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Generalized synchronization of chaotic systems by pure error dynamics and elaborate Lyapunov function

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ABSTRACT

The generalized synchronization is studied by applying pure error dynamics and elaborate Lyapunov function in this paper. Generalized synchronization can be obtained by pure error dynamics without auxiliary numerical simulation, instead of current mixed error dynamics in which master state variables and slave state variables are presented. The elaborate Lyapunov function is applied rather than the current plain square sum Lyapunov function, deeply weakening the power of Lyapunov direct method. The scheme is successfully applied to both autonomous and nonautonomous double Mathieu systems with numerical simulations.

1. Introduction

Chaos synchronization has been applied in secure communication [1,2], biological systems [3,4], and many other fields [5–25]. One of the intricate types of chaos synchronization is generalized synchronization, which has been extensively investigated recently [26–33]. The generalized synchronization is studied by applying pure error dynamics and elaborate Lyapunov function in this paper.

The pure error dynamics can be analyzed theoretically without auxiliary numerical simulation, whereas the aid of additional numerical simulation is unavoidable for current mixed error dynamics in which master state variables and slave state variables are presented, while their maximum values must be determined by simulation [34–38]. Besides, the elaborate Lyapunov function is applied rather than current plain square sum Lyapunov function, $V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^T\mathbf{e}$, which is currently used for convenience. However, the Lyapunov function can be chosen in a variety of forms for different systems. Restricting Lyapunov function to square sum makes a long river brook-like, and greatly weakens the power of Lyapunov direct method.

Based on the Lyapunov direct method [39], generalized synchronization is achieved and a systematic method of designing Lyapunov function is proposed. The technique is successfully applied to both autonomous and nonautonomous double Mathieu systems. This paper is organized as follows. In Section 2, the method of designing Lyapunov function is presented,

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and generalized synchronization is obtained. Section 3 contains the examples of autonomous and nonautonomous double Mathieu systems, and numerical simulations show the feasibility of the proposed method. Finally, the conclusions are drawn.

2. Design of Lyapunov function

Consider the master and slave nonlinear dynamic systems described by

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{2.1}$$

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) + \mathbf{u}(t, \mathbf{x}, \mathbf{y}) \tag{2.2}$$

where $\mathbf{x}, \mathbf{y} \in R^n$ are master and slave state vectors, $\mathbf{f} : R_+ \times R^n \to R^n$ is a nonlinear vector function, and $\mathbf{u} : R_+ \times R^n \times R^n \to R^n$ is controller vector.

Generalized synchronization means that there is a functional relation $\mathbf{y} = \mathbf{g}(t, \mathbf{x})$ between master and slave states as time goes to infinity, where $\mathbf{g} : R_+ \times R^n \to R^n$ is a continuously differentiable vector function. Define $\mathbf{e} = \mathbf{y} - \mathbf{g}(t, \mathbf{x})$ as the generalized synchronization error vector, and the error dynamics can be obtained:

$$\begin{aligned}
\dot{\mathbf{e}} &= \dot{\mathbf{y}} - \dot{\mathbf{g}}(t, \mathbf{x}) \\
&= \dot{\mathbf{y}} - \frac{\partial \mathbf{g}(t, \mathbf{x})}{\partial \mathbf{x}}\dot{\mathbf{x}} - \frac{\partial \mathbf{g}(t, \mathbf{x})}{\partial t} \\
&= \mathbf{f}(t, \mathbf{y}) - \frac{\partial \mathbf{g}(t, \mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(t, \mathbf{x}) - \frac{\partial \mathbf{g}(t, \mathbf{x})}{\partial t} + \mathbf{u}(t, \mathbf{x}, \mathbf{y}).
\end{aligned} \tag{2.3}$$

Based on Lyapunov direct method [39], the scheme of generalized synchronization and the procedure of designing Lyapunov function are described as follows:

Step 1. Construct a Lyapunov function

$$\begin{aligned}
V(t, \mathbf{e}) &= \frac{1}{2}\mathbf{e}^T \boldsymbol{\Lambda}(t)\mathbf{e} \\
&= \frac{1}{2}\lambda_{11}(t)e_1^2 + \frac{1}{2}\lambda_{22}(t)e_2^2 + \cdots + \frac{1}{2}\lambda_{nn}(t)e_n^2
\end{aligned} \tag{2.4}$$

where $\boldsymbol{\Lambda}(t) = [\lambda_{ii}(t)] \in R^{n\times n}$ is an unknown continuously differentiable positive definite diagonal matrix to be designed. Its derivative is

$$\begin{aligned}
\dot{V}(t, \mathbf{e}) &= \dot{\mathbf{e}}^T \boldsymbol{\Lambda}(t)\mathbf{e} + \frac{1}{2}\mathbf{e}^T \dot{\boldsymbol{\Lambda}}(t)\mathbf{e} \\
&= \lambda_{11}(t)e_1\dot{e}_1 + \lambda_{22}(t)e_2\dot{e}_2 + \cdots + \lambda_{nn}(t)e_n\dot{e}_n + \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2 + \frac{1}{2}\dot{\lambda}_{22}(t)e_2^2 + \cdots + \frac{1}{2}\dot{\lambda}_{nn}(t)e_n^2.
\end{aligned} \tag{2.5}$$

Step 2. Eq. (2.5) can be rewritten in the following form:

$$\begin{aligned}
\dot{V}(t, \mathbf{e}) &= G_1(\lambda_{11}, \dot{\lambda}_{11})e_1^2 + G_2(\lambda_{22}, \dot{\lambda}_{22})e_2^2 + \cdots + G_n(\lambda_{nn}, \dot{\lambda}_{nn})e_n^2 \\
&\quad + [H_1(\lambda_{11}, \ldots, \lambda_{nn}, x_1, \ldots, x_n, y_1, \ldots, y_n, t) + \lambda_{11}u_1]e_1 \\
&\quad + [H_2(\lambda_{11}, \ldots, \lambda_{nn}, x_1, \ldots, x_n, y_1, \ldots, y_n, t) + \lambda_{22}u_2]e_2 \\
&\quad + \cdots + [H_n(\lambda_{11}, \ldots, \lambda_{nn}, x_1, \ldots, x_n, y_1, \ldots, y_n, t) + \lambda_{nn}u_n]e_n
\end{aligned} \tag{2.6}$$

where $G_i(\lambda_{ii}, \dot{\lambda}_{ii})$ and $H_i(\lambda_{11}, \ldots, \lambda_{nn}, x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ $(i = 1, 2, \ldots, n)$ are continuous differentiable functions, u_i $(i = 1, 2, \ldots, n)$ are controllers to be determined.

Step 3. Eq. (2.6) may be classified as two general forms: (1) All $G_i(\lambda_{ii}, \dot{\lambda}_{ii})$ depend on $\lambda_{ii}(t)$ and $\dot{\lambda}_{ii}(t)$, (2) Some of $G_j(\lambda_{jj}, \dot{\lambda}_{jj})$ depend on $\lambda_{jj}(t)$ and $\dot{\lambda}_{jj}(t)$, the remaining $G_k(\lambda_{kk}, \dot{\lambda}_{kk})$ depend only on $\dot{\lambda}_{kk}(t)$.

Form (1) All $G_i(\lambda_{ii}, \dot{\lambda}_{ii})$ depend on $\lambda_{ii}(t)$ and $\dot{\lambda}_{ii}(t)$.

Step 4. Design the controllers u_i such that

$$H_i(\lambda_{11}, \ldots, \lambda_{nn}, x_1, \ldots, x_n, y_1, \ldots, y_n, t) + \lambda_{ii}u_i = 0 \quad (i = 1, 2, \ldots, n) \tag{2.7}$$

i.e., current mixed error dynamics has been replaced by pure error dynamics. By Eq. (2.7), we design the controllers u_i such that λ_{ii} $(i = 1, 2, \ldots, n)$ are linear function of each other with positive coefficients.

Step 5. Design $\lambda_{11}(t), \lambda_{22}(t), \ldots, \lambda_{nn}(t)$ such that

$$\forall t \geq 0, \quad 0 < \lambda_{mii} \leq \lambda_{ii}(t) \leq \lambda_{Mii} \quad (i = 1, 2, \ldots, n) \tag{2.8}$$

where $\lambda_{m\,ii}, \lambda_{M\,ii}$ are positive constants, and

$$\forall t \geq 0, \quad G_i(\lambda_{ii}, \dot{\lambda}_{ii}) < 0 \quad (i = 1, 2, \ldots, n) \tag{2.9}$$

then the Lyapunov function can be obtained and the generalized synchronization is achieved according to the Lyapunov direct method.

Form (2) Some of $G_j(\lambda_{jj}, \dot{\lambda}_{jj})$ depend on $\lambda_{jj}(t)$ and $\dot{\lambda}_{jj}(t)$, and the remaining $G_k(\lambda_{kk}, \dot{\lambda}_{kk})$ depend only on $\dot{\lambda}_{kk}(t)$.

Step 4. Assume

$$\forall k, \quad \lambda_{kk}(t) = 1 \tag{2.10}$$

$$\forall k, \quad H_k(\lambda_{11}, \ldots, \lambda_{nn}, x_1, \ldots, x_n, y_1, \ldots, y_n, t) + \lambda_{kk}(t)u_k = -e_k \tag{2.11}$$

$$\forall j, \quad H_j(\lambda_{11}, \ldots, \lambda_{nn}, x_1, \ldots, x_n, y_1, \ldots, y_n, t) + \lambda_{jj}(t)u_j = 0 \tag{2.12}$$

i.e., current mixed error dynamics has been replaced by pure error dynamics, and appropriately design the controllers u_i $(i = 1, 2, \ldots, n)$ and $\lambda_{jj}(t)$ such that

$$\forall t \geq 0, \quad 0 < \lambda_{mjj} \leq \lambda_{jj}(t) \leq \lambda_{Mjj} \tag{2.13}$$

where $\lambda_{m\,jj}, \lambda_{M\,jj}$ are positive constants, and

$$\forall t \geq 0, \quad G_j(\lambda_{jj}, \dot{\lambda}_{jj}) < 0 \tag{2.14}$$

then the Lyapunov function can be obtained and the generalized synchronization is achieved according to the Lyapunov direct method.

3. Generalized synchronization of double Mathieu systems

In this section, the functional relation between master and slave states is $y_i = g_i(t, x_i) = \alpha(t)x_i + \beta(t)$ $(i = 1, 2, \ldots, n)$. To demonstrate the use of the proposed method, two examples of autonomous and nonautonomous double Mathieu systems are presented.

3.1. Regular and chaotic dynamics of autonomous and nonautonomous double Mathieu systems

The nonlinear damped Mathieu system is [40,41]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a(1 + \sin \omega t)x_1 - (1 + \sin \omega t)x_1^3 - ax_2. \end{aligned} \tag{3.1}$$

An autonomous double Mathieu system can be constructed by mutual linear coupling of two Mathieu systems:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a(1 + x_4)x_1 - (1 + x_4)x_1^3 - ax_2 + bx_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -(1 + x_2)x_3 - a(1 + x_2)x_3^3 - ax_4 + bx_1. \end{aligned} \tag{3.2}$$

The parameters in simulation are $a = 0.5$, $b = 1$–1.254, and the initial condition is $x_1(0) = 0.1, x_2(0) = 0.1, x_3(0) = 0.2, x_4(0) = 0.2$. The phase portraits, Poincaré maps, bifurcation diagram, and Lyapunov exponents are shown in Fig. 1. It can be observed that the motion is period 1 for $b = 1.1$, period 4 for $b = 1.243$, and period 8 for $b = 1.246$. For $b = 1.24$, the motion is chaotic.

A nonautonomous double Mathieu system can also be constructed by mutual linear coupling of two Mathieu systems:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a(1 + \sin \omega t)x_1 - (1 + \sin \omega t)x_1^3 - ax_2 + bx_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -(1 + \sin \omega t)x_3 - a(1 + \sin \omega t)x_3^3 - ax_4 + bx_1. \end{aligned} \tag{3.3}$$

The parameters in simulation are $a = 0.5$, $b = 0.9$–1, $\omega = 1$, and the initial condition is $x_1(0) = 0.1, x_2(0) = 0.1, x_3(0) = 0.2, x_4(0) = 0.2$. The phase portraits, Poincaré maps, bifurcation diagram, and Lyapunov exponents are shown in Fig. 2. It can be observed that the motion is period 1 for $b = 0.9$, period 2 for $b = 0.93$, and period 4 for $b = 0.934$. For $b = 1$, the motion is chaotic.

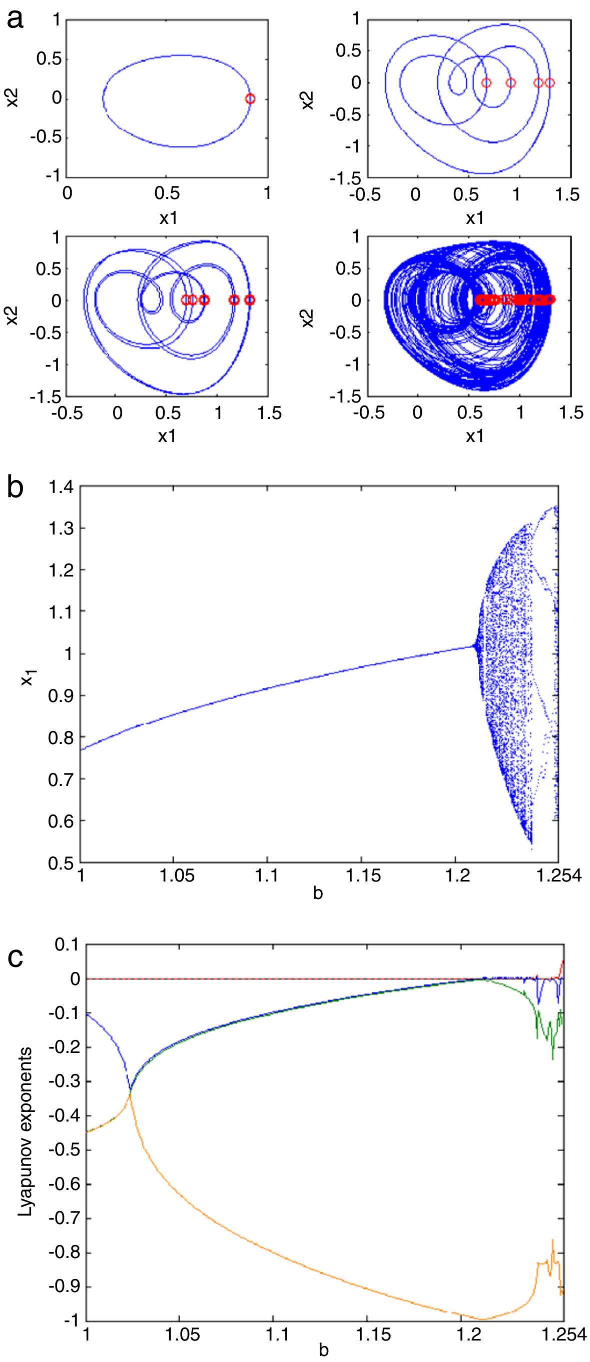

Fig. 1. (a) Phase portraits and Poincaré maps (b) Bifurcation diagram (c) Lyapunov exponents for autonomous double Mathieu system.

3.2. *Generalized synchronization of autonomous double Mathieu systems*

The master and slave autonomous double Mathieu systems can be described by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a(1+x_4)x_1 - (1+x_4)x_1^3 - ax_2 + bx_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -(1+x_2)x_3 - a(1+x_2)x_3^3 - ax_4 + bx_1
\end{aligned} \tag{3.4}$$

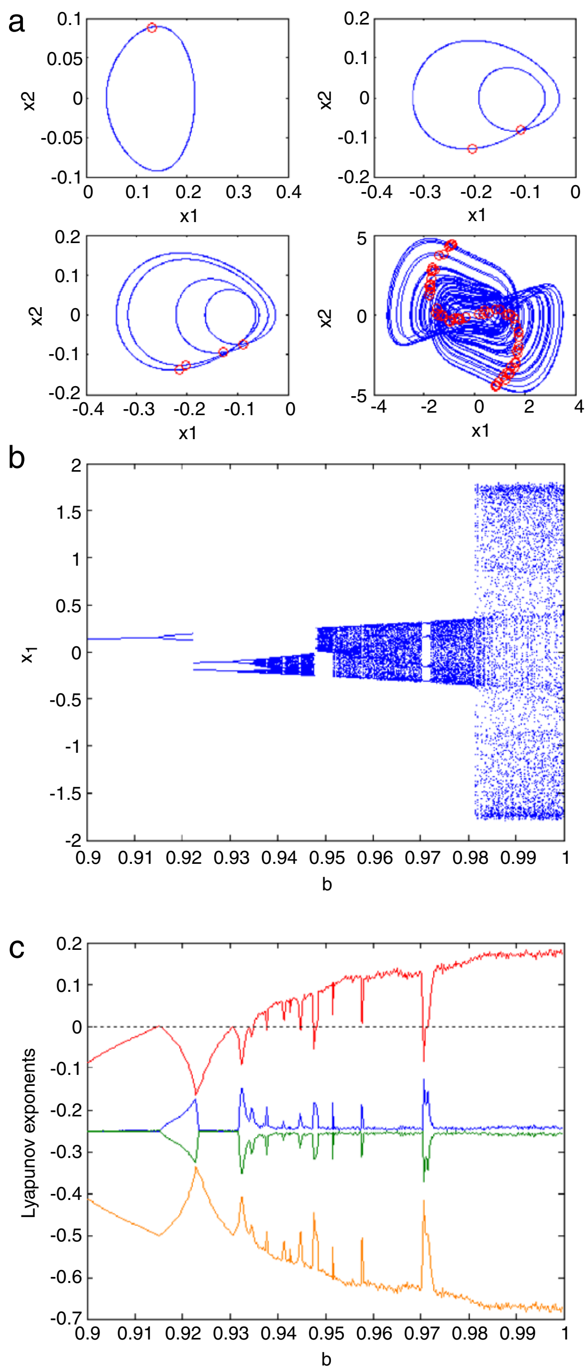

Fig. 2. (a) Phase portraits and Poincaré maps (b) Bifurcation diagram (c) Lyapunov exponents for nonautonomous double Mathieu system.

$$\begin{aligned}
\dot{y}_1 &= y_2 + u_1 \\
\dot{y}_2 &= -a(1+y_4)y_1 - (1+y_4)y_1^3 - ay_2 + by_3 + u_2 \\
\dot{y}_3 &= y_4 + u_3 \\
\dot{y}_4 &= -(1+y_2)y_3 - a(1+y_2)y_3^3 - ay_4 + by_1 + u_4.
\end{aligned} \tag{3.5}$$

The parameters in simulation are $a = 0.5$, $b = 1.24$, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $x_4(0) = 0.2$, $y_1(0) = 0.3$, $y_2(0) = 0.3$, $y_3(0) = 0.4$, $y_4(0) = 0.4$.

Let $e_i = y_i - \alpha(t)x_i - \beta(t)$ $(i = 1, \ldots, 4)$ and subtract Eq. (3.4) from Eq. (3.5), then the error dynamics can be obtained:

$$\begin{aligned}
\dot{e}_1 &= e_2 - \dot{\alpha}(t)x_1 + \beta(t) - \dot{\beta}(t) + u_1 \\
\dot{e}_2 &= -ae_1 - ae_2 + be_3 - a(y_1y_4 - \alpha(t)x_1x_4) - [(1+y_4)y_1^3 - \alpha(t)(1+x_4)x_1^3] \\
&\quad - \dot{\alpha}(t)x_2 + (b-2a)\beta(t) - \dot{\beta}(t) + u_2 \\
\dot{e}_3 &= e_4 - \dot{\alpha}(t)x_3 + \beta(t) - \dot{\beta}(t) + u_3 \\
\dot{e}_4 &= -e_3 - ae_4 + be_1 - (y_2y_3 - \alpha(t)x_2x_3) - a[(1+y_2)y_3^3 - \alpha(t)(1+x_2)x_3^3] \\
&\quad - \dot{\alpha}(t)x_4 + (b-a-1)\beta(t) - \dot{\beta}(t) + u_4.
\end{aligned} \tag{3.6}$$

Step 1. Construct a Lyapunov function

$$\begin{aligned}
V(t, \mathbf{e}) &= \frac{1}{2}\mathbf{e}^T \mathbf{\Lambda}(t)\mathbf{e} \\
&= \frac{1}{2}\lambda_{11}(t)e_1^2 + \frac{1}{2}\lambda_{22}(t)e_2^2 + \frac{1}{2}\lambda_{33}(t)e_3^2 + \frac{1}{2}\lambda_{44}(t)e_4^2.
\end{aligned} \tag{3.7}$$

Its derivative is

$$\begin{aligned}
\dot{V}(t, \mathbf{e}) &= \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2 + \lambda_{11}(t)e_1\dot{e}_1 + \frac{1}{2}\dot{\lambda}_{22}(t)e_2^2 + \lambda_{22}(t)e_2\dot{e}_2 \\
&\quad + \frac{1}{2}\dot{\lambda}_{33}(t)e_3^2 + \lambda_{33}(t)e_3\dot{e}_3 + \frac{1}{2}\dot{\lambda}_{44}(t)e_4^2 + \lambda_{44}(t)e_4\dot{e}_4.
\end{aligned} \tag{3.8}$$

Step 2. Eq. (3.8) can be rewritten in the following form

$$\begin{aligned}
\dot{V}(t, \mathbf{e}) &= G_1(\lambda_{11}, \dot{\lambda}_{11})e_1^2 + G_2(\lambda_{22}, \dot{\lambda}_{22})e_2^2 + G_3(\lambda_{33}, \dot{\lambda}_{33})e_3^2 + G_4(\lambda_{44}, \dot{\lambda}_{44})e_4^2 \\
&\quad + [H_1(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{11}u_1]e_1 \\
&\quad + [H_2(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{22}u_2]e_2 \\
&\quad + [H_3(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{33}u_3]e_3 \\
&\quad + [H_4(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{44}u_4]e_4
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
&G_1(\lambda_{11}, \dot{\lambda}_{11}) = \frac{1}{2}\dot{\lambda}_{11}(t) - \lambda_{11}(t) \\
&G_2(\lambda_{22}, \dot{\lambda}_{22}) = \frac{1}{2}\dot{\lambda}_{22}(t) - a\lambda_{22}(t) \\
&G_3(\lambda_{33}, \dot{\lambda}_{33}) = \frac{1}{2}\dot{\lambda}_{33}(t) - \lambda_{33}(t) \\
&G_4(\lambda_{44}, \dot{\lambda}_{44}) = \frac{1}{2}\dot{\lambda}_{44}(t) - a\lambda_{44}(t) \\
&H_1(\lambda_{11}, \ldots, t) = \lambda_{11}(t)[-\dot{\alpha}(t)x_1 + \beta(t) - \dot{\beta}(t) + e_1] + b\lambda_{44}(t)e_4 \\
&H_2(\lambda_{11}, \ldots, t) = \lambda_{11}(t)e_1 + \lambda_{22}(t)[-ae_1 - a(y_4y_1 - \alpha(t)x_4x_1) - ((1+y_4)y_1^3 - \alpha(t)(1+x_4)x_1^3) \\
&\qquad - \dot{\alpha}(t)x_2 + (b-2a)\beta(t) - \dot{\beta}(t)] \\
&H_3(\lambda_{11}, \ldots, t) = b\lambda_{22}(t)e_2 + \lambda_{33}(t)[-\dot{\alpha}(t)x_3 + \beta(t) - \dot{\beta}(t) + e_3] \\
&H_4(\lambda_{11}, \ldots, t) = \lambda_{33}(t)e_3 + \lambda_{44}(t)[-e_3 - (y_2y_3 - \alpha(t)x_2x_3) - a((1+y_2)y_3^3 - \alpha(t)(1+x_2)x_3^3) \\
&\qquad - \dot{\alpha}(t)x_4 + (b-a-1)\beta(t) - \dot{\beta}(t)].
\end{aligned} \tag{3.10}$$

Step 3. Since all $G_i(\lambda_{ii}, \dot{\lambda}_{ii})$ depend on $\lambda_{ii}(t)$ and $\dot{\lambda}_{ii}(t)$ $(i = 1, \ldots, 4)$, Eq. (3.9) can be classified as form (1).
Step 4. Design the controllers

$$\begin{aligned}
u_1 &= -y_1 - by_4 + (\alpha(t) + \dot{\alpha}(t))x_1 + b\alpha(t)x_4 + b\beta(t) + \dot{\beta}(t) \\
u_2 &= a(y_1y_4 - \alpha(t)x_1x_4) + (1+y_4)y_1^3 - \alpha(t)(1+x_4)x_1^3 + \dot{\alpha}(t)x_2 - (b-2a)\beta(t) + \dot{\beta}(t) \\
u_3 &= -by_2 - y_3 + (\alpha(t) + \dot{\alpha}(t))x_3 + b\alpha(t)x_2 + b\beta(t) + \dot{\beta}(t) \\
u_4 &= y_2y_3 - \alpha(t)x_2x_3 + a(1+y_2)y_3^3 - \alpha(t)(1+x_2)x_3^3 + \left(1 - \frac{1}{a}\right)y_3 - \left(1 - \frac{1}{a}\right)\alpha(t)x_3 \\
&\quad + \dot{\alpha}(t)x_4 - \left(b - a - \frac{1}{a}\right)\beta(t) + \dot{\beta}(t)
\end{aligned} \tag{3.11}$$

such that

$$H_i(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{ii}(t)u_i = 0 \quad (i = 1, \ldots, 4) \tag{3.12}$$

and λ_{ii} $(i = 1, \ldots, 4)$ are linear function of each other with positive coefficients:

$$\begin{aligned}
&\lambda_{11}(t) = \lambda_{44}(t) \\
&\lambda_{22}(t) = \frac{1}{a}\lambda_{11}(t) \\
&\lambda_{33}(t) = \frac{1}{a}\lambda_{11}(t).
\end{aligned} \tag{3.13}$$

Now, the mixed error dynamics is replaced by pure error dynamics:

$$\dot{V}(t, \mathbf{e}) = G_1(\lambda_{11}, \dot{\lambda}_{11})e_1^2 + G_2(\lambda_{22}, \dot{\lambda}_{22})e_2^2 + G_3(\lambda_{33}, \dot{\lambda}_{33})e_3^2 + G_4(\lambda_{44}, \dot{\lambda}_{44})e_4^2. \tag{3.14}$$

Step 5. Design

$$\begin{aligned}
&\lambda_{11}(t) = \frac{1}{1 + e^{-t}} \\
&\lambda_{22}(t) = \frac{1}{a(1 + e^{-t})} \\
&\lambda_{33}(t) = \frac{1}{a(1 + e^{-t})} \\
&\lambda_{44}(t) = \frac{1}{1 + e^{-t}}
\end{aligned} \tag{3.15}$$

such that

$$\begin{aligned}
&\forall t \geq 0, \quad 0 < \lambda_{m11}(t) = \frac{1}{2} \leq \lambda_{11}(t) \leq \lambda_{M11}(t) = 1 \\
&\forall t \geq 0, \quad 0 < \lambda_{m22}(t) = \frac{1}{2a} \leq \lambda_{22}(t) \leq \lambda_{M22}(t) = \frac{1}{a} \\
&\forall t \geq 0, \quad 0 < \lambda_{m33}(t) = \frac{1}{2a} \leq \lambda_{33}(t) \leq \lambda_{M33}(t) = \frac{1}{a} \\
&\forall t \geq 0, \quad 0 < \lambda_{m44}(t) = \frac{1}{2} \leq \lambda_{44}(t) \leq \lambda_{M44}(t) = 1
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\forall t \geq 0, \quad G_1(\lambda_{11}, \dot{\lambda}_{11}) &= \frac{1}{2}\dot{\lambda}_{11}(t) - \lambda_{11}(t) \\
&= \frac{-2 - e^{-t}}{2(1 + e^{-t})^2} < 0 \\
\forall t \geq 0, \quad G_2(\lambda_{22}, \dot{\lambda}_{22}) &= \frac{1}{2}\dot{\lambda}_{22}(t) - a\lambda_{22}(t) \\
&= \frac{-2a + (1 - 2a)e^{-t}}{2a(1 + e^{-t})^2} \\
&= \frac{-1}{(1 + e^{-t})^2} < 0 \quad (\because a = 0.5 \text{ in simulation}) \\
\forall t \geq 0, \quad G_3(\lambda_{33}, \dot{\lambda}_{33}) &= \frac{1}{2}\dot{\lambda}_{33}(t) - \lambda_{33}(t) \\
&= \frac{-2 - e^{-t}}{2a(1 + e^{-t})^2} \\
&= \frac{-2 - e^{-t}}{(1 + e^{-t})^2} < 0 \quad (\because a = 0.5 \text{ in simulation}) \\
\forall t \geq 0, \quad G_4(\lambda_{44}, \dot{\lambda}_{44}) &= \frac{1}{2}\dot{\lambda}_{44}(t) - \lambda_{44}(t) \\
&= \frac{-2a + (1 - 2a)e^{-t}}{2(1 + e^{-t})^2} \\
&= \frac{-1}{2(1 + e^{-t})^2} < 0 \quad (\because a = 0.5 \text{ in simulation})
\end{aligned} \tag{3.17}$$

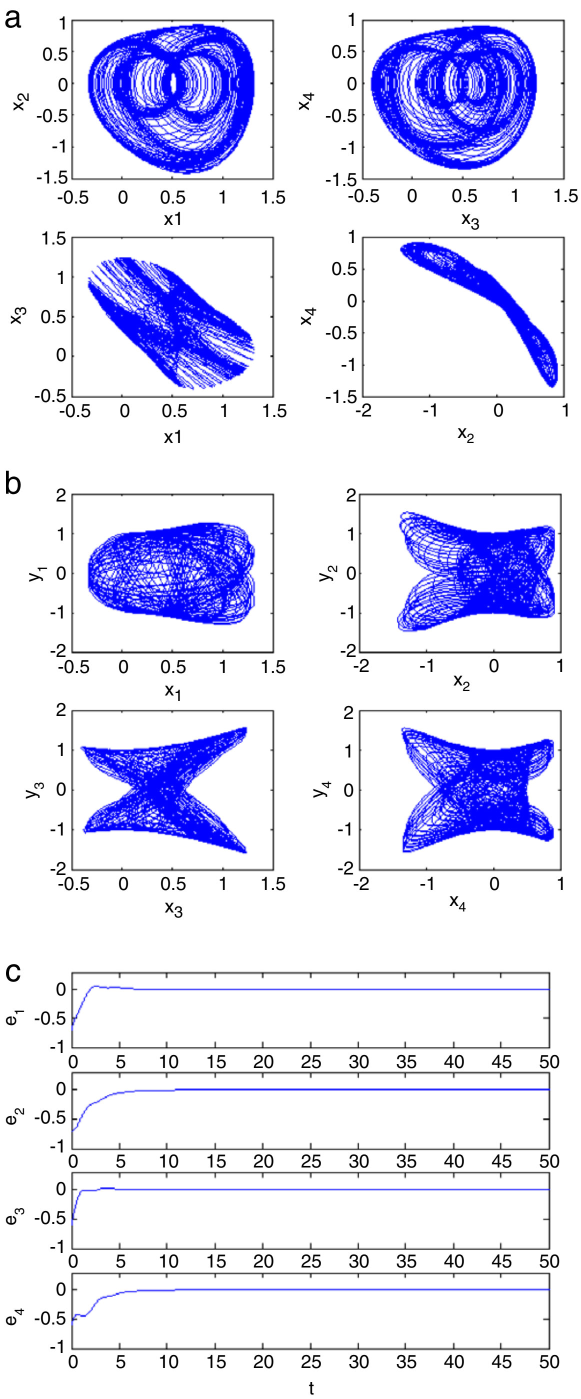

Fig. 3. (a) Phase portraits of master system (b) Phase portraits of x_i to y_i $(i = 1, \ldots, 4)$ when generalized synchronization is obtained (c) Time history of errors.

then the Lyapunov function can be obtained

$$V(t, \mathbf{e}) = \frac{1}{2(1+e^{-t})}e_1^2 + \frac{1}{2a(1+e^{-t})}e_2^2 + \frac{1}{2a(1+e^{-t})}e_3^2 + \frac{1}{2(1+e^{-t})}e_4^2 \tag{3.18}$$

and

$$\dot{V}(t, \mathbf{e}) = -\frac{2+e^{-t}}{2(1+e^{-t})^2}e_1^2 - \frac{1}{(1+e^{-t})^2}e_2^2 - \frac{2+e^{-t}}{(1+e^{-t})^2}e_3^2 - \frac{1}{2(1+e^{-t})^2}e_4^2. \tag{3.19}$$

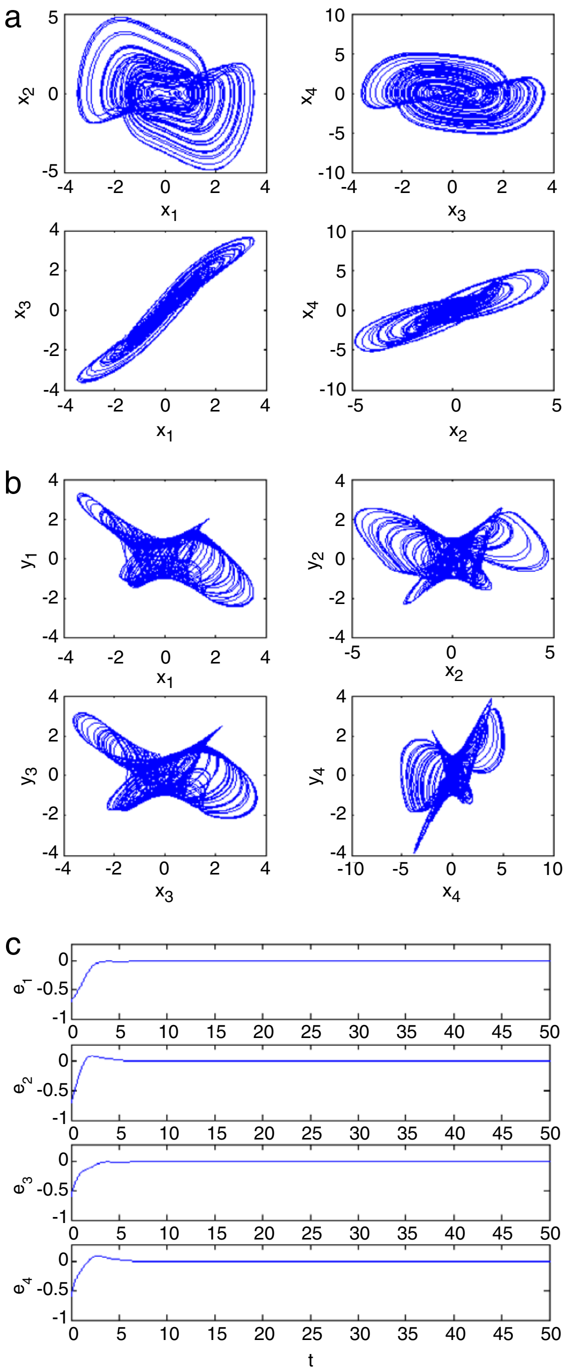

Fig. 4. (a) Phase portraits of master system (b) Phase portraits of x_i to y_i $(i = 1, \dots, 4)$ when generalized synchronization is obtained (c) Time history of errors.

Since Lyapunov global asymptotical stability theorem is satisfied, the global generalized synchronization is achieved. $\alpha(t) = \sin\omega t$, $\beta(t) = \cos\omega t$ are chosen in simulation, and the results are shown in Fig. 3.

3.3. Generalized synchronization of nonautonomous double Mathieu systems

The master and slave nonautonomous double Mathieu systems can be described by

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + bx_3\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= -(1+\sin\omega t)x_3 - a(1+\sin\omega t)x_3^3 - ax_4 + bx_1
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\dot{y}_1 &= y_2 + u_1\\
\dot{y}_2 &= -a(1+\sin\omega t)y_1 - (1+\sin\omega t)y_1^3 - ay_2 + by_3 + u_2\\
\dot{y}_3 &= y_4 + u_3\\
\dot{y}_4 &= -(1+\sin\omega t)y_3 - a(1+\sin\omega t)y_3^3 - ay_4 + by_1 + u_4.
\end{aligned} \tag{3.21}$$

The parameters in simulation are $a = 0.5$, $b = 1$, $\omega = 1$, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $x_4(0) = 0.2$, $y_1(0) = 0.3$, $y_2(0) = 0.3$, $y_3(0) = 0.4$, $y_4(0) = 0.4$.

Let $e_i = y_i - \alpha(t)x_i - \beta(t)$ $(i = 1, \ldots, 4)$ and subtract Eq. (3.20) from Eq. (3.21), then the error dynamics can be obtained:

$$\begin{aligned}
\dot{e}_1 &= e_2 - \dot{\alpha}(t)x_1 + \beta(t) - \dot{\beta}(t) + u_1\\
\dot{e}_2 &= -a(1+\sin\omega t)e_1 - ae_2 + be_3 - (1+\sin\omega t)(y_1^3 - \alpha(t)x_1^3) - \dot{\alpha}(t)x_2\\
&\quad + (-a(1+\sin\omega t) - a + b)\beta(t) - \dot{\beta}(t) + u_2\\
\dot{e}_3 &= e_4 - \dot{\alpha}(t)x_3 + \beta(t) - \dot{\beta}(t) + u_3\\
\dot{e}_4 &= (1+\sin\omega t)e_3 - ae_4 + be_1 - a(1+\sin\omega t)(y_3^3 - \alpha(t)x_3^3) - \dot{\alpha}(t)x_4\\
&\quad + (-(1+\sin\omega t) - a + b)\beta(t) - \dot{\beta}(t) + u_4.
\end{aligned} \tag{3.22}$$

Step 1. Construct a Lyapunov function

$$\begin{aligned}
V(t, \mathbf{e}) &= \frac{1}{2}\mathbf{e}^T\mathbf{\Lambda}(t)\mathbf{e}\\
&= \frac{1}{2}\lambda_{11}(t)e_1^2 + \frac{1}{2}\lambda_{22}(t)e_2^2 + \frac{1}{2}\lambda_{33}(t)e_3^2 + \frac{1}{2}\lambda_{44}(t)e_4^2.
\end{aligned} \tag{3.23}$$

Its derivative is

$$\begin{aligned}
\dot{V}(t, \mathbf{e}) &= \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2 + \lambda_{11}(t)e_1\dot{e}_1 + \frac{1}{2}\dot{\lambda}_{22}(t)e_2^2 + \lambda_{22}(t)e_2\dot{e}_2\\
&\quad + \frac{1}{2}\dot{\lambda}_{33}(t)e_3^2 + \lambda_{33}(t)e_3\dot{e}_3 + \frac{1}{2}\dot{\lambda}_{44}(t)e_4^2 + \lambda_{44}(t)e_4\dot{e}_4.
\end{aligned} \tag{3.24}$$

Step 2. Eq. (3.24) can be rewritten in the following form

$$\begin{aligned}
\dot{V}(t, \mathbf{e}) &= G_1(\lambda_{11}, \dot{\lambda}_{11})e_1^2 + G_2(\lambda_{22}, \dot{\lambda}_{22})e_2^2 + G_3(\lambda_{33}, \dot{\lambda}_{33})e_3^2 + G_4(\lambda_{44}, \dot{\lambda}_{44})e_4^2\\
&\quad + [H_1(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{11}u_1]e_1\\
&\quad + [H_2(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{22}u_2]e_2\\
&\quad + [H_3(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{33}u_3]e_3\\
&\quad + [H_4(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{44}u_4]e_4
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
&G_1(\lambda_{11}, \dot{\lambda}_{11}) = \frac{1}{2}\dot{\lambda}_{11}(t)\\
&G_2(\lambda_{22}, \dot{\lambda}_{22}) = \frac{1}{2}\dot{\lambda}_{22}(t) - a\lambda_{22}(t)\\
&G_3(\lambda_{33}, \dot{\lambda}_{33}) = \frac{1}{2}\dot{\lambda}_{33}(t)\\
&G_4(\lambda_{44}, \dot{\lambda}_{44}) = \frac{1}{2}\dot{\lambda}_{44}(t) - a\lambda_{44}(t)\\
&H_1(\lambda_{11}, \ldots, t) = \lambda_{11}(t)[-\dot{\alpha}(t)x_1 + \beta(t) - \dot{\beta}(t)] + b\lambda_{44}(t)e_4\\
&H_2(\lambda_{11}, \ldots, t) = \lambda_{11}(t)e_1 + \lambda_{22}(t)[-a(1+\sin\omega t)e_1 - (1+\sin\omega t)(y_1^3 - \alpha(t)x_1^3) - \dot{\alpha}(t)x_2\\
&\qquad + (-a(1+\sin\omega t) - a + b)\beta(t) - \dot{\beta}(t)]\\
&H_3(\lambda_{11}, \ldots, t) = b\lambda_{22}(t)e_2 + \lambda_{33}(t)[-\dot{\alpha}(t)x_3 + \beta(t) - \dot{\beta}(t)]\\
&H_4(\lambda_{11}, \ldots, t) = \lambda_{33}(t)e_3 + \lambda_{44}(t)[-(1+\sin\omega t)e_3 - a(1+\sin\omega t)(y_3^3 - \alpha(t)x_3^3) - \dot{\alpha}(t)x_4\\
&\qquad + (-(1+\sin\omega t) - a + b)\beta(t) - \dot{\beta}(t)].
\end{aligned} \tag{3.26}$$

Step 3. Since some of $G_j(\lambda_{jj}, \dot{\lambda}_{jj})$ depend on $\lambda_{jj}(t)$ and $\dot{\lambda}_{jj}(t)$ $(j = 2, 4)$, the remaining $G_k(\lambda_{kk}, \dot{\lambda}_{kk})$ depend only on $\dot{\lambda}_{kk}(t)$ $(k = 1, 3)$, Eq. (3.26) can be classified as form (2).
Step 4. Assume

$$\begin{aligned}\lambda_{11}(t) &= 1\\ \lambda_{33}(t) &= 1\end{aligned} \tag{3.27}$$

$$\begin{aligned}&H_1(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{11}(t)u_1 = -e_1\\ &H_3(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{33}(t)u_3 = -e_3\end{aligned} \tag{3.28}$$

$$\begin{aligned}&H_2(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{22}(t)u_2 = 0\\ &H_4(\lambda_{11}, \ldots, \lambda_{44}, x_1, \ldots, x_4, y_1, \ldots, y_4, t) + \lambda_{44}(t)u_4 = 0\end{aligned} \tag{3.29}$$

and appropriately design the controllers u_i $(i = 1, \ldots, 4)$ and $\lambda_{22}(t), \lambda_{44}(t)$

$$\begin{aligned}
u_1 &= -y_1 - \frac{b}{2+\sin\omega t}y_4 + (\alpha(t) + \dot{\alpha}(t))x_1 + \frac{b\alpha(t)}{2+\sin\omega t}x_4 + \frac{b\beta(t)}{2+\sin\omega t} + \dot{\beta}(t)\\
u_2 &= -ay_1 + a\alpha(t)x_1 + \dot{\alpha}(t)x_2 + (1+\sin\omega t)(y_1^3 - \alpha(t)x_1^3) + (a\sin\omega t + 3a - b)\beta(t) + \dot{\beta}(t)\\
u_3 &= -y_3 - \frac{b}{2+\sin\omega t}y_2 + (\alpha(t) + \dot{\alpha}(t))x_3 + \frac{b\alpha(t)}{2+\sin\omega t}x_2 + \frac{b\beta(t)}{2+\sin\omega t} + \dot{\beta}(t)\\
u_4 &= -y_3 + \alpha(t)x_3 + \dot{\alpha}(t)x_4 + a(1+\sin\omega t)(y_3^3 - \alpha(t)x_3^3) + (\sin\omega t + a - b + 2)\beta(t) + \dot{\beta}(t)
\end{aligned} \tag{3.30}$$

$$\begin{aligned}\lambda_{22}l(t) &= \frac{1}{a(2+\sin\omega t)}\\ \lambda_{44}(t) &= \frac{1}{2+\sin\omega t}\end{aligned} \tag{3.31}$$

such that

$$\begin{aligned}&\forall t \geq 0, \quad 0 < \lambda_{m22} = \frac{1}{3a} \leq \lambda_{22}(t) \leq \lambda_{M22} = \frac{1}{a}\\ &\forall t \geq 0, \quad 0 < \lambda_{m44} = \frac{1}{3} \leq \lambda_{44}(t) \leq \lambda_{M44} = 1\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\forall t \geq 0, \quad G_2(\lambda_{22}, \dot{\lambda}_{22}) &= \frac{1}{2}\dot{\lambda}_{22}(t) - a\lambda_{22}(t)\\
&= \frac{-(4a + 2a\sin\omega t + \omega\cos\omega t)}{2a(2+\sin\omega t)^2}\\
&= \frac{-(2+\sin t + \cos t)}{(2+\sin t)^2} < 0 \quad (\because a = 0.5, \omega = 1 \text{ in simulation})\\
\forall t \geq 0, \quad G_4(\lambda_{44}, \dot{\lambda}_{44}) &= \frac{1}{2}\dot{\lambda}_{44}(t) - a\lambda_{44}(t)\\
&= \frac{-(4a + 2a\sin\omega t + \omega\cos\omega t)}{2(2+\sin\omega t)^2}\\
&= \frac{-(2+\sin t + \cos t)}{2(2+\sin t)^2} < 0 \quad (\because a = 0.5, \omega = 1 \text{ in simulation}).
\end{aligned} \tag{3.33}$$

Now, the mixed error dynamics is replaced by pure error dynamics:

$$\dot{V}(t, \mathbf{e}) = [G_1(\lambda_{11}, \dot{\lambda}_{11}) - \lambda_{11}]e_1^2 + G_2(\lambda_{22}, \dot{\lambda}_{22})e_2^2 + [G_3(\lambda_{33}, \dot{\lambda}_{33}) - \lambda_{33}]e_3^2 + G_4(\lambda_{44}, \dot{\lambda}_{44})e_4^2. \tag{3.34}$$

Then the Lyapunov function can be obtained

$$V(t, \mathbf{e}) = \frac{1}{2}e_1^2 + \frac{1}{2a(2+\sin\omega t)}e_2^2 + \frac{1}{2}e_3^2 + \frac{1}{2(2+\sin\omega t)}e_4^2 \tag{3.35}$$

and

$$\dot{V}(t, \mathbf{e}) = -e_1^2 - \frac{2+\sin t + \cos t}{(2+\sin t)^2}e_2^2 - e_3^2 - \frac{2+\sin t + \cos t}{2(2+\sin t)^2}e_4^2. \tag{3.36}$$

Since Lyapunov global asymptotical stability theorem is satisfied, the global generalized synchronization is achieved. $\alpha(t) = \sin\omega t, \beta(t) = \cos\omega t$ are chosen in simulation, and the results are shown in Fig. 4.

4. Conclusions

The generalized synchronization is studied by applying pure error dynamics and elaborate Lyapunov function in this paper. By classification of the forms of $\dot{V}(t, \mathbf{e})$, the complexity of designing suitable Lyapunov function is reduced greatly. The proposed method is effectively applied to both autonomous and nonautonomous double Mathieu systems.

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