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應用數學系

博 士 論 文

完全多部圖之線性 k 蔭度

Linear k-arboricity of Complete Multipartite Graphs

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摘　要

如果圖形 G 的一些子圖能讓 G 的每一個邊恰好只出現在它們的其中之一，則這些子圖被稱為是圖形 G 的一個*分解*。到目前為止，在圖形的分解這個研究領域上，已經有不少有趣的結果和問題被發表。而這篇論文所要探討的是其中的一個問題，我們稱做是*線性 k 蔭度問題*。

一個完全由長度不大於 k 的路徑所構成的圖形，我們稱之為*線性 k 森林*。而一個圖形 G 所能分解成線性 k 森林的最少數量則稱為圖形 G 的*線性 k 蔭度*。因此，當一個圖形給定後，它的線性 k 蔭度為何，就是我們所謂的線性 k 蔭度問題。

對於線性 k 蔭度所闡述的概念，我們可以將之視為圖形理論裡*邊著色*課題的一種延伸性想法，以及對*線性蔭度*課題更深入詳盡的探究。而所謂的線性蔭度，其實是將線性 k 蔭度的定義裡有關於路徑長度的限制去除。

在西元 1982 年，針對一個圖形 G 的線性 k 蔭度的上界，有兩位學者提出了一個重要猜測。而對於這個猜測的驗證，迄今也發表了許多的結果在文獻裡。譬如當圖形 G 是一個立方圖、一個樹、一個完全圖、或者是一個均衡完全二部圖，以及某些特定的 k 值。

在這篇論文裡，我們會確定均衡完全二部圖、完全圖、和部份的均衡完全多部圖的線性 3 蔭度。同時，對於部份的完全二部圖、完全圖、和均衡完全多部圖，我們也會提供它們的線性 2 蔭度的值。而所有我們獲得的結果，在相同的條件下，會剛好驗證上述的猜測。

此外，在這篇論文裡，我們還探討了一個關於*位元排列網路*的問題。我們會證明從一個位元排列網路N的有向線圖所建構出的新網路$G(N)^+$，它的結構將依然是一個位元排列網路。我們也會給定一個簡易的運算式，它能夠從N的特徵向量來導出$G(N)^+$的特徵向量。這個運算式可以幫助我們去獲得位元排列網路彼此之間的關聯性。

Linear k-arboricity of Complete Multipartite Graphs

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Abstract

A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. There are many interesting results and problems in this area. In this thesis, we study a special case of graph decomposition, called the *linear k-arboricity problem*.

A *linear k-forest* is a graph whose components are paths with lengths at most k. The minimum number of linear k-forests needed to decompose a graph G is the *linear k-arboricity* of G, denoted $la_k(G)$. Thus, the linear k-arboricity problem is what the value $la_k(G)$ should be when a graph G is given.

The notion of linear k-arboricity is a natural generalization of *edge coloring* and also a refinement of the concept of *linear arboricity* in which the paths have no length constraints.

In 1982, Habib and Peroche made the following conjecture:

Conjecture. If G is a graph with maximum degree $\Delta(G)$ and $k \geq 2$, then

$$la_k(G) \leq \begin{cases} \left\lceil \frac{\Delta(G)\cdot|V(G)|}{2\left\lfloor \frac{k\cdot|V(G)|}{k+1} \right\rfloor} \right\rceil & \text{if } \Delta(G) = |V(G)| - 1 \text{ and} \\ \left\lceil \frac{\Delta(G)\cdot|V(G)|+1}{2\left\lfloor \frac{k\cdot|V(G)|}{k+1} \right\rfloor} \right\rceil & \text{if } \Delta(G) < |V(G)| - 1. \end{cases}$$

So far, in the literature, quite a few results on the verification of this conjecture have been obtained. For example, when G is a cubic graph, tree, complete graph, or balanced complete bipartite graph, and k is small or $k \geq \lceil \frac{|V(G)|}{2} \rceil - 1$.

In this thesis, we determine the linear 3-arboricity of balanced complete bipartite graphs, complete graphs, and parts of balanced complete multipartite graphs. We also give some substantial results about the linear 2-arboricity of complete bipartite graphs, complete graphs, and balanced complete multipartite graphs. The results obtained are coherent with the corresponding cases of the conjecture mentioned above.

Furthermore, in this thesis, we study a problem on the *bit permutation network*. We prove that if N is an s-stage d-nary bit permutation network with d^n inputs (outputs), then a new network $L(N)^+$ obtained from the *line digraph* of N is an $(s+1)$-stage d-nary bit permutation network with d^{n+1} inputs (outputs). We also give a simple (but not trivial) formula to determine the characteristic vector of $L(N)^+$ from the characteristic vector of N. This formula can help us to obtain relations between some well-studied bit permutation networks.

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Chapter 1

Fundamental Concepts

In this chapter, we shall list the basic notations, terminologies, and definitions on graph theory and the mathematical theory of switching networks, which are the excerpts from two textbooks, one by Douglas B. West [28] and the other by Frank K. Hwang [16]. We also give an overview of this thesis.

1.1 Graphs

A **graph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its **endpoints**. We **draw** a graph on paper by placing each vertex at a point and representing each edge by a curve joining the locations of its endpoints.

A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same pair of endpoints. A **simple graph** is a graph having no loops or multiple edges. In this case an edge is determined by its endpoints, so we can view an edge as an unordered pair of vertices. Thus a simple graph can be specified by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices and writing $e = uv$ (or $e = vu$) for an edge e with endpoints u and v.

The **order** of a graph G, written $|V(G)|$, is the number of vertices in G. The **size** of a graph G, written $|E(G)|$, is the number of edges in G. A graph G is **finite** if its vertex set and edge set are finite, i.e., $|V(G)|$ and $|E(G)|$ are well-defined nonnegative integers. The **null graph** is the graph whose vertex set and edge set are empty.

Figure 1.1 is a drawing of a finite simple graph. The vertex set is $\{u, v, w, x, y\}$, and the edge set is $\{uv, uw, ux, vx, vw, xw, xy\}$.

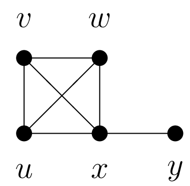

Figure 1.1: A drawing of a finite simple graph.

We adopt the convention that **every graph mentioned in this thesis is finite and simple**. Besides, all statements should be considered only for graphs with a nonempty set of vertices.

If vertex v is an endpoint of edge e, then v and e are **incident**. The **degree** of vertex v in a loopless graph G, written $d_G(v)$, is the number of edges incident to v. The maximum degree is $\Delta(G)$ and the minimum degree is $\delta(G)$. A vertex is **odd** (**even**) when its degree is odd (even). An **isolated vertex** is a vertex of degree 0.

When u and v are the endpoints of an edge, they are **adjacent** and are **neighbors**. The **neighborhood** of v in G, written $N_G(v)$, is the set of vertices adjacent to v. Furthermore, two edges are **incident** if they have one endpoint in common.

A **matching** in a graph G is a set of non-loop edges with no shared endpoints. The vertices incident to the edges of a matching M are **saturated** by M; the others are **unsaturated** (we say M-*saturated* and M-*unsaturated*). A **perfect matching** in a graph is a matching that saturates every vertex. A matching is a set of edges, so its **size** is the number of edges.

A k**-edge-coloring** of a graph G is a labelling $f : E(G) \to S$, where $|S| = k$. The labels are **colors**; the edges of one color form a **color class**. A k-edge-coloring is **proper** if incident edges have different labels; that is, if each color class is a matching. A graph is k**-edge-colorable** if it has a proper k-edge-coloring. The **chromatic index** $\chi'(G)$ of a graph G is the least k such that G is k-edge-colorable.

A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and the assignment of endpoints to edges in H is the same as in G, written $H \subseteq G$.

For example, the graph in Figure 1.2 is a subgraph of the graph in Figure 1.1.

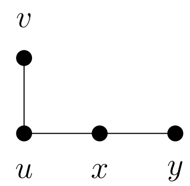

Figure 1.2: A subgraph of the graph in Figure 1.1.

A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A **cycle** is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The path and cycle with n vertices are denoted P_n and C_n, respectively; an n-**cycle** is a cycle with n vertices. A cycle C_n is **odd** (**even**) when n is odd (even). A path in a graph G is a subgraph of G that is a path (similarly for cycles).

A **walk** is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A **trail** is a walk with no repeated edges. A u,v-**walk** or u,v-**trail** has first vertex u and last vertex v; these are its **endpoints**. A u,v-**path** is a path whose vertices of degree 1 (its **endpoints**) are u and v; the others are **internal vertices**.

The **length** of a walk, trail, path, or cycle is its number of edges. In a simple graph, a walk (or trail) is completely specified by its ordered list of vertices. We usually name a path, cycle, trail, or walk in a simple graph by listing only its vertices in order, even though it consists of both vertices and edges.

A graph G is **connected** if it has a u,v-path whenever $u, v \in V(G)$ (otherwise, G is **disconnected**). If G has a u,v-path, then u is **connected to** v in G. A **maximal** connected subgraph of G is a subgraph that is connected and is not contained in any other connected subgraph of G. The **components** of a graph G are its maximal connected subgraphs. Components are pairwise disjoint; no two share a vertex. A component (or a graph) is **trivial** if it has no edges; otherwise it is **nontrivial**.

We write $G - e$ or $G - M$ for the subgraph of G obtained by deleting an edge e or a set of edges M. We write $G - v$ or $G - S$ for the subgraph obtained by deleting a vertex v or a set of vertices S. Note that when we obtain a subgraph by deleting a vertex, it must be a graph, so deleting the vertex also deletes all edges incident to it.

Suppose that $V' \subseteq V(G)$ and $E' \subseteq E(G)$. The subgraph of G **induced by** V', written $G[V']$, is the subgraph of G consists of V' as its vertex set and all edges in G whose endpoints are contained in V'. Similarly, the subgraph of G **induced by** E', written $G[E']$, is the subgraph of G consists of E' as its edge set and all vertices in G which are the endpoints of edges in E'. We say that $G[V']$ is an **induced subgraph** of G and $G[E']$ is an **edge-induced subgraph** of G.

The **union** of graphs $G_1, G_2, \ldots, G_k$, written $G_1 \cup G_2 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$. When a graph G is expressed as the union of two or more subgraphs, an edge of G can belong to many of them. If any edge in the union G of $G_1, G_2, \ldots, G_k$ is contained only by one of $G_1, G_2, \ldots, G_k$, then we say G is an **edge-disjoint union**. If G and H are two graphs with disjoint vertex sets, then the graph obtained by taking the union of G and H is the **disjoint union** or **sum**, written $G + H$.

1.2 Directed Graphs

In general, a relation on S can be any set of ordered pairs in $S \times S$. For such relations, we need a more general model.

A **directed graph** or **digraph** D is a triple consisting of a **vertex set** $V(D)$, an **edge set** $E(D)$, and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the **tail** of the edge, and the second is the **head**; together, they are the **endpoints**. The terms “head” and “tail” come from the arrows used to draw digraphs. As with graphs, we assign each vertex a point in the plane and each edge a curve joining its endpoints. When drawing a digraph, we give the curve a direction from the tail to the head. Figure 1.3 shows a digraph D with vertex set $V(D) = \{a, b, c, d, e, f\}$ and edge set $E(D) = \{(a, b), (b, c), (c, d), (d, e), (e, a), (f, a)\}$.

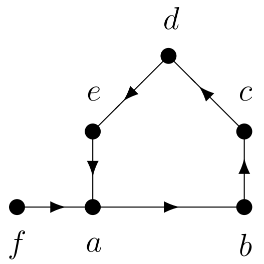

Figure 1.3: A digraph D.

When a digraph models a relation, each ordered pair is the (head, tail) pair for at most one edge. In this setting as with simple graphs, we ignore the technicality of a function assigning endpoints to edges and simply treat an edge as an ordered pair of vertices.

In a digraph, a **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same ordered pair of endpoints. A digraph is **simple** if each ordered pair is the head and tail of at most one edge; one loop may be present at each vertex.

In a simple digraph, we write uv for an edge with tail u and head v. If there is an edge from u to v, then v is a **successor** of u, and u is a **predecessor** of v. We write $u \to v$ for "there is an edge from u to v".

A digraph is a **path** if it is a simple digraph whose vertices can be linearly ordered so that there is an edge with tail u and head v if and only if v immediately follows u in the vertex ordering. A **cycle** is defined similarly using an ordering of the vertices on a circle. We often use the same names for corresponding concepts in the graph and digraph models. Also, a graph G can be modelled using a digraph D in which each edge $uv \in E(G)$ is replaced with $uv, vu \in E(D)$. In this way, results about digraphs can be applied to graphs. Since the notion of "edge" in digraphs extends the notion of "edge" in graphs, using the same name makes sense.

The **underlying graph** of a digraph D is the graph G obtained by treating the edges of D as unordered pairs; the vertex set and edge set remain the same, and the endpoints of an edge are the same in G as in D, but in G they become an unordered pair. Figure 1.4 shows a digraph D and its underlying graph G.

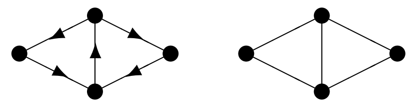

Figure 1.4: A digraph D and its underlying graph G.

The definitions of **subgraph** and **union** are the same for graphs and digraphs. A digraph is **weakly connected** if its underlying graph is connected. A digraph is **strong connected** or **strong** if for each *ordered pair* u, v of vertices, there is a path from u to v. The **strong components** of a digraph are its maximal strong subgraphs.

In a digraph, we use the same notation for number of vertices and number of edges as in graphs. The notation for vertex degrees incorporates the distinction between heads and tails of edges. Let v be a vertex in a digraph. The **outdegree** $d^+(v)$ is the number of edges with tail v. The **indegree** $d^-(v)$ is the number of edges with head v. The **out-neighborhood** or **successor set** $N^+(v)$ is $\{x \in V(G) : v \to x\}$. The **in-neighborhood** or **predecessor set** $N^-(v)$ is $\{x \in V(G) : x \to v\}$. The minimum and maximum indegree $\delta^-(G)$ and $\Delta^-(G)$; for outdegree we use $\delta^+(G)$ and $\Delta^+(G)$.

The definitions of **trail** and **walk** are the same in graphs and digraphs when we list edges as ordered pairs of vertices. In a digraph, the successive edges must "follow the arrows". In a walk $v_0, e_1, v_1, \ldots, e_k, v_k$, the edge e_i has tail v_{i-1} and head v_i.

There are n^2 ordered pairs of elements that can be formed from a vertex set of size n. A simple digraph allows loops but uses each ordered pair at most once as an edge. Thus there are n^2 ordered pairs that may or may not be present as edges. Hence, there are 2^{n^2} simple digraphs with vertices $v_1, v_2, \ldots, v_n$.

Sometimes we want to forbid loops. An **orientation** of a graph G is a digraph D obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for each edge $xy \in E(G)$. An **oriented graph** is an orientation of a simple graph. The number of oriented graphs with vertices $v_1, v_2, \ldots, v_n$ is $3^{\binom{n}{2}}$.

1.3 Special Types of Graphs

A graph G is **regular** if $\Delta(G) = \delta(G)$. It is k**-regular** if the common degree is k. A **cubic graph** is a graph that is regular of degree 3. An **even graph** is a graph with vertex degrees all even.

An **independent set** in a graph is a set of pairwise nonadjacent vertices. A graph G is **bipartite** if $V(G)$ is the union of two disjoint (possibly empty) independent sets called **partite sets** of G. A **bipartition** of G is a specification of two disjoint independent sets in G whose union is $V(G)$. The statement "let G be a bipartite graph with bipartition X, Y" specifies one such partition. An X, Y**-bigraph** G, written $G(X, Y)$, is a bipartite graph with bipartition X, Y.

A **complete bipartite graph** is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the partite sets have sizes r and s, the complete bipartite graph is denoted $K_{r,s}$. Such a graph is called a **balanced complete bipartite graph** and denoted $K_{n,n}$ if $r = s = n$. Figure 1.5 shows a balanced complete bipartite graph $K_{2,2}$.

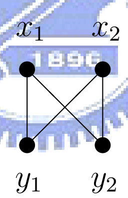

Figure 1.5: A balanced complete bipartite graph $K_{2,2}$.

A graph G is m**-partite** if $V(G)$ can be partitioned into m (possibly empty) independent sets called **partite sets** of G. This generalizes the idea of bipartite graphs, which are 2-partite.

The **chromatic number** of a graph G, written $\chi(G)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors. Vertices given the same color must form an independent set, so $\chi(G)$ is the minimum number of independent sets needed to partition $V(G)$. A graph is m-partite if and only if its chromatic number is at most m.

A **complete m-partite graph** G is an m-partite graph such that the edge $uv \in E(G)$ if and only if u and v are in different partite sets. When $m \geq 2$, we write $K_{n_1,n_2,\ldots,n_m}$ for the complete m-partite graph with partite sets of sizes $n_1, n_2, \ldots, n_m$. Moreover, if $n_1 = n_2 = \cdots = n_m = n$, then it is called a **balanced complete m-partite graph** and denoted $K_{m(n)}$.

A **balanced complete multipartite graph** is a balanced complete m-partite graph with $m \geq 2$. A **complete graph** is a simple graph whose vertices are pairwise adjacent; the complete graph with m vertices is denoted K_m. We can also view a complete graph K_m as a balanced complete m-partite graph $K_{m(n)}$ with $n = 1$.

A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A **tree** is a connected acyclic graph. A **leaf** is a vertex of degree 1. A **spanning subgraph** of G is a subgraph with vertex set $V(G)$. A **spanning tree** is a spanning subgraph that is a tree. A tree is a connected forest, and every component of a forest is a tree. A **star** is a tree consisting of one vertex adjacent to all the others. The star of order n is the complete bipartite graph $K_{1,n-1}$.

The **line graph** of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G. Substituting "digraph" for "graph" in this sentence yields the definition of **line digraph**. For graphs, e and f share a vertex; for digraphs, the head of e must be the tail of f. Figure 1.6 shows a graph G and its line digraph $L(G)$; a digraph D and its line digraph $L(D)$.

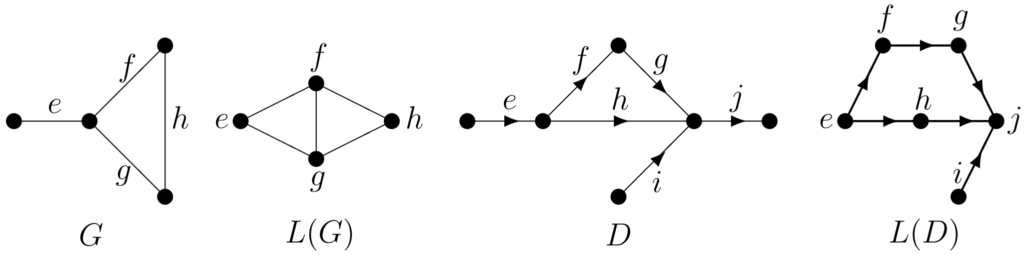

Figure 1.6: G and its line graph $L(G)$; D and its line digraph $L(D)$.

Finally, for $x \in \mathbb{R}$, the **floor** $\lfloor x \rfloor$ is the greatest integer that is at most x. The **ceiling** $\lceil x \rceil$ is the smallest integer that is at least x.

1.4 Switching Networks

The need of a switching network came from the requirements to interconnect pairs of telephones. At first, when there were not so many phones, a direct wire was installed between every two phones. However, with the increase in the number of phones, the transmission cost of these wires became overbearing and the notion of **switching** was born. Every phone in a given locality was then connected to a "switching" center where the wires from these phones were interconnected through a network called **switching networks**. Later, it was reinvented for the parallel computer to interconnect processors with memories. Currently, it is intended for many other applications such as data transmission, video rental, conference calls, and broadcast. It is safe to say that the need of switching networks is expanding fast.

A switching network can interconnect either one group of users, called a **1-sided network**, or two groups, called a **2-sided network**. The dominant applications and theory for switching networks are 2-sided. For many applications, the two sides represent two different types of entities; so input x connecting to output y is not the same as input y connecting to output x. Note that a 2-sided network can be used as a 1-sided network by putting the same type of entities on both sides, although this is less economical from the switching viewpoint. In this thesis, we will only deal with 2-sided networks.

In the 2-sided case we assume that the network has a set of **input terminals** and a set of **output terminals**, while the former generate **requests** to be connected to the latter through the network. Theoretically, an input terminal can request to be connected to any output terminal, just as one phone can call any other phone. Therefore the network must provide access from any input terminal to every output terminal. Furthermore, once a connection is established, it could last for a period of time, while other input terminals may generate their own requests during this period. What a switching network does is to simultaneously connect these requests, the pattern constantly changing by some terminals hanging up and others making new requests.

The basic components of a switching network are **crossbar switches**, or just **crossbars**, and **links** which connect crossbars. A crossbar with n **inlets** and m **outlets**, denoted X_{nm}, is said of **size** $n \times m$. Inlets (outlets) on the same crossbar are called **co-inlets** (**co-outlets**). Any one-to-one mapping between the inlets and the outlets of a crossbar is considered routable, i.e., a crossbar is nonblocking.

Some crossbars are connected to the outside world. For a 2-sided network, one set of such crossbars will be called **input crossbars** and the other set **output crossbars**. The links on an input (output) crossbar linking to the outside world are called **inputs** (**outputs**) of the network, and often drawn by open-ended lines. They are also referred to as **external links**, while other links are **internal links**.

An (N, M)**-network** has N inputs and M outputs. If $M = N$, then it is called an N**-network**. Although a request is originally generated by a pair of input-output, it can be treated as if generated by a pair of input-output crossbars since the crossbar is nonblocking. A request is connected by a path in the network, while two connections do not block each other if their paths are link-disjoint.

In an s-stage network, the crossbars are lined up into s columns, each called a **stage**. Sometimes s is not specified and the network is called a **multistage interconnection network** (MIN). Crossbars in the same stage have the same size. Links exist only between crossbars in adjacent stages. A link between a crossbar in stage i and a crossbar in stage $i+1$ connects an outlet of the former to an inlet of the latter.

Crossbars in the first (last) stage are the input (output) crossbars and its inlets (outlets) are the input (output) terminals, sometimes just called inputs (outputs) of the network connected to external lines. The notation for an s-stage network is that stage i has r_i crossbars of size $n_i \times m_i$. Necessarily, $r_i m_i = r_{i+1} n_{i+1}$ for $i = 1, 2, \ldots, s-1$. Figure 1.7 shows a 3-stage network with 8 inputs and 6 outputs, where $r_1 = r_2 = 4$, $r_3 = 2$, $n_1 = 2$, $m_1 = 3$, $n_2 = 3$, $m_2 = 1$, $n_3 = 2$, $m_3 = 3$, and a crossbar is represented by a square.

A d**-nary network** (MIN) is simply a network (a MIN) using only crossbars of size $d \times d$. In a d-nary MIN of size N, a power of d, it is customary to use the notation $n = \log_d N$. Note that in a d-nary MIN every stage has the same number of crossbars.

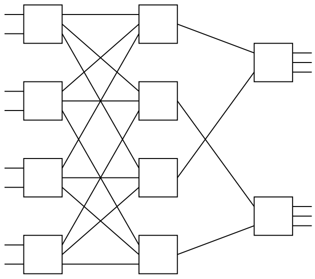

Figure 1.7: A 3-stage MIN.

By treating a crossbar as a vertex and a link as an edge, a switching network is very much like a digraph except that each input (output) crossbar has external links dangling without connecting to any vertex and hence cannot be considered as edges. To remedy this irregularity, the graph theorist prefers to define a true digraph, called a line digraph, from a network by converting each link as a vertex including the inputs and the outputs, while a crosspoint connecting two links in the network becomes an edge in this digraph. Note that a crossbar is represented by a complete bipartite subgraph whose recognizability may depend on the drawing of the line digraph. Figure 1.8 shows the line digraph of the network in Figure 1.7.

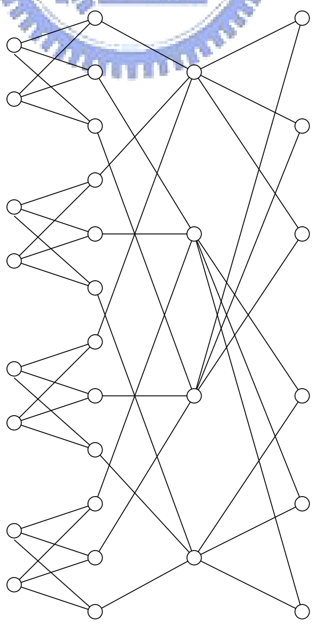

Figure 1.8: The line digraph of the network in Figure 1.7.

1.5 Overview

The first purpose of this thesis is to determine the linear k-arboricity of a complete multipartite graph. The second purpose of this thesis is to characterize a new network obtained from the line digraph of a bit permutation network. We give an overview of this thesis in the following:

In Chapter 1, we list the basic notations, terminologies, and definitions on graph theory and the mathematical theory of switching networks.

Chapter 2 is an introduction of the linear k-arboricity problem. This problem has been conjectured that it is NP-complete for any fixed k. However, it is solvable for some classes of graphs, such as cubic graphs, trees, complete graphs, or balanced complete bipartite graphs, and some values of k. Hence, we state the corresponding results which have been determined.

In Chapter 3, we consider the linear 3-arboricity problem on balanced complete bipartite graphs, complete graphs, and balanced complete multipartite graphs. We find the linear 3-arboricity of balanced complete bipartite graphs and complete graphs. We also give some substantial results when G is a balanced complete multipartite graph.

In Chapter 4, we discuss the linear 2-arboricity problem on complete bipartite graphs, complete graphs, and balanced complete multipartite graphs. We give some substantial results for each class of the graphs above. It is worthy of mentioning that we point out that some computing errors happened in the proof of a result previously [3] and we give a revised result.

In Chapter 5, we first introduce the concept of bit permutation networks. Then we list some results about bit permutation networks which are equivalent. Finally, we characterize the network obtained from the line digraph of a bit permutation network.

Chapter 6 makes a conclusion, besides stating the results obtained on the linear k-arboricity problem and bit permutation networks, some unsolved questions that we concern most are also mentioned.

Chapter 2

The Linear k-arboricity Problem

A **decomposition** of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition $G_1, G_2, \ldots, G_d$, then we say G can be decomposed into $G_1, G_2, \ldots, G_d$ or $G_1, G_2, \ldots, G_d$ decompose G. There are many interesting results and problems in this area. A good survey of them is provided by Chung and Graham [8]. In this thesis, we will study a special case of graph decomposition, called the **linear k-arboricity problem**.

2.1 Introduction

A **linear k-forest** is a graph whose components are paths with lengths at most k. The **linear k-arboricity** of a graph G, denoted $la_k(G)$, is the minimum number of linear k-forests needed to decompose G. Then, the linear k-arboricity problem is what the value $la_k(G)$ should be when a graph G is given. For example, Figure 2.1 shows that the graph K_4 can be decomposed into two linear 3-forests. Thus $la_3(K_4) \leq 2$. In fact, $la_3(K_4) = 2$.

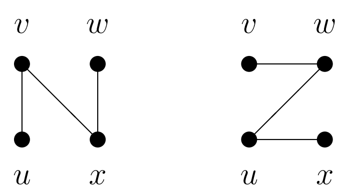

Figure 2.1: Two linear 3-forests in K_4.

The notion of linear k-arboricity was defined by Habib and Peroche in [11]. It is a natural generalization of edge coloring. Recall that the chromatic index of a graph G, written $\chi'(G)$, is the least k such that G is k-edge-colorable. Clearly, a linear 1-forest is induced by a matching and $la_1(G) = \chi'(G)$.

Linear k-arboricity is also a refinement of the concept of **linear arboricity**, which is the minimum number of **linear forests** needed to decompose a graph G and denoted $la(G)$. A linear forest is a graph in which every component is a path with no length constraints. The idea of linear arboricity was introduced earlier by Harary [14].

Next, we describe some properties of $la_k(G)$.

Lemma 2.1.1. *If G is a graph of order n, then $la(G) = la_{n-1}(G) \leq la_{n-2}(G) \leq \cdots \leq la_2(G) \leq la_1(G) = \chi'(G) \leq \Delta(G) + 1$.*

Lemma 2.1.2. *If H is a subgraph of G, then $la_k(H) \leq la_k(G)$.*

Lemma 2.1.3. *If a graph G is the edge-disjoint union of two subgraphs G_1 and G_2, then $la_k(G) \leq la_k(G_1) + la_k(G_2)$.*

Lemma 2.1.4. *If a graph G is the disjoint union of two graphs G_1 and G_2, then $la_k(G) = \max\{la_k(G_1), la_k(G_2)\}$.*

Lemma 2.1.5. $la_k(G) \geq \max\left\{\left\lceil \frac{\Delta(G)}{2} \right\rceil, \left\lceil \frac{|E(G)|}{\left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor} \right\rceil\right\}$.

Lemmas 2.1.1 $\sim$ 2.1.4 are evident by the definition of linear k-arboricity. In particular, since edges sharing a vertex need different colors, $\chi'(G) \geq \Delta(G)$. Vizing [27] proved that $\Delta(G) + 1$ colors suffice when G is simple. Hence $\Delta(G) \leq la_1(G) = \chi'(G) \leq \Delta(G) + 1$ in Lemma 2.1.1. We shall use Lemmas 2.1.2 $\sim$ 2.1.4 frequently without an explicit reference. Since any vertex of a linear k-forest in a graph G has degree at most 2 and a linear k-forest in G has at most $\left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor$ edges, we have Lemma 2.1.5.

In the rest of this chapter, we will state some results which have been proved.

2.2 The Known Results

In 1981, Holyer [15] obtained the result that determining $\chi'(G)$ (or $la_1(G)$) of a graph G is NP-complete. Next year, Peroche [22] also proved the NP-completeness of determining $la(G)$. Further, in 1984, Bermond et al. [2] showed that determining whether $la_3(G) = 2$ is NP-complete for a cubic graph G with $|V(G)| \equiv 0 \pmod 4$ and hence conjectured that it is NP-complete to determine $la_k(G)$ for a graph G and any fixed k. Therefore, the linear k-arboricity problem seems to be difficult.

In 1982, Habib and Peroche [12] made the following important conjecture:

Conjecture 2.2.1. *If G is a graph with maximum degree $\Delta(G)$ and $k \geq 2$, then*

$$la_k(G) \leq \begin{cases} \left\lceil \frac{\Delta(G)\cdot|V(G)|}{2\left\lfloor \frac{k\cdot|V(G)|}{k+1} \right\rfloor} \right\rceil & \text{if } \Delta(G) = |V(G)| - 1 \text{ and} \\ \left\lceil \frac{\Delta(G)\cdot|V(G)|+1}{2\left\lfloor \frac{k\cdot|V(G)|}{k+1} \right\rfloor} \right\rceil & \text{if } \Delta(G) < |V(G)| - 1. \end{cases}$$

This conjecture contains Akiyama's conjecture [1] that $la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ and gives an upper bound about the linear k-arboricity of a graph G. So far, quite a few results on the verification of Conjecture 2.2.1 have been obtained in the literature. For example, when G is a cubic graph, tree, complete graph, or balanced complete bipartite graph, and the value k is small or $k \geq \lceil \frac{|V(G)|}{2} \rceil - 1$. In what follows, we will state them in detail.

In 1984, Bermond et al. [2] proved that if G is a graph with maximum degree $\Delta(G)$, then $la_k(G) \leq \Delta(G)$ for any $k \geq 2$. By using this result and Lemma 2.1.5, it is not difficult to know that the linear 2-arboricity of a cubic graph is equal to 3. Moreover, in [2], Bermond et al. also showed that:

Theorem 2.2.2. *If G is a cubic graph with $la_3(G) = 2$, then $|V(G)| \equiv 0 \pmod 4$.*

Hence, for each cubic graph G with $|V(G)| \equiv 2 \pmod 4$, $la_3(G) = 3$. However, it's a pity that the determination of $la_3(G)$ is NP-complete for cubic graphs G with $|V(G)| \equiv 0 \pmod 4$. Finally, Bermond et al. conjectured that $la_5(G) = 2$ if G is a cubic graph.

In 1996, Jackson and Wormald [19] asked a relative question "is it true that $la_4(G) = 2$ for all cubic graphs G with at least eight vertices?" They also showed that if G is a cubic graph and $k \geq 18$, then $la_k(G) = 2$. In 1999, Thomassen [24] proved $la_k(G) \leq 2$ for a cubic graph G and $k \geq 5$. This result is best possible.

Next, we study the linear k-arboricity of trees from an algorithmic point of view. Habib and Peroche [11] showed the first result along this line. They gave an algorithm to prove that if T is a tree with exactly one vertex of maximum degree 2θ, then $la_2(T) \leq \theta$. Using this as the induction basis, they then gave a characterization for a tree T with maximum degree 2θ to have $la_2(T) = \theta$. However, Chang [5] pointed out that this characterization has a flaw. He then presented a linear-time algorithm for determining whether a tree T satisfies $la_2(T) \leq \theta$ and gave a new characterization for a tree T with maximum degree 2θ to have $la_2(T) = \theta$. As for general k, Chang [5] also proved:

Theorem 2.2.3. *If T is a tree with $\Delta(T) = 2\theta - 1$ then $la_k(T) = \theta$ for $k \geq 2$. If T is a tree with $\Delta(T) = 2\theta$ then $\theta \leq la_k(T) \leq \theta + 1$ for $k \geq 2$.*

So, it remains to determine whether $la_k(T)$ is θ or $\theta + 1$ when $\Delta(T) = 2\theta$. Latterly, in [6], Chang et al. gave a linear-time algorithm for answering whether a tree T satisfies $la_k(T) \leq \theta$ for a fixed k.

Now, let's focus on another class of graphs. In 1984, Bermond et al. [2] determined the linear 2-arboricity of complete graphs. They had the following result:

Theorem 2.2.4. *For $m \not\equiv 10, 11 \pmod{12}$, $la_2(K_m) = \left\lceil \frac{m(m-1)}{2\lfloor \frac{2m}{3} \rfloor} \right\rceil$.*

Bermond et al. also said that if $la_2(K_m) = \left\lceil \frac{m(m-1)}{2\lfloor \frac{2m}{3} \rfloor} \right\rceil$ for $m \equiv 11 \pmod{12}$, then $la_2(K_m) = \left\lceil \frac{m(m-1)}{2\lfloor \frac{2m}{3} \rfloor} \right\rceil$ for $m \equiv 10 \pmod{12}$. This statement can be proved by Lemmas 2.1.2 and 2.1.5. Let $m = 12t + 11$ for any $t \geq 0$, then $\left\lceil \frac{m(m-1)}{2\lfloor \frac{2m}{3} \rfloor} \right\rceil = 9t + 8$. Since $K_{12t+10} \subseteq K_{12t+11}$, if $la_2(K_{12t+11}) = 9t + 8$, then $la_2(K_{12t+10}) \leq 9t + 8$ by Lemma 2.1.2. However, $\left\lceil \frac{m(m-1)}{2\lfloor \frac{2m}{3} \rfloor} \right\rceil$ is also equal to $9t + 8$ when $m = 12t + 10$ for any $t \geq 0$.

Hence, $la_2(K_m) \le \left\lceil \frac{m(m-1)}{2\lfloor \frac{2m}{3} \rfloor} \right\rceil$ for $m \equiv 10 \pmod{12}$ if $la_2(K_m) = \left\lceil \frac{m(m-1)}{2\lfloor \frac{2m}{3} \rfloor} \right\rceil$ for $m \equiv 11 \pmod{12}$. On the other hand, by Lemma 2.1.5, $la_2(K_m) \ge \left\lceil \frac{m(m-1)}{2\lfloor \frac{2m}{3} \rfloor} \right\rceil$ for $m \equiv 10 \pmod{12}$.

In 1991, Chen et al. [3] derived a similar result about the linear 2-arboricity of a complete graph K_m by using the ideas from latin squares. They had:

Theorem 2.2.5. *$la_2(K_{3u}) = \left\lceil \frac{3(3u-1)}{4} \right\rceil$, $la_2(K_{3u+1}) = \left\lceil \frac{3(3u+1)}{4} \right\rceil$ and $la_2(K_{3u+2}) = \left\lceil \frac{(3u+2)(3u-1)}{2(2u+1)} \right\rceil$ except possibly if $3u+1 \in \{49, 52, 58\}$.*

In [3], Chen et al. indicated the fact that $la_2(K_{12t+11}) = 9t + 9$ for any $t \ge 0$. However, some computing errors happened in its proof. In Chapter 4 of this thesis, we will show that $la_2(K_{12t+10})$ and $la_2(K_{12t+11})$ are equal to $9t + 8$ for any $t \ne 4$, which provide the answers of the unsolved cases in Theorem 2.2.4.

In 1994, by using similar ideas from latin squares, Fu and Huang [10] also gave the following result about the linear 2-arboricity of a balanced complete bipartite graph $K_{n,n}$.

Theorem 2.2.6. *$la_2(K_{n,n}) = \lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \rceil$.*

It is worthy of noting that most of the results mentioned above on $la_k(G)$ of a graph G have the same property that k is small. Therefore, finally, we state the following results obtained by Chen and Huang [4] on $la_k(K_m)$ for $k \ge \lceil \frac{m}{2} \rceil - 1$ and on $la_k(K_{n,n})$ for $k \ge n - 1$.

Theorem 2.2.7. *Suppose $m > i \ge 2$ and let $\lceil \frac{m}{i} \rceil - 1 \le k \le \lceil \frac{m}{i-1} \rceil - 2$. Then $la_k(K_m) \ge \lceil \frac{m(m-1)}{2(m-i)} \rceil$, and the equality holds in case that $i = 2$.*

Theorem 2.2.8. *Suppose $2n > i \ge 2$ and let $\lceil \frac{2n}{i} \rceil - 1 \le k \le \lceil \frac{2n}{i-1} \rceil - 2$. Then $la_k(K_{n,n}) \ge \lceil \frac{n^2}{2n-i} \rceil$, and the equality holds in case that $i = 2$.*

Chapter 3

Linear 3-arboricity of Balanced Complete Multipartite Graphs

In this chapter, we study the linear 3-arboricity problem on balanced complete bipartite graphs, complete graphs, and balanced complete multipartite graphs. The results obtained are coherent with the corresponding cases of Conjecture 2.2.1.

3.1 Preliminary Lemmas

Assume that G and H are graphs. A spanning subgraph F of G is called an H**-factor** if each component of F is isomorphic to H. If G is expressible as an edge-disjoint union of H-factors, then this union is called an H**-factorization** of G.

Furthermore, we say that a 1**-factor** of a graph G is a spanning 1-regular subgraph of G. A 1-factor and a perfect matching are almost the same thing. The precise distinction is that "1-factor" is a spanning 1-regular subgraph of G, while "perfect matching" is the set of edges in such a subgraph. A decomposition of a regular graph G into 1-factors is a 1**-factorization** of G. A graph with a 1-factorization is 1**-factorable**.

Let $G(X, Y)$ be a bipartite graph with bipartition $X = \{x_j \mid j = 0, 1, \ldots, r-1\}$, $Y = \{y_j \mid j = 0, 1, \ldots, s-1\}$, and $|Y| = s \geq r = |X|$. We define the **bipartite difference** of an edge $x_p y_q$ in $G(X, Y)$ as the value $q - p \pmod{s}$. For example, the bipartite differences of $x_1 y_2$ and $x_3 y_0$ in a complete bipartite graph $K_{4,7}$ are 1 and 4.

It is not difficult to see that an edge subset in $G(X,Y)$ containing the edges of the same bipartite difference must be a matching. In particular, the edge subset is also a perfect matching if $G(X,Y)$ is a balanced complete bipartite graph $K_{s,s}$. Moreover, we can partition the edge set of $G(X,Y)$ (or $K_{s,s}$) into s edge-disjoint matchings such that each matching is consisting of edges with the same bipartite difference $\ell \in \{0,1,\ldots,s-1\}$ and the edges in different matchings have different bipartite differences.

The following lemmas are essential to obtain our results.

Lemma 3.1.1. *[23] K_m has a K_3-factorization if and only if $m \equiv 3 \pmod 6$.*

Lemma 3.1.2. *[13] K_m has a K_4-factorization if and only if $m \equiv 4 \pmod{12}$.*

Lemma 3.1.3. *A complete graph with even order K_{2u} has a 1-factorization in which there are $2u-1$ 1-factors.*

Proof. We can obtain simply the 1-factors of K_{2u} from a circle and u chords in it. Let the $2u-1$ vertices be placed equally spaced round a circle, and label them $0,1,\ldots,2u-2$; also label the center $2u-1$. The 1-factor with label $i+1$ are then induced by an edge joining vertices i and $2u-1$, and by parallel edges joining the other vertices in pairs. Figure 3.1 shows the case of four vertices. □

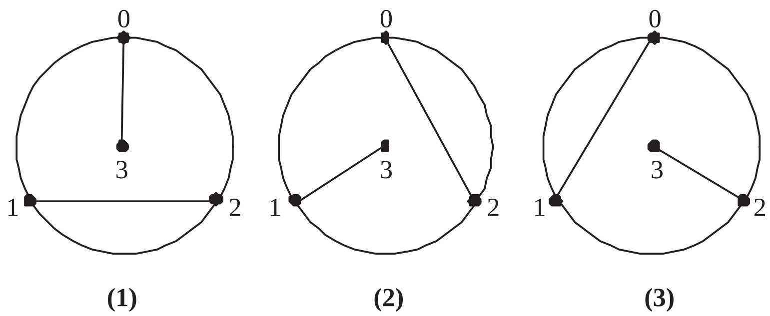

Figure 3.1: A 1-factorization of K_4.

Lemma 3.1.4. *If a graph G has an H-factorization with r H-factors, then $la_k(G) \leq r \cdot la_k(H)$.*

Proof. Since an H-factor of G is a spanning subgraph of G whose components are all isomorphic to H, the linear k-arboricity of every H-factor of G is then equal to $la_k(H)$ by Lemma 2.1.4. Since G has an H-factorization with r H-factors, therefore, $la_k(G) \leq r \cdot la_k(H)$ by Lemma 2.1.3. □

3.2 Balanced Complete Bipartite Graphs

In this section, we study the linear 3-arboricity of a balanced complete bipartite graph $K_{n,n}$. We start with the results of smaller orders.

Lemma 3.2.1. $la_3(K_{6,6}) = 4$.

Proof. Assume that the vertices of two partite sets in $K_{6,6}$ are $x_0, x_1, \ldots, x_5$ and $y_0, y_1, \ldots, y_5$. Then we observe that two edges with bipartite difference 0 (or bipartite difference 2) and one edge with bipartite difference 1 can form a path of length 3, such as $y_0x_0y_1x_1$ (or $x_0y_2x_1y_3$). Thus, the edges with bipartite differences $0, 1, 2$ in $K_{6,6}$ can produce two linear 3-forests $\{y_jx_jy_{j+1}x_{j+1} | \; j = 0, 2, 4\}$, $\{x_jy_{j+2}x_{j+1}y_{j+3} | \; j = 0, 2, 4\}$, as shown in Figure 3.2. Note that the index of each vertex is modulo 6.

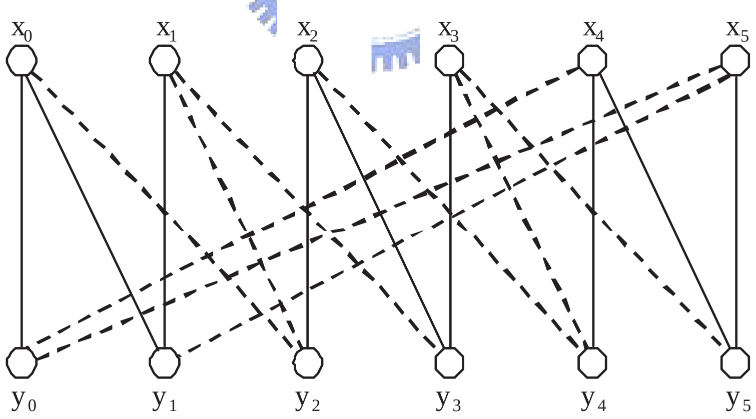

Figure 3.2: Two linear 3-forests in $K_{6,6}$.

Similarly, the edges with bipartite differences $3, 4, 5$ in $K_{6,6}$ also can produce two other linear 3-forests $\{y_{j+3}x_jy_{j+4}x_{j+1} | \; j = 0, 2, 4\}$, $\{x_jy_{j+5}x_{j+1}y_{j+6} | \; j = 0, 2, 4\}$.

Hence, $la_3(K_{6,6}) \leq 4$. We construct the array in Figure 3.3 to show this bound. The entry ω in row x_γ and column y_δ means that the edge $x_\gamma y_\delta$ appears in the linear 3-forest labelled by ω. On the other hand, by Lemma 2.1.5, $la_3(K_{6,6}) \geq \left\lceil \frac{36}{\lfloor \frac{3\cdot 12}{4} \rfloor} \right\rceil = 4$. □

	y_0	y_1	y_2	y_3	y_4	y_5
x_0	1	1	2	3	3	4
x_1	4	1	2	2	3	4
x_2	3	4	1	1	2	3
x_3	3	4	4	1	2	2
x_4	2	3	3	4	1	1
x_5	2	2	3	4	4	1

Figure 3.3: The array shows that $la_3(K_{6,6}) \leq 4$.

Lemma 3.2.2. $la_3(K_{7,7}) = 5$.

Proof. Assume that the vertices of two partite sets in $K_{7,7}$ are $x_0, x_1, \ldots, x_6$ and $y_0, y_1, \ldots, y_6$. Due to the observation mentioned in the proof of Lemma 3.2.1, then the edges with bipartite differences $0, 1, 2$ in $K_{7,7}$ can produce two linear 3-forests $\{x_6y_6\} \cup \{y_jx_jy_{j+1}x_{j+1} | \ j = 0, 2, 4\}$, $\{x_6y_1\} \cup \{x_jy_{j+2}x_{j+1}y_{j+3} | \ j = 0, 2, 4\}$ except the edge x_6y_0 with bipartite difference 1 which is not being used, as shown in Figure 3.4. We call the edges x_6y_6 and x_6y_1 **base edges** because we can construct the whole linear 3-forests from them. Similarly, the edges with bipartite differences $3, 4, 5$ in $K_{7,7}$ also can produce two other linear 3-forests $\{x_5y_1\} \cup \{y_{j+3}x_jy_{j+4}x_{j+1} | \ j = 6, 1, 3\}$, $\{x_5y_3\} \cup \{x_jy_{j+5}x_{j+1}y_{j+6} | \ j = 6, 1, 3\}$ except the edge x_5y_2 with bipartite difference 4 which is not being used. Note that the index of each vertex is modulo 7.

Now, let the edges x_6y_0, x_5y_2 which are not being used and all edges with bipartite difference 6 in $K_{7,7}$ form the last linear 3-forest. It is consisting of three isolated edges and two paths of length 3. Thus, $la_3(K_{7,7}) \leq 5$ and the array in Figure 3.5 shows this bound. On the other hand, by Lemma 2.1.5, $la_3(K_{7,7}) \geq \left\lceil \frac{49}{\lfloor \frac{3\cdot 14}{4} \rfloor} \right\rceil = 5$. □

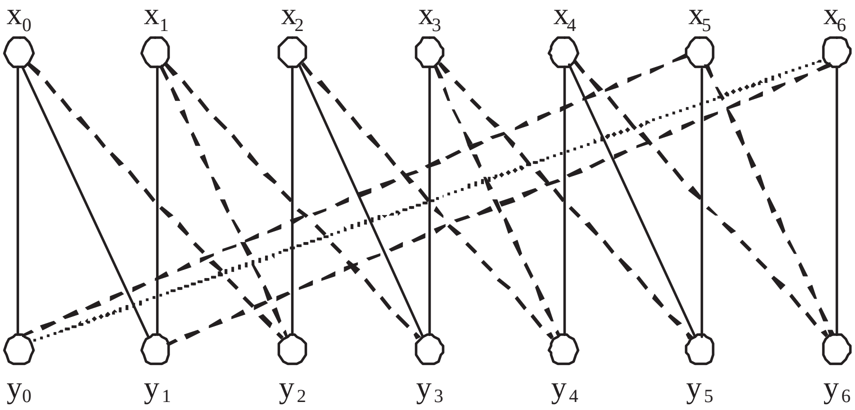

Figure 3.4: Two linear 3-forests and one isolated edge in $K_{7,7}$.

	y_0	y_1	y_2	y_3	y_4	y_5	y_6
x_0	1	1	2	3	4	4	5
x_1	5	1	2	2	3	3	4
x_2	4	5	1	1	2	3	4
x_3	3	4	5	1	2	2	3
x_4	3	4	4	5	1	1	2
x_5	2	3	5	4	5	1	2
x_6	5	2	3	3	4	5	1

Figure 3.5: The array shows that $la_3(K_{7,7}) \leq 5$.

In what follows, we consider the general cases of n.

Proposition 3.2.3. *$la_3(K_{n,n}) = \frac{2n}{3}$ if $n \equiv 0 \pmod 6$.*

Proof. From the proof of Lemma 3.2.1, we observe that if n is even, then the edges with bipartite differences $\epsilon, \epsilon+1, \epsilon+2$ in $K_{n,n}$ for any ϵ can produce two linear 3-forests. Hence, the edges with bipartite differences from 0 to $n-1$ in $K_{n,n}$ can generate $\left(\frac{n}{3}\right) \cdot 2 = \frac{2n}{3}$ linear 3-forests. On the other hand, by Lemma 2.1.5, $la_3(K_{n,n}) \geq \left\lceil \frac{n^2}{\lfloor \frac{3n}{2} \rfloor} \right\rceil = \frac{2n}{3}$ if $n \equiv 0 \pmod 6$. □

Proposition 3.2.4. *$la_3(K_{n,n}) = \left\lceil \frac{2n}{3} \right\rceil$ if $n \equiv 4 \pmod 6$.*

Proof. First, by using the method in the proof of Proposition 3.2.3, the edges with bipartite differences from 0 to $n-2$ in $K_{n,n}$ can generate $\left(\frac{n-1}{3}\right) \cdot 2 = \frac{2(n-1)}{3}$ linear 3-forests. Next, the edges with bipartite difference $n-1$ in $K_{n,n}$ can uniquely produce a linear 3-forest. Thus $la_3(K_{n,n}) \leq \frac{2(n-1)}{3} + 1 = \frac{2n+1}{3} = \lceil \frac{2n}{3} \rceil$ if $n \equiv 4 \pmod 6$. On the other hand, by Lemma 2.1.5, $la_3(K_{n,n}) \geq \left\lceil \frac{n^2}{\lfloor \frac{3n}{2} \rfloor} \right\rceil = \lceil \frac{2n}{3} \rceil$ if $n \equiv 4 \pmod 6$. □

Proposition 3.2.5. $la_3(K_{n,n}) = \lceil \frac{2n}{3} \rceil$ *if* $n \equiv 2 \pmod 6$.

Proof. The edges with bipartite differences from 0 to $n-3$ in $K_{n,n}$ can generate $\left(\frac{n-2}{3}\right) \cdot 2 = \frac{2(n-2)}{3}$ linear 3-forests. The edges with bipartite differences $n-2$ and $n-1$ in $K_{n,n}$ can produce different linear 3-forests respectively. Thus $la_3(K_{n,n}) \leq \frac{2(n-2)}{3} + 2 = \frac{2n+2}{3} = \lceil \frac{2n}{3} \rceil$ if $n \equiv 2 \pmod 6$. On the other hand, by Lemma 2.1.5, $la_3(K_{n,n}) \geq \left\lceil \frac{n^2}{\lfloor \frac{3n}{2} \rfloor} \right\rceil = \lceil \frac{2n}{3} \rceil$ if $n \equiv 2 \pmod 6$. □

Proposition 3.2.6. $la_3(K_{n,n}) = \lceil \frac{2n}{3} \rceil$ *if* $n \equiv 5 \pmod 6$.

Proof. By Proposition 3.2.3, $la_3(K_{n,n}) \leq la_3(K_{n+1,n+1}) = \frac{2(n+1)}{3} = \frac{2n+2}{3} = \lceil \frac{2n}{3} \rceil$ if $n \equiv 5 \pmod 6$. On the other hand, by Lemma 2.1.5, $la_3(K_{n,n}) \geq \left\lceil \frac{n^2}{\lfloor \frac{3n}{2} \rfloor} \right\rceil = \lceil \frac{2n}{3} \rceil$ if $n \equiv 5 \pmod 6$. □

Proposition 3.2.7. $la_3(K_{n,n}) = \lceil \frac{2n+2}{3} \rceil$ *if* $n \equiv 3 \pmod 6$.

Proof. By Proposition 3.2.4, $la_3(K_{n,n}) \leq la_3(K_{n+1,n+1}) = \left\lceil \frac{2(n+1)}{3} \right\rceil = \lceil \frac{2n+2}{3} \rceil$. On the other hand, by Lemma 2.1.5, $la_3(K_{n,n}) \geq \left\lceil \frac{n^2}{\lfloor \frac{3n}{2} \rfloor} \right\rceil = \lceil \frac{2n+2}{3} \rceil$ if $n \equiv 3 \pmod 6$. □

Proposition 3.2.8. $la_3(K_{n,n}) = \lceil \frac{2n}{3} \rceil$ *if* $n \equiv 1 \pmod 6$.

Proof. Assume that the vertices of two partite sets X, Y in $K_{n,n}$ are $x_0, x_1, \ldots, x_{n-1}$ and $y_0, y_1, \ldots, y_{n-1}$. First, from the proof of Lemma 3.2.2, we observe that if n is odd, then the edges with bipartite differences $\epsilon, \epsilon+1, \epsilon+2$ in $K_{n,n}$ for any ϵ can produce two linear 3-forests except one edge with bipartite difference $\epsilon+1$ which is not being used. Thus, the edges with bipartite differences from 0 to $n-2$ in $K_{n,n}$ can generate $\left(\frac{n-1}{3}\right) \cdot 2 = \frac{2(n-1)}{3}$ linear 3-forests except $\frac{n-1}{3}$ edges which are not being used.

Next, without loss of generality, suppose that those $\frac{2(n-1)}{3}$ linear 3-forests are constructed from the base edges in $\{x_{n-j}y_{n-j+3(j-1)}, x_{n-j}y_{n-j+3(j-1)+2} | \ 1 \leq j \leq \frac{n-1}{3}\}$, where the index of each vertex is modulo n. Then, the set of those $\frac{n-1}{3}$ edges which are not being used is a matching $\{x_{n-j}y_{n-j+3(j-1)+1} |\ 1 \le j \le \frac{n-1}{3}\}$, denoted M_1. Moreover, the set of edges with bipartite difference $n-1$ in $K_{n,n}$ is a perfect matching $\{x_j y_{j-1} |\ 0 \le j \le n-1\}$, denoted M_2.

In what follows, we want to show that the edges of M_1 and M_2 can produce a linear 3-forest together. Since the endpoints u, v of an edge in M_1 are incident to two other edges e_1, e_2 in M_2, it suffices to prove that the endpoints of e_1, e_2 except u, v are not the endpoints of edges in M_1.

Since the endpoints in partite set X of edges in M_1 are $x_{n-1}, x_{n-2}, \ldots, x_{\frac{2(n-1)}{3}+1}$, they are adjacent to the endpoints $y_{n-2}, y_{n-3}, \ldots, y_{\frac{2(n-1)}{3}}$ of edges in M_2. Similarly, the endpoints in partite set Y of edges in M_1 are $y_0, y_2, \ldots, y_{\frac{2(n-1)}{3}-2}$, which are adjacent to the endpoints $x_1, x_3, \ldots, x_{\frac{2(n-1)}{3}-1}$ of edges in M_2. Note that the endpoints mentioned above are distinct. Hence, each component of the subgraph induced by $M_1 \cup M_2$ is a path of length at most 3 and then we have a linear 3-forest in $K_{n,n}$.

Therefore, $la_3(K_{n,n}) \le \frac{2(n-1)}{3} + 1 = \frac{2n+1}{3} = \lceil \frac{2n}{3} \rceil$ if $n \equiv 1 \pmod 6$. On the other hand, by Lemma 2.1.5, $la_3(K_{n,n}) \ge \left\lceil \frac{n^2}{\lfloor \frac{3n}{2} \rfloor} \right\rceil = \lceil \frac{2n}{3} \rceil$ if $n \equiv 1 \pmod 6$. □

From the propositions given above, we determine the linear 3-arboricity of $K_{n,n}$ for any n and conclude the work of this section with the following theorem.

Theorem 3.2.9.

$$la_3(K_{n,n}) = \left\lceil \frac{n^2}{\lfloor \frac{3n}{2} \rfloor} \right\rceil = \begin{cases} \lceil \frac{2n}{3} \rceil & \textit{if} \ \ n \equiv 0, 1, 2, 4, 5 \pmod 6, \\ \lceil \frac{2n+2}{3} \rceil & \textit{if} \ \ n \equiv 3 \pmod 6. \end{cases}$$

3.3 Complete Graphs

In this section, we study the linear 3-arboricity of a complete graph K_m and the results on $K_{n.n}$ will give us great help. We start with the case $m = 8$ of K_m.

Lemma 3.3.1. $la_3(K_8) = 5$.

Proof. Assume that the vertices of K_8 are $v_0, v_1, \ldots, v_7$. First, let the perfect matching $\{v_{2i}v_{2i+1} | \ 0 \leq i \leq 3\}$ of K_8 be denoted M. Then, for $0 \leq i \leq 3$, we define N_i as the set of one edge $v_{2i}v_{2i+1}$ and its endpoints v_{2i}, v_{2i+1}. Thus K_8 can be viewed as K_4 with nodes N_i for $0 \leq i \leq 3$ and unordered pairs of nodes (N_α, N_β) for $0 \leq \alpha \neq \beta \leq 3$. The notation (N_α, N_β) also means the 4-cycle consisting of the edges $v_{2\alpha}v_{2\beta}$, $v_{2\alpha}v_{2\beta+1}$, $v_{2\alpha+1}v_{2\beta}$, and $v_{2\alpha+1}v_{2\beta+1}$ in original K_8.

From the proof of Lemma 3.1.3, we know that K_4 has a 1-factorization, in which there are three different 1-factors and each 1-factor owns two disjoint unordered pairs of nodes. For example, the 1-factor with label 1 has unordered pairs of nodes (N_0, N_3) and (N_1, N_2). Then, from this 1-factor, we observe that the subgraph consisting of two paths $v_6v_0v_7v_1$ in (N_0, N_3) and $v_2v_4v_3v_5$ in (N_1, N_2) is a linear 3-forest in original K_8, labelled by 1. However, each of (N_0, N_3) and (N_1, N_2) has an edge which is not being used to construct the linear 3-forest with label 1, they are v_6v_1 and v_2v_5.

Figure 3.6 shows the linear 3-forest with label 1 in original K_8. Similarly, the other 1-factors of K_4 can produce two other linear 3-forests in original K_8, labelled by 2 and 3, except the edges v_2v_7, v_0v_5, v_4v_7, and v_0v_3 not being used.

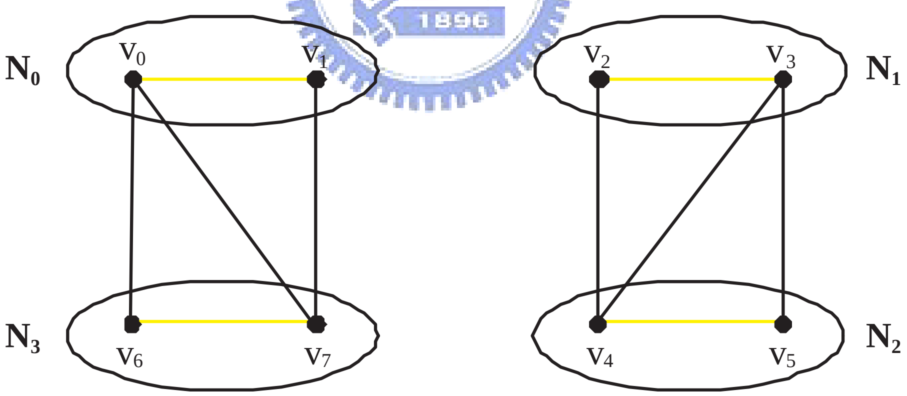

Figure 3.6: A linear 3-forest in K_8.

Now, let $G(X, Y)$ be a bipartite graph with bipartition $X = \{v_0\ (= x_0), v_2\ (= x_1), v_4\ (= x_2), v_6\ (= x_3)\}$ and $Y = \{v_1\ (= y_0), v_3\ (= y_1), v_5\ (= y_2), v_7\ (= y_3)\}$. Then those edges which are not being used are exactly all of edges with bipartite difference 1 in $G(X, Y)$ and half of edges with bipartite difference 2 in $G(X, Y)$.

We observe that the edges which are half of edges with bipartite difference 2 in $G(X, Y)$ and all edges of M can produce a linear 3-forest labelled by 4 in original K_8, as shown in Figure 3.7(1). Moreover, the edges with bipartite difference 1 in $G(X, Y)$ also can produce a linear 3-forest labelled by 5 in original K_8 because they form a perfect matching, as shown in Figure 3.7(2). Therefore, $la_3(K_8) \leq 5$. We construct the array in Figure 3.8 to show this bound. On the other hand, by Lemma 2.1.5, $la_3(K_8) \geq \left\lceil \frac{28}{\lfloor \frac{3 \cdot 8}{4} \rfloor} \right\rceil = 5$. □

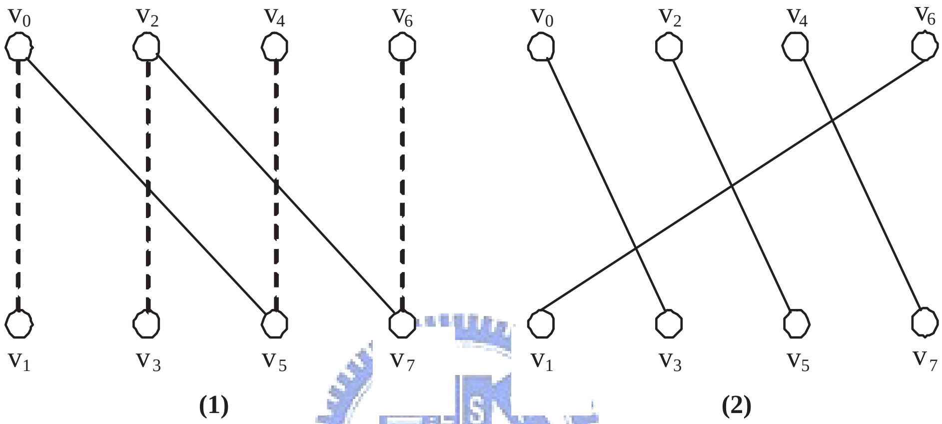

Figure 3.7: Two linear 3-forests in K_8.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_0		4	1	5	2	4	3	3
v_1	4		1	1	2	2	5	3
v_2	1	1		4	3	5	2	4
v_3	5	1	4		3	3	2	2
v_4	2	2	3	3		4	1	5
v_5	4	2	5	3	4		1	1
v_6	3	5	2	2	1	1		4
v_7	3	3	4	2	5	1	4	

Figure 3.8: The array shows that $la_3(K_8) \leq 5$.

Lemma 3.3.2. $la_3(K_{10}) = 7$.

Proof. Assume that the vertices of K_{10} are $v_0, v_1, \ldots, v_9$. First, let the matching $\{v_{2i}v_{2i+1} | \ 0 \leq i \leq 3\}$ of K_{10} be denoted M. Then, for $0 \leq i \leq 3$, we define N_i as the set $\{v_{2i}, v_{2i+1}, v_{2i}v_{2i+1}\}$. Thus K_{10} can be viewed as K_6 with nodes v_8, v_9, N_i for $0 \leq i \leq 3$ and unordered pairs of nodes (v_8, N_α), (v_9, N_β), (N_α, N_β) for $0 \leq \alpha \neq \beta \leq 3$.

From the proof of Lemma 3.1.3 (by placing $v_8, N_0, \ldots, N_3$ equally spaced round a circle and v_9 the center), K_6 has a 1-factorization in which there are five different 1-factors and each 1-factor owns three disjoint unordered pairs of nodes. For example, the 1-factor with label 1 has (v_8, v_9), (N_0, N_3), and (N_1, N_2). From this 1-factor, we can then construct a linear 3-forest labelled by 1 in original K_{10}, as shown in Figure 3.9. However, the edges v_6v_1 in (N_0, N_3) and v_2v_5 in (N_1, N_2) are not being used.

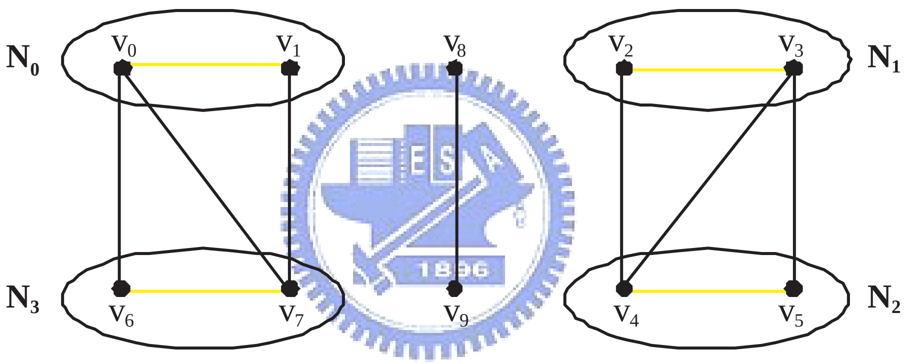

Figure 3.9: A linear 3-forest in K_{10}.

Similarly, the other 1-factors of K_6 can produce four other linear 3-forests labelled by $2, 3, 4, 5$ in original K_{10} except the edges v_4v_7, v_0v_5, v_2v_7, and v_0v_3 not being used. Figure 3.10 shows the linear 3-forest with label 2 in original K_{10}.

Finally, from the proof of Lemma 3.3.1, the six edges above which are not being used and all edges of M can produce two other linear 3-forests labelled by 6 and 7 in original K_{10}. Hence, $la_3(K_{10}) \leq 7$. On the other hand, by Lemma 2.1.5, $la_3(K_{10}) \geq \left\lceil \frac{45}{\lfloor \frac{3 \cdot 10}{4} \rfloor} \right\rceil = 7$. $\square$

In what follows, we consider the general cases of m.

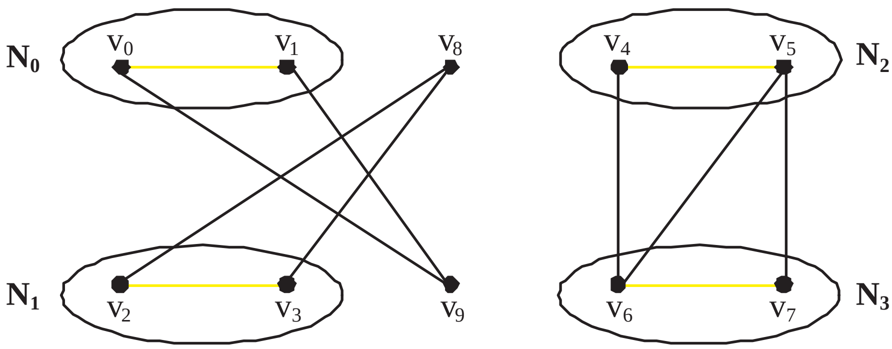

Figure 3.10: Another linear 3-forest in K_{10}.

Proposition 3.3.3. $la_3(K_m) = \lceil \frac{2m-2}{3} \rceil$ *if* $m \equiv 0, 4, 8 \pmod{12}$.

Proof. Assume that the vertices of K_m are $v_0, v_1, \ldots, v_{m-1}$. First, let the perfect matching $\{v_{2i}v_{2i+1} | \ 0 \le i \le \frac{m}{2} - 1\}$ of K_m be denoted M. Then, for $0 \le i \le \frac{m}{2} - 1$, we define N_i as the set $\{v_{2i}, v_{2i+1}, v_{2i}v_{2i+1}\}$. Thus K_m can be viewed as $K_{\frac{m}{2}}$ with nodes N_i for $0 \le i \le \frac{m}{2} - 1$ and unordered pairs of nodes (N_α, N_β) for $0 \le \alpha \ne \beta \le \frac{m}{2} - 1$.

From the proof of Lemma 3.1.3, $K_{\frac{m}{2}}$ can be decomposed into $\frac{m}{2} - 1$ different 1-factors and each 1-factor owns $\frac{m}{4}$ disjoint unordered pairs of nodes. Since each unordered pair of nodes in $K_{\frac{m}{2}}$ is composed of a path with length 3 and one edge in original K_m, then a 1-factor of $K_{\frac{m}{2}}$ can produce one linear 3-forest in original K_m except $\frac{m}{4}$ edges which are not being used. Hence, from the $\frac{m}{2} - 1$ 1-factors of $K_{\frac{m}{2}}$, we obtain $\frac{m}{2} - 1$ linear 3-forests in original K_m except $(\frac{m}{2} - 1) \cdot \frac{m}{4}$ edges not being used.

Now, let $G(X, Y)$ be a bipartite graph with bipartition $X = \{v_{2i} \ (= x_i) | \ 0 \le i \le \frac{m}{2} - 1\}$ and $Y = \{v_{2i+1} \ (= y_i) | \ 0 \le i \le \frac{m}{2} - 1\}$. Then those edges which are not being used are exactly all of edges with bipartite differences $1, 2, \ldots, \frac{m}{4} - 1$ in $G(X, Y)$ and half of edges with bipartite difference $\frac{m}{4}$ in $G(X, Y)$.

We observe that the edges which are half of edges with bipartite difference $\frac{m}{4}$ in $G(X, Y)$ and all edges of M can produce a linear 3-forest in original K_m. Moreover, since the size of X (or Y) is even and $|X| = |Y|$, the edges with bipartite differences $\epsilon, \epsilon + 1, \epsilon + 2$ in $G(X, Y)$ for any ϵ can produce two linear 3-forests from the proof of Proposition 3.2.3. Thus, the edges with bipartite differences $1, 2, \ldots, \frac{m}{4} - 1$ in $G(X, Y)$ can generate $\left\lceil (\frac{\frac{m}{4} - 1}{3}) \cdot 2 \right\rceil = \lceil \frac{m-4}{6} \rceil$ other linear 3-forests in original K_m.

Therefore, $la_3(K_m) \le (\frac{m}{2}-1)+1+\lceil\frac{m-4}{6}\rceil = \lceil\frac{2m-2}{3}\rceil$. On the other hand, by Lemma 2.1.5, $la_3(K_m) \ge \left\lceil \frac{m\cdot(m-1)}{2\cdot\lfloor\frac{3m}{4}\rfloor} \right\rceil = \lceil\frac{2m-2}{3}\rceil$ if $m \equiv 0,4,8 \pmod{12}$. $\square$

Proposition 3.3.4. $la_3(K_m) = \lceil\frac{2m}{3}\rceil$ *if* $m \equiv 2,6,10 \pmod{12}$.

Proof. Assume that the vertices of K_m are $v_0, v_1, \ldots, v_{m-1}$. First, let the matching $\{v_{2i}v_{2i+1} | \ 0 \le i \le \frac{m-2}{2}-1\}$ of K_m be denoted M. Then, for $0 \le i \le \frac{m-2}{2}-1$, we define N_i as the set $\{v_{2i}, v_{2i+1}, v_{2i}v_{2i+1}\}$. Thus K_m can be viewed as $K_{\frac{m+2}{2}}$ with nodes v_{m-2}, v_{m-1}, N_i for $0 \le i \le \frac{m-2}{2}-1$ and unordered pairs of nodes (v_{m-2}, N_α), (v_{m-1}, N_β), (N_α, N_β) for $0 \le \alpha \ne \beta \le \frac{m-2}{2}-1$.

Since $m \equiv 2,6,10 \pmod{12}$, then $\frac{m+2}{2} \equiv 0,2,4 \pmod 6$. From the proof of Lemma 3.1.3 (by placing $v_{m-2}, N_0, N_1, \ldots, N_{\frac{m-2}{2}-1}$ equally spaced round a circle and v_{m-1} the center), $K_{\frac{m+2}{2}}$ has a 1-factorization in which there are $\frac{m+2}{2}-1$ different 1-factors and each 1-factor owns $\frac{m+2}{4}$ disjoint unordered pairs of nodes. However, an unordered pair of nodes (N_α, N_β) in $K_{\frac{m+2}{2}}$ is composed of a path with length 3 and one edge in original K_m. Hence, each 1-factor of $K_{\frac{m+2}{2}}$ can produce one linear 3-forest in original K_m and leaves $\frac{m+2}{4}-2$ edges which are not being used except the 1-factor with label 1 which contains the unordered pair of nodes (v_{m-2}, v_{m-1}) leaves $\frac{m+2}{4}-1$ edges not being used. Therefore, from the $\frac{m+2}{2}-1$ 1-factors of $K_{\frac{m+2}{2}}$, we obtain $\frac{m+2}{2}-1$ linear 3-forests in original K_m except $(\frac{m+2}{2}-1)\cdot(\frac{m+2}{4}-2)+1$ edges not being used.

Now, let $G(X,Y)$ be a bipartite graph with bipartition $X = \{v_{2i} \ (= x_i) | \ 0 \le i \le \frac{m-2}{2}-1\}$ and $Y = \{v_{2i+1} \ (= y_i) | \ 0 \le i \le \frac{m-2}{2}-1\}$. Then those edges which are not being used are exactly all of edges with bipartite differences $1, 2, \ldots, \frac{m-2}{4}-1$ in $G(X,Y)$ and half of edges with bipartite difference $\frac{m-2}{4}$ in $G(X,Y)$.

We observe that the edges which are half of edges with bipartite difference $\frac{m-2}{4}$ in $G(X,Y)$ and all edges of M can produce a linear 3-forest in original K_m. Moreover, since the size of X (or Y) is even and $|X| = |Y|$, the edges with bipartite differences $\epsilon, \epsilon+1, \epsilon+2$ in $G(X,Y)$ for any ϵ can produce two linear 3-forests from the proof of Proposition 3.2.3. Thus, the edges with bipartite differences $1, 2, \ldots, \frac{m-2}{4}-1$ in $G(X,Y)$ can generate $\left\lceil (\frac{\frac{m-2}{4}-1}{3})\cdot 2 \right\rceil = \lceil\frac{m-6}{6}\rceil$ other linear 3-forests in original K_m.

Therefore, $la_3(K_m) \le (\frac{m+2}{2} - 1) + 1 + \lceil \frac{m-6}{6} \rceil = \lceil \frac{2m}{3} \rceil$. On the other hand, by Lemma 2.1.5, $la_3(K_m) \ge \left\lceil \frac{m\cdot(m-1)}{2\cdot\lfloor \frac{3m}{4} \rfloor} \right\rceil = \lceil \frac{2m}{3} \rceil$ if $m \equiv 2, 6, 10 \pmod{12}$. □

Proposition 3.3.5. $la_3(K_m) = \lceil \frac{2m}{3} \rceil$ *if* $m \equiv 1, 9 \pmod{12}$.

Proof. Assume that the vertices of K_m are $v_0, v_1, \ldots, v_{m-1}$. First, let the matching $\{v_{2i}v_{2i+1} |\ 0 \le i \le \frac{m-1}{2} - 1\}$ of K_m be denoted M. Then, for $0 \le i \le \frac{m-1}{2} - 1$, we define N_i as the set $\{v_{2i}, v_{2i+1}, v_{2i}v_{2i+1}\}$. Thus K_m can be viewed as the union of $K_{1,\frac{m-1}{2}}$ and $K_{\frac{m-1}{2}}$. The star $K_{1,\frac{m-1}{2}}$ has nodes v_{m-1}, N_i for $0 \le i \le \frac{m-1}{2} - 1$ and unordered pairs of nodes (v_{m-1}, N_i) for $0 \le i \le \frac{m-1}{2} - 1$; the complete graph $K_{\frac{m-1}{2}}$ has nodes N_i for $0 \le i \le \frac{m-1}{2} - 1$ and unordered pairs of nodes (N_α, N_β) for $0 \le \alpha \ne \beta \le \frac{m-1}{2} - 1$.

Since $\frac{m-1}{2}$ is even, from the proof of Lemma 3.1.3 (by placing $N_0, N_1, \ldots, N_{\frac{m-1}{2}-2}$ equally spaced round a circle and $N_{\frac{m-1}{2}-1}$ the center), $K_{\frac{m-1}{2}}$ has a 1-factorization in which there are $\frac{m-1}{2} - 1$ different 1-factors and each 1-factor owns $\frac{m-1}{4}$ disjoint unordered pairs of nodes. It is worthy of mentioning that each 1-factor of $K_{\frac{m-1}{2}}$ has at most one unordered pair of nodes (N_i, N_{i+1}) for some $i \in \{0, 1, \ldots, \frac{m-1}{2} - 3\}$. Moreover, an unordered pair of nodes (N_i, N_{i+1}) in $K_{\frac{m-1}{2}}$ is composed of a path $v_{2i}v_{2i+2}v_{2i+1}v_{2i+3}$ and one edge $v_{2i}v_{2i+3}$ in original K_m.

Hence, as the proof of the propositions previously, each 1-factor of $K_{\frac{m-1}{2}}$ can produce one linear 3-forest in original K_m except $\frac{m-1}{4}$ edges which are not being used. So, from the $\frac{m-1}{2} - 1$ 1-factors of $K_{\frac{m-1}{2}}$, we obtain $\frac{m-1}{2} - 1$ linear 3-forests in original K_m except $(\frac{m-1}{2} - 1) \cdot (\frac{m-1}{4})$ edges not being used.

Next, for each linear 3-forest obtained from a 1-factor has (N_i, N_{i+1}) for some $i \in \{0, 1, \ldots, \frac{m-1}{2} - 3\}$, we replace the path $v_{2i}v_{2i+2}v_{2i+1}v_{2i+3}$ in (N_i, N_{i+1}) by another path $v_{2i+3}v_{2i}v_{m-1}v_{2i+1}$ in (N_i, N_{i+1}) and (v_{m-1}, N_i). For example, consider the linear 3-forest in K_{13} obtained from the 1-factor has (N_0, N_1), we replace the path $v_0v_2v_1v_3$ in (N_0, N_1) by $v_3v_0v_{12}v_1$ in (N_0, N_1) and (v_{12}, N_0), as shown in Figure 3.11.

Then, let the replaced paths $v_{2i}v_{2i+2}v_{2i+1}v_{2i+3}$ in (N_i, N_{i+1}) for $i = 0, 2, \ldots, \frac{m-1}{2} - 4$ and another path $v_{m-2}v_{m-5}v_{m-1}v_{m-4}$ in $(N_{\frac{m-1}{2}-2}, N_{\frac{m-1}{2}-1})$ and $(v_{m-1}, N_{\frac{m-1}{2}-2})$ form a linear 3-forest in original K_m.

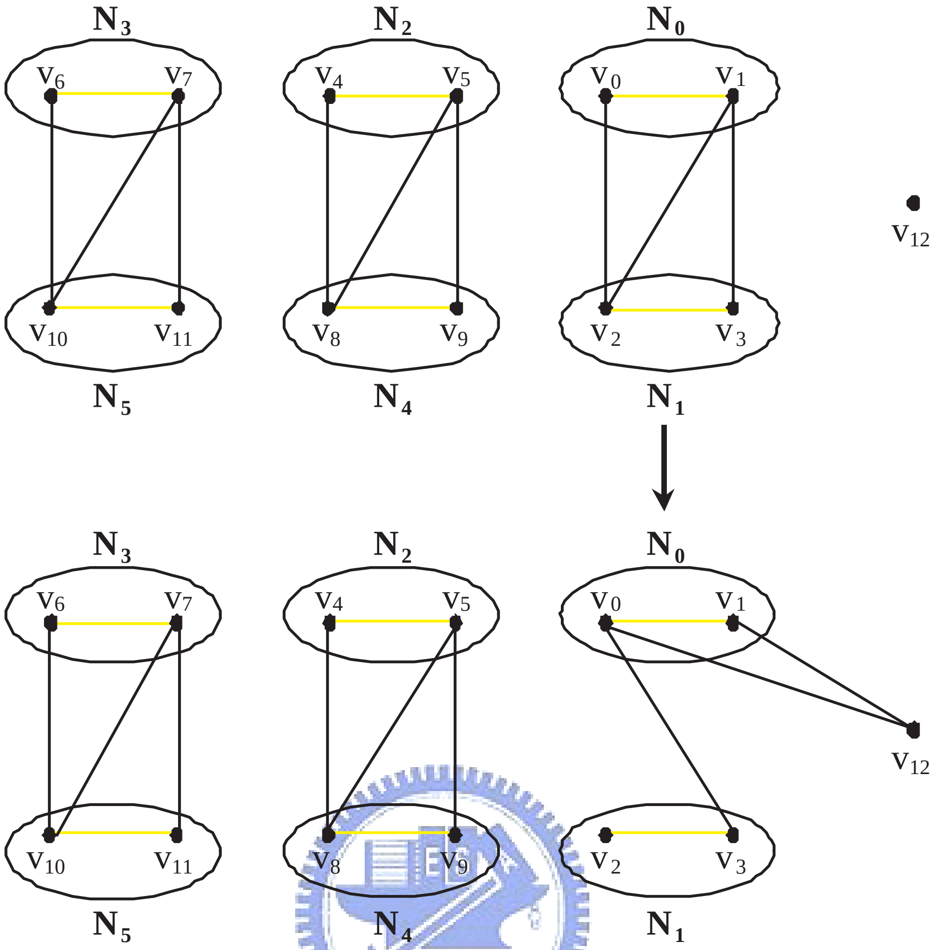

Figure 3.11: The process of replacing the paths of length 3.

Also let the replaced paths $v_{2i}v_{2i+2}v_{2i+1}v_{2i+3}$ in (N_i, N_{i+1}) for $i = 1, 3, \ldots, \frac{m-1}{2} - 3$ and another path $v_1 v_{m-3} v_{m-1} v_{m-2}$ in $(N_{\frac{m-1}{2}-1}, N_0)$ and $(v_{m-1}, N_{\frac{m-1}{2}-1})$ form a linear 3-forest in original K_m. Thus, the edges appear in $K_{1,\frac{m-1}{2}}$ and the edges appear in $(N_{\frac{m-1}{2}-1}, N_0)$, (N_i, N_{i+1}) for $0 \leq i \leq \frac{m-1}{2} - 2$ of $K_{\frac{m-1}{2}}$ are all being used.

Now, let $G(X,Y)$ be a bipartite graph with bipartition $X = \{v_{2i}\ (= x_i) |\ 0 \leq i \leq \frac{m-1}{2} - 1\}$ and $Y = \{v_{2i+1}\ (= y_i) |\ 0 \leq i \leq \frac{m-1}{2} - 1\}$. Then those edges which are not being used are exactly all of edges with bipartite differences $2, 3, \ldots, \frac{m-1}{4} - 1$ in $G(X,Y)$ and half of edges with bipartite difference $\frac{m-1}{4}$ in $G(X,Y)$.

We observe that the edges which are half of edges with bipartite difference $\frac{m-1}{4}$ in $G(X,Y)$ and all edges of M can produce a linear 3-forest in original K_m. Moreover, since the size of X (or Y) is even and $|X| = |Y|$, from the proof of Proposition 3.2.3, the edges with bipartite differences $\epsilon, \epsilon+1, \epsilon+2$ in $G(X,Y)$ for any ϵ can produce two linear 3-forests. Hence, the edges with bipartite differences $2, 3, \ldots, \frac{m-1}{4} - 1$ in $G(X,Y)$ can generate $\left\lceil (\frac{\frac{m-1}{4}-2}{3}) \cdot 2 \right\rceil = \lceil \frac{m-9}{6} \rceil$ other linear 3-forests in original K_m.

Therefore, $la_3(K_m) \le (\frac{m-1}{2} - 1) + 2 + 1 + \lceil \frac{m-9}{6} \rceil = \lceil \frac{2m}{3} \rceil$. On the other hand, by Lemma 2.1.5, $la_3(K_m) \ge \left\lceil \frac{m \cdot (m-1)}{2 \cdot \lfloor \frac{3m}{4} \rfloor} \right\rceil = \lceil \frac{2m}{3} \rceil$ if $m \equiv 1, 9 \pmod{12}$. □

Proposition 3.3.6. $la_3(K_m) = \lceil \frac{2m}{3} \rceil$ *if* $m \equiv 3, 7 \pmod{12}$.

Proof. By Proposition 3.3.3, $la_3(K_m) \le la_3(K_{m+1}) = \left\lceil \frac{2(m+1)-2}{3} \right\rceil = \lceil \frac{2m}{3} \rceil$. On the other hand, by Lemma 2.1.5, $la_3(K_m) \ge \left\lceil \frac{m(m-1)}{2 \lfloor \frac{3m}{4} \rfloor} \right\rceil = \lceil \frac{2m}{3} \rceil$ if $m \equiv 3, 7 \pmod{12}$. □

Proposition 3.3.7. $la_3(K_m) = \lceil \frac{2m}{3} \rceil$ *if* $m \equiv 5 \pmod{12}$.

Proof. By Proposition 3.3.4, $la_3(K_m) \le la_3(K_{m+1}) = \left\lceil \frac{2(m+1)}{3} \right\rceil = \lceil \frac{2m+2}{3} \rceil = \lceil \frac{2m}{3} \rceil$ if $m \equiv 5 \pmod{12}$. On the other hand, by Lemma 2.1.5, $la_3(K_m) \ge \left\lceil \frac{m(m-1)}{2 \lfloor \frac{3m}{4} \rfloor} \right\rceil = \lceil \frac{2m}{3} \rceil$ if $m \equiv 5 \pmod{12}$. □

Proposition 3.3.8. $la_3(K_m) = \lceil \frac{2m-2}{3} \rceil$ *if* $m \equiv 11 \pmod{12}$.

Proof. Assume that the vertices of K_m are $v_0, v_1, \ldots, v_{m-1}$. First, let the matching $\{v_{2i}v_{2i+1} | \; 0 \le i \le \frac{m-1}{2} - 1\}$ of K_m be denoted M. Then, for $0 \le i \le \frac{m-1}{2} - 1$, we define N_i as the set $\{v_{2i}, v_{2i+1}, v_{2i}v_{2i+1}\}$. Thus K_m can be viewed as $K_{\frac{m+1}{2}}$ with nodes v_{m-1}, N_i for $0 \le i \le \frac{m-1}{2} - 1$ and unordered pairs of nodes (v_{m-1}, N_α), (N_α, N_β) for $0 \le \alpha \ne \beta \le \frac{m-1}{2} - 1$.

Since $\frac{m+1}{2}$ is even, from the proof of Lemma 3.1.3 (by placing $N_0, N_1, \ldots, N_{\frac{m-1}{2}-1}$ equally spaced round a circle and v_{m-1} the center), $K_{\frac{m+1}{2}}$ has a 1-factorization in which there are $\frac{m+1}{2} - 1$ different 1-factors and each 1-factor owns $\frac{m+1}{4}$ disjoint unordered pairs of nodes. It is worthy of mentioning that each 1-factor of $K_{\frac{m+1}{2}}$ contains exactly one unordered pair of nodes (v_{m-1}, N_i) for some $i \in \{0, ..., \frac{m-1}{2} - 1\}$.

Hence, as the proof of the propositions previously, a 1-factor of $K_{\frac{m+1}{2}}$ can produce one linear 3-forest in original K_m except $\frac{m+1}{4} - 1$ edges which are not being used. So, from the $\frac{m+1}{2} - 1$ 1-factors of $K_{\frac{m+1}{2}}$, we obtain $\frac{m+1}{2} - 1$ linear 3-forests in original K_m except $(\frac{m+1}{2} - 1) \cdot (\frac{m+1}{4} - 1)$ edges not being used.

Now, let $G(X, Y)$ be a bipartite graph with bipartition $X = \{v_{2i}\ (= x_i) |\ 0 \leq i \leq \frac{m-1}{2} - 1\}$ and $Y = \{v_{2i+1}\ (= y_i) |\ 0 \leq i \leq \frac{m-1}{2} - 1\}$. Then those edges which are not being used and the edges of M are exactly all of edges with bipartite differences $0, 1, \ldots, \lfloor \frac{m-1}{4} \rfloor$ in $G(X, Y)$. Since the size of X (or Y) is odd and $|X| = |Y|$, from the proof of Proposition 3.2.8, the edges with bipartite differences $\epsilon, \epsilon + 1, \epsilon + 2$ in $G(X, Y)$ for any ϵ can produce two linear 3-forests except one edge with bipartite difference $\epsilon + 1$ which is not being used. Thus, the edges with bipartite differences $0, 1, \ldots, \lfloor \frac{m-1}{4} \rfloor$ in $G(X, Y)$ can generate $\left\lceil (\frac{\lfloor \frac{m-1}{4} \rfloor + 1}{3}) \cdot 2 \right\rceil = \lceil \frac{m+1}{6} \rceil$ other linear 3-forests labelled by $1, 2, \ldots, \lceil \frac{m+1}{6} \rceil$ except $\lceil \frac{m+1}{12} \rceil$ edges which are still not being used.

For example, let's consider K_m with $m = 23$. Then the partite sets of $G(X, Y)$ are $X = \{v_0, v_2, \ldots, v_{20}\}$ and $Y = \{v_1, v_3, \ldots, v_{21}\}$. Moreover, the edges with bipartite differences $0, 1, \ldots, 5$ in $G(X, Y)$ can produce four linear 3-forests labelled by $1, 2, 3, 4$ except two edges $v_{18}v_5$, $v_{20}v_1$ with bipartite differences 4 and 1 respectively which are still not being used, as shown in Figure 3.12.

	v_1	v_3	v_5	v_7	v_9	v_{11}	v_{13}	v_{15}	v_{17}	v_{19}	v_{21}
v_0	1	1	2	3	4	4					
v_2		1	2	2	3	3	4				
v_4			1	1	2	3	4	4			
v_6				1	2	2	3	3	4		
v_8					1	1	2	3	4	4	
v_{10}						1	2	2	3	3	4
v_{12}	4						1	1	2	3	4
v_{14}	3	4						1	2	2	3
v_{16}	3	4	4						1	1	2
v_{18}	2	3		4						1	2
v_{20}		2	3	3	4						1

Figure 3.12: The array shows four linear 3-forests in $G(X, Y)$.

Next, we plan to put the $\lceil \frac{m+1}{12} \rceil$ edges which are still not being used into the linear 3-forest with label 1 and interchange its edges with the edges of another linear 3-forests such that no more linear 3-forests needed to decompose K_m. The linear 3-forest with label 1 is consisting of the paths $v_{2i+1}v_{2i}v_{2i+3}v_{2i+2}$ for all $i = 0, 2, ..., \frac{m-1}{2} - 3$ and the base edge $v_{m-3}v_{m-2}$.

We start by putting the first edge $v_{m-3}v_1$ which is still not being used into the linear 3-forest with label 1, then it produce a path $P = v_{m-2}v_{m-3}v_1v_0v_3v_2$, which can not be a component of a linear 3-forest. So, we interchange the edge $v_{m-3}v_{m-2}$ in P with another edge $v_{m-2}v_{m-1}$ in the linear 3-forest constructed from the 1-factor contains $(v_{m-1}, N_{\frac{m-1}{2}-1})$. Again, we interchange the edge v_2v_3 in P with another edge v_3v_{m-1} in the linear 3-forest constructed from the 1-factor contains (v_{m-1}, N_1) and move the edge v_3v_{m-1} into the linear 3-forest with label 2.

Since the linear 3-forest with label 2 is consisting of the paths $v_{2i}v_{2i+5}v_{2i+2}v_{2i+7}$ for all $i = 0, 2, ..., \frac{m-1}{2} - 3$ and the base edge $v_{m-3}v_3$, that movement creates a path $v_{m-3}v_3v_{m-1}$ and we have a new linear 3-forest with label 2. Moreover, the steps above let the length of P become 3. Hence, we also have a new linear 3-forest with label 1, which is consisting of paths with length 3 and one edge $v_{m-2}v_{m-1}$. Note that the index of each vertex is modulo m.

Without loss of generality, for $2 \le \ell \le \lceil \frac{m+1}{12} \rceil$, we assume that the ℓth edge still not being used is $v_{m-5-4(\ell-2)}v_{5+2(\ell-2)}$, abbreviated to $v_{m-4\ell+3}v_{2\ell+1}$. Then, for $2 \le \ell \le \lceil \frac{m+1}{12} \rceil$, we put the ℓth edge $v_{m-4\ell+3}v_{2\ell+1}$ still not being used sequentially into the linear 3-forest with label 1 according to the following rules.

If ℓ is even, then we interchange the edge $v_{m-4\ell+3}v_{m-4\ell+4}$ in the linear 3-forest with label 1 with another edge $v_{m-4\ell+3}v_{m-1}$ in the linear 3-forest constructed from the 1-factor contains $(v_{m-1}, N_{\frac{m-4\ell+3}{2}})$ and move the edge $v_{m-4\ell+3}v_{m-1}$ into the linear 3-forest with label $2\ell-1$. We also interchange the edge $v_{2\ell+2}v_{2\ell+3}$ in the linear 3-forest with label 1 with another edge $v_{2\ell+3}v_{m-1}$ in the linear 3-forest constructed from the 1-factor contains $(v_{m-1}, N_{\frac{2\ell+2}{2}})$ and move the edge $v_{2\ell+3}v_{m-1}$ into the linear 3-forest with label 2ℓ.

If ℓ is odd, then we interchange the edge $v_{2\ell-2}v_{2\ell-1}$ in the linear 3-forest with label 1 with another edge $v_{2\ell-1}v_{m-1}$ in the linear 3-forest constructed from the 1-factor contains $(v_{m-1}, N_{\frac{2\ell-2}{2}})$ and move the edge $v_{2\ell-1}v_{m-1}$ into the linear 3-forest with label $2\ell - 1$. We also interchange the edge $v_{m-4\ell+3}v_{m-4\ell+4}$ in the linear 3-forest with label 1 with another edge $v_{m-4\ell+3}v_{m-1}$ in the linear 3-forest constructed from the 1-factor contains $(v_{m-1}, N_{\frac{m-4\ell+3}{2}})$ and move the edge $v_{m-4\ell+3}v_{m-1}$ into the linear 3-forest with label 2ℓ.

By using the above method recursively until all edges which are still not being used have been putted completely into the linear 3-forest with label 1, we can find that each component in the linear 3-forest with label 1 and the other linear 3-forests is a path of length at most three. For example, Figure 3.13 shows the linear 3-forest with label 1 in K_m with $m = 35$.

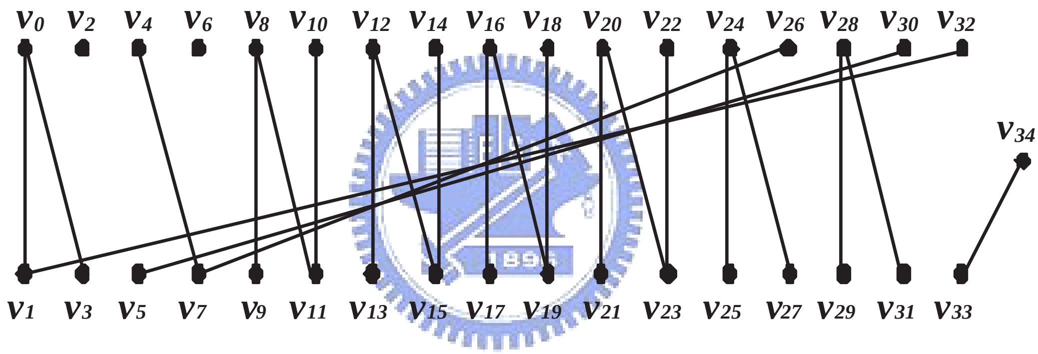

Figure 3.13: A linear 3-forest in K_{35}.

Therefore, we have $la_3(K_m) \leq (\frac{m+1}{2} - 1) + \lceil \frac{m+1}{6} \rceil = \lceil \frac{2m-1}{3} \rceil = \lceil \frac{2m-2}{3} \rceil$ if $m \equiv 11 \pmod{12}$. On the other hand, by Lemma 2.1.5, $la_3(K_m) \geq \left\lceil \frac{m \cdot (m-1)}{2 \cdot \lfloor \frac{3m}{4} \rfloor} \right\rceil = \lceil \frac{2m-2}{3} \rceil$ if $m \equiv 11 \pmod{12}$. □

From the propositions given above, we determine the linear 3-arboricity of K_m for any m and conclude the work of this section with the following theorem.

Theorem 3.3.9.

$$la_3(K_m) = \left\lceil \frac{m(m-1)}{2\lfloor \frac{3m}{4} \rfloor} \right\rceil = \begin{cases} \lceil \frac{2m-2}{3} \rceil & \textit{if} \ \ m \equiv 0, 4, 8, 11 \pmod{12}, \\ \lceil \frac{2m}{3} \rceil & \textit{if} \ \ m \equiv 1, 2, 3, 5, 6, 7, 9, 10 \pmod{12}. \end{cases}$$

3.4 Balanced Complete Multipartite Graphs

In this section, we study the linear 3-arboricity of a balanced complete multipartite graph $K_{m(n)}$ with $mn \equiv 0 \pmod 4$. Before we go any further, we need some more lemmas.

Let $P_{\alpha(\beta)}$ be an α-partite graph such that each partite set V_i has β vertices for all $i \in \{0, 1, \ldots, \alpha - 1\}$ and the edge $uv \in E(P_{\alpha(\beta)})$ if and only if $u \in V_w$ and $v \in V_{w+1}$ where $w \in \{0, 1, \ldots, \alpha - 2\}$.

Lemma 3.4.1. *$la_k(P_{k+1(s)}) = s$.*

Proof. First, for all $i \in \{0, 1, \ldots, k\}$, assume that the vertices of partite set V_i in $P_{k+1(s)}$ are $v_{i[0]}, v_{i[1]}, \ldots, v_{i[s-1]}$. Then, let the ℓth linear k-forest be the set of P_{k+1}'s $\{v_{0[j]} v_{1[j+(\ell-1)]} \ldots v_{k[j+k(\ell-1)]} \mid j = 0, 1, \ldots, s-1\}$ for all $\ell \in \{1, 2, \ldots, s\}$. Note that the index y of each vertex $v_{x[y]}$ is modulo s. It is not difficult to check that the edges in the linear k-forests above are distinct and exactly all of the edges in $P_{k+1(s)}$. Thus $la_k(P_{k+1(s)}) = s$. □

Lemma 3.4.2. *$la_k(K_{m(sn)}) \leq s \cdot la_k(K_{m(n)})$.*

Proof. We can obtain $K_{m(sn)}$ from $K_{m(n)}$ by replacing each edge of $K_{m(n)}$ with $K_{s,s}$. Hence, a path P_λ in a linear k-forest of $K_{m(n)}$ corresponds to a λ-partite subgraph $P_{\lambda(s)}$ of $K_{m(sn)}$, where $2 \leq \lambda \leq k+1$. Moreover, $la_k(P_{\lambda(s)}) \leq la_k(P_{k+1(s)})$ for all $2 \leq \lambda \leq k+1$. Therefore, $la_k(K_{m(sn)}) \leq la_k(P_{k+1(s)}) \cdot la_k(K_{m(n)}) = s \cdot la_k(K_{m(n)})$ by Lemmas 3.1.4 and 3.4.1. □

Lemma 3.4.3. *If $n \equiv 0 \pmod{2^\sigma}$ where $\sigma \geq 1$, then $K_{m(n)}$ has a $K_{\frac{n}{2^\sigma},\frac{n}{2^\sigma}}$-factorization and there are $2^\sigma(m-1)$ $K_{\frac{n}{2^\sigma},\frac{n}{2^\sigma}}$-factors in it.*

Proof. We prove this lemma by using induction on the number σ. Assume $\sigma = 1$. From Lemma 3.1.3 (by replacing each edge of K_{2m} with $K_{\frac{n}{2},\frac{n}{2}}$), the graph $K_{2m(\frac{n}{2})}$ has a $K_{\frac{n}{2},\frac{n}{2}}$-factorization in which there are $2m-1$ $K_{\frac{n}{2},\frac{n}{2}}$-factors. Moreover, $K_{2m(\frac{n}{2})}$ is the union of $K_{m(n)}$ and a $K_{\frac{n}{2},\frac{n}{2}}$-factor of $K_{2m(\frac{n}{2})}$. Hence, $K_{m(n)}$ has a $K_{\frac{n}{2},\frac{n}{2}}$-factorization and there are $2m-2 = 2(m-1)$ $K_{\frac{n}{2},\frac{n}{2}}$-factors in it. This provides the basis.

For the induction step, suppose $\sigma = h+1 \geq 2$. The induction hypothesis is that $K_{m(n)}$ has a $K_{\frac{n}{2^h},\frac{n}{2^h}}$-factorization in which there are $2^h(m-1)$ $K_{\frac{n}{2^h},\frac{n}{2^h}}$-factors. Since a $K_{\frac{n}{2^h},\frac{n}{2^h}}$-factor can be decomposed into two $K_{\frac{n}{2^{h+1}},\frac{n}{2^{h+1}}}$-factors, then $K_{m(n)}$ has a $K_{\frac{n}{2^{h+1}},\frac{n}{2^{h+1}}}$-factorization and there are $2\cdot 2^h(m-1) = 2^{h+1}(m-1)$ $K_{\frac{n}{2^{h+1}},\frac{n}{2^{h+1}}}$-factors in it. Therefore, by mathematical induction, the assertion holds. □

Now, we are ready to prove the main results on $la_3(K_{m(n)})$.

Proposition 3.4.4. $la_3(K_{m(n)}) \leq \frac{2(m-1)n}{3}$ *if* $m \equiv 0 \pmod 2$ *and* $n \equiv 0 \pmod 6$.

Proof. From Lemma 3.1.3 (by replacing each edge of K_m with $K_{n,n}$), $K_{m(n)}$ has a $K_{n,n}$-factorization and there are $m-1$ $K_{n,n}$-factors in it. Hence, $la_3(K_{m(n)}) \leq (m-1)\cdot la_3(K_{n,n}) = (m-1)\cdot\frac{2n}{3} = \frac{2(m-1)n}{3}$ by Lemma 3.1.4 and Theorem 3.2.9. □

Proposition 3.4.5. $la_3(K_{m(n)}) \leq \frac{2(m-1)n}{3}$ *if* $n \equiv 0 \pmod{12}$.

Proof. From Lemma 3.4.3, $K_{m(n)}$ has a $K_{\frac{n}{2},\frac{n}{2}}$-factorization in which there are $2m-2$ $K_{\frac{n}{2},\frac{n}{2}}$-factors. Therefore, $la_3(K_{m(n)}) \leq (2m-2)\cdot la_3(K_{\frac{n}{2},\frac{n}{2}}) = (2m-2)\cdot\frac{2\left(\frac{n}{2}\right)}{3} = \frac{2(m-1)n}{3}$ by Lemma 3.1.4 and Theorem 3.2.9. □

Proposition 3.4.6. $la_3(K_{m(n)}) \leq \frac{2(m-1)n}{3}$ *if* $m \equiv 4 \pmod{12}$.

Proof. From Lemma 3.1.2, K_m has a K_4-factorization and there are $\frac{|E(K_m)|}{\left(\frac{|V(K_m)|}{4}\right)\cdot 6} = \frac{m-1}{3}$ K_4-factors in it. Since $la_3(K_4) = 2$, from Lemmas 3.1.4 and 3.4.2, $la_3(K_{m(n)}) \leq n\cdot la_3(K_m) \leq n\cdot\frac{m-1}{3}\cdot la_3(K_4) = \frac{2(m-1)n}{3}$. □

Proposition 3.4.7. $la_3(K_{m(n)}) \leq \frac{2(m-1)n}{3}$ *if* $m \equiv 1 \pmod 3$ *and* $n \equiv 0 \pmod 4$.

Proof. Since $4m \equiv 4 \pmod{12}$, from Lemma 3.1.2, K_{4m} has a K_4-factorization and there are $\frac{|E(K_{4m})|}{\left(\frac{|V(K_{4m})|}{4}\right)\cdot 6} = \frac{4m-1}{3}$ K_4-factors in it. Moreover, K_{4m} is the union of $K_{m(4)}$ and one K_4-factor of K_{4m}. Hence, $K_{m(4)}$ has a K_4-factorization in which there are $\frac{4m-1}{3}-1$ K_4-factors. By Lemmas 3.1.4 and 3.4.2, $la_3(K_{m(n)}) \leq \frac{n}{4}\cdot la_3(K_{m(4)}) \leq \frac{n}{4}\cdot\left(\frac{4m-1}{3}-1\right)\cdot la_3(K_4) = \frac{n}{4}\cdot\frac{8(m-1)}{3} = \frac{2(m-1)n}{3}$. □

Proposition 3.4.8. $la_3(K_{m(n)}) \leq \frac{2(m-1)n}{3}$ *if* $m \equiv 0 \pmod 4$ *and* $n \equiv 0 \pmod 3$.

Proof. Dividing all m partite sets of $K_{m(n)}$ into $\frac{m}{4}$ disjoint collections of four partite sets shows that $K_{m(n)}$ is the union of $K_{\frac{m}{4}(4n)}$ and one $K_{4(n)}$-factor of $K_{m(n)}$. Since $4n \equiv 0 \pmod{12}$, by Propositions 3.4.5 and 3.4.6, $la_3(K_{m(n)}) \le la_3(K_{4(n)}) + la_3(K_{\frac{m}{4}(4n)}) \le \frac{2(4-1)n}{3} + \frac{2\left(\frac{m}{4}-1\right)(4n)}{3} = \frac{2(m-1)n}{3}$. □

Proposition 3.4.9. *$la_3(K_{m(n)}) \le \frac{2(m-1)n}{3}$ if $m \equiv 10 \pmod{12}$ and $n \equiv 0 \pmod 2$.*

Proof. From Lemma 3.1.3 (by replacing each edge of K_m with $K_{2,2}$), $K_{m(2)}$ has a $K_{2,2}$-factorization and there are $m-1$ $K_{2,2}$-factors in it. Moreover, since $K_{2,2}$ is consisting of a path P_4 and one isolated edge, then a linear 3-forest can be induced by the set of P_4's in all $K_{2,2}$ of any $K_{2,2}$-factor in $K_{m(2)}$. Therefore, we obtain $m-1$ linear 3-forests from the $m-1$ $K_{2,2}$-factors of $K_{m(2)}$. Now, we want to show that the isolated edges in those $K_{2,2}$ of $K_{2,2}$-factors in $K_{m(2)}$ also produce linear 3-forests.

For all $i \in \{0, 1, \ldots, m-1\}$, let the vertices of partite set V_i in $K_{m(2)}$ be denoted $v_{i[0]}$ and $v_{i[1]}$. Without loss of generality, we assume that all isolated edges in those $K_{2,2}$ of $K_{2,2}$-factors in $K_{m(2)}$ are the edges of $\frac{m}{2}-1$ perfect matchings $U_1, U_2, \ldots, U_{\frac{m}{2}-1}$ and a matching $M_{\frac{m}{2}}$ in $K_{m(2)}$, where $U_\ell = \left\{v_{i[0]}v_{i+\ell[1]} \mid i = 0, 1, \ldots, m-1\right\}$ for $\ell \in \{1, \ldots, \frac{m}{2}-1\}$ and $M_{\frac{m}{2}} = \left\{v_{i[0]}v_{i+\frac{m}{2}[1]} \mid i = 0, 2, \ldots, m-2\right\}$. Then, the edges of $U_1, U_2, \ldots, U_{\frac{m}{2}-2}$ can generate $\frac{2\left(\frac{m}{2}-2\right)}{3}$ linear 3-forests from the proof of Proposition 3.2.3 and the edges of $U_{\frac{m}{2}-1}, M_{\frac{m}{2}}$ also produce a linear 3-forest. Thus, $la_3(K_{m(n)}) \le \frac{n}{2} \cdot la_3(K_{m(2)}) \le \frac{n}{2} \cdot [(m-1) + \frac{2\left(\frac{m}{2}-2\right)}{3} + 1] = \frac{2(m-1)n}{3}$ by Lemma 3.4.2. □

Concluding the conditions of the pair (m, n) in the propositions given above, we find that $mn \equiv 0 \pmod 4$ and $(m-1)n \equiv 0 \pmod 3$. On the other hand, by Lemma 2.1.5, it is easy to show that $la_3(K_{m(n)}) \ge \frac{2(m-1)n}{3}$ if $mn \equiv 0 \pmod 4$ and $(m-1)n \equiv 0 \pmod 3$. Therefore, we have the following:

Corollary 3.4.10. *$la_3(K_{m(n)}) = \frac{2(m-1)n}{3}$ when $mn \equiv 0 \pmod 4$ and $(m-1)n \equiv 0 \pmod 3$.*

It is worthy of noting that, in 1999, Muthusamy and Paulraja [21] showed that:

Theorem 3.4.11. *For $k = p+1 > 3$ and p is a prime, $K_{m(n)}$ has a P_k-factorization if and only if $mn \equiv 0 \pmod{k}$ and $2(k-1) \mid k(m-1)n$.*

From the definitions of the linear $(k-1)$-arboricity and a P_k-factorization of a graph, we know that if a graph G has a P_k-factorization then $la_{k-1}(G)$ is equal to $\frac{k\cdot|E(G)|}{(k-1)\cdot|V(G)|}$, which is the number of P_k-factors required to decompose G. Therefore, what we have proved gives an independent proof of the case $k=4$ of Theorem 3.4.11.

Next, we consider the cases when $K_{m(n)}$ does not have a P_4-factorization.

Proposition 3.4.12. *$la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ if $m \equiv 0,4,6,8 \pmod{12}$ and $n \equiv 4 \pmod{6}$.*

Proof. From Lemma 3.1.3 (by replacing each edge of K_m with $K_{n,n}$), $K_{m(n)}$ has a $K_{n,n}$-factorization in which there are $m-1$ $K_{n,n}$-factors. Hence, from the proof of Proposition 3.2.3, the edges with bipartite differences $1, 2, \ldots, n-1$ in those $K_{n,n}$ of $K_{n,n}$-factors in $K_{m(n)}$ can generate $(m-1)\cdot(\frac{2(n-1)}{3})$ linear 3-forests.

Moreover, it is not difficult to see that the subgraph induced by the set of edges with bipartite difference 0 in those $K_{n,n}$ of $K_{n,n}$-factors in $K_{m(n)}$ is exactly a K_m-factor. Therefore, by Theorem 3.3.9 and $\lceil \frac{2m}{3} \rceil = \lceil \frac{2m-2}{3} \rceil$ if $m \equiv 6 \pmod{12}$, we have that $la_3(K_{m(n)}) \leq (m-1)\cdot(\frac{2(n-1)}{3}) + la_3(K_m) = (m-1)\cdot(\frac{2(n-1)}{3}) + \lceil \frac{2m-2}{3} \rceil = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

Proposition 3.4.13. *$la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ if $m \equiv 2 \pmod{6}$ and $n \equiv 0 \pmod{2}$.*

Proof. Dividing all m partite sets of $K_{m(n)}$ into $\frac{m}{2}$ disjoint pairs of two partite sets shows that $K_{m(n)}$ is the union of $K_{\frac{m}{2}(2n)}$ and one $K_{n,n}$-factor of $K_{m(n)}$. Since $\frac{m}{2} \equiv 1 \pmod{3}$ and $2n \equiv 0 \pmod{4}$, by Theorem 3.2.9 and Proposition 3.4.7, $la_3(K_{m(n)}) \leq la_3(K_{n,n}) + la_3(K_{\frac{m}{2}(2n)}) \leq \lceil \frac{2n}{3} \rceil + \frac{2(\frac{m}{2}-1)(2n)}{3} = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

Proposition 3.4.14. *$la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ if $m \equiv 0 \pmod{6}$ and $n \equiv 2 \pmod{6}$.*

Proof. From Lemma 3.1.3 (by replacing each edge of K_m with $K_{n,n}$), $K_{m(n)}$ has a $K_{n,n}$-factorization in which there are $m-1$ $K_{n,n}$-factors. Hence, from the proof of Proposition 3.2.3, the edges with bipartite differences $2, 3, \ldots, n-1$ in those $K_{n,n}$ of $K_{n,n}$-factors in $K_{m(n)}$ can generate $(m-1)\cdot(\frac{2(n-2)}{3})$ linear 3-forests.

Moreover, the edges with bipartite differences $0, 1$ in those $K_{n,n}$ of $K_{n,n}$-factors in $K_{m(n)}$ also can produce $m-1$ linear 3-forests except half of the edges with bipartite difference 1 in those $K_{n,n}$ of $K_{n,n}$-factors in $K_{m(n)}$ which are not being used. Thus, in what follows, we want to show that those edges which are not being used also produce linear 3-forests.

For all $i \in \{0, 1, \ldots, m-1\}$, let the vertices of partite set V_i in $K_{m(n)}$ be denoted $v_{i[0]}, v_{i[1]}, \ldots, v_{i[n-1]}$. Without loss of generality, we assume that those edges which are not being used are the edges of $\frac{m}{2}-1$ perfect matchings $U_1, U_2, \ldots, U_{\frac{m}{2}-1}$ and a matching $M_{\frac{m}{2}}$ in $K_{m(n)}$, where

$$U_\ell = \left\{ v_{i[j]} v_{i+\ell[j+1]} \middle| \; i \in \{0, 1, \ldots, m-1\}, j \in \{1, 3, \ldots, n-1\} \right\}$$

for all $\ell \in \{1, \ldots, \frac{m}{2}-1\}$ and

$$M_{\frac{m}{2}} = \left\{ v_{i[j]} v_{i+\frac{m}{2}[j+1]} \middle| \; i \in \{0, 1, \ldots, \tfrac{m}{2}-1\}, j \in \{1, 3, \ldots, n-1\} \right\}.$$

Then, from the proof of Proposition 3.2.3, the edges of $U_1, U_2, \ldots, U_{\frac{m}{2}-1}, M_{\frac{m}{2}}$ can generate $\frac{2 \cdot \frac{m}{2}}{3}$ linear 3-forests in $K_{m(n)}$. Therefore, $la_3(K_{m(n)}) \leq (m-1) \cdot (\frac{2(n-2)}{3}) + (m-1) + \frac{2 \cdot \frac{m}{2}}{3} = \frac{2(m-1)n+1}{3} = \lceil \frac{2(m-1)n}{3} \rceil$. $\square$

Proposition 3.4.15. *$la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ if $m \equiv 3 \pmod 6$ and $n \equiv 4 \pmod{12}$.*

Proof. By Lemma 3.4.3, $K_{m(n)}$ has a $K_{\frac{n}{2},\frac{n}{2}}$-factorization in which there are $2m-2$ $K_{\frac{n}{2},\frac{n}{2}}$-factors. Hence, from the proof of Proposition 3.2.3, the edges with bipartite differences $2, 3, \ldots, \frac{n}{2}-1$ in those $K_{\frac{n}{2},\frac{n}{2}}$ of $K_{\frac{n}{2},\frac{n}{2}}$-factors in $K_{m(n)}$ can generate $(2m-2) \cdot (\frac{2(\frac{n}{2}-2)}{3})$ linear 3-forests.

Moreover, the edges with bipartite differences $0, 1$ in those $K_{\frac{n}{2},\frac{n}{2}}$ of $K_{\frac{n}{2},\frac{n}{2}}$-factors in $K_{m(n)}$ also can produce $2m-2$ linear 3-forests except half of the edges with bipartite difference 1 in those $K_{\frac{n}{2},\frac{n}{2}}$ of $K_{\frac{n}{2},\frac{n}{2}}$-factors in $K_{m(n)}$ which are not being used.

Therefore, in what follows, we want to show that those edges which are not being used also produce linear 3-forests. Since $K_{2m(\frac{n}{2})}$ is the union of $K_{m(n)}$ and one $K_{\frac{n}{2},\frac{n}{2}}$-factor of $K_{2m(\frac{n}{2})}$, for convenience, we consider this question on $K_{2m(\frac{n}{2})}$.

For all $i \in \{0,1,\ldots,2m-1\}$, let the vertices of partite set V_i in $K_{2m(\frac{n}{2})}$ be denoted $v_{i[0]}, v_{i[1]}, \ldots, v_{i[\frac{n}{2}-1]}$. Without loss of generality, we assume that those edges which are not being used are the edges of $m-2$ perfect matchings $U_2, U_3, \ldots, U_{m-1}$ and two matchings M_1, M_m in $K_{2m(\frac{n}{2})}$, where

$$\begin{aligned} M_1 = & \left\{ v_{i[j]}v_{i+1[j+1]} \middle| \, i \in \{1,3,\ldots,2m-1\}, j \in \{1,3,\ldots,\tfrac{n}{2}-1\} \right\}, \\ U_\ell = & \left\{ v_{i[j]}v_{i+\ell[j+1]} \middle| \, i \in \{0,1,\ldots,2m-1\}, j \in \{1,3,\ldots,\tfrac{n}{2}-1\} \right\} \end{aligned}$$

for all $\ell \in \{2,3,\ldots,m-1\}$ and

$$M_m = \left\{ v_{i[j]}v_{i+m[j+1]} \middle| \, i \in \{0,2,\ldots,2m-2\}, j \in \{1,3,\ldots,\tfrac{n}{2}-1\} \right\}.$$

Then (i) the edges of M_1 and U_2 can produce a linear 3-forest; (ii) the edges of $U_3, U_4, \ldots, U_{m-1}$ can generate $\frac{2(m-3)}{3}$ linear 3-forests from the proof of Proposition 3.2.3; (iii) the edges of M_m can produce a linear 3-forest. Hence, $la_3(K_{m(n)}) \leq (2m-2) \cdot \left(\frac{2\left(\frac{n}{2}-2\right)}{3} \right) + (2m-2) + \left(2 + \frac{2(m-3)}{3} \right) = \frac{2(m-1)n+2}{3} = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

Proposition 3.4.16. *$la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ if $m \equiv 5 \pmod 6$ and $n \equiv 4 \pmod{12}$.*

Proof. It is similar to the proof of Proposition 3.4.15 except the following: (i) The edges of M_1 and M_m can produce a linear 3-forest; (ii) the edges of $U_2, U_3, \ldots, U_{m-1}$ can generate $\frac{2(m-2)}{3}$ linear 3-forests from the proof of Proposition 3.2.3. Therefore, $la_3(K_{m(n)}) \leq (2m-2) \cdot \left(\frac{2\left(\frac{n}{2}-2\right)}{3} \right) + (2m-2) + \left(1 + \frac{2(m-2)}{3} \right) = \frac{2(m-1)n+1}{3} = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

Proposition 3.4.17. *$la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ if $m \equiv 3 \pmod 6$ and $n \equiv 8 \pmod{12}$.*

Proof. By Lemma 3.4.3, $K_{m(n)}$ has a $K_{\frac{n}{4},\frac{n}{4}}$-factorization in which there are $4m-4$ $K_{\frac{n}{4},\frac{n}{4}}$-factors. Hence, from the proof of Proposition 3.2.3, the edges with bipartite differences $2, 3, \ldots, \frac{n}{4}-1$ in those $K_{\frac{n}{4},\frac{n}{4}}$ of $K_{\frac{n}{4},\frac{n}{4}}$-factors in $K_{m(n)}$ can generate $(4m-4) \cdot \left(\frac{2\left(\frac{n}{4}-2\right)}{3} \right)$ linear 3-forests.

Moreover, the edges with bipartite differences $0, 1$ in those $K_{\frac{n}{4},\frac{n}{4}}$ of $K_{\frac{n}{4},\frac{n}{4}}$-factors in $K_{m(n)}$ also can produce $4m-4$ linear 3-forests except half of the edges with bipartite difference 1 in those $K_{\frac{n}{4},\frac{n}{4}}$ of $K_{\frac{n}{4},\frac{n}{4}}$-factors in $K_{m(n)}$ which are not being used.

Therefore, in what follows, we want to show that those edges which are not being used also produce linear 3-forests. Since $K_{4m(\frac{n}{4})}$ is the union of $K_{m(n)}$ and three $K_{\frac{n}{4},\frac{n}{4}}$-factors of $K_{4m(\frac{n}{4})}$, for convenience, we consider this question on $K_{4m(\frac{n}{4})}$.

For all $i \in \{0, 1, \ldots, 4m-1\}$, let the vertices of partite set V_i in $K_{4m(\frac{n}{4})}$ be denoted $v_{i[0]}, v_{i[1]}, \ldots, v_{i[\frac{n}{4}-1]}$. Without loss of generality, we assume that those edges which are not being used are the edges of $2m-4$ perfect matchings $U_4, U_5, \ldots, U_{2m-1}$ and four matchings M_1, M_2, M_3, M_{2m} in $K_{4m(\frac{n}{4})}$, where

$$\begin{aligned} M_1 = &\ \left\{ v_{i[j]}v_{i+1[j+1]} \middle| \ i \in \{3, 7, \ldots, 4m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\} \right\}, \\ M_2 = &\ \left\{ v_{i[j]}v_{i+2[j+1]} \middle| \ i \in \{2, 3, 6, 7, \ldots, 4m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\} \right\}, \end{aligned}$$

$$M_3 = \left\{ v_{i[j]}v_{i+3[j+1]} \middle| \ i \in \{1, 2, 3, 5, 6, 7, \ldots, 4m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\} \right\},$$

$$U_\ell = \ \left\{ v_{i[j]}v_{i+\ell[j+1]} \middle| \ i \in \{0, 1, \ldots, 4m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\} \right\}$$

for all $\ell \in \{4, 5, \ldots, 2m-1\}$ and

$$M_{2m} = \left\{ v_{i[j]}v_{i+2m[j+1]} \middle| \ i \in \{0, 1, 4, 5, \ldots, 4m-3\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\} \right\}.$$

Then (i) the edges of M_1, a subset $\{v_{i[j]}v_{i+3[j+1]} | \ i = 2, 6, \ldots, 4m-2, j = 1, 3, \ldots, \frac{n}{4}-1\}$ of M_3, and U_4 can produce a linear 3-forest; (ii) the edges of M_2, a subset $\{v_{i[j]}v_{i+3[j+1]} | \ i = 1, 3, \ldots, 4m-1, j = 1, 3, \ldots, \frac{n}{4}-1\}$ of M_3, and M_{2m} can produce a linear 3-forest; (iii) the edges of $U_5, U_6, \ldots, U_{2m-2}$ can generate $\frac{2(2m-6)}{3}$ linear 3-forests from the proof of Proposition 3.2.3; (iv) the edges of U_{2m-1} can produce a linear 3-forest. Hence, $la_3(K_{m(n)}) \leq (4m-4) \cdot \left(\frac{2(\frac{n}{4}-2)}{3} \right) + (4m-4) + \left(3 + \frac{2(2m-6)}{3} \right) = \frac{2(m-1)n+1}{3} = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. $\square$

Proposition 3.4.18. *$la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ if $m \equiv 5 \pmod 6$ and $n \equiv 8 \pmod{12}$.*

Proof. It is similar to the proof of Proposition 3.4.17 except the following: (i) The edges of M_1 and M_3 can produce a linear 3-forest; (ii) the edges of M_2 and U_4 can produce a linear 3-forest; (iii) the edges of $U_5, U_6, \ldots, U_{2m-1}$ and M_{2m} can generate $\frac{2(2m-4)}{3}$ linear 3-forests from the proof of Proposition 3.2.3. Hence, $la_3(K_{m(n)}) \leq (4m-4) \cdot \left(\frac{2(\frac{n}{4}-2)}{3} \right) + (4m-4) + \left(2 + \frac{2(2m-4)}{3} \right) = \frac{2(m-1)n+2}{3} = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. $\square$

Proposition 3.4.19. $la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ *if* $m \equiv 0, 8 \pmod{12}$ *and* $n \equiv 1, 5 \pmod 6$.

Proof. Dividing all m partite sets of $K_{m(n)}$ into $\frac{m}{4}$ disjoint collections of four partite sets shows that $K_{m(n)}$ is the union of $K_{\frac{m}{4}(4n)}$ and one $K_{4(n)}$-factor of $K_{m(n)}$. Since $\frac{m}{4} \equiv 0, 2 \pmod 3$ and $4n \equiv 4, 8 \pmod{12}$, from Propositions 3.4.6 and 3.4.12 $\sim$ 3.4.18, $la_3(K_{m(n)}) \leq la_3(K_{4(n)}) + la_3(K_{\frac{m}{4}(4n)}) \leq \frac{2(4-1)n}{3} + \left\lceil \frac{2\left(\frac{m}{4}-1\right)(4n)}{3} \right\rceil = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

From the propositions given above, we have that $la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ if $mn \equiv 0 \pmod 4$. On the other hand, by Lemma 2.1.5, $la_3(K_{m(n)}) \geq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ if $mn \equiv 0 \pmod 4$. Hence, we determine the linear 3-arboricity of $K_{m(n)}$ for $mn \equiv 0 \pmod 4$ and conclude the work of this section with the following theorem.

Theorem 3.4.20.

$$la_3(K_{m(n)}) = \left\lceil \frac{2(m-1)n}{3} \right\rceil \text{ when } mn \equiv 0 \pmod 4.$$

Concluding Remark. By using the ideas in this section, we can also find $la_3(K_{m(n)})$ for quite a few other cases when $mn \equiv 2 \pmod 4$. But, we are not able to finish the whole part at this moment due to several stubborn subcases. As for the cases when mn is odd, they are expected to be more difficult.

We remark finally that the work about the linear 3-arboricity of balanced complete multipartite graphs presented in this section will appear in [29].

Chapter 4

Linear 2-arboricity of Complete Multipartite Graphs

In this chapter, we study the linear 2-arboricity problem on complete bipartite graphs, complete graphs, and balanced complete multipartite graphs. The results obtained are coherent with the corresponding cases of Conjecture 2.2.1.

4.1 Complete Bipartite Graphs

Let $K_{r,s}$ denote a complete bipartite graph with partite sets of sizes r and s. If $r = s = n$, then such a graph is called a balanced complete bipartite graph and denoted $K_{n,n}$. Without loss of generality, we assume that $s \geq r$.

In Chapter 2, we had mentioned that the following result by Fu and Huang [10] about the linear 2-arboricity of $K_{n,n}$.

Theorem 4.1.1. $la_2(K_{n,n}) = \left\lceil \frac{n^2}{\lfloor \frac{4n}{3} \rfloor} \right\rceil$.

Naturally, we would like to determine the linear 2-arboricity of $K_{r,s}$ when $s > r$. So, we begin with the case $s \geq 2r$ of $K_{r,s}$.

Theorem 4.1.2. *If* $s \geq 2r$, *then* $la_2(K_{r,s}) = \lceil \frac{s}{2} \rceil$.

Proof. Assume that the partite sets of $K_{r,s}$ are $X = \{x_0, x_1, \ldots, x_{r-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{s-1}\}$. For $0 \leq j \leq \lceil \frac{s}{2} \rceil - 1$, we define N_j as the set $\{y_{2j}, y_{2j+1}\}$ except $N_{\lceil \frac{s}{2} \rceil - 1} = \{y_{s-1}\}$ when s is odd. Then $K_{r,s}$ can be viewed as $K_{r,\lceil \frac{s}{2} \rceil}$ with nodes x_i, N_j and unordered pairs of nodes (x_i, N_j) for $0 \leq i \leq r-1$ and $0 \leq j \leq \lceil \frac{s}{2} \rceil - 1$.

Moreover, since each unordered pair of nodes (x_i, N_j) in $K_{r,\lceil \frac{s}{2} \rceil}$ is composed of a path $y_{2j}x_iy_{2j+1}$ in original $K_{r,s}$ except $(x_i, N_{\lceil \frac{s}{2} \rceil - 1})$ which is an edge x_iy_{s-1} when s is odd. Therefore, for $\ell = 0, 1, \ldots, \lceil \frac{s}{2} \rceil - 1$, the unordered pairs of nodes with bipartite difference ℓ in $K_{r,\lceil \frac{s}{2} \rceil}$ can produce a linear 2-forest in original $K_{r,s}$. Hence, $la_2(K_{r,s}) \leq \lceil \frac{s}{2} \rceil$. On the other hand, if $s \geq 2r$, then it is not difficult to see that a linear 2-forest in $K_{r,s}$ has at most $2r$ edges and then $la_2(K_{r,s}) \geq \lceil \frac{|E(K_{r,s})|}{2r} \rceil = \lceil \frac{rs}{2r} \rceil = \lceil \frac{s}{2} \rceil$. □

In what follows, we consider the cases when $2r \geq s > r$. First, let P_n be a path with n vertices. An earlier work of Ushio [25] had shown the following:

Theorem 4.1.3. *$K_{r,s}$ has a P_3-factorization if and only if* (i) $r + s \equiv 0 \pmod 3$, (ii) $2s \geq r$, (iii) $2r \geq s$, and (iv) $\frac{3rs}{2(r+s)}$ is an integer.

Recall that if a graph G has a P_k-factorization then $la_{k-1}(G)$ is equal to $\frac{k \cdot |E(G)|}{(k-1) \cdot |V(G)|}$. Thus, by Theorem 4.1.3, we have the following corollary:

Corollary 4.1.4. *Assume $2r \geq s \geq r$. If $r + s \equiv 0 \pmod 3$ and $\frac{3rs}{2(r+s)}$ is an integer, then $la_2(K_{r,s}) = \frac{3rs}{2(r+s)}$.*

Next, we assume that the graph $K_{r,s}$ does not have a P_3-factorization and let $s = 2r - t$. Then $r > t \geq 0$, i.e., $r \geq t + 1$.

Proposition 4.1.5. *If $3\ell + 2 \geq t \geq 3\ell$ and $r \geq \lambda t + 1$, then $la_2(K_{r,2r-t}) \geq r - \ell + \lceil \frac{\lambda(3\ell+2)-2\ell-1}{2\lambda(3\ell+2)-2\ell} \cdot \ell \rceil$.*

Proof. If $3\ell + 2 \geq t \geq 3\ell$, then $\lfloor \frac{2|V(K_{r,2r-t})|}{3} \rfloor = \lfloor \frac{2(3r-t)}{3} \rfloor = \lfloor 2r - \frac{2t}{3} \rfloor = \lfloor 2r - t + \frac{t}{3} \rfloor = 2r - t + \ell$. By Lemma 2.1.5, $la_2(K_{r,2r-t}) \geq \lceil \frac{|E(K_{r,2r-t})|}{\lfloor \frac{2|V(K_{r,2r-t})|}{3} \rfloor} \rceil = \lceil \frac{r \cdot (2r-t)}{2r-t+\ell} \rceil = \lceil r - \frac{r\ell}{2r-t+\ell} \rceil = \lceil r - \ell + (\frac{r-t+\ell}{2r-t+\ell} \cdot \ell) \rceil = r - \ell + \lceil \frac{r-t+\ell}{2r-t+\ell} \cdot \ell \rceil$. Since $3\ell + 2 \geq t \geq 3\ell$ and $r \geq \lambda t + 1$, we have $\frac{r-t+\ell}{2r-t+\ell} \geq \frac{\lambda t+1-t+\ell}{2\lambda t+2-t+\ell} = \frac{(\lambda-1)t+\ell+1}{(2\lambda-1)t+\ell+2} \geq \frac{(\lambda-1)(3\ell+2)+\ell+1}{(2\lambda-1)(3\ell+2)+\ell+2} = \frac{\lambda(3\ell+2)-2\ell-1}{2\lambda(3\ell+2)-2\ell}$. Hence, $la_2(K_{r,2r-t}) \geq r - \ell + \lceil \frac{\lambda(3\ell+2)-2\ell-1}{2\lambda(3\ell+2)-2\ell} \cdot \ell \rceil$. □

Corollary 4.1.6. *If $3\ell + 2 \geq t \geq 3\ell$ and $r \geq t + 1$, then $la_2(K_{r,2r-t}) \geq r - \ell + \lceil \frac{\ell}{4} \rceil$.*

Proof. From Proposition 4.1.5 and let $\lambda = 1$, then $\frac{\lambda(3\ell+2)-2\ell-1}{2\lambda(3\ell+2)-2\ell} = \frac{\ell+1}{4(\ell+1)} = \frac{1}{4}$. □

Finally, we conclude the work of this section with the following theorem.

Theorem 4.1.7. *If* $5 \geq t \geq 0$ *and* $r \geq t+1$*, then* $la_2(K_{r,2r-t}) = r$*.*

Proof. From Corollary 4.1.6 and let $\ell = 0, 1$, then we have $la_2(K_{r,2r-t}) \geq r$. On the other hand, by Theorem 4.1.2, we know that $la_2(K_{r,2r}) = r$. Thus, $la_2(K_{r,2r-t}) \leq la_2(K_{r,2r}) = r$. □

4.2 Complete Graphs

In Chapter 2, we had mentioned the following result by Chen et al. [3] about the linear 2-arboricity of a complete graph K_m.

Proposition 4.2.1. $la_2(K_{12t+11}) = 9t+9$ *for any* $t \geq 0$*.*

However, the answer $9t+9$ of $la_2(K_{12t+11})$ is wrong, because some computing errors happened in its proof. Hence, in this section, we will give a revised result that $la_2(K_{12t+10}) = la_2(K_{12t+11}) = 9t+8$ for any $t \neq 4$. Moreover, this result also solve a problem raised by Bermond et al. [2] almost completely.

Before we go any further, we need some more definitions. Let $S = \{1, 2, \ldots, \nu\}$ be a set of ν elements. A **latin square of order** ν is a $\nu \times \nu$ array in which each cell contains a single element from S, such that each element occurs exactly once in each row and exactly once in each column. If in a latin square L of order ν the r^2 cells defined by r rows and r columns form a latin square of order r it is a latin **subsquare of** L. A latin square $L = [\ell_{ij}]$ is said to be **symmetric** if $\ell_{ij} = \ell_{ji}$ for all $1 \leq i, j \leq \nu$.

An **incomplete latin square** $\text{ILS}(\nu; b_1, b_2, \ldots, b_\kappa)$ is a $\nu \times \nu$ array A with entries from a set B of size ν, where $B_i \subseteq B$ for $1 \leq i \leq \kappa$ with $|B_i| = b_i$, and $B_i \cap B_j = \emptyset$ for $1 \leq i, j \leq \kappa$. Moreover,

1. each cell of A is empty or contains an element of B;
2. the subarrays indexed by $B_i \times B_i$ are empty (these subarrays are **holes**); and
3. the elements in row or column b are exactly those of $B - B_i$ if $b \in B_i$, and of B otherwise.

A **partitioned complete latin square** $\text{PILS}(\nu; b_1, b_2, \ldots, b_\kappa)$ is an incomplete latin square with $b_1 + b_2 + \cdots + b_\kappa = \nu$. Figure 4.1 is an example of a symmetric $\text{PILS}(8; 2, 2, 2, 2)$.

		8	6	7	3	4	5
		5	7	4	8	3	6
8	5			1	7	6	2
6	7			8	2	5	1
7	4	1	8			2	3
3	8	7	2			1	4
4	3	6	5	2	1		
5	6	2	1	3	4		

Figure 4.1: An example of a symmetric PILS$(8; 2, 2, 2, 2)$.

It is worthy of noting that, in 1987, Fu [9] proved that:

Theorem 4.2.2. *A symmetric partitioned complete latin square PILS*$(2\kappa; 2, 2, \dots, 2)$ *exists for each* $\kappa \geq 3$.

Next, we want to show some lemmas. For convenience, the vertices in K_m are denoted $v_0, v_1, \dots, v_{m-1}$.

Lemma 4.2.3. $la_2(K_{11}) = 8$.

Proof. We construct the array in Figure 4.2 to show that $la_2(K_{11}) \leq 8$. The entry ω in row v_γ and column v_δ means that the edge $v_\gamma v_\delta$ appears in the linear 2-forest labelled by ω. On the other hand, by Lemma 2.1.5, $la_2(K_{11}) \geq \lceil \frac{55}{\lfloor \frac{2 \cdot 11}{3} \rfloor} \rceil = 8$. □

Lemma 4.2.4. $la_2(K_{12} - M) = 8$ *where* M *is a matching of size* 3 *in* K_{12}.

Proof. Without loss of generality, let the matching M be the set $\{v_1v_4, v_6v_{10}, v_7v_{11}\}$ in K_{12}. Then the array in Figure 4.3 shows that $la_2(K_{12} - M) \leq 8$. On the other hand, by Lemma 2.1.5, $la_2(K_{12} - M) \geq \lceil \frac{63}{\lfloor \frac{2 \cdot 12}{3} \rfloor} \rceil = 8$. □

Lemma 4.2.5. $la_2(K_{35}) = 26$.

Proof. The array in Figure 4.4 shows that $la_2(K_{35}) \leq 26$. Since it is symmetric, we omit the entries of half the array. On the other hand, by Lemma 2.1.5, $la_2(K_{35}) \geq \lceil \frac{595}{\lfloor \frac{2 \cdot 35}{3} \rfloor} \rceil = 26$. □

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_0		2	1	7	8	3	6	5	6	4	5
v_1	2		1	3	3	2	5	4	5	6	8
v_2	1	1		5	2	3	4	6	7	7	8
v_3	7	3	5		2	1	4	6	8	8	7
v_4	8	3	2	2		1	7	8	4	5	6
v_5	3	2	3	1	1		8	7	4	5	6
v_6	6	5	4	4	7	8		3	1	2	2
v_7	5	4	6	6	8	7	3		2	1	4
v_8	6	5	7	8	4	4	1	2		3	3
v_9	4	6	7	8	5	5	2	1	3		1
v_{10}	5	8	8	7	6	6	2	4	3	1	

Figure 4.2: The array shows that $la_2(K_{11}) \leq 8$.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
v_0		2	1	3	8	2	6	5	6	4	5	4
v_1	2		1	3		7	5	4	5	6	8	6
v_2	1	1		2	3	3	4	6	7	5	8	7
v_3	3	3	2		2	1	4	6	8	5	7	8
v_4	8		3	2		1	7	8	4	7	6	5
v_5	2	7	3	1	1		8	7	4	8	6	5
v_6	6	5	4	4	7	8		2	1	3		2
v_7	5	4	6	6	8	7	2		1	3	4	
v_8	6	5	7	8	4	4	1	1		2	3	3
v_9	4	6	5	5	7	8	3	3	2		2	1
v_{10}	5	8	8	7	6	6		4	3	2		1
v_{11}	4	6	7	8	5	5	2		3	1	1	

Figure 4.3: The array shows that $la_2(K_{12} - \{v_1v_4, v_6v_{10}, v_7v_{11}\}) \leq 8$.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}	v_{20}	v_{21}	v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	v_{28}	v_{29}	v_{30}	v_{31}	v_{32}	v_{33}	v_{34}
v_0		2	1	3	8	2	6	5	6	4	5	4	11	26	25	14	10	16	23	20	17	22	19	13	15	14	12	11	9	24	25	21	22	18	19
v_1			1	3	10	7	5	4	5	6	8	6	20	17	13	23	22	19	14	11	26	10	16	24	25	21	22	18	19	17	15	14	12	11	9
v_2				2	3	3	4	6	7	5	8	7	10	19	11	25	26	14	13	22	16	17	23	20	17	15	14	12	11	9	24	25	21	22	18
v_3					2	1	4	6	8	5	7	8	26	23	10	11	18	25	20	17	14	16	13	22	19	17	15	14	12	11	9	24	25	21	22
v_4						1	7	8	4	7	6	5	22	16	20	13	17	23	25	9	19	26	14	11	24	25	21	22	18	19	17	15	14	12	11
v_5							8	7	4	8	6	5	19	13	26	10	23	11	22	16	25	14	20	17	18	19	17	15	14	12	11	9	24	25	21
v_6								2	1	3	13	2	17	14	22	20	16	13	11	26	23	19	25	10	9	24	25	21	22	18	19	17	15	14	12
v_7									1	3	4	16	14	11	16	17	25	22	26	23	20	13	10	19	12	11	9	24	25	21	22	18	19	17	15
v_8										2	3	3	23	20	19	26	13	10	17	14	11	25	21	16	22	18	19	17	15	14	12	11	9	24	25
v_9											2	1	13	22	23	19	20	26	16	25	10	11	17	14	21	22	18	19	17	15	14	12	11	9	24
v_{10}												1	16	25	17	22	14	20	10	19	12	23	11	26	11	9	24	25	21	22	18	19	17	15	14
v_{11}													25	10	14	15	11	17	19	13	22	20	26	23	14	12	11	9	24	25	21	22	18	19	17
v_{12}														2	1	3	8	2	6	5	6	4	5	4	18	23	21	26	24	9	10	12	13	15	16
v_{13}															1	3	19	7	5	4	5	6	8	6	21	26	24	9	10	12	13	15	16	20	18
v_{14}																2	3	3	4	6	7	5	8	7	20	18	23	21	26	24	9	10	12	13	15
v_{15}																	2	1	4	6	8	5	7	8	16	20	18	23	21	26	24	9	10	12	13
v_{16}																		1	7	8	4	7	6	5	23	21	26	24	9	10	12	13	15	16	20
v_{17}																			8	7	4	8	6	5	15	16	20	18	23	21	26	24	9	10	12
v_{18}																				2	1	3	22	2	24	9	10	12	13	15	16	20	18	23	21
v_{19}																					1	3	4	25	10	12	13	15	16	20	18	23	21	26	24
v_{20}																						2	3	3	13	15	16	20	18	23	21	26	24	9	10
v_{21}																							2	1	12	13	15	16	20	18	23	21	26	24	9
v_{22}																								1	26	24	9	10	12	13	15	16	20	18	23
v_{23}																									9	10	12	13	15	16	20	18	23	21	26
v_{24}																										2	1	7	8	3	6	5	6	4	5
v_{25}																											1	3	3	2	5	4	5	6	8
v_{26}																												5	2	3	4	6	7	7	8
v_{27}																													2	1	4	6	8	8	7
v_{28}																														1	7	8	4	5	6
v_{29}																															8	7	4	5	6
v_{30}																																3	1	2	2
v_{31}																																	2	1	4
v_{32}																																		3	3
v_{33}																																			1
v_{34}																																			

Figure 4.4: The symmetric array shows that $la_2(K_{35}) \leq 26$.

Lemma 4.2.6. $la_2(K_{12,12}) = 9$.

Proof. The array in Figure 4.5 shows that $la_2(K_{12,12}) \leq 9$. On the other hand, by Lemma 2.1.5, $la_2(K_{12,12}) \geq \lceil \frac{144}{\lfloor \frac{2 \cdot 24}{3} \rfloor} \rceil = 9$. □

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
x_0	3	9	8	7	1	5	1	2	4	6	2	3
x_1	6	1	6	7	8	9	4	5	3	2	4	5
x_2	8	1	9	9	4	5	6	7	3	2	7	8
x_3	2	5	2	3	4	3	6	1	8	7	9	1
x_4	4	9	5	5	3	6	2	8	6	7	1	4
x_5	7	5	7	8	3	8	2	4	9	9	1	6
x_6	9	3	1	1	8	2	7	4	2	3	5	6
x_7	1	6	8	4	6	4	7	3	5	5	9	2
x_8	1	8	4	6	9	7	9	3	7	8	5	2
x_9	5	2	4	6	2	9	3	8	1	1	3	7
x_{10}	9	4	3	2	5	1	5	6	8	4	6	7
x_{11}	5	7	3	2	7	1	8	9	4	6	8	9

Figure 4.5: The array shows that $la_2(K_{12,12}) \leq 9$.

Lemma 4.2.7. $la_2(K_{11,12} \cup G[M]) = 9$ *where* M *is a matching of size* 3 *in* K_{12}.

Proof. Without loss of generality, let the matching M be the set $\{v_1v_4, v_6v_{10}, v_7v_{11}\}$ in K_{12}. Then the array in Figure 4.6 shows that $la_2(K_{11,12} \cup G[M]) \leq 9$. On the other hand, by Lemma 2.1.5, $la_2(K_{11,12} \cup G[M]) \geq \lceil \frac{135}{\lfloor \frac{2 \cdot 23}{3} \rfloor} \rceil = 9$. □

Now, we are ready to obtain the main results.

Proposition 4.2.8. $la_2(K_{12t+11}) = 9t + 8$ *for any* $t \geq 0$ *and* t *is odd.*

Proof. First, we partition the vertex set of K_{12t+11} into $t+1$ disjoint subsets $S_0, S_1, \ldots, S_t$, where $S_i = \{v_{i[0]}, v_{i[1]}, \ldots, v_{i[11]}\}$ for all $i = 0, 1, \ldots, t-1$ and $S_t = \{x_0, x_1, \ldots, x_{10}\}$. Hence, the subgraph of K_{12t+11} induced by S_i is a K_{12} or a K_{11}, for

			7						8	9		
	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
x_0	3	9	8	7	1	5	1	2	4	6	2	3
x_1	6	1	6	7	8	9	4	5	3	2	4	5
x_2	8	1	9	9	4	5	6	7	3	2	7	8
x_3	2	5	2	3	4	3	6	1	8	7	9	1
x_4	4	9	5	5	3	6	2	8	6	7	1	4
x_5	7	5	7	8	3	8	2	4	9	9	1	6
x_6	9	3	1	1	8	2	7	4	2	3	5	6
x_7	1	6	8	4	6	4	7	3	5	5	9	2
x_8	1	8	4	6	9	7	9	3	7	8	5	2
x_9	5	2	4	6	2	9	3	8	1	1	3	7
x_{10}	9	4	3	2	5	1	5	6	8	4	6	7

Figure 4.6: The array shows that $la_2(K_{11,12} \cup G[\{v_1v_4, v_6v_{10}, v_7v_{11}\}]) \leq 9$.

all $i = 0, 1, \ldots, t$. More precisely, the edges of K_{12t+11} can be partitioned into two classes, one is the edges in K_{12} or K_{11} and the other is the edges in $K_{12,12}$ or $K_{11,12}$.

Next, we want to show that $la_2(K_{12t+11}) = 9t + 8$. Since $t + 1$ is even, from Lemma 3.1.3, K_{12t+11} can be decomposed into t $K_{12,12}$-factors in each of which there exists one component is $K_{11,12}$, and a K_{12}-factor in which there exists one component is K_{11}. Then, for the K_{12}-factor, the edges of $K_{12} - \{v_{i[1]}v_{i[4]}, v_{i[6]}v_{i[10]}, v_{i[7]}v_{i[11]}\}$ for $0 \leq i \leq t-1$ in those K_{12} and the edges of K_{11} can produce eight linear 2-forests from Lemmas 4.2.3 and 4.2.4. Moreover, since the edges $v_{i[1]}v_{i[4]}$, $v_{i[6]}v_{i[10]}$, $v_{i[7]}v_{i[11]}$ in each K_{12} of the K_{12}-factor are not being used, then we unite them with the corresponding component $K_{11,12}$ of each $K_{12,12}$-factor. Hence, for each $K_{12,12}$-factor, the edges in $K_{11,12} \cup G[\{v_{i[1]}v_{i[4]}, v_{i[6]}v_{i[10]}, v_{i[7]}v_{i[11]}\}]$ and the edges in those $K_{12,12}$ can produce nine linear 2-forests from Lemmas 4.2.6 and 4.2.7. Therefore, from the t $K_{12,12}$-factors and a K_{12}-factor of K_{12t+11}, we have that $la_2(K_{12t+11}) \leq 9t + 8$. On the other hand, by Lemma 2.1.5, $la_2(K_{12t+11}) \geq \lceil \frac{(12t+11)(12t+10)}{2\lfloor \frac{2(12t+11)}{3} \rfloor} \rceil = 9t + 8$. This concludes the proof. □

Proposition 4.2.9. *$la_2(K_{12t+11}) = 9t + 8$ for any $t \geq 6$ and t is even.*

Proof. We prove this proposition by using the techniques on latin squares proposed by Chen et al. [3]. First, let the 35×35 array in Figure 4.4 be partitioned into four subarrays P, Q, Q^T, R as shown in Figure 4.7, where P, Q, and R are 24×24, 24×11, and 11×11 arrays respectively. Moreover, let the 12×12 array in Figure 4.5 also be denoted W.

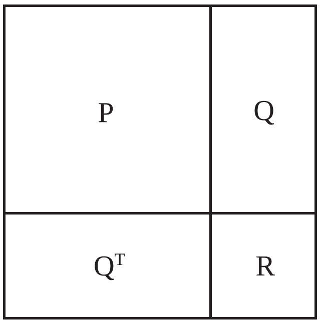

Figure 4.7: Four subarrays of the array in Figure 4.4.

Next, since $t \geq 6$ and t is even, from Theorem 4.2.2, we can find a symmetric PILS$(2\kappa; 2, 2, \ldots, 2)$ such that $t = 2\kappa$. We use $L = [\ell_{ij}]$ to denote this symmetric PILS$(2\kappa; 2, 2, \ldots, 2)$. Then, from L, we can construct a $(12t + 11) \times (12t + 11)$ symmetric array L' as shown in Figure 4.8 to show that $la_2(K_{12t+11}) \leq 9t + 8$, where

1. B_x is a 24×24 array, for $1 \leq x \leq \kappa$;
2. the entry $B_x(r, s)$ in B_x equals $P(r, s)$ in P if $P(r, s) \in \{1, 2, \ldots, 8\}$, for $1 \leq x \leq \kappa$;
3. $B_x(r, s) = P(r, s) + (x - 1) \cdot 18$ if $P(r, s) \notin \{1, 2, \ldots, 8\}$, for $1 \leq x \leq \kappa$;
4. the 12×12 array $C_{ij} = W + 8 + (\ell_{ij} - 1) \cdot 9$, for $1 \leq i, j \leq 2\kappa$;
5. the 24×11 array $D_x = Q + (x - 1) \cdot 18$, for $1 \leq x \leq \kappa$; and
6. the 11×11 array $E = R$.

On the other hand, by Lemma 2.1.5, $la_2(K_{12t+11}) \geq \lceil \frac{(12t+11)(12t+10)}{2\lfloor \frac{2(12t+11)}{3} \rfloor} \rceil = 9t + 8$. □

Theorem 4.2.10. *$la_2(K_{12t+10}) = la_2(K_{12t+11}) = 9t + 8$ for any $t \neq 4$.*

Proof. By Lemma 4.2.5 and Propositions 4.2.8 $\sim$ 4.2.9, $la_2(K_{12t+11}) = 9t + 8$ for any $t \neq 4$. Moreover, $9t + 8 = la_2(K_{12t+11}) \geq la_2(K_{12t+10}) \geq \lceil \frac{(12t+10)(12t+9)}{2\lfloor \frac{2(12t+10)}{3} \rfloor} \rceil = 9t + 8$. □

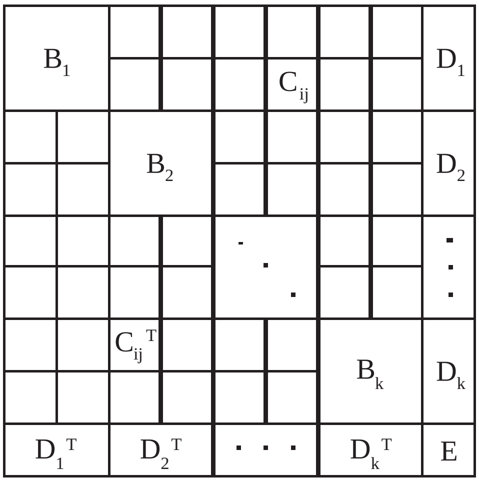

Figure 4.8: A $(12t + 11) \times (12t + 11)$ symmetric array.

4.3 Balanced Complete Multipartite Graphs

In 1989, Ushio and Tsuruno [26] showed the following result on balanced complete multipartite graphs $K_{m(n)}$.

Theorem 4.3.1. *$K_{m(n)}$ has a P_3-factorization if and only if $mn \equiv 0 \pmod 3$ and $(m-1)n \equiv 0 \pmod 4$.*

Recall that if a graph G has a P_k-factorization then $la_{k-1}(G)$ is equal to $\frac{k \cdot |E(G)|}{(k-1) \cdot |V(G)|}$. Thus, by Theorem 4.3.1, we have the following corollary:

Corollary 4.3.2. *$la_2(K_{m(n)}) = \frac{3(m-1)n}{4}$ when $mn \equiv 0 \pmod 3$ and $(m-1)n \equiv 0 \pmod 4$.*

In what follows, we consider the cases when $K_{m(n)}$ does not have a P_3-factorization and begin with the case $m = 3$ of $K_{m(n)}$.

Lemma 4.3.3. *$la_2(K_{3(n)}) = \lceil \frac{3n}{2} \rceil$ if $n \equiv 1 \pmod 2$.*

Proof. Assume that the partite sets of $K_{3(n)}$ are $V_0 = \{v_{0[0]}, v_{0[1]}, \ldots, v_{0[n-1]}\}$, $V_1 = \{v_{1[0]}, v_{1[1]}, \ldots, v_{1[n-1]}\}$, and $V_2 = \{v_{2[0]}, v_{2[1]}, \ldots, v_{2[n-1]}\}$. First, for all $0 \le \alpha \ne \beta \le 2$, let the balanced complete bipartite subgraph of $K_{3(n)}$ induced by V_α and V_β be denoted $G(V_\alpha, V_\beta)$.

Then, for any $\epsilon \in \{0, 1, ..., n-2\}$, we observe that the edges with bipartite differences $\epsilon, \epsilon+1$ in all of $G(V_1, V_2)$, $G(V_2, V_3)$, and $G(V_3, V_1)$ can produce three linear 2-forests. Hence, the edges with bipartite differences $1, 2, \ldots, n-1$ in all of $G(V_1, V_2)$, $G(V_2, V_3)$, and $G(V_3, V_1)$ can generate $(\frac{n-1}{2}) \cdot 3$ linear 2-forests, which are $\{v_{0[j]}v_{1[j+1+2r]}v_{2[j+2+4r]} | j = 0, 1, \ldots, n-1\}$, $\{v_{2[j]}v_{0[j+1+2r]}v_{1[j+3+4r]} | j = 0, \ldots, n-1\}$, and $\{v_{1[j]}v_{2[j+2+2r]}v_{0[j+4+4r]} | j = 0, 1, \ldots, n-1\}$ for all $r \in \{0, 1, \ldots, \frac{n-1}{2} - 1\}$. Note that the index y of each vertex $v_{x[y]}$ is modulo n. For example, Figure 4.9 shows that the edges with bipartite differences $1, 2$ in all of $G(V_1, V_2)$, $G(V_2, V_3)$, and $G(V_3, V_1)$ can produce three linear 2-forests in $K_{3(7)}$.

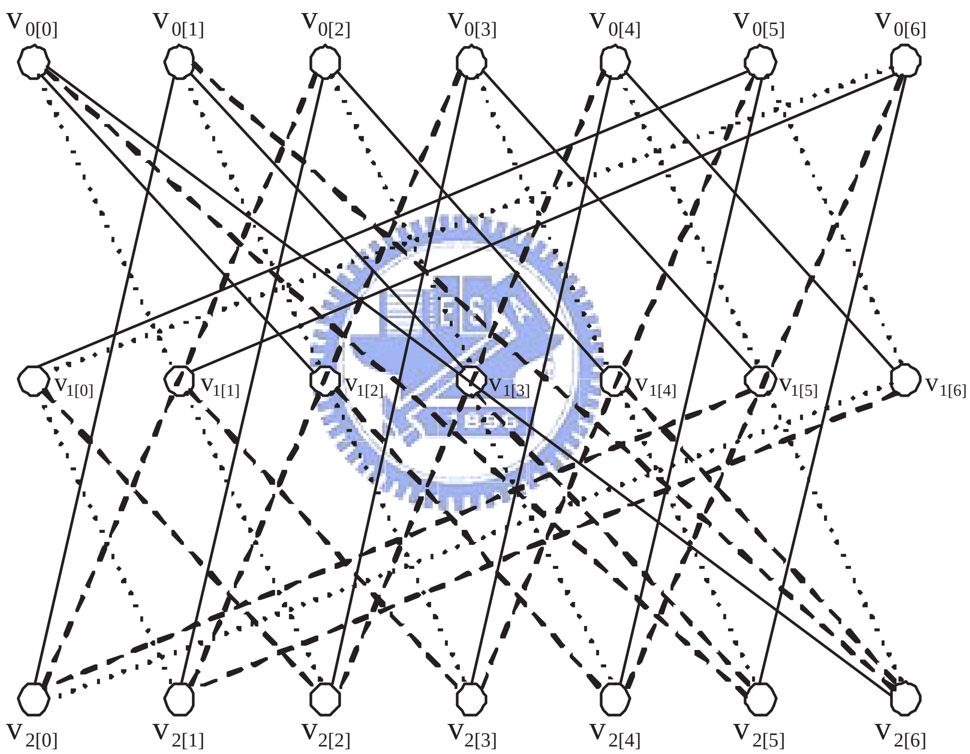

Figure 4.9: Three linear 2-forests in $K_{3(7)}$.

Moreover, the disjoint 3-cycles induced by the edges with bipartite difference 0 in all of $G(V_1, V_2)$, $G(V_2, V_3)$, and $G(V_3, V_1)$ can be decomposed into two linear 2-forests $\{v_{0[j]}v_{1[j]}v_{2[j]} | j = 0, 1, \ldots, n-1\}$ and $\{v_{2[j]}v_{0[j]} | j = 0, 1, \cdots, n-1\}$. Thus, $la_2(K_{3(n)}) \leq \frac{3(n-1)}{2} + 2 = \frac{3n+1}{2} = \lceil \frac{3n}{2} \rceil$ if $n \equiv 1 \pmod 2$.

On the other hand, from Lemma 2.1.5, $la_2(K_{3(n)}) \geq \lceil \frac{3n}{2} \rceil$ if $n \equiv 1 \pmod 2$. $\square$

Proposition 4.3.4. *$la_2(K_{m(n)}) = \left\lceil \frac{3(m-1)n}{4} \right\rceil$ if $m \equiv 3 \pmod{12}$ and $n \equiv 1 \pmod 2$.*

Proof. Assume that the partite sets of $K_{m(n)}$ are denoted $V_0, V_1, \ldots, V_{m-1}$. First, for all $0 \leq \alpha \neq \beta \leq m-1$, let the balanced complete bipartite subgraph of $K_{m(n)}$ induced by V_α and V_β be denoted $G(V_\alpha, V_\beta)$. Moreover, from Lemma 3.1.1 (by replacing each edge of K_m with $K_{n,n}$), $K_{m(n)}$ has a $K_{3(n)}$-factorization in which there are $\frac{|E(K_m)|}{(\frac{|V(K_m)|}{3})\cdot 3} = \frac{m-1}{2}$ $K_{3(n)}$-factors.

Then, from the proof of Lemma 4.3.3, we know that the edges with bipartite differences $1, 2, \ldots, n-1$ in all of $G(V_\alpha, V_\beta)$ for $0 \leq \alpha \neq \beta \leq m-1$ can generate $(\frac{m-1}{2}) \cdot (\frac{3(n-1)}{2})$ linear 2-forests. Also, it is not difficult to see that the subgraph induced by the set of edges with bipartite difference 0 in all of $G(V_\alpha, V_\beta)$ for $0 \leq \alpha \neq \beta \leq m-1$ is exactly a K_m-factor. Thus, by Theorem 2.2.4, $la_2(K_{m(n)}) \leq \left(\frac{m-1}{2} \cdot \frac{3(n-1)}{2}\right) + la_2(K_m) = \frac{3(m-1)(n-1)}{4} + \left\lceil \frac{3(m-1)}{4} \right\rceil = \left\lceil \frac{3(m-1)n}{4} \right\rceil$. On the other hand, from Lemma 2.1.5, $la_2(K_{m(n)}) \geq \left\lceil \frac{3(m-1)n}{4} \right\rceil$ if $m \equiv 3 \pmod{12}$ and $n \equiv 1 \pmod 2$. $\square$

Proposition 4.3.5. *$la_2(K_{m(n)}) = \left\lceil \frac{3(m-1)n}{4} \right\rceil$ if $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 6$.*

Proof. Dividing all m partite sets of $K_{m(n)}$ into $\frac{m}{2}$ disjoint pairs of two partite sets shows that $K_{m(n)}$ is the union of $K_{\frac{m}{2}(2n)}$ and one $K_{n,n}$-factor of $K_{m(n)}$. Since $\frac{m}{2} \equiv 0 \pmod 2$ and $2n \equiv 0 \pmod{12}$, from Theorem 4.1.1 and Corollary 4.3.2, $la_2(K_{m(n)}) \leq la_2(K_{n,n}) + la_2(K_{\frac{m}{2}(2n)}) = \lceil \frac{3n}{4} \rceil + \frac{3\left(\frac{m}{2}-1\right)(2n)}{4} = \left\lceil \frac{3(m-1)n}{4} \right\rceil$.

On the other hand, from Lemma 2.1.5, $la_2(K_{m(n)}) \geq \left\lceil \frac{3(m-1)n}{4} \right\rceil$ if $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 6$. $\square$

Proposition 4.3.6. *$la_2(K_{m(n)}) = \left\lceil \frac{3(m-1)n}{4} \right\rceil$ if $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 3$.*

Proof. Dividing all m partite sets of $K_{m(n)}$ into $\frac{m}{2}$ disjoint pairs of two partite sets shows that $K_{m(n)}$ is the union of $K_{\frac{m}{2}(2n)}$ and one $K_{n,n}$-factor of $K_{m(n)}$. Since $\frac{m}{2} \equiv 1 \pmod 2$ and $2n \equiv 0 \pmod 6$, from Theorem 4.1.1 and Corollary 4.3.2, $la_2(K_{m(n)}) \leq la_2(K_{n,n}) + la_2(K_{\frac{m}{2}(2n)}) = \lceil \frac{3n}{4} \rceil + \frac{3\left(\frac{m}{2}-1\right)(2n)}{4} = \left\lceil \frac{3(m-1)n}{4} \right\rceil$.

On the other hand, from Lemma 2.1.5, $la_2(K_{m(n)}) \geq \left\lceil \frac{3(m-1)n}{4} \right\rceil$ if $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 3$. $\square$

Proposition 4.3.7. *$la_2(K_{m(n)}) = \left\lceil \frac{3(m-1)n}{4} \right\rceil$ if $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 2$.*

Proof. Dividing all m partite sets of $K_{m(n)}$ into $\frac{m}{3}$ disjoint collections of three partite sets shows that $K_{m(n)}$ is the union of $K_{\frac{m}{3}(3n)}$ and one $K_{3(n)}$-factor of $K_{m(n)}$. Since $3n \equiv 0 \pmod 6$, from Corollary 4.3.2 and Propositions 4.3.5 $\sim$ 4.3.6, $la_2(K_{m(n)}) \leq la_2(K_{3(n)}) + la_2(K_{\frac{m}{3}(3n)}) = \frac{3n}{2} + \left\lceil \frac{3\left(\frac{m}{3}-1\right)(3n)}{4} \right\rceil = \left\lceil \frac{3(m-1)n}{4} \right\rceil$. On the other hand, from Lemma 2.1.5, $la_2(K_{m(n)}) \geq \left\lceil \frac{3(m-1)n}{4} \right\rceil$ if $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 2$. □

Concluding Remark. The main goal of this section is to determine $la_2(K_{m(n)})$ when $mn \equiv 0 \pmod 3$. However, we are not able to finish the whole part at this moment due to several stubborn subcases. We expect to settle the rest cases in the near future.

Chapter 5

Bit Permutation Networks

In this chapter, we will prove that if N is an s-stage d-nary bit permutation network with d^n inputs (outputs), then $L(N)^+$, a new network obtained from the line digraph of N, is an $(s+1)$-stage d-nary bit permutation network with d^{n+1} inputs (outputs). Furthermore, we give a simple (but not trivial) formula to determine the characteristic vector of $L(N)^+$ from N's characteristic vector. Finally, we use this formula to obtain relations between some well-studied bit permutation networks.

5.1 Introduction

In 1999, Chang et al. [7] proposed the notion of a **bit permutation network** which is an s-stage interconnection network composed of d^{n-1} $d \times d$ crossbar switches in each stage. This class of networks includes *Beneš network*, *Omega network*, *Banyan network*, *Baseline network*, and their *extra-stage* versions, namely, most of the multi-stage interconnection networks.

Suppose that the d^{n-1} crossbars in a stage are each labelled by a distinct d-nary $(n-1)$-bit vector. Chang et al. [7] showed that an s-stage d-nary bit permutation network N with d^n inputs (outputs) can be characterized by an $(s-1)$-bit vector $(k_1, k_2, \ldots, k_{s-1})$, where $k_t = j \in \{1, 2, \ldots, n-1\}$ means that N is topologically equivalent to a network whose linking pattern between stage t and $t+1$ consists of d^{n-2} disjoint complete bipartite graphs and each such graph connects all crossbars in stage t and $t+1$ having the same d-nary $(n-1)$-bit vectors except bit j.

Figure 5.1 shows a bit permutation network with characteristic vector $(3, 1, 2)$ and is topologically equivalent to the network in Figure 5.2.

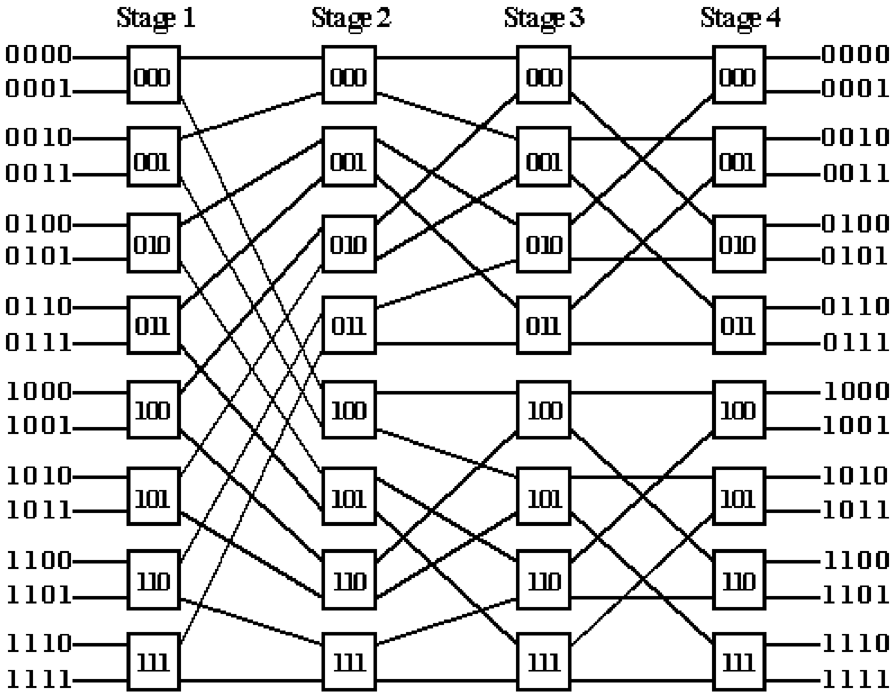

Figure 5.1: A bit permutation network $N_2(4; u, v, f_1, f_2, f_3)$.

The line digraph $L(N)$ of a multistage interconnection network N is obtained by taking the links in N (including the inputs and the outputs) as vertices in $L(N)$, and an edge directed from vertex u to vertex v exists in $L(N)$ if link u is incident to and precedes link v in N (see Figure 5.3). Note that vertices in the same stage of $L(N)$ are ordered according to the starting endpoints of their corresponding links in N and we omit the directions of edges in $L(N)$ because they are all from left to right.

Let $L(N)^+$ be obtained from $L(N)$ by adding d inlets (outlets) to each of those vertices which are inputs (outputs) in N. By interpreting vertices as crossbars, then $L(N)^+$ can also be viewed as a multistage interconnection network as shown in Figure 5.4. It is well-known that being crosstalk-free (each crossbar carries at most one path) is an essential property for photonic switching, which uses optical fiber instead of electric wire as the transmission media. Lea [20] also observed that if two paths are link-disjoint in N, then their corresponding paths are vertex-disjoint in $L(N)$.

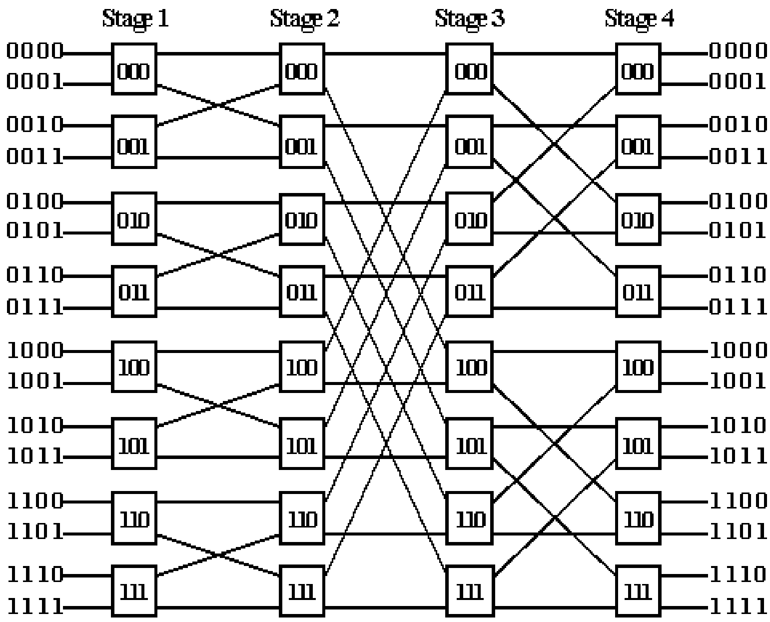

Figure 5.2: A bit permutation network $N_2(4; I_3, I_1, I_2)$.

Moreover, Hwang and Lin [17] gave formulas relating the nonblocking properties of N to the crosstalk-free nonblocking properties of $L(N)^+$. Therefore, it is of interest to know that if N is a bit permutation network, what kind of network $L(N)^+$ is.

5.2 Preliminary Lemmas

Consider a multistage interconnection network with s stages of d^{n-1} crossbars of size $d \times d$. For all $i \in \{0, 1, \ldots, d^{n-1} - 1\}$, let the ith crossbar in each stage be labelled by i in the d-nary $(n-1)$-bit vector. We define a bit-j group as the set of crossbars in a stage with identical labels except the jth bit. Such a group will also be labelled by a d-nary $(n-1)$-bit vector which is identical to any member in the group except that bit j is replaced by the symbol x_0, which stands for the set $\{0, 1, \ldots, d-1\}$. Then Chang et al. [7] called an s-stage d-nary interconnection network a bit permutation network if the links between stage t and $t+1$ are always from a bit-u_t group Z to a bit-v_t group Z', where Z' is a permutation of Z, for $t = 1, 2, \ldots, s-1$.

In what follows, for our purpose, we will restate the main results proved by Chang et al. [7] in a slightly different way (and provide proofs for justification).

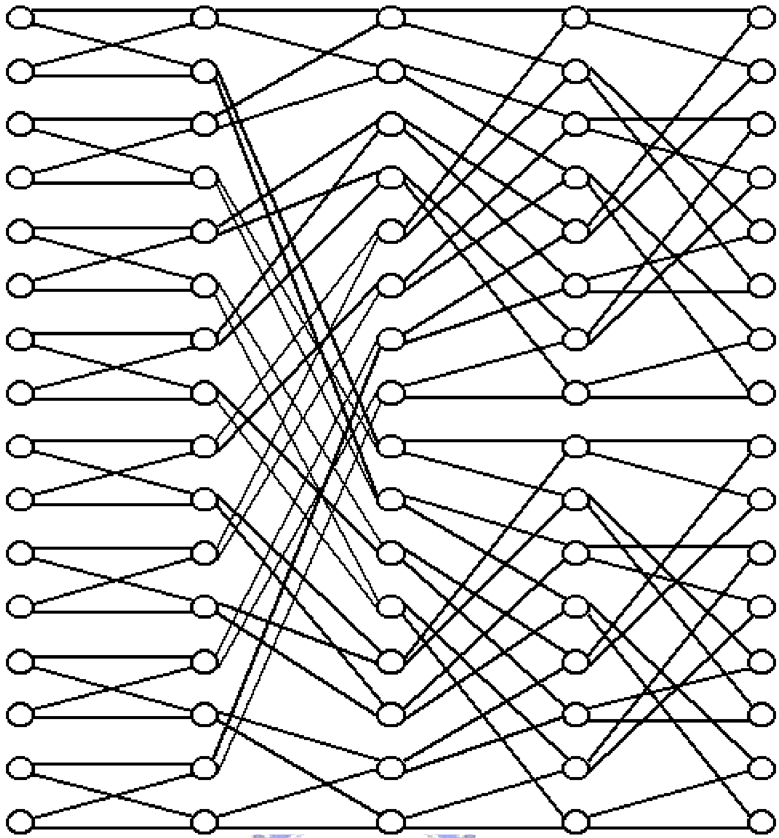

Figure 5.3: The line digraph $L(N)$ obtained from the network in Figure 5.1.

First, assume that N is an s-stage d-nary bit permutation network with d^n inputs (outputs). Let f_t, $t = 1, 2, \ldots, s-1$, denote the group linking function which shows that the links between stage t and $t+1$ of N are from a bit-u_t group to a bit-v_t group. We also define two functions u and v from $\{1, 2, \ldots, s-1\}$ to $\{1, 2, \ldots, n-1\}$ to include the values u_t and v_t for all $1 \leq t \leq s-1$. Then, N can be represented by $N_d(n; u, v, f_1, f_2, \ldots, f_{s-1})$. It is worthy of mentioning that f_t is a permutation of $\{1, 2, \ldots, n-1\}$ and $(f_t)^{-1}(u_t) = v_t$.

For example, Figure 5.1 shows a bit permutation network with 16 inputs (outputs) and crossbar i in stage t is labelled by i in the binary 3-bit vector (x_1, x_2, x_3), where $1 \leq t \leq 4$ and $x_1, x_2, x_3 \in \{0, 1\}$. Moreover, the links are from a bit-3 group (x_1, x_2, x_0) in stage 1 to a bit-1 group (x_0, x_1, x_2) in stage 2, from a bit-2 group (x_1, x_0, x_3) in stage 2 to a bit-3 group (x_1, x_3, x_0) in stage 3, and from a bit-2 group (x_1, x_0, x_3) in stage 3 to a bit-2 group (x_1, x_0, x_3) in stage 4, where x_0 stands for $\{0, 1\}$.

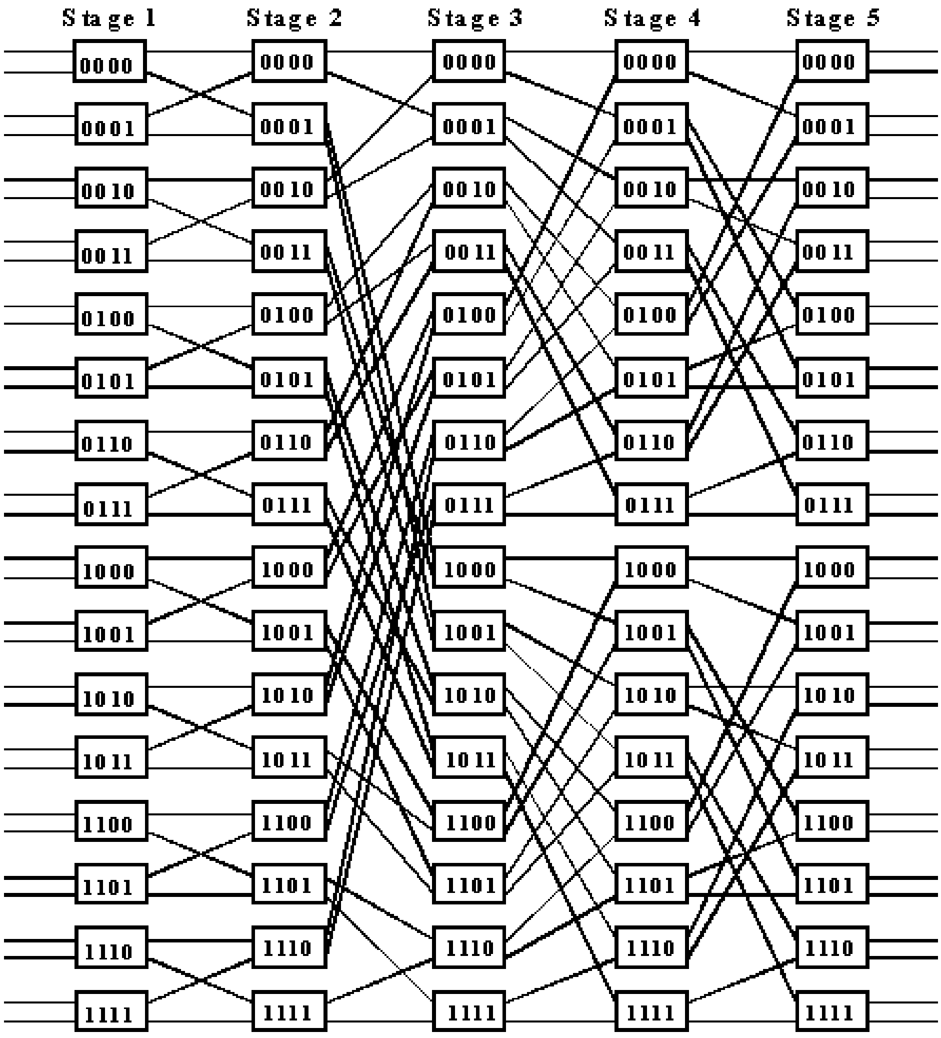

Figure 5.4: The network $L(N)^+$ obtained from the network in Figure 5.1.

Thus,

$$u_1 = 3, v_1 = 1, f_1(1) = 3, f_1(2) = 1, f_1(3) = 2,$$

$$u_2 = 2, v_2 = 3, f_2(1) = 1, f_2(2) = 3, f_2(3) = 2,$$

$$u_3 = 2, v_3 = 2, f_3(1) = 1, f_3(2) = 2, f_3(3) = 3.$$

In this paper, we shall use the **cycle** notation for permutations, that is, the cycle $(i_1, i_2, \ldots, i_n)$ represents the permutation f with $f(i_1) = i_2, f(i_2) = i_3, \ldots, f(i_{n-1}) = i_n, f(i_n) = i_1$, and the cycle (i) represents f with $f(i) = i$. Then, f_1 can be represented by $(1, 3, 2)$; f_2, by $(1)(2, 3)$; and f_3, by $(1)(2)(3)$.

Note that an s-stage d-nary network is completely determined by the linking patterns between adjacent stages. Two such networks are called **equivalent** if the linking patterns between adjacent stages of the two networks are identical.

Theorem 5.2.1. *If there exist permutations $g_1, g_2, \ldots, g_s$ on $\{1, 2, \ldots, n-1\}$ such that $u'_t = (g_t)^{-1}(u_t)$, $v'_t = (g_{t+1})^{-1}(v_t)$, and $f'_t = (g_t)^{-1} \circ f_t \circ g_{t+1}$ for $t = 1, 2, \ldots, s-1$, then two bit permutation networks $N_d(n; u, v, f_1, f_2, \ldots, f_{s-1})$ and $N_d(n; u', v', f'_1, f'_2, \ldots, f'_{s-1})$ are equivalent.*

Proof. For all $i = 1, 2, \ldots, s$, we define the bijection φ_i from the set of crossbars in stage i of $N_d(n; u, v, f_1, f_2, \ldots, f_{s-1})$ to another set of crossbars in stage i of $N_d(n; u', v', f'_1, f'_2, \ldots, f'_{s-1})$ as $\varphi_i((x_1, x_2, \ldots, x_{n-1})) = (x_{g_i(1)}, x_{g_i(2)}, \ldots, x_{g_i(n-1)})$. In other words, $\varphi_i((x_1, x_2, \ldots, x_{n-1})) = (x'_1, x'_2, \ldots, x'_{n-1})$, where $x'_j = x_{g_i(j)}$ for all $j = 1, 2, \ldots, n-1$.

To see $N_d(n; u, v, f_1, f_2, \ldots, f_{s-1})$ and $N_d(n; u', v', f'_1, f'_2, \ldots, f'_{s-1})$ are equivalent, we need to check that $\varphi_1, \varphi_2, \ldots, \varphi_s$ are link-preserving. Suppose that the links between stage t and $t+1$ of $N_d(n; u, v, f_1, f_2, \ldots, f_{s-1})$ are from a bit-u_t group $(x_1, x_2, \ldots, x_{u_t}, \ldots, x_{n-1})$ to a bit-v_t group $(y_1, y_2, \ldots, y_{v_t}, \ldots, y_{n-1})$, where $y_j = x_{f_t(j)}$ for all $j = 1, 2, \ldots, n-1$. Then,

$$\varphi_t((x_1, x_2, \ldots, x_{u_t}, \ldots, x_{n-1})) = (x'_1, x'_2, \ldots, x'_{(g_t)^{-1}(u_t)}, \ldots, x'_{n-1}),$$

$$\varphi_{t+1}((y_1, y_2, \ldots, y_{v_t}, \ldots, y_{n-1})) = (y'_1, y'_2, \ldots, y'_{(g_{t+1})^{-1}(v_t)}, \ldots, y'_{n-1}),$$

where $x'_j = x_{g_t(j)}$ and $y'_j = y_{g_{t+1}(j)}$ for all $j = 1, 2, \ldots, n-1$.

Since $y'_j = y_{g_{t+1}(j)} = x_{f_t \circ g_{t+1}(j)} = x_{g_t \circ f'_t(j)} = x'_{f'_t(j)}$ for all $j = 1, 2, \ldots, n-1$, then $(y'_1, y'_2, \ldots, y'_{(g_{t+1})^{-1}(v_t)}, \ldots, y'_{n-1}) = (x'_{f'_t(1)}, x'_{f'_t(2)}, \ldots, x'_{f'_t((g_{t+1})^{-1}(v_t))}, \ldots, x'_{f'_t(n-1)})$ and there exist indeed links from a bit-$(g_t)^{-1}(u_t)$ group $(x'_1, x'_2, \ldots, x'_{(g_t)^{-1}(u_t)}, \ldots, x'_{n-1})$ to a bit-$(g_{t+1})^{-1}(v_t)$ group $(x'_{f'_t(1)}, x'_{f'_t(2)}, \ldots, x'_{f'_t((g_{t+1})^{-1}(v_t))}, \ldots, x'_{f'_t(n-1)})$ between stage t and $t+1$ of $N_d(n; u', v', f'_1, f'_2, \ldots, f'_{s-1})$.

Therefore, through the comparisons by those bijections $\varphi_1, \varphi_2, \ldots, \varphi_s$, the linking patterns between adjacent stages of the two networks are identical. □

Let I denote the identity permutation $(1)(2)\cdots(n-1)$ and $N_d(n; I_{k_1}, \ldots, I_{k_{s-1}})$ denote the bit permutation network $N_d(n; u, v, f_1, \ldots, f_{s-1})$ with $f_t = I$ and $u_t = v_t = k_t$ for all t. While Chang et al. [7] proved that $N_d(n; u, v, f_1, \ldots, f_{s-1})$ is equivalent to $N_d(n; I_{k_1}, \ldots, I_{k_{s-1}})$ for some $(k_1, \ldots, k_{s-1})$, we give an explicit formula to compute k_t for all $t = 1, 2, \ldots, s-1$.

Theorem 5.2.2. *A bit permutation network $N_d(n; u, v, f_1, \ldots, f_{s-1})$ is equivalent to $N_d(n; I_{k_1}, \ldots, I_{k_{s-1}})$, where $k_1 = u_1$ and $k_t = (f_1 \circ \cdots \circ f_{t-1})(u_t)$ for $t = 2, \ldots, s-1$.*

Proof. First, by setting $g_2 = (f_1)^{-1}$ and $g_t = I$ for all $t = 1, 3, 4, \ldots, s-1$, we have that $N_d(n; u, v, f_1, f_2, \ldots, f_{s-1})$ is equivalent to $N_d(n; u', v', I_{k_1}, f'_2, \ldots, f'_{s-1})$ from Theorem 5.2.1, where $u'_2 = f_1(u_2)$, $v'_2 = v_2 = (f'_2)^{-1}(u'_2)$, $f'_2 = f_1 \circ f_2$, $u'_t = u_t$, $v'_t = v_t$, and $f'_t = f_t$ for all $t = 3, 4, \ldots, s-1$.

Next, assume $N_d(n; u, v, f_1, \ldots, f_{s-1})$ and $N_d(n; u', v', I_{k_1}, \ldots, I_{k_{j-1}}, f'_j, \ldots, f'_{s-1})$ are equivalent, where $u'_j = (f_1 \circ \cdots \circ f_{j-1})(u_j)$, $v'_j = v_j = (f'_j)^{-1}(u'_j)$, $f'_j = f_1 \circ \cdots \circ f_j$, $u'_t = u_t$, $v'_t = v_t$, and $f'_t = f_t$ for $t = j+1, j+2, \ldots, s-1$. Similarly, by setting $g_t = I$ except $g_{j+1} = (f'_j)^{-1}$, then $N_d(n; u', v', I_{k_1}, \ldots, I_{k_{j-1}}, f'_j, \ldots, f'_{s-1})$ is equivalent to $N_d(n; u'', v'', I_{k_1}, \ldots, I_{k_{j-1}}, I_{k_j}, f''_{j+1}, \ldots, f''_{s-1})$ from Theorem 5.2.1, where $u''_{j+1} = (f_1 \circ \cdots \circ f_j)(u_{j+1})$, $v''_{j+1} = v_{j+1} = (f''_{j+1})^{-1}(u''_{j+1})$, $f''_{j+1} = f_1 \circ \cdots \circ f_{j+1}$, $u''_t = u_t$, $v''_t = v_t$, and $f''_t = f_t$ for $t = j+2, j+3, \ldots, s-1$. Thus, $N_d(n; u, v, f_1, \ldots, f_{s-1})$ is equivalent to $N_d(n; u'', v'', I_{k_1}, \ldots, I_{k_{j-1}}, I_{k_j}, f''_{j+1}, \ldots, f''_{s-1})$, where $u''_{j+1} = (f_1 \circ \cdots \circ f_j)(u_{j+1})$, $v''_{j+1} = v_{j+1} = (f''_{j+1})^{-1}(u''_{j+1})$, $f''_{j+1} = f_1 \circ \cdots \circ f_{j+1}$, $u''_t = u_t$, $v''_t = v_t$, and $f''_t = f_t$ for $t = j+2, j+3, \ldots, s-1$. $\square$

For convenience, we shall use $(k_1, \ldots, k_{s-1})$ as a short notation for the network $N_d(n; I_{k_1}, \ldots, I_{k_{s-1}})$. By Theorem 5.2.2, we say that a bit permutation network $N_d(n; u, v, f_1, \ldots, f_{s-1})$ can be characterized by an $(s-1)$-bit vector $(k_1, \ldots, k_{s-1})$.

Theorem 5.2.3. *If g is a permutation on $\{1, \ldots, n-1\}$, then $N_d(n; I_{k_1}, \ldots, I_{k_{s-1}})$ is equivalent to $N_d(n; I_{g(k_1)}, \ldots, I_{g(k_{s-1})})$.*

Proof. Choose all g_t as $(g)^{-1}$ in Theorem 5.2.1. Since $g \circ I_{k_t} \circ (g)^{-1} = I_{g(k_t)}$, the assertion holds. $\square$

5.3 The Main Results

Let N be an s-stage d-nary bit permutation network with d^n inputs (outputs). It is not difficult to see that $L(N)^+$ is an $(s+1)$-stage d-nary crossbar network with d^{n+1} inputs (outputs). We show that $L(N)^+$ is also a bit permutation network and how the group linking functions of N determine those of $L(N)^+$.

Theorem 5.3.1. *If a bit permutation network N is represented by $N_d(n; u, v, f_1, \ldots, f_{s-1})$, then $L(N)^+$ is a bit permutation network represented by $N_d(n+1; u^*, v^*, h_1, \ldots, h_s)$, where $u_1^* = v_1^* = n$, h_1 is the identity permutation $(1)\cdots(n)$, $u_t^* = u_{t-1}$, $v_t^* = n$, and h_t is the same as f_{t-1} except $h_t(n) = u_{t-1}$ and $h_t(v_{t-1}) = n$ for $t = 2, 3, \ldots, s$.*

Proof. First, for all $j = 1, 2, \ldots, d^n - 1$ and any $t \in \{1, 2, \ldots, s\}$, let the jth link which is incident to some crossbar in stage t of N be labelled by j in the d-nary n-bit vector $(x_1, \ldots, x_n)$. Note that the links are ordered according to the starting endpoints of them. Then, through the construction rules of $L(N)^+$, we find that the relation between links which are incident to crossbars in stage t of N is equal to the group linking function h_t between stage t and $t+1$ of $L(N)^+$ for any $t \in \{1, 2, \ldots, s\}$.

In stage 1 of N, since the links $(x_1, \ldots, x_{n-1}, x_0)$ are incident to and precede the links $(x_1, \ldots, x_{n-1}, x_0)$, where $x_0 \in \{0, 1, \ldots, d-1\}$, then $u_1^* = v_1^* = n$ and h_1 is equal to $(1)\cdots(n)$. For $t = 2, \ldots, s$, if the permutation f_{t-1} of N is from a bit-u_{t-1} group $(x_1, \ldots, x_{u_{t-1}-1}, x_0, x_{u_{t-1}+1}, \ldots, x_{n-1})$ to a bit-v_{t-1} group $(x_{f_{t-1}(1)}, \ldots, x_{f_{t-1}(v_{t-1}-1)}, x_0, x_{f_{t-1}(v_{t-1}+1)}, \ldots, x_{f_{t-1}(n-1)})$, then in stage t of N, the links $(x_1, \ldots, x_{u_{t-1}-1}, x_0, x_{u_{t-1}+1}, x_{u_{t-1}+2}, \ldots, x_n)$ must be incident to and precede the links $(x_{f_{t-1}(1)}, \ldots, x_{f_{t-1}(v_{t-1}-1)}, x_n, x_{f_{t-1}(v_{t-1}+1)}, \ldots, x_{f_{t-1}(n-1)}, x_0)$, where $x_0 \in \{0, 1, \ldots, d-1\}$. Hence, for all $t = 2, \ldots, s$, $u_t^* = u_{t-1}$, $v_t^* = n$, and h_t is the same as f_{t-1} except $h_t(n) = u_{t-1}$ and $h_t(v_{t-1}) = n$. Moreover, the above statements show that $L(N)^+$ is a bit permutation network. $\square$

Theorem 5.3.2. *If the characteristic vector of a bit permutation network N with d^n inputs (outputs) is $(k_1, \ldots, k_{s-1})$, then $L(N)^+$'s characteristic vector is $(l_1, \ldots, l_s)$, where $l_1 = n$ and $l_t = k_{t-1}$ if $k_{t-1} \notin \{k_1, \ldots, k_{t-2}\}$ or $l_t = l_i$ if $k_{t-1} \in \{k_1, \ldots, k_{t-2}\}$ for all $t = 2, 3, \ldots, s$, where $i = \max\{j \mid k_j = k_{t-1}, 1 \le j \le t-2\}$.*

Proof. Without loss of generality, let N be represented by $N_d(n; u, v, f_1, \ldots, f_{s-1})$. Since the characteristic vector of $N_d(n; u, v, f_1, \ldots, f_{s-1})$ is $(k_1, \ldots, k_{s-1})$, where $k_t \in \{1, \ldots, n-1\}$ for all $t = 1, 2, \ldots, s-1$, by Theorems 5.2.2 and 5.3.1, we can prove that the characteristic vector of $L(N_d(n; I_{k_1}, \ldots, I_{k_{j-1}}, f_j, \ldots, f_{s-1}))^+$ is equal to the characteristic vector of $L(N_d(n; I_{k_1}, \ldots, I_{k_{j-1}}, I_{k_j}, f'_{j+1}, \ldots, f'_{s-1}))^+$ for all $j = 1, \ldots, s-1$. Hence, the two characteristic vectors of $L(N_d(n; u, v, f_1, \ldots, f_{s-1}))^+$ $(= L(N)^+)$ and $L(N_d(n; I_{k_1}, \ldots, I_{k_{s-1}}))^+$ are identical.

By Theorem 5.3.1, $L(N_d(n; I_{k_1}, \ldots, I_{k_{s-1}}))^+$ is a bit permutation network and can be represented by $N_d(n+1; u^*, v^*, h_1, \ldots, h_s)$, where $u_1^* = v_1^* = n, h_1 = (1) \cdots (n), u_t^* = k_{t-1}, v_t^* = n$, and $h_t = (1) \cdots (k_{t-1} - 1)(k_{t-1} + 1) \cdots (n-1)(k_{t-1}, n)$ for $t = 2, \ldots, s$. Note that $h_t(c) = c$ if $c \notin \{k_{t-1}, n\}$ for $c = 1, \ldots, n$ and $t = 1, \ldots, s$. Therefore, by Theorem 5.2.2, the characteristic vector of $L(N_d(n; I_{k_1}, \ldots, I_{k_{s-1}}))^+$ is $(l_1, \ldots, l_s)$, where $l_1 = n$ and $l_t = (h_1 \circ \cdots \circ h_{t-1})(k_{t-1})$ for $t = 2, \ldots, s$. If $k_{t-1} \notin \{k_1, \ldots, k_{t-2}\}$, then $l_t = k_{t-1}$. If $k_{t-1} \in \{k_1, \ldots, k_{t-2}\}$, then $l_t = (h_1 \circ \cdots \circ h_{t-1})(k_{t-1}) = (h_1 \circ \cdots \circ h_{i+1})(k_{t-1}) = (h_1 \circ \cdots \circ h_{i+1})(k_i) = (h_1 \circ \cdots \circ h_i)(n) = (h_1 \circ \cdots \circ h_{i-1})(k_{i-1}) = l_i$, where $i = \max\{j \mid k_j = k_{t-1}, 1 \leq j \leq t-2\}$. □

For example, if the characteristic vector of a bit permutation network N with d^4 inputs (outputs) is $(1, 3, 3, 2, 2, 3, 1, 3, 1, 1, 2, 3, 2, 2, 1)$, then the characteristic vector of $L(N)^+$ is $(4, 1, 3, 1, 2, 1, 3, 4, 1, 3, 1, 2, 4, 1, 4, 3)$. Here, $l_1 = n = 4$, $l_2 = k_1 = 1$, $l_3 = k_2 = 3$, and $l_4 = l_2 = 1$ since $k_3 = 3 = k_2$. The formula obtained from Theorem 5.3.2 is useful for some well-studied bit permutation networks.

Let us consider the network obtained by adding k extra stages to *Banyan network* with 2^n inputs (outputs) and by specifying that the extra k stages should be identical to the first k stages (denote this way of adding extra stages by F). Represent the above network by $BY_F(k, n)$. If the extra k stages are identical to the mirror image of the first k stages, then denote such network by $BY_{F^{-1}}(k, n)$. Figure 5.5 shows the network $BY_F(1, 4)$.

Theorem 5.3.3. *The network $L(BY_F(k, n))^+$, $0 \leq k \leq n$, is equivalent to another network $BY_F(k, n+1)$, where F can be replaced by F^{-1}.*

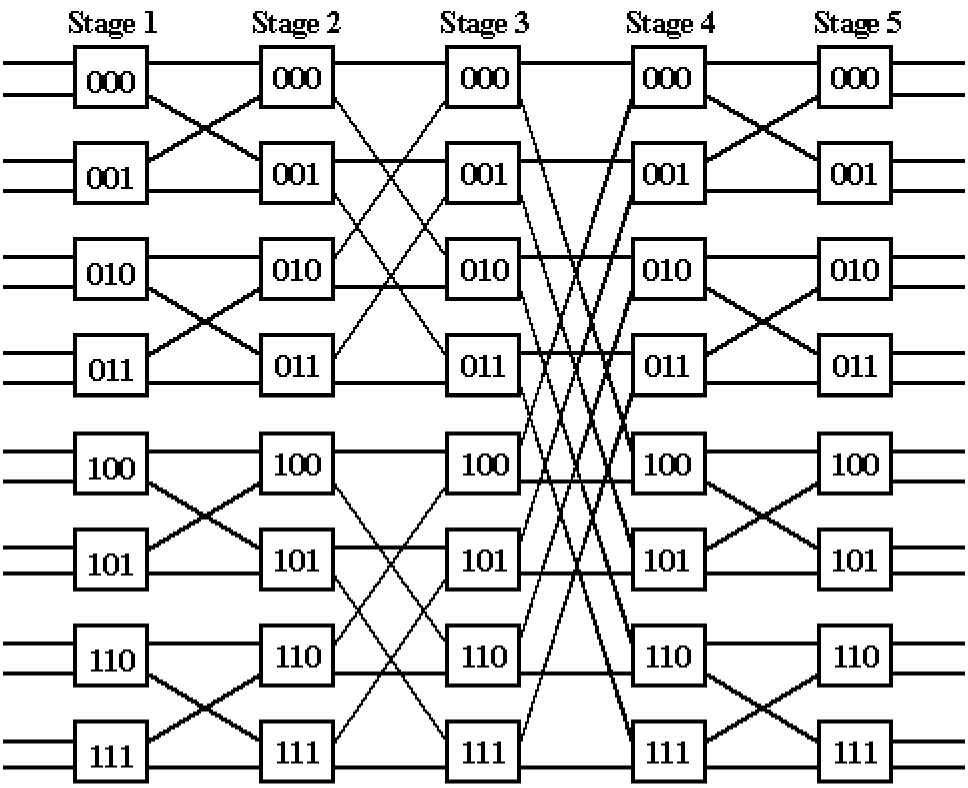

Figure 5.5: The network $BY_F(1, 4)$.

Proof. Since $BY_F(k, n)$ can be represented by $N_2(n; I_{n-1}, I_{n-2}, \ldots, I_1, I_{n-1}, I_{n-2}, \ldots, I_{n-k})$, from Theorem 5.3.2, the characteristic vector of $L(BY_F(k, n))^+$ is $(n, n-1, n-2, \ldots, 1, n, n-1, \ldots, n-k+1)$. This means that $L(BY_F(k, n))^+$ is equivalent to the network $N_2(n+1; I_n, I_{n-1}, I_{n-2}, \ldots, I_1, I_n, I_{n-1}, \ldots, I_{n-k+1})$. Hence, $L(BY_F(k, n))^+$ is equivalent to $BY_F(k, n+1)$. Similarly, we have the result if F is replaced by F^{-1}. $\square$

Let W^{-1} denote the inverse network of W, i.e., the network obtained from W by reversing the order of the stages. Then it is not difficult to have the following result.

Theorem 5.3.4. *The network $L(BY_F^{-1}(k, n))^+$, $0 \le k \le n$, is equivalent to another network $BY_F^{-1}(k, n+1)$, where F can be replaced by F^{-1}.*

Proof. Since $BY_F^{-1}(k, n)$ is represented by $N_2(n; I_1, I_2, \ldots, I_{n-1}, I_1, I_2, \ldots, I_k)$, from Theorem 5.3.2, the characteristic vector of $L(BY_F^{-1}(k, n))^+$ is $(n, 1, 2, \ldots, n-1, n, 1, \ldots, k-1)$.

Then, by Theorem 5.2.3, we find that the permutation $g = (1, 2, \ldots, n)$ can let $N_2(n+1; I_n, I_1, I_2, \ldots, I_{n-1}, I_n, I_1, \ldots, I_{k-1})$ be equivalent to $N_2(n+1; I_1, I_2, I_3, \ldots, I_n, I_1, I_2, \ldots, I_k)$. Hence, $L(BY_F^{-1}(k,n))^+$ is equivalent to $BY_F^{-1}(k, n+1)$. Moreover, if F is replaced by F^{-1}, then we can also obtain the similar result. $\square$

Theorem 5.3.4 was crucially used in [17] to prove the crosstalk-free property of $BY_{F^{-1}}^{-1}(k,n)$ essential to photonic switching.

Concluding Remark. The work on bit permutation networks presented in this chapter has been published in [18] which is a joint work with Professor Frank K. Hwang.

Chapter 6

Conclusion

This thesis studies the linear k-arboricity problem on complete bipartite graphs, complete graphs, and balanced complete multipartite graphs.

For complete bipartite graphs $K_{r,s}$, we first determine the linear 3-arboricity of $K_{n,n}$ for any n. Then, we show that if $s \geq 2r$ then $la_2(K_{r,s}) = \lceil \frac{s}{2} \rceil$ and if $5 \geq t \geq 0$ and $r \geq t+1$ then $la_2(K_{r,2r-t}) = r$.

For complete graphs K_m, we first determine the linear 3-arboricity of K_m for any m. Then, we give a result that $la_2(K_{12t+10}) = la_2(K_{12t+11}) = 9t+8$ for any $t \neq 4$, which solve a problem raised in an earlier paper [2] almost completely.

For balance complete multipartite graphs $K_{m(n)}$, we first determine the linear 3-arboricity of $K_{m(n)}$ for $mn \equiv 0 \pmod 4$. Then, we give some substantial results about the linear 2-arboricity of $K_{m(n)}$.

However, there are still many questions remain unsolved. We describe below some of them that we concern most.

(1) Prove $la_2(K_{59}) = 44$.

(2) Find the answer of $la_3(K_{m(n)})$ for any m and n.

(3) Find the answer of $la_2(K_{m(n)})$ for any m and n.

(4) Find the answer of $la_2(K_{r,s})$ for any $2r > s > r$.

Furthermore, in this thesis, we study a problem on the bit permutation network. We prove that if N is an s-stage d-nary bit permutation network with d^n inputs (outputs), then a new network $L(N)^+$ obtained from the line digraph of N is an $(s+1)$-stage d-nary bit permutation network with d^{n+1} inputs (outputs). We also give a simple (but not trivial) formula to determine the characteristic vector of $L(N)^+$ from the characteristic vector of N. This formula can help us to obtain relations between some well-studied bit permutation networks.

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