).

We now present a $D(\alpha, r)$-stability criterion for the system (1) as follows.

Lemma 1: [6] If the following inequality (2) holds, all the poles of the system (1) are within the specified disk $D(\alpha, r)$:

$$||A - \alpha I|| < r. \tag{2}$$

III. Problem Formulation

Consider the following discrete multiparameter singularly perturbed system which is referred as the R-model [7]:

$$\begin{aligned} x(k+1) &= A_0 x(k) + A_{01} z_1(k) + \cdots + A_{0N} Z_N(k) \\ z_1(k+1) &= \varepsilon_1 A_{10} x(k) + \varepsilon_1 A_{11} z_1(k) + \cdots + \varepsilon_1 A_{1N} z_N(k) \\ \vdots \quad & \quad \vdots \quad\quad \vdots \quad\quad\quad \vdots \\ z_N(k+1) &= \varepsilon_N A_{N0} x(k) + \varepsilon_N A_{N1} z_1(k) + \cdots + \varepsilon_N A_{NN} z_N(k) \end{aligned} \tag{3}$$

in which A_0 is assumed to be nonsingular. System (3) can be obtained from the slow sampling rate model as a result of discretization or sampled-data control of the singularly perturbed continuous-time systems [3]. The small positive scalars $\varepsilon_1, \cdots, \varepsilon_N$ are N singular perturbation parameters which often occur naturally due to the presence of small parameters in the various physical systems, e.g., in power system model the singular perturbation parameters can represent machine reactances or transients in voltage regulators, in the industrial control systems they may represent time constants of drives and actuators and in the nuclear reactor models they are due to fast neutrons, etc. In many real systems, these singular perturbation parameters are of the same order and do not allow the multitime scale assumption [8]. Accordingly, the ratios of $\varepsilon_1, \cdots, \varepsilon_N$ are assumed to be bounded by some positive constants m_{ij}, M_{ij}:

$$m_{ij} \le \frac{\varepsilon_i}{\varepsilon_j} \le M_{ij}, \qquad i, j = 1, \cdots, N \tag{4}$$

that is, the possible values of $\varepsilon = (\varepsilon_1 \cdots \varepsilon_N)^T$ are restricted to a cone $H \subset R^N$. Such an assumption allows that the convergence results are sought as $||\varepsilon|| \to 0$ in H to guarantee that they hold for all sufficiently small $\varepsilon \in H$ [2]. The system (3) can be rewritten as follows:

$$\begin{aligned} x(k+1) &= A_{0s} x(k) + A_{0f} z(k), \qquad x(0) = x_0, \\ z(k+1) &= \mu(\varepsilon) G^{-1}(\varepsilon) A_{Ns} x(k) + \mu(\varepsilon) G^{-1}(\varepsilon) A_{Nf} z_{(k)}, \\ z(0) &= z_0 \end{aligned} \tag{5}$$

where

$$\begin{aligned} z(k) &= (z_1^T(k) \cdots z_N^T(k))^T \\ A_{0s} &= A_0, A_{0f} = [A_{01} \cdots A_{0N}] \\ A_{Ns} &= \begin{bmatrix} A_{10} \\ \vdots \\ A_{N0} \end{bmatrix}, \quad A_{Nf} = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \vdots & \vdots \\ A_{N1} & \vdots & A_{NN} \end{bmatrix}, \\ G(\varepsilon) &\equiv \text{block diag} \left[\left(\frac{\mu(\varepsilon)}{\varepsilon_1} \right) I_1 \cdots \left(\frac{\mu(\varepsilon)}{\varepsilon_N} \right) I_N \right] \quad \text{and} \\ \mu(\varepsilon) &\equiv ||\varepsilon||. \end{aligned} \tag{6}$$

In view of (4), the matrix $G^{-1}(\varepsilon)$ in (5) is bounded for all $\varepsilon \in H, m_i \le (\varepsilon_i/\mu) \le M_i$, where m_i, M_i depend on m_{ij}, M_{ij}. Hence, the system (5) becomes a single-parameter form except that G and μ depend on ε and the new singular perturbation parameter is μ instead of ε.

Remark 1: For convenience, the symbols $G(\varepsilon)$ and $\mu(\varepsilon)$ are replaced with G and μ, respectively, in the remainder of this brief.

The zero-order approximation of the system (5) can be written as [9]:

$$x_s(k+1) = A_{0s} x_s(k), \tag{7a}$$

$$\begin{aligned} z_f(k+1) &= \mu [G^{-1} A_{Nf} - G^{-1} A_{Ns} A_{0s}^{-1} A_{0f}] z_f(k) \\ &= \mu G^{-1} [A_{Nf} - A_{Ns} A_{0s}^{-1} A_{0f}] z_f(k) \end{aligned} \tag{7b}$$

with the initial conditions

$$x_s(0) = x_0 + A_{0s}^{-1} A_{0f} z_0, \quad z_f(0) = z_0 \tag{8}$$

and the approximate solution of system (5) is

$$\begin{aligned} x(k, \mu) &= x_s(k) - A_{0s}^{-1} A_{0f} z_f(k) + O(\mu), \\ z(k, \mu) &= z_f(k) + O(\mu). \end{aligned} \tag{9}$$

Here, x_s is the slow state and z_f is the fast state; the systems (7a) and (7b) are called the slow and fast subsystems of the original system (5), respectively. In view of (9), we can see that the response of the original system (5) is dominated by the dynamics of the slow and fast states. Hence, if the slow and fast subsystems are both D-stable, then so is the original system for sufficiently small singular perturbation parameters. The corresponding theoretical consequence is stated in the following theorem.

Lemma 2: If the slow and fast subsystems (7a) and (7b) are both $D(\alpha, r)$-stable, then the original system (5) is also $D(\alpha, r)$-stable for sufficiently small μ.

Proof: Following the similar procedure as that in Lemma 1 of [2], system (5) can be transformed into

$$\begin{aligned} \begin{bmatrix} \overline{x}(k+1) \\ \overline{z}(k+1) \end{bmatrix} &= \begin{bmatrix} A_{0s} + O(\mu) & 0 \\ 0 & \mu G^{-1}(A_{Nf} - A_{Ns} A_{0s}^{-1} A_{0f}) + O(\mu^2) \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \overline{x}(k) \\ \overline{z}(k) \end{bmatrix}. \end{aligned} \tag{10}$$

Comparing (7) and (10), it is obvious that if the slow and fast subsystems (7a) and (7b) are both $D(\alpha, r)$-stable (i.e., all the eigenvalues of A_{0s} and $\mu G^{-1}(A_{Nf} - A_{Ns} A_{0s}^{-1} A_{0f})$ are within the disk $D(\alpha, r)$), then the original system (5) is $D(\alpha, r)$-stable for sufficiently small μ.

Remark 2: The significance of the singular perturbation parameters lies in their effects on the deviation of the original system from its corresponding model—the reduced system. Additionally, the deviation can be improved as the singular perturbation parameters decrease. In this study, the reduced system is a valid model only for certain values of μ, in which both the original system and the reduced system are $D(\alpha, r)$-stable. In the next section, an upper bound of μ is derived such that the validity of the reduced system can be assured if the singular perturbation parameters are within this bound.

IV. Finding an Upper Bound μ^*

The main purpose of this brief, which will be presented in this section, is to use an algorithm to find an upper bound μ^* of the singular perturbation parameters for the $D(\alpha, r)$-stability analysis. Before proceeding to derive the main result, some useful lemmas are given in the following.

Lemma 3: [10] Let a matrix $E(z) \in \Re_\infty^{m \times n}$ with $\Re_\infty^{m \times n}$ denoting the set of $m \times n$ matrices whose elements are proper stable rational functions, then

$$\sup_{z \in \Omega} ||E(z)|| = \sup_{|z| \ge 1} ||E(z)|| = \sup_{\theta \in [0, 2\pi]} ||E(e^{j\theta})||$$

where

$$\Omega = \{z = re^{j\theta}, \theta \in [0, 2\pi], |r| \geq 1\}.$$

Since $E(z)$ is analytic for $z \in \Omega$, this norm is well defined.

Lemma 4: [10] If $E(z) \in \Re_{\infty}^{n\times n}$ and $||E(z)|| < 1, \forall |z| \geq 1$, then $[I - E(z)]^{-1} \in \Re_{\infty}^{n\times n}$.

After reviewing the above lemmas, we are in the position to derive the main result.

Theorem 1: Given the original discrete system (5) and the reduced system (7), in which the slow subsystem (7a) is assumed to be $D(\alpha, r)$-stable (i.e., all the eigenvalues of A_{0s} are within the disk $D(\alpha, r)), D(\alpha, r)$-stability (with $r > |\alpha|$) of the reduced system (7) can imply that of the original system (5) for all $\mu \in (0, \mu^*)$ where μ^* are determined according to the following steps:

i) find the supreme value of μ, called μ_1^*, such that

$$||\mu(A_{Nf} - A_{Ns}A_{0s}^{-1}A_{0f})|| < r - |\alpha| \tag{11}$$

ii) compute (12) shown at the bottom of the page.

iii) choose

$$\mu^* = \min(\mu_1^*, \mu_2^*). \tag{13}$$

Proof: i) Considering the matrix in (7b), we have

$$\begin{aligned} &||\mu G^{-1}(A_{Nf} - A_{Ns}A_{0s}^{-1}A_{0f}) - \alpha I|| \\ &\quad \leq ||G^{-1}|| \, ||\mu(A_{Nf} - A_{Ns}A_{0s}^{-1}A_{0f})|| + |\alpha| \\ &\quad \leq ||\mu(A_{Nf} - A_{Ns}A_{0s}^{-1}A_{0f})|| + |\alpha|. \\ &\qquad (\because \; ||G^{-1}|| \leq 1). \end{aligned}$$

Thus, if μ^*1 is chosen such that

$$||\mu(A_{Nf} - A_{Ns}A_{0s}^{-1}A_{0f}|| < r - |\alpha|, \quad \forall \mu \in (0, \mu_1^*),$$

the following inequality is obtained

$$||\mu G^{-1}(A_{Nf} - A_{Ns}A_{0s}^{-1}A_{0f}) - \alpha I|| < r \quad \forall \mu \in (0, \mu_1^*). \tag{14}$$

Therefore, according to Lemma 1 and the assumption of $D(\alpha, r)$-stability of the slow subsystem (7a), we conclude that the reduced system (7) is $D(\alpha, r)$-stable for all $\mu \in (0, \mu_1^*)$.

ii) Applying z-transform to the original system (5), yields

$$\begin{aligned} X(z) &= (zI - A_{0s})^{-1}A_{0f}Z(z) + (zI - A_{0s})^{-1}x_0 \\ Z(z) &= \Psi^{-1}(z)z_0 + \Psi^{-1}(z)\mu G^{-1}A_{Ns}(zI - A_{0s})^{-1}x_0 \end{aligned} \tag{15}$$

where

$$\Psi(z) \equiv [zI - \mu G^{-1}A_{Nf} - \mu G^{-1}A_{Ns}(zI - A_{0s})^{-1}A_{0f}].$$

Since the slow subsystem (7a) is assumed to be $D(\alpha, r)$-stable, all poles of $(zI - A_{0s})^{-1}$ are inside the disk $D(\alpha, r)$. Therefore, to let all the poles of $Z(z)$ be within the disk $D(\alpha, r)$ (and so are those of $X(z)$), we only need to find the condition which guarantees that all the poles of $\Psi^{-1}(z)$ are within the disk $D(\alpha, r)$. Moreover, since

$$\begin{aligned} &\Psi^{-1}(z) \\ &\quad = z^{-1}\{I - \mu z^{-1}G^{-1}[A_{Nf} + A_{Ns}(zI - A_{0s})^{-1}A_{0f}]\}^{-1} \\ &\quad \equiv z^{-1}[I - \phi(z)]^{-1} \end{aligned} \tag{16}$$

and the pole of the term z^{-1} in (16) is $z = 0$ which is inside the disk $D(\alpha, r)$ ($\because \; r > |\alpha|$). Consequently, if all poles of the term $[I - \phi(z)]^{-1}$ in (16) lie inside the disk $D(\alpha, r)$, then $\Psi^{-1}(z)$ has all poles lying inside the disk $D(\alpha, r)$. Let $(z - \alpha)/r$ be replaced by a variable g (i.e., $z = rg + \alpha$), then the term $[I - \phi(z)]^{-1}$ becomes $[I - \phi_g(g)]^{-1}$ where

$$\begin{aligned} \phi_g(g) \equiv \mu(rg + \alpha)^{-1}G^{-1}\{&A_{Nf} + A_{Ns}[(rg + \alpha)I \\ &- A_{0s}]^{-1}A_{0f}\}. \end{aligned}$$

It is obvious that $\phi_g(g) \in \Re_{\infty}^{n\times n}$. Furthermore, taking norm on $\phi_g(g)$, we have

$$\begin{aligned} ||\phi_g(g)|| &\leq \mu||G^{-1}|| \, ||(rg + \alpha)^{-1}\{A_{Nf} + A_{Ns}[(rg + \alpha)I \\ &\qquad - A_{0s}]^{-1}A_{0f}\}|| \\ &\leq \mu||(rg + \alpha)^{-1}\{A_{Nf} + A_{Ns}[(rg + \alpha)I \\ &\qquad - A_{0s}]^{-1}A_{0f}\}|| \end{aligned} \tag{17}$$

($\because \; ||G^{-1}|| \leq 1$). It can be seen from (17) that if (see (18) at the bottom of the page then $||\phi_g(g)|| < 1, \forall |g| \geq 1$. Consequently, according to Lemma 4, we have $[I - \phi_g(g)]^{-1} \in \Re_{\infty}^{n\times n}$ Subsequently, based on Lemma 3, the condition (18) is quivalent to (19), shown at the bottom of the page.

In other words, if $\mu < \mu_2^*$, then $[I - \phi_g(g)]^{-1} \in \Re_{\infty}^{n\times n}$ and then all poles of the term $[I - \phi(z)]^{-1}$ in (16) lie inside the disk $D(\alpha, r)$. Therefore, the original system (5) is $D(\alpha, r)$-stable.

iii) The smaller of the two values μ_1^* and μ_2^* is chosen such that the μ-bound can satisfy the $D(\alpha, r)$-stability criteria. □

Remark 3: In principle, any norm can be used in our results. However, the choice of norm affects the conservatism of the bound μ^*. As there is no explicit information to indicate the conservatism of the bound μ^* obtained by using various norms, the norm which is easier to compute is thus first used. In some cases, however, resorting to other norms to obtain a less conservative upper bound may be desirable. In other words, the choice of norm depends not only on the convenience of computation but also on the conservatism of the bound μ^*.

V. Example

In this section, an example of a multiparameter singularly perturbed system is given to illustrate how to find an upper bound of the singular perturbation parameters, μ^*, such that $D(\alpha, r)$-stability

$$\mu_2^* = \frac{1}{\sup_{\theta \in [0, 2\pi]} ||(re^{j\theta} + \alpha)\{A_{Nf} + A_{Ns}[(re^{j\theta} + \alpha)I - A_{0s}]^{-1}A_{0f}\}||} \tag{12}$$

$$\mu < \frac{1}{\sup_{|g| \geq 1} ||(rg + \alpha)^{-1}\{A_{Nf} + A_{Ns}[(rg + \alpha)I - A_{0s}]^{-1}A_{0f}\}||} \tag{18}$$

$$\mu < \frac{1}{\sup_{\theta \in [0, 2\pi]} ||(re^{j\theta} + \alpha)^{-1}\{A_{Nf} + A_{Ns}[(re^{j\theta} + \alpha)I - A_{0s}]^{-1}A_{0f}\}||} = \mu_2^*. \tag{19}$$

of the original system can be inferred from the analysis of their corresponding reduced systems.

Consider a discrete dynamic system with three singular perturbation parameters described by the following equations:

$$\begin{aligned} x_1(k+1) &= -0.1x_1(k) - 0.02z_1(k) + 0.06z_2(k) \\ &\quad + 0.05z_3(k) \\ x_2(k+1) &= 0.2x_1(k) - 0.03x_2(k) + 0.001z_1(k) \\ &\quad + 0.004z_2(k) + 0.003z_3(k) \\ z_1(k+1) &= 1.2\varepsilon_1 x_1(k) + \varepsilon_1 x_2(k) + 0.6\varepsilon_1 z_1(k) \\ &\quad + 0.47\varepsilon_1 z_2(k) + 1.5\varepsilon_1 z_3(k) \\ z_2(k+1) &= 0.5\varepsilon_2 x_1(k) + 0.7\varepsilon_2 x_2(k) + 0.45\varepsilon_2 z_1(k) \\ &\quad + 0.71\varepsilon_2 z_2(k) + \varepsilon_2 z_3(k) \\ z_3(k+1) &= -0.4\varepsilon_3 x_1(k) + 0.2\varepsilon_3 x_2(k) - 1.1\varepsilon_3 z_1(k) \\ &\quad + 0.8\varepsilon_3 z_2(k) + 0.25\varepsilon_3 z_3(k). \end{aligned} \tag{20}$$

According to (5), the system (20) can be rewritten as

$$\begin{aligned} x(k+1) &= A_{0s}x(k) + A_{0f}z(k) \\ z(k+1) &= \mu(\varepsilon)G^{-1}(\varepsilon)A_{Ns}x(k) + \mu(\varepsilon)G^{-1}(\varepsilon) \\ &\quad \cdot A_{Nf}z(k) \end{aligned} \tag{21}$$

where

$$\begin{aligned} x(k) &= \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \end{bmatrix} \\ \varepsilon &= (\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3)^T \\ G(\varepsilon) &= \operatorname{diag}\left[\frac{\mu(\varepsilon)}{\varepsilon_1} \quad \frac{\mu(\varepsilon)}{\varepsilon_2} \quad \frac{\mu(\varepsilon)}{\varepsilon_3}\right], \quad \mu(\varepsilon) = \|\varepsilon\| \\ A_{0s} &= \begin{bmatrix} -0.1 & 0 \\ 0.02 & -0.03 \end{bmatrix}, \\ A_{0f} &= \begin{bmatrix} -0.02 & 0.06 & 0.05 \\ 0.001 & 0.004 & 0.003 \end{bmatrix} \\ A_{Ns} &= \begin{bmatrix} 1.2 & 1 \\ 0.5 & 0.7 \\ -0.4 & 0.2 \end{bmatrix}, \\ A_{Nf} &= \begin{bmatrix} 0.6 & 0.47 & 1.5 \\ 0.45 & 0.71 & 1 \\ -1.1 & 0.8 & 0.25 \end{bmatrix}. \end{aligned} \tag{22}$$

Suppose that the time-domain specifications of the system (20) are given as follows:

a) overshoot $\leq 15\%$, or equivalently, damping ratio $\zeta \geq 0.5$; (23a)

b) rise time ≤ 8 s, or equivalently, natural frequency $\omega_n \geq 0.3125$; (23b)

c) settling time ≤ 20 s, or equivalently, all poles less than 0.8 (the sampling interval $T = 1$ s). (23c)

As these constraints (a)–(c) may be interpreted as pole locations inside the specified disk D(0.3, 0.46) [11], it is preferable to find an upper bound of μ, called μ^*, such that D(0.3, 0.46)-stability of the reduced system can imply that of the original system (20) for all $\mu \in (0, \mu^*)$.

It is obvious that A_{0s} is D(0.3, 0.46)-stable and nonsingular and hence satisfies the assumption in Theorem 1. And then we can follow the design algorithm proposed in Theorem 1 to find an upper bound of the singular perturbation parameters.

i) Let

$$A_r \equiv A_{Nf} - A_{Ns}A_{0s}^{-1}A_{0f} = \begin{bmatrix} 0.26 & 1.7233 & 2.5333 \\ 0.28 & 1.3833 & 1.5333 \\ -1.04 & 0.6667 & 0.1367 \end{bmatrix}.$$

Based on (11), we have (by using Euclidean norm)

$$\mu\|A_r\| = \mu\lambda_{\max}^{1/2}(A_r^T A_r) < r - |\alpha| = 0.46 - 0.3 = 0.16,$$

where

$$\|A_r\| = \lambda_{\max}^{1/2}(A_r^T A_r) = 3.739.$$

This implies $\mu < 0.0428$. Therefore, we choose $\mu_1^* = 0.0428$.

ii) According to (12), we have $\mu_2^* = 0.05945$.

iii) Based on (13), we choose $\mu^* = \min(\mu_1^*, \mu_2^*) = 0.0428$.

In order to verify this result, a set of singular perturbation parameters is chosen as follows:

$$\begin{aligned} &\varepsilon_1 = 0.02, \quad \varepsilon_2 = 0.025, \quad \varepsilon_3 = 0.027, \quad \text{i.e.} \\ &\varepsilon = (\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3)^T = (0.02 \quad 0.025 \quad 0.027)^T. \end{aligned}$$

From here, we can establish that $\mu = \|\varepsilon\| \cong 0.0419 < 0.0428 = \mu^*$ Using this set of singular perturbation parameters, the matrix in fast subsystem (7b) is written as

$$A_{fs} = \mu G^{-1}A_r = \begin{bmatrix} 0.0052 & 0.0345 & 0.0507 \\ 0.007 & 0.0346 & 0.0388 \\ -0.0281 & 0.018 & 0.0037 \end{bmatrix}.$$

Since the eigenvalues of the matrix A_{fs} are $0.0057 \pm j0.0202, 0.032$; that is, all the poles of the system (7b) lie inside the specific disk D(0.3, 0.46), the reduced system (7) is thus D(0.3, 0.46)-stable.

Moreover, the original system (20) can be rewritten as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ z_1(k+1) \\ z_2(k+1) \\ z_3(k+1) \end{bmatrix} = \overline{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ z_1(k) \\ z_2(k) \\ z_3(k) \end{bmatrix}$$

in which

$$\overline{A} = \begin{bmatrix} -0.1 & 0 & -0.02 & 0.06 & 0.05 \\ 0.02 & -0.03 & 0.001 & 0.004 & 0.003 \\ 0.024 & 0.02 & 0.012 & 0.0094 & 0.03 \\ 0.0125 & 0.0175 & 0.0113 & 0.0177 & 0.025 \\ -0.0108 & 0.0054 & -0.0297 & 0.0216 & 0.0068 \end{bmatrix}.$$

The eigenvalues of $\overline{A}$ are $-0.0961, -0.0364, 0.0056 \pm j0.0201, 0.0278$; indicating that all the poles of system (20) are within the disk D(0.3, 0.46). Hence, the original system (20) is also D(0.3, 0.46)-stable and then meets the time-domain specifications (23a)–(23c) as well. Furthermore, the first two eigenvalues of $\overline{A}$ are close to those of A_{0s} (matrix of the slow subsystem) and the remaining three eigenvalues are also close to those of A_{fs} (matrix of the fast subsystem). Hence, D(0.3, 0.46)-stability of the slow and fast subsystems can imply that of the original system (20). This justifies our result.

Remark 4: If 1-norm and ∞-norm are adopted, then μ^* are found to be 0.0379 and 0.0354, respectively. Although adopting 1-norm (∞-norm) will make the computation quite easy by dispensing with troublesome eigenvalue evaluation, a more conservative upper bound is obtained and thus 1-norm (∞-norm) is not considered in this example.

VI. Conclusion

In this brief, we consider a discrete multiparameter singularly perturbed system which can be transformed into a form similar to that of a discrete single-parameter singularly perturbed system. It has been shown that the D-stability of the original system can be investigated by establishing that of the reduced system, provided that the singular perturbation parameters are sufficiently small. An algorithm is then proposed for finding an upper bound of the norm of the multiparameter vector $\varepsilon = (\varepsilon_1 \cdots \varepsilon_N)^T$. Within this bound, the D-stability of the reduced system implies that of the original system.

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References

[1] P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*. New York: Academic, 1986.
[2] M. S. Mahmoud, "Stabilization of discrete systems with multiple-time scales," *IEEE Trans. Automat. Contr.*, vol. 31, pp. 159–162, 1986.
[3] T. H. S. Li and J. H. Li, "Stabilization bound of discrete two-time-scale systems," *Syst. Contr. Lett.*, vol. 18, pp. 479–489, 1992.
[4] H. K. Khalil and P. V. Kokotovic, "Control of linear systems with multiparameter singular perturbations," *Automatica*, vol. 15, pp. 197–207, 1979.
[5] E. H. Abed, "On multiparameter singularly perturbed discrete-time systems," in *Proc. IEEE Conf. Decision and Contr.*, Los Angeles, CA, Dec. 1987, pp. 2104–2105.
[6] R. J. Wang and W. J. Wang, "Disk stability and robustness for discrete-time systems with multiple time-delays," *Control-Theory and Advanced Technol.*, vol. 10, pp. 1505–1513, 1995.
[7] D. S. Naidu and A. K. Rao, *Singular Perturbation Analysis of Discrete Control Systems*. Berlin, Germany: Springer-Verlag, 1985.
[8] H. K. Khalil and P. V. Kokotovic, "D-stability and multi-parameter singular perturbation," *SIAM J. Contr. Optimiz.*, vol. 17, pp. 56–65, 1979.
[9] H. Kando and T. Iwazumi, "Stabilizing feedback controllers for singularly perturbed discrete systems," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-14, pp. 903–911, 1984.
[10] M. Vidyasagar, *Control System Synthesis*. Cambridge, MA: M.I.T. Press, 1985.
[11] S. H. Lee and T. T. Lee, "Optimal pole assignment for a discrete linear regulator with constant disturbances," *Int. J. Contr.*, vol. 45, pp. 161–168, 1987.

An Analog Scheme for Fixed Point Computation—Part I: Theory

Vivek S. Borkar and K. Soumyanath

***Abstract*— An analog system for fixed point computation is described. The system is derived from a continuous time analog of the classical over-relaxed fixed point iteration. The dynamical system is proved to converge for nonexpansive mappings under all p norms, $p \in (1, \infty]$. This extends previously established results to not necessarily differentiable maps which are nonexpansive under the ∞-norm. The system will always converge to a single fixed point in a connected set of fixed points. This allows the system to function as a complementary paradigm to energy minimization techniques for optimization in the analog domain. It is shown that the proposed technique is applicable to a large class of dynamic programming computations.**

I. Introduction

Many problems in optimization theory and numerical analysis can be posed as problems of finding a fixed point of a map F from a finite dimensional vector space into itself. Often these maps are nonexpansive with respect to a suitable norm, i.e., the distance between the images of two distinct points under F does not exceed the distance between the points themselves. Such mappings are ubiquitous and arise naturally in solving linear systems of equations, some recursive schemes for nonlinear programming, dynamic programming and certain formulations of network flow problems. The classical approach to finding fixed points under nonexpansive maps is to set up the recursion

$$x_{n+1} = F(x_n), \; n \geq 0$$

where x_0 is arbitrary.

The over-relaxed version of the above, with a relaxation parameter $\gamma \in (0, 1]$, is given by

$$x_{n+1} = (1 - \gamma)(x_n) + \gamma F(x_n).$$

It can be shown that, under certain conditions [3], both the above and its over relaxed version, converge to a fixed point x^* of $F(x)$, when F is a nonexpansive map. The above over-relaxation can be rewritten as

$$\frac{x_{n+1} - x_n}{\gamma} = F(x_n) - x_n, \; \gamma \geq 0$$

This suggests an analog or continuous time version:

$$\dot{x}\,(t) = F(x(t)) - x(t), t \geq 0. \tag{1}$$

This is the coupled dynamical system we study in this brief. A schematic of the computation element for a given x_i is shown in Fig. 1. The set $N(i)$ is the index set of the "neighbors" of component x_i, in the sense that computation of F_i requires knowledge of x_j where $j \in N(i)$.

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