Pinching theorems for conformal classes of Willmore surfaces in the unit n-sphere

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Abstract

Let x be an immersion of a compact Willmore surface M into the n-dimensional unit sphere S^n. In this thesis we first consider the Willmore surfaces in the unit 3-sphere, and establish an integral inequality for the square of the length of the trace free part of the second fundamental form and the mean curvature. Based on this integral inequality, we characterize the totally umbilical spheres and the Clifford torus by a certain pinching condition. We then introduce a conformal invariant quantity which is formulated in terms of the square of the length of the trace free part of the second fundamental form and the mean curvature, and prove that if this quantity is bounded above by that value of the Clifford torus then $x(M)$ is either a totally umbilical sphere or a conformal Clifford torus. As for the case $n = 3$, we also characterize the totally umbilical spheres and the Veronese surface by a pinching condition for the case $n \geq 4$. Analogous to the case $n = 3$, we then introduce a conformal invariant quantity, and prove that if this quantity is bounded above by that value of the Veronese surface then $x(M)$ is either a totally umbilical sphere or a conformal Veronese surface.

Contents

Chapter 1

Introduction

For an immersion $x : M \hookrightarrow S^n$ of a compact surface M into the n-dimensional unit sphere S^n, we use (h_{ij}^α) to denote the second fundamental form of M, and let $H^\alpha = \sum h_{ii}^\alpha$ be the $\alpha-$component of the mean curvature vector $\mathbb{H}$. It is convenient to denote by $\phi_{ij}^\alpha = h_{ij}^\alpha - \frac{H^\alpha}{2}\delta_{ij}$ the trace free part of the second fundamental form, and let $\Phi = \sum(\phi_{ij}^\alpha)^2$. We shall denote by H the length of the mean curvature vector $\mathbb{H}$ when $n \geq 4$, and by $H = \sum h_{ii}$ the mean curvature when $n = 3$.

The Willmore functional is defined by

$$W(x) = \int_M \Phi,$$

where the integration is with respect to the area measure of M. This functional is preserved if we move M via conformal transformations of S^n. The critical points of W are called Willmore surfaces. In the case $n \geq 4$, they satisfy the Euler-Lagrange equation

$$\Delta H^\alpha + \sum \phi_{ij}^\alpha \phi_{ij}^\beta H^\beta = 0,$$

where Δ is the Laplacian in the normal bundle NM (see [15]). For the case $n = 3$ (see [2]), the corresponding equation is given by

$$\Delta H + \Phi H = 0.$$

Since the mean curvature depends on the second derivatives of x, this is a fourth order equation. The simplest examples of Willmore surfaces are minimal surfaces in

S^n. However the set of Willmore surfaces turns out to be larger than that of minimal surfaces (see [11]).

For M being a minimal submanifold in the n-dimensional unit sphere S^n, there are vast estimates for the square of the length of the second fundamental form. Significant works in this direction have been obtained by Simons (see [14]), Chern, do Carmo and Kobayashi (see [3]), Peng and Terng (see [12]) and the references cited therein. One expects that similar results are also valid for Willmore surfaces (see [8]). Based on this idea, Li proved that if M is a compact Willmore surface in the 3-dimensional unit sphere S^3 satisfying $0 \leq \Phi \leq 2$, then $x(M)$ is the totally umbilical sphere or the Clifford torus. He also proved that if M is a compact Willmore surface in the n-dimensional unit sphere S^n satisfying $0 \leq \Phi \leq \frac{4}{3}$ when $n \geq 4$, then $x(M)$ is the totally umbilical sphere or the Veronese surface (see [7] and [8]). These results are analogous to that of Chern, do Carmo and Kobayashi in the case of minimal surfaces. As a special case of their result, they proved that if $n = 3$, $H = 0$ and $0 \leq \Phi \leq 2$, then $x(M)$ is the equatorial sphere or the Clifford torus, and if $n \geq 4$, $H = 0$ and $0 \leq \Phi \leq \frac{4}{3}$, then $x(M)$ is the equatorial sphere or the Veronese surface (see [3]).

For M being a hypersurface with constant mean curvature in the n-dimensional unit sphere S^n, Alencar and do Carmo obtained a pinching constant which depends on the mean curvature (see [1]). For submanifolds with parallel mean curvature vector in spheres, the above theorem was extended to higher codimension by Santos and Fontenele(see [13] and [5]).

Because in general a Willmore surface is not minimal, it is interesting to find an upper estimate for Φ including the mean curvature.

This thesis is divided by two parts. In the first part, we shall consider the Willmore surfaces in the unit 3-sphere. Our starting point is to find an upper estimate for Φ which includes the mean curvature.

Theorem A. Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere S^3. Then

$$\int_M \Phi(2 + \frac{H^2}{4} - \Phi) \leq 0.$$

In particular, if

$$0 \leq \Phi \leq 2 + \frac{H^2}{4},$$

then either $\Phi = 0$ and M is totally umbilical or $\Phi = 2 + \frac{H^2}{4}$ and M is the Clifford torus.

Just as the results of Li, the result mentioned above does not characterize any non-minimal Willmore surface in S^3 except the totally umbilical spheres. However, the estimate is sharp in the sense that for every given positive ϵ, there is a compact Willmore surface M in S^3 satisfying $0 < \Phi \leq 2 + \frac{1}{4}H^2 + \epsilon$ but it is not the Clifford torus. Such examples can be constructed by using the method given in Chapter 5.

For characterizing a non-minimal Willmore surface and the conformal classes of Willmore surfaces, for each immersion x of M into the unit 3-sphere S^3, we consider the infimum of maximum values of $\Phi - \frac{1}{4}H^2$ obtained by composition of x with g where g ranges over all conformal transformations of S^3. We show that this conformal invariant characterizes the totally umbilical sphere and the conformal classes of the Clifford torus. Since the conformal group G of the ambient space S^3 is not compact, the proof involves some new tricks. More precisely, we shall prove the following theorem:

Theorem B. Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere S^3. If

$$inf_{g \in G} max_{g \circ x(M)} (\Phi_g - \frac{1}{4}H_g^2) \leq 2,$$

where G is the conformal group of the ambient space S^3, Φ_g and H_g are the square of the length of the trace free part of the second fundamental form and the mean curvature of the immersion $g \circ x$ respectively, then $x(M)$ is either a totally umbilical sphere or a conformal Clifford torus.

From the above theorem, we obtain immediately the following.

Corollary 1. Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere S^3. If

$$inf_{g \in G} max_{g \circ x(M)} (\Phi_g) \leq 2,$$

then M is either a totally umbilical sphere or a conformal Clifford torus.

In the second part, we shall consider the case $n \geq 4$. As the case $n = 3$, we first find an upper estimate for Φ which includes the mean curvature.

Theorem C. Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n, $n \geq 4$. If

$$0 \leq \Phi \leq \frac{2}{3} + \frac{1}{8}H^2 + \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4},$$

then either $\Phi = 0$ and M is totally umbilical or $\Phi = \frac{2}{3} + \frac{1}{8}H^2 + \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4}$. In the latter case, $n = 4$ and M is the Veronese surface.

From the above theorem, we obtain immediately the following.

Corollary 2. Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n, $n \geq 4$. If

$$0 \leq \Phi \leq \frac{4}{3} + \frac{1}{6}H^2,$$

then either $\Phi = 0$ and M is totally umbilical or $\Phi = \frac{4}{3} + \frac{1}{6}H^2$. In the latter case, $n = 4$ and M is the Veronese surface.

It is remarkable that the Veronese surface is the minimal surface in the 4-dimensional unit sphere S^4 satisfying $\Phi = \frac{4}{3}$ (see [3]). As the case $n = 3$, the above theorem does not characterize any non-minimal Willmore surface except the totally umbilical spheres. However, the estimate is sharp in the sense that for every given positive ϵ, there is a compact Willmore surface M in S^4 satisfying $0 < \Phi \leq \frac{2}{3} + \frac{1}{8}H^2 + \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4} + \epsilon$ but it is not the Veronese surface. Such examples can be constructed by using the method given in Chapter 5.

For characterizing a non-minimal Willmore surface, for each immersion x of M into the unit n-sphere S^n, we consider the infimum of maximum values of

$$\Phi - \frac{1}{8}H^2 - \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4}$$

obtained by composition of x with g where g ranges over all conformal mappings of S^n. This conformal invariant depends on the immersion x. We show that this conformal invariant characterizes the totally umbilical sphere and the conformal class of the Veronese surface. The following is the main result in the case $n \geq 4$.

Theorem D. Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n, $n \geq 4$. If

$$inf_{g\in G} max_{g\circ x(M)}(\Phi_g - \frac{1}{8}H_g^2 - \sqrt{\frac{4}{9} + \frac{1}{6}H_g^2 + \frac{1}{96}H_g^4}\,) \leq \frac{2}{3},$$

where G is the conformal group of the ambient space S^n, Φ_g and H_g are the square of the length of the trace free part of the second fundamental form and the mean curvature of the immersion $g \circ x$ respectively, then $x(M)$ is either a totally umbilical sphere or a conformal Veronese surface.

As an immediate consequence of the above theorem, the pinching condition can be simplified as follows.

Corollary 3. Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n, $n \geq 4$. If

$$inf_{g\in G} max_{g\circ x(M)}(\Phi_g - \frac{1}{6}H_g^2\,) \leq \frac{4}{3},$$

then $x(M)$ is either a totally umbilical sphere or a conformal Veronese surface.

The main idea of the proof of Theorem D is close to that of Theorem B. However, the proof requires some careful modifications in progress. In these proofs, we consider a minimizing sequence g_m in G. If this minimizing sequence is convergent in G, the assertion follows from Theorems A and C. Otherwise, we shall show that M must be totally umbilical.

This thesis is organized as follows. In Chapter 2 we introduce some notations and auxiliary lemmas about Willmore surfaces. In Chapter 3 we consider the case $n = 3$, and prove Theorems A and B. In Chapter 4, the case $n \geq 4$ is dealt and Theorems C and D are proved. Finally, in Chapter 5 we construct certain examples which show our upper bound estimates for Φ, Theorems A and C, may fail to be true if we make a slight change in the pinching conditions.

Chapter 2

Preliminaries

In this chapter we shall introduce some notations and auxiliary lemmas.

2.1 Notations

Let $x : M \hookrightarrow S^n$ be an immersed surface in the n-dimensional unit sphere S^n. We choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ in S^n, so that when restricted to $x(M)$ the vectors e_1, e_2 are tangent to $x(M)$, and $\{e_3, \cdots, e_n\}$ is a local frame field in the normal bundle NM of M. Let $\{\omega_1, \cdots, \omega_n\}$ denote the dual coframe field in S^n. We shall use the following ranges of indices

$$1 \leq i, j, k, \cdots \leq 2; \qquad 3 \leq \alpha, \beta, \gamma, \cdots \leq n.$$

Then the structure equations are given by

$$\begin{aligned} dx &= \sum \omega_i e_i, \\ de_i &= \sum \omega_{ij} e_j + \sum h^\alpha_{ij} \omega_j e_\alpha - \omega_i x, \\ de_\alpha &= -\sum h^\alpha_{ij} \omega_j e_i + \sum \omega_{\alpha\beta} e_\beta, \qquad h^\alpha_{ij} = h^\alpha_{ji}, \end{aligned}$$

where ω_{ij} and $\omega_{\alpha\beta}$ are the connection forms and (h^α_{ij}) is the second fundamental form of M. From the structure equations of M, the Gauss equations are then given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{2.1}$$

$$R_{ik} = \delta_{ik} + \sum H^\alpha h^\alpha_{ik} - \sum h^\alpha_{ij} h^\alpha_{jk}, \tag{2.2}$$

$$2K = 2 + H^2 - S, \tag{2.3}$$

$$R_{\alpha\beta ij} = \sum (h^\alpha_{ik} h^\beta_{kj} - h^\alpha_{jk} h^\beta_{ki}), \tag{2.4}$$

were K is the Gaussian curvature of M, $S = \sum (h_{ij}^\alpha)^2$ is the square of the length of the second fundamental form, $\mathbb{H} = \sum H^\alpha e_\alpha = \sum h_{ii}^\alpha e_\alpha$ is the mean curvature vector, and $H = \sqrt{\sum (h_{ii}^\alpha)^2}$ is the length of the mean curvature vector of M.

As M is two-dimensional surface, we have

$$\begin{aligned} R_{ijkl} &= K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ &= \frac{1}{2}(2 + H^2 - S)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \end{aligned}$$

The covariant derivative ∇h_{ij}^α of the second fundamental form h_{ij}^α of M with components h_{ijk}^α is defined by

$$\sum h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum h_{kj}^\alpha \omega_{ki} + \sum h_{ik}^\alpha \omega_{kj} + \sum h_{ij}^\beta \omega_{\beta\alpha},$$

and the covariant derivative $\nabla^2 h_{ij}^\alpha$ of ∇h_{ij}^α with components h_{ijkl}^α is defined by

$$\sum h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum h_{ljk}^\alpha \omega_{li} + \sum h_{ilk}^\alpha \omega_{lj} + \sum h_{ijl}^\alpha \omega_{lk} + \sum h_{ijk}^\beta \omega_{\beta\alpha}.$$

Then the Codazzi equation and the Ricci formula are given by

$$h_{ijk}^\alpha - h_{ikj}^\alpha = 0, \tag{2.5}$$

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum h_{mj}^\alpha R_{mikl} + \sum h_{im}^\alpha R_{mjkl} + \sum h_{ij}^\beta R_{\beta\alpha kl}. \tag{2.6}$$

The Laplacian $\triangle h_{ij}^\alpha$ of h_{ij}^α is defined by

$$\triangle h_{ij}^\alpha = \sum h_{ijkk}^\alpha.$$

From the Codazzi equation and the Ricci formula we get

$$\begin{aligned} \Delta h_{ij}^\alpha &= \sum h_{ijkk}^\alpha = \sum h_{kijk}^\alpha \\ &= \sum h_{kikj}^\alpha + \sum h_{li}^\alpha R_{lkjk} + \sum h_{kl}^\alpha R_{lijk} + \sum h_{ki}^\beta R_{\beta\alpha jk} \\ &= \sum h_{kkij}^\alpha + \sum h_{li}^\alpha R_{lkjk} + \sum h_{kl}^\alpha R_{lijk} + \sum h_{ki}^\beta R_{\beta\alpha jk} \\ &= H_{ij}^\alpha + \sum h_{li}^\alpha R_{lkjk} + \sum h_{kl}^\alpha R_{lijk} + \sum h_{ki}^\beta R_{\beta\alpha jk}. \end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2}\Delta S &= \frac{1}{2}\Delta \sum (h_{ij}^\alpha)^2 \ = \sum (h_{ijk}^\alpha)^2 + \sum h_{ij}^\alpha \Delta h_{ij}^\alpha \\
&= \sum (h_{ijk}^\alpha)^2 + \sum h_{ij}^\alpha (H_{ij}^\alpha + \sum h_{li}^\alpha R_{lkjk} + \sum h_{kl}^\alpha R_{lijk} + \sum h_{ki}^\beta R_{\beta\alpha jk}) \\
&= \sum (h_{ijk}^\alpha)^2 + \sum h_{ij}^\alpha H_{ij}^\alpha + \sum (h_{ij}^\alpha h_{li}^\alpha R_{lkjk} + h_{ij}^\alpha h_{kl}^\alpha R_{lijk}) + \sum h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk} \\
&= \sum (h_{ijk}^\alpha)^2 + \sum h_{ij}^\alpha H_{ij}^\alpha \\
&\quad + K \sum [h_{ij}^\alpha h_{li}^\alpha (\delta_{lj}\delta_{kk} - \delta_{lk}\delta_{kj}) + h_{ij}^\alpha h_{kl}^\alpha (\delta_{lj}\delta_{ik} - \delta_{lk}\delta_{ij})] \\
&\quad + \sum [\sum (h_{1i}^\alpha h_{i2}^\beta - h_{2i}^\alpha h_{i1}^\beta)] R_{\beta\alpha 12} \\
&= \sum (h_{ijk}^\alpha)^2 + \sum h_{ij}^\alpha H_{ij}^\alpha + K(2S - H^2) - \sum (R_{\beta\alpha 12})^2.
\end{aligned}$$

Let ϕ_{ij}^α denote the tensor $h_{ij}^\alpha - \frac{H^\alpha}{2}\delta_{ij}$, and $\Phi = \sum(\phi_{ij}^\alpha)^2$ the square of the length of the trace free tensor ϕ_{ij}^α. Then we have

$$\phi_{ijk}^\alpha - \phi_{ikj}^\alpha = \frac{H_j^\alpha}{2}\delta_{ik} - \frac{H_k^\alpha}{2}\delta_{ij}, \tag{2.7}$$

$$\phi_{ijkl}^\alpha - \phi_{ijlk}^\alpha = \sum \phi_{mj}^\alpha R_{mikl} + \sum \phi_{im}^\alpha R_{mjkl} + \sum \phi_{ij}^\beta R_{\beta\alpha kl}, \tag{2.8}$$

$$R_{\alpha\beta ij} = \sum (\phi_{ik}^\alpha \phi_{kj}^\beta - \phi_{jk}^\alpha \phi_{ki}^\beta), \tag{2.9}$$

and $\Phi = S - \frac{H^2}{2}$. We replace Δh_{ij}^α by $\Delta \phi_{ij}^\alpha$. Hence we can get

$$\begin{aligned}
\Delta \phi_{ij}^\alpha &= \sum \phi_{ijkk}^\alpha = \sum (\phi_{ikj}^\alpha + \frac{H_j^\alpha}{2}\delta_{ik} - \frac{H_k^\alpha}{2}\delta_{ij})_k \\
&= \sum \phi_{kijk}^\alpha + \sum \frac{H_{jk}^\alpha}{2}\delta_{ik} - \sum \frac{H_{kk}^\alpha}{2}\delta_{ij} \\
&= \sum \phi_{kikj}^\alpha + \sum \phi_{mi}^\alpha R_{mkjk} + \sum \phi_{km}^\alpha R_{mijk} + \sum \phi_{ki}^\beta R_{\beta\alpha jk} + \frac{H_{ji}^\alpha}{2} \\
&\quad - \frac{\Delta H^\alpha}{2}\delta_{ij} \\
&= \sum (\phi_{kki}^\alpha + \frac{H_i^\alpha}{2}\delta_{kk} - \frac{H_k^\alpha}{2}\delta_{ki})_j \\
&\quad + K \sum [\phi_{mi}^\alpha (\delta_{mj}\delta_{kk} - \delta_{mk}\delta_{kj}) + \phi_{km}^\alpha (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij})] \\
&\quad + \sum \phi_{ki}^\beta R_{\beta\alpha jk} + \frac{H_{ji}^\alpha}{2} - \frac{\Delta H^\alpha}{2}\delta_{ij}.
\end{aligned}$$

where Δ is the Laplacian in the normal bundle NM. Since

$$
\begin{aligned}
H^\alpha_{ji} &= \sum h^\alpha_{kkji} = \sum (h^\alpha_{kkij} + \sum h^\alpha_{mk} R_{mkji} + \sum h^\alpha_{km} R_{mkji} + \sum h^\beta_{kk} R_{\beta\alpha ji}) \\
&= H^\alpha_{ij} + 2 \sum h^\alpha_{mk} R_{mkji} + \sum H^\beta R_{\beta\alpha ji} \\
&= H^\alpha_{ij} + 2K \sum h^\alpha_{mk} (\delta_{mj}\delta_{ki} - \delta_{mi}\delta_{kj}) + \sum H^\beta R_{\beta\alpha ji} \\
&= H^\alpha_{ij} + 2K \sum (h^\alpha_{ji} - h^\alpha_{ij}) + \sum H^\beta R_{\beta\alpha ji} \\
&= H^\alpha_{ij} + \sum H^\beta R_{\beta\alpha ji},
\end{aligned}
$$

we have

$$
\begin{aligned}
\Delta \phi^\alpha_{ij} &= \sum (\phi^\alpha_{kki} + \frac{H^\alpha_i}{2}\delta_{kk} - \frac{H^\alpha_k}{2}\delta_{ki})_j \\
&\quad + K \sum [\phi^\alpha_{mi}(\delta_{mj}\delta_{kk} - \delta_{mk}\delta_{kj}) + \phi^\alpha_{km}(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij})] \\
&\quad + \sum \phi^\beta_{ki} R_{\beta\alpha jk} + \frac{H^\alpha_{ji}}{2} - \frac{\Delta H^\alpha}{2}\delta_{ij} \\
&= \sum \phi^\alpha_{kkij} + H^\alpha_{ij} - \frac{H^\alpha_{ij}}{2} + 2K\phi^\alpha_{ij} - K \sum \phi^\alpha_{kk}\delta_{ij} + \sum \phi^\beta_{ki} R_{\beta\alpha jk} \\
&\quad + \frac{H^\alpha_{ij}}{2} + \sum \frac{H^\beta}{2} R_{\beta\alpha ji} - \frac{\Delta H^\alpha}{2}\delta_{ij} \\
&= \sum \phi^\alpha_{kkij} + H^\alpha_{ij} + (2 + H^2 - S)\phi^\alpha_{ij} + \sum \phi^\beta_{ki} R_{\beta\alpha jk} \\
&\quad - (1 + \frac{H^2}{2} - \frac{S}{2}) \sum \phi^\alpha_{kk}\delta_{ij} + \sum \frac{H^\beta}{2} R_{\beta\alpha ji} - \frac{\Delta H^\alpha}{2}\delta_{ij} \\
&= \sum \phi^\alpha_{kkij} + H^\alpha_{ij} + (2 + \frac{H^2}{2} - \Phi)\phi^\alpha_{ij} + \sum \phi^\beta_{ki} R_{\beta\alpha jk} \\
&\quad - (1 + \frac{H^2}{4} - \frac{\Phi}{2}) \sum \phi^\alpha_{kk}\delta_{ij} + \sum \frac{H^\beta}{2} R_{\beta\alpha ji} - \frac{\Delta H^\alpha}{2}\delta_{ij}.
\end{aligned}
$$

Since $\sum \phi^\alpha_{kk} = 0$, it follows that

$$
\begin{aligned}
\frac{1}{2}\Delta \Phi &= \sum (\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij} \Delta \phi^\alpha_{ij} \\
&= \sum (\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij} H^\alpha_{ij} + (2 + \frac{H^2}{2} - \Phi)\Phi \\
&\quad + \sum \phi^\alpha_{ij}\phi^\beta_{ki} R_{\beta\alpha jk} + \sum \frac{H^\beta}{2} \phi^\alpha_{ij} R_{\beta\alpha ji} \\
&= \sum (\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij} H^\alpha_{ij} + (2 + \frac{H^2}{2} - \Phi)\Phi \\
&\quad + \sum (\phi^\alpha_{1i}\phi^\beta_{i2} - \phi^\alpha_{2i}\phi^\beta_{i1}) R_{\beta\alpha 12} + \sum \frac{H^\beta}{2} (\phi^\alpha_{12} R_{\beta\alpha 21} + \phi^\alpha_{21} R_{\beta\alpha 12}) \\
&= \sum (\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij} H^\alpha_{ij} + (2 + \frac{H^2}{2} - \Phi)\Phi - \sum R^2_{\alpha\beta 12}.
\end{aligned}
$$

2.2 Auxiliary lemmas

We shall establish some basic lemmas about Willmore surfaces in this section.

Lemma 2.2.1. $\frac{1}{2}\Delta\Phi = \sum(\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij}H^\alpha_{ij} + \Phi(2+\frac{H^2}{2}-\Phi) - \sum R^2_{\alpha\beta 12}$.
If $n=3$, *then* $\frac{1}{2}\Delta\Phi = \sum \phi^2_{ijk} + \sum \phi_{ij}H_{ij} + \Phi(2+\frac{H^2}{2}-\Phi)$.

Proof. By section 2.1, we have

$$\begin{aligned}
\Delta\phi^\alpha_{ij} &= \sum \phi^\alpha_{ijkk} \\
&= \sum \phi^\alpha_{kkij} + H^\alpha_{ij} - \frac{\Delta H^\alpha}{2}\delta_{ij} + (2+\frac{H^2}{2}-\Phi)\phi^\alpha_{ij} + \sum \phi^\beta_{ki}R_{\beta\alpha jk} \\
&\quad -(1+\frac{H^2}{4}-\frac{\Phi}{2})\sum \phi^\alpha_{kk}\delta_{ij} + \sum_\beta \frac{H^\beta}{2}R_{\beta\alpha ji},
\end{aligned}$$

where Δ is the Laplacian in the normal bundle. Since $\sum \phi^\alpha_{kk} = 0$, it follows that

$$\begin{aligned}
\frac{1}{2}\Delta\Phi &= \sum(\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij}\phi^\alpha_{ijkk} \\
&= \sum(\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij}H^\alpha_{ij} + (2+\frac{H^2}{2}-\Phi)\Phi \\
&\quad + \sum \phi^\alpha_{ij}\phi^\beta_{ki}R_{\beta\alpha jk} + \sum \frac{H^\beta}{2}\phi^\alpha_{ij}R_{\beta\alpha ji} \\
&= \sum(\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij}H^\alpha_{ij} + (2+\frac{H^2}{2}-\Phi)\Phi \\
&\quad + \sum(\phi^\alpha_{1i}\phi^\beta_{i2} - \phi^\alpha_{2i}\phi^\beta_{i1})R_{\beta\alpha 12} + \sum \frac{H^\beta}{2}(\phi^\alpha_{12}R_{\beta\alpha 21} + \phi^\alpha_{21}R_{\beta\alpha 12}) \\
&= \sum(\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij}H^\alpha_{ij} + \Phi(2+\frac{H^2}{2}-\Phi) - \sum R^2_{\alpha\beta 12}.
\end{aligned}$$

The case of $n=3$ is clear from the above argument. □

Lemma 2.2.2. $\sum \phi^\alpha_{ijj}H^\alpha_i = \frac{1}{2}\sum |\nabla H^\alpha|^2$, *where* $\sum |\nabla H^\alpha|^2 = \sum (H^\alpha_i)^2$.

Proof. It is an immediate consequence of the fact that

$$\begin{aligned}
\sum \phi^\alpha_{ijj} &= \sum \phi^\alpha_{jij} \\
&= \sum(\phi^\alpha_{jji} + \frac{H^\alpha_i}{2}\delta_{jj} - \frac{H^\alpha_j}{2}\delta_{ij}) \\
&= H^\alpha_i - \frac{H^\alpha_i}{2} = \frac{H^\alpha_i}{2}.
\end{aligned}$$

$$\sum \phi^\alpha_{ijj}H^\alpha_i = \sum \frac{(H^\alpha_i)^2}{2} = \frac{1}{2}\sum |\nabla H^\alpha|^2.$$

□

Lemma 2.2.3. $\sum(\phi^\alpha_{ijk})^2 \geq \frac{1}{4}\sum|\nabla H^\alpha|^2$. *The equality holds if and only if* $\phi^\alpha_{111} = \phi^\alpha_{122} = \frac{H^\alpha_1}{4}$ *and* $\phi^\alpha_{211} = \phi^\alpha_{222} = \frac{H^\alpha_2}{4}$, *for all* α.

Proof. Since $0 = \phi^\alpha_{11} + \phi^\alpha_{22}$, we therefore have $\phi^\alpha_{111} = -\phi^\alpha_{221}$ and $\phi^\alpha_{112} = -\phi^\alpha_{222}$, which implies

$$\begin{aligned}
\sum(\phi^\alpha_{ijk})^2 &= \sum [\,(\phi^\alpha_{111})^2 + (\phi^\alpha_{112})^2 + 2(\phi^\alpha_{121})^2 + 2(\phi^\alpha_{122})^2 + (\phi^\alpha_{221})^2 + (\phi^\alpha_{222})^2\,] \\
&= \sum 2\,[\,(\phi^\alpha_{111})^2 + (\phi^\alpha_{222})^2 + (\phi^\alpha_{211})^2 + (\phi^\alpha_{122})^2)\,] \\
&\geq \sum [\,(\phi^\alpha_{111} + \phi^\alpha_{122})^2 + (\phi^\alpha_{222} + \phi^\alpha_{211})^2\,] \\
&= \sum [\,(\phi^\alpha_{111} + \phi^\alpha_{221} + \frac{H^\alpha_1}{2})^2 + (\phi^\alpha_{222} + \phi^\alpha_{112} + \frac{H^\alpha_2}{2})^2\,] \\
&= \sum [\,(\frac{H^\alpha_1}{2})^2 + (\frac{H^\alpha_2}{2})^2)\,] \\
&= \frac{1}{4}\sum|\,\nabla H^\alpha\,|^2 .
\end{aligned}$$

Equality holds if and only if $\phi^\alpha_{111} = \phi^\alpha_{122}$ and $\phi^\alpha_{222} = \phi^\alpha_{211}$. Since

$$\begin{aligned}
\phi^\alpha_{122} &= \phi^\alpha_{111} = -\phi^\alpha_{221} \\
&= -(\phi^\alpha_{212} + \frac{H^\alpha_2}{2}\delta_{12} - \frac{H^\alpha_1}{2}\delta_{22}) \\
&= -\phi^\alpha_{122} + \frac{H^\alpha_1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\phi^\alpha_{211} &= \phi^\alpha_{222} = -\phi^\alpha_{112} \\
&= -(\phi^\alpha_{121} + \frac{H^\alpha_1}{2}\delta_{12} - \frac{H^\alpha_2}{2}\delta_{11}) \\
&= -\phi^\alpha_{211} + \frac{H^\alpha_2}{2},
\end{aligned}$$

equality case is clear from the above argument. □

By use of the Willmore surface equation and Stokes' theorem, we have

Lemma 2.2.4. *Let* M *be a compact Willmore surface in the unit sphere* S^n. *Then*

$$\int_M \sum|\nabla H^\alpha|^2 = \int_M \sum_{ij}(\sum_\alpha \phi^\alpha_{ij}H^\alpha)^2.$$

In particular, if $n = 3$, *then*

$$\int_M |\nabla H|^2 = \int_M \Phi H^2.$$

Proof.

$$\begin{aligned}\int_M \sum |\nabla H^\alpha|^2 &= -\int_M \sum H^\alpha \Delta H^\alpha \\ &= \int_M \sum \phi^\alpha_{ij}\phi^\beta_{ij} H^\alpha H^\beta \\ &= \int_M \sum_{ij} (\sum_\alpha \phi^\alpha_{ij} H^\alpha)^2.\end{aligned}$$

In particular, if $n = 3$, then

$$\int_M |\nabla H|^2 = \int_M \Phi H^2.$$

□

Lemma 2.2.5. *For* $n = 3$, $\Phi \sum \phi^2_{ijk} = \frac{|\nabla\Phi|^2}{2} + \Phi \frac{|\nabla H|^2}{2} - \sum \phi_{ij} H_i \Phi_j$.

Proof. Since $\sum \phi_{ii} = 0$, we have $\phi_{11i} = -\phi_{22i}$, for all i. It follows that

$$\sum \phi^2_{ijk} = 2(\phi^2_{111} + \phi^2_{112} + \phi^2_{121} + \phi^2_{122}).$$

At a point p of M, we can rotate the frame so $\phi_{11} = \sqrt{\frac{\Phi}{2}}$, $\phi_{12} = \phi_{21} = 0$ and $\phi_{22} = -\sqrt{\frac{\Phi}{2}}$, we have

$$\Phi_i = 2\sum \phi_{kl}\phi_{kli} = 2\sqrt{\frac{\Phi}{2}}(\phi_{11i} - \phi_{22i}) = 2\sqrt{2\Phi}\phi_{11i},$$

for all i. Using $\sum \phi_{ijj} = \frac{H_i}{2}$, we have

$$\begin{aligned}&| \nabla\Phi |^2 = 8\Phi(\phi^2_{111} + \phi^2_{112}),\\ &\sum \phi_{ij} H_i \Phi_j = 4\Phi[\phi_{111}(\phi_{111} + \phi_{122}) - \phi_{112}(\phi_{121} - \phi_{112})],\\ &| \nabla H |^2 = 2[(\phi_{111} + \phi_{122})^2 + (\phi_{121} - \phi_{112})^2],\end{aligned}$$

at p. The proof is then straightforward. □

Lemma 2.2.6. *If* $\sum (x^\alpha)^2 + (y^\alpha)^2 = \frac{\Phi}{2}$, $\sum (z^\alpha)^2 = z^2$ *and* c *is a nonnegative constant, then* $(\sum x^\alpha z^\alpha)^2 + (\sum y^\alpha z^\alpha)^2 + 16c\sum(x^\alpha)^2\sum(y^\alpha)^2 - 16c(\sum x^\alpha y^\alpha)^2 \le f(\Phi, z)$, *where* $f(\Phi, z) = c(\Phi + \frac{z^2}{8c})^2$, *if* c *is positive and* $\Phi > \frac{z^2}{8c}$; $f(\Phi, z) = \frac{1}{2}\Phi z^2$, *otherwise. The equality of the first case holds if and only if one of the following three cases holds*

(1) $A=0$, $B^2=\frac{z^2}{4}(\Phi+\frac{z^2}{8c})$, $\xi=\frac{1}{4}(\Phi-\frac{z^2}{8c})$, $\eta=\frac{1}{4}(\Phi+\frac{z^2}{8c})$, $\zeta=0$ *and* $z^\alpha=4\frac{By^\alpha}{\Phi+\frac{z^2}{8c}}$,
(2) $A^2=\frac{z^2}{4}(\Phi+\frac{z^2}{8c})$, $B=0$, $\xi=\frac{1}{4}(\Phi+\frac{z^2}{8c})$, $\eta=\frac{1}{4}(\Phi-\frac{z^2}{8c})$, $\zeta=0$ *and* $z^\alpha=4\frac{Ax^\alpha}{\Phi+\frac{z^2}{8c}}$,
(3)$A^2+B^2=\frac{z^2}{4}(\Phi+\frac{z^2}{8c})$, $A^2-B^2=4c(\Phi+\frac{z^2}{8c})(\xi-\eta)$, $AB=4c(\Phi+\frac{z^2}{8c})\zeta$, $\xi\eta-\zeta^2=\frac{1}{16}(\Phi+\frac{z^2}{8c})(\Phi-\frac{z^2}{8c})$ *and* $z^\alpha=4\frac{Ax^\alpha+By^\alpha}{\Phi+\frac{z^2}{8c}}$, *where* $A=\sum x^\alpha z^\alpha$, $B=\sum y^\alpha z^\alpha$, $\xi=\sum(x^\alpha)^2$, $\eta=\sum(y^\alpha)^2$ *and* $\zeta=\sum x^\alpha y^\alpha$.

Proof. We first observe that the result follows by direct estimate for the cases of $c=0$, $z=0$, $\Phi=0$ or $\xi\eta-\zeta^2=0$. Without loss of generality, we may assume that c, z, Φ and $\xi\eta-\zeta^2$ are positive. By using the Lagrange multiplier technique, we get that

$$\begin{aligned} Az^\alpha+16c\eta x^\alpha-16c\zeta y^\alpha+\mu x^\alpha &= 0,\\ Bz^\alpha+16c\xi y^\alpha-16c\zeta x^\alpha+\mu y^\alpha &= 0,\\ Ax^\alpha+By^\alpha+\nu z^\alpha &= 0, \end{aligned}$$

for all α. Multiplying the these equations by x^β, y^β and z^β, we find that

$$\begin{aligned} A^2+16c(\xi\eta-\zeta^2)+\mu\xi &= 0,\\ B^2+16c(\xi\eta-\zeta^2)+\mu\eta &= 0,\\ AB+\mu\zeta &= 0,\\ Az^2+16cA\eta-16cB\zeta+\mu A &= 0,\\ Bz^2+16cB\xi-16cA\zeta+\mu B &= 0,\\ A\xi+B\zeta+\nu A &= 0,\\ A\zeta+B\eta+\nu B &= 0,\\ A^2+B^2+\nu z^2 &= 0, \end{aligned}$$

and thus

$$\mu=-\frac{2}{\Phi}(A^2+B^2+32c(\xi\eta-\zeta^2)),$$

and

$$\nu=-\frac{A^2+B^2}{z^2}.$$

After making the substitutions of μ and ν, the Lagrange conditions can be rewritten as

$$\begin{aligned}
A^2+16c(\xi\eta-\zeta^2) &= \frac{2\xi}{\Phi}(A^2+B^2+32c(\xi\eta-\zeta^2)),\\
B^2+16c(\xi\eta-\zeta^2) &= \frac{2\eta}{\Phi}(A^2+B^2+32c(\xi\eta-\zeta^2)),\\
AB &= \frac{2\zeta}{\Phi}(A^2+B^2+32c(\xi\eta-\zeta^2)),\\
Az^2+16cA\eta-16cB\zeta &= \frac{2A}{\Phi}(A^2+B^2+32c(\xi\eta-\zeta^2)),\\
Bz^2+16cB\xi-16cA\zeta &= \frac{2B}{\Phi}(A^2+B^2+32c(\xi\eta-\zeta^2)),\\
z^2(A\xi+B\zeta) &= A(A^2+B^2),\\
z^2(A\zeta+B\eta) &= B(A^2+B^2).
\end{aligned}$$

Case 1. $A=B=0$. The only points that can give rise to a local maximum value $c\Phi^2$ are $\xi=\eta=\frac{\Phi}{4}$ and $\zeta=0$. We note that $c\Phi^2\le\frac{1}{2}\Phi z^2$ if $\Phi\le\frac{z^2}{8c}$.

Case 2. $A=0$ but $B\neq 0$. In this case the third equation gives $\zeta=0$. If $\xi\neq 0$, then the side condition $\xi+\eta=\frac{\Phi}{2}$, the first and fifth equations imply $\xi=\frac{1}{2}(\frac{\Phi}{2}-\frac{z^2}{16c})$ and $\eta=\frac{1}{2}(\frac{\Phi}{2}+\frac{z^2}{16c})$. This case occurs only when $\Phi>\frac{z^2}{8c}$. It follows from the last equation that $B^2=\frac{z^2}{4}(\Phi+\frac{z^2}{8c})$, and therefore that the function takes on the value $c(\Phi+\frac{z^2}{8c})^2$. If $\xi=0$, then the assertion follows from the simple case of $\xi\eta-\zeta^2=0$.

Case 3. $A\neq 0$ but $B=0$. The argument is similar to Case 2.

Case 4. $A\neq 0$ and $B\neq 0$. It flows from the sixth and seventh equations that

$$\begin{aligned}
\xi &= \frac{1}{z^2}(A^2+B^2)-\frac{B}{A}\zeta,\\
\eta &= \frac{1}{z^2}(A^2+B^2)-\frac{A}{B}\zeta.
\end{aligned}$$

The side condition $\xi+\eta=\frac{\Phi}{2}$ then gives

$$\frac{\zeta}{AB}=\frac{2}{z^2}-\frac{\Phi}{2(A^2+B^2)}.$$

On the other hand, we know from the third, fourth and sixth equations that

$$\frac{AB}{\zeta}=z^2+8c\Phi-\frac{16c}{z^2}(A^2+B^2).$$

Comparing these two equations, we find that A^2+B^2 satisfies a quadratic equation, and by solving it, we obtain $A^2+B^2=\frac{1}{2}\Phi z^2$ or $\frac{z^2}{4}(\Phi+\frac{z^2}{8c})$.

To find the value of $\xi\eta-\zeta^2$, the third equation gives

$$\frac{2}{\Phi}(A^2+B^2+32c(\xi\eta-\zeta^2)) = z^2+8c\Phi-\frac{16c}{z^2}(A^2+B^2).$$

If $A^2+B^2=\frac{1}{2}\Phi z^2$, then $c(\xi\eta-\zeta^2)=0$. There are nothing to prove. Thus we may assume $A^2+B^2=\frac{z^2}{4}(\Phi+\frac{z^2}{8c})$. In this case, we have $c(\xi\eta-\zeta^2)=\frac{c}{16}(\Phi+\frac{z^2}{8c})(\Phi-\frac{z^2}{8c})$. This case occurs only when $\Phi>\frac{z^2}{8c}$. Combining with the first and second equations, we then obtain $A^2-B^2=4c(\Phi+\frac{z^2}{8c})(\xi-\eta)$. The third equation implies $AB=4c(\Phi+\frac{z^2}{8c})\zeta$. Equalities cases are clear from the above argument. $\square$

Let $D_{n+1}=\{x\in\mathbb{R}^{n+1}:|x|<1\}$ be the open unit ball in $\mathbb{R}^{n+1}$ and G the conformal group of S^n. For each $g\in D_{n+1}$, we introduce the mapping, also denote by g, $g:S^n\to S^n$ given by

$$g(x)=\frac{x+(\lambda+\mu<x,g>)g}{\lambda(1+<x,g>)},$$

where $\lambda=\frac{1}{\sqrt{1-|g|^2}}$ and $\mu=\frac{\lambda^2}{\lambda+1}$. We know that each conformal transformation of S^n can be expressed by $T\circ g$, where T is an orthogonal transformation of S^n and $g\in D_{n+1}$ (see [9] and [10]).

Let $x:M\hookrightarrow S^n$ be a compact Willmore surface. It follows that for each $g\in D_{n+1}$, $\bar{x}=g\circ x$ is also a compact Willmore surface. Then we have

$$\begin{aligned}\bar{e}_A &= \lambda(1+<x,g>)e_A\\ \bar{\omega}_A &= \frac{1}{\lambda(1+<x,g>)}\omega_A\\ \bar{\omega}_{AB} &= \omega_{AB}+(log\frac{1}{\lambda(1+<x,g>)})_A\omega_B-(log\frac{1}{\lambda(1+<x,g>)})_B\omega_A,\end{aligned}$$

where $1\le A,B\le n$. The new induced first fundamental form of $\bar{x}$ may be written in terms of the original induced first fundamental form as

$$d\bar{s}^2=\frac{1}{\lambda^2(1+<x,g>)^2}ds^2.$$

Furthermore, the second fundamental forms of $\bar{x}$ and x are related by

$$\bar{h}^\alpha_{ij}=\lambda[(1+<x,g>)h^\alpha_{ij}+<e_\alpha,g>\delta_{ij}].$$

We recite some relationships of corresponding quantities between $\bar{x}$ and x as follows

Lemma 2.2.7. *The new $\bar{H}$, $\bar{\Phi}$ and its derivatives can be expressed in terms of that of original as follows*

1. $\bar{H}^\alpha = \lambda[(1+ < x, g >)H^\alpha + 2 < e_\alpha, g >]$.
2. $\bar{H}^\alpha_i = \lambda^2(1+ < x, g >)[(1+ < x, g >)H^\alpha_i - 2\sum \phi^\alpha_{ij} < e_j, g >]$.
3. $\bar{\phi}^\alpha_{ij} = \lambda(1+ < x, g >)\phi^\alpha_{ij}$.
4. $\bar{\Phi} = \lambda^2(1+ < x, g >)^2\Phi$.
5. $\bar{\phi}^\alpha_{ijk} = \lambda^2(1+ < x, g >)[(1+ < x, g >)\phi^\alpha_{ijk} + \phi^\alpha_{ij} < e_k, g > +\phi^\alpha_{jk} < e_i, g > +\phi^\alpha_{ki} < e_j, g > - \sum \phi^\alpha_{lj} < e_l, g > \delta_{ki} - \sum \phi^\alpha_{il} < e_l, g > \delta_{jk}]$.

Proof. (1)It follows from the induced second fundamental forms that

$$\begin{aligned}
\bar{H}^\alpha &= \sum \bar{h}^\alpha_{ii} \\
&= \lambda[(1+ < x, g >) \sum h^\alpha_{ii} + \sum < e_\alpha, g > \delta_{ii}] \\
&= \lambda[(1+ < x, g >)H^\alpha + 2 < e_\alpha, g >].
\end{aligned}$$

(2)By using of the structure equations, we have

$$\begin{aligned}
< x, g >_i &= < e_i, g >, \\
< e_\alpha, g >_i &= -\sum \phi^\alpha_{ij} < e_j, g > -\frac{H^\alpha}{2} < e_i, g > .
\end{aligned}$$

Since

$$\sum h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum h^\alpha_{kj}\omega_{ki} + \sum h^\alpha_{ik}\omega_{kj} + \sum h^\beta_{ij}\omega_{\beta\alpha},$$

we have that

$$
\begin{aligned}
d\bar{H}^{\alpha} &= \sum d\bar{h}^{\alpha}_{ii} = \sum(\bar{h}^{\alpha}_{iik}\bar{\omega}_k - 2\bar{h}^{\alpha}_{ki}\bar{\omega}_{ki} - \bar{h}^{\beta}_{ii}\bar{\omega}_{\beta\alpha}) \\
&= \frac{1}{\lambda(1+<x,g>)}\sum \bar{H}^{\alpha}_k\omega_k - 2\lambda\sum[\ (1+<x,g>)h^{\alpha}_{ki} \\
&\quad\ +<e_{\alpha},g>\delta_{ki}\ ](\omega_{ki} - \frac{<e_k,g>}{1+<x,g>}\omega_i + \frac{<e_i,g>}{1+<x,g>}\omega_k) - \sum \bar{H}^{\beta}\omega_{\beta\alpha} \\
&= \frac{1}{\lambda(1+<x,g>)}\sum \bar{H}^{\alpha}_k\omega_k - 2\lambda(1+<x,g>)\sum h^{\alpha}_{ki}\omega_{ki} \\
&\quad -\lambda\sum[(1+<x,g>)H^{\beta} + 2<e_{\beta},g>]\omega_{\beta\alpha} \\
&= \frac{1}{\lambda(1+<x,g>)}\sum \bar{H}^{\alpha}_k\omega_k - 2\lambda(1+<x,g>)\sum \phi^{\alpha}_{ki}\omega_{ki} \\
&\quad -\lambda\sum[(1+<x,g>)H^{\beta} + 2<e_{\beta},g>]\omega_{\beta\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
d\bar{H}^{\alpha} &= d\{\ \lambda[(1+<x,g>)H^{\alpha} + 2<e_{\alpha},g>]\ \} \\
&= \lambda\sum <x,g>_k\ \omega_k H^{\alpha} + \lambda(1+<x,g>)\sum dh^{\alpha}_{ii} + 2\lambda <de_{\alpha},g> \\
&= \lambda\sum <x,g>_k\ \omega_k H^{\alpha} + \lambda(1+<x,g>)\sum(h^{\alpha}_{iik}\omega_k - 2h^{\alpha}_{ki}\omega_{ki} - h^{\beta}_{ii}\omega_{\beta\alpha}) \\
&\quad -2\lambda\sum h^{\alpha}_{ki} <e_i,g>\omega_k + 2\lambda\sum <e_{\beta},g>\omega_{\alpha\beta} \\
&= \lambda <e_k,g>\omega_k H^{\alpha} + \lambda(1+<x,g>)\sum H^{\alpha}_k\omega_k \\
&\quad -2\lambda(1+<x,g>)\sum(\phi^{\alpha}_{ki} + \frac{H^{\alpha}}{2}\delta_{ki})\omega_{ki} \\
&\quad -\lambda(1+<x,g>)\sum H^{\beta}\omega_{\beta\alpha} - 2\lambda\sum(\phi^{\alpha}_{ki} + \frac{H^{\alpha}}{2}\delta_{ki}) <e_i,g>\omega_k \\
&\quad +2\lambda\sum <e_{\beta},g>\omega_{\alpha\beta} \\
&= \lambda(1+<x,g>)\sum H^{\alpha}_k\omega_k - 2\lambda(1+<x,g>)\sum \phi^{\alpha}_{ki}\omega_{ki} \\
&\quad -\lambda(1+<x,g>)\sum H^{\beta}\omega_{\beta\alpha} - 2\lambda\sum \phi^{\alpha}_{ki} <e_i,g>\omega_k + 2\lambda\sum <e_{\beta},g>\omega_{\alpha\beta}.
\end{aligned}
$$

Hence

$$
\bar{H}^{\alpha}_i = \lambda^2(1+<x,g>)[(1+<x,g>)H^{\alpha}_i - 2\sum \phi^{\alpha}_{ij} <e_j,g>].
$$

(3)The proof is straightforward by

$$\begin{aligned}\bar{\phi}^{\alpha}_{ij} &= \bar{h}^{\alpha}_{ij} - \frac{1}{2}\bar{H}^{\alpha}\delta_{ij} \\ &= \lambda[(1+<x,g>)h^{\alpha}_{ij} + <e_{\alpha},g>\delta_{ij}] \\ &\quad -\frac{1}{2}\lambda[(1+<x,g>)H^{\alpha} + 2<e_{\alpha},g>]\delta_{ij} \\ &= \lambda(1+<x,g>)(h^{\alpha}_{ij} - \frac{1}{2}H^{\alpha}\delta_{ij}) \\ &= \lambda(1+<x,g>)\phi^{\alpha}_{ij}.\end{aligned}$$

(4)By (3), we have

$$\bar{\Phi} = \sum(\bar{\phi}^{\alpha}_{ij})^2 = \lambda^2(1+<x,g>)^2\sum(\phi^{\alpha}_{ij})^2 = \lambda^2(1+<x,g>)^2\Phi.$$

(5)Since

$$\begin{aligned}d\bar{h}^{\alpha}_{ij} &= \sum(\bar{h}^{\alpha}_{ijk}\bar{\omega}_k - \bar{h}^{\alpha}_{ik}\bar{\omega}_{kj} - \bar{h}^{\alpha}_{kj}\bar{\omega}_{ki} - \bar{h}^{\beta}_{ij}\bar{\omega}_{\beta\alpha}) \\ &= \frac{1}{\lambda(1+<x,g>)}\sum\bar{h}^{\alpha}_{ijk}\omega_k - \lambda\sum[\,(1+<x,g>)h^{\alpha}_{ik} \\ &\quad + <e_{\alpha},g>\delta_{ik}\,](\omega_{kj} - \frac{<e_k,g>}{1+<x,g>}\omega_j + \frac{<e_j,g>}{1+<x,g>}\omega_k) \\ &\quad -\lambda\sum[\,(1+<x,g>)h^{\alpha}_{kj} \\ &\quad + <e_{\alpha},g>\delta_{kj}\,](\omega_{ki} - \frac{<e_k,g>}{1+<x,g>}\omega_i + \frac{<e_i,g>}{1+<x,g>}\omega_k) \\ &\quad -\lambda\sum[\,(1+<x,g>)h^{\beta}_{ij} + <e_{\beta},g>\delta_{ij}\,]\omega_{\beta\alpha} \\ &= \frac{1}{\lambda(1+<x,g>)}\sum\bar{h}^{\alpha}_{ijk}\omega_k - \lambda(1+<x,g>)\sum h^{\alpha}_{ik}\omega_{kj} + \lambda\sum<e_k,g>h^{\alpha}_{ik}\omega_j \\ &\quad -\lambda\sum<e_j,g>h^{\alpha}_{ik}\omega_k - \lambda(1+<x,g>)\sum h^{\alpha}_{kj}\omega_{ki} + \lambda\sum<e_k,g>h^{\alpha}_{kj}\omega_i \\ &\quad -\lambda\sum<e_i,g>h^{\alpha}_{kj}\omega_k - \lambda(1+<x,g>)\sum h^{\beta}_{ij}\omega_{\beta\alpha} - \lambda\sum<e_{\beta},g>\delta_{ij}\omega_{\beta\alpha} \\ &= \sum\{\,\frac{1}{\lambda(1+<x,g>)}\bar{h}^{\alpha}_{ijk} + \lambda<e_l,g>h^{\alpha}_{il}\delta_{jk} - \lambda<e_j,g>h^{\alpha}_{ik} \\ &\quad +\lambda<e_l,g>h^{\alpha}_{lj}\delta_{ki} - \lambda<e_i,g>h^{\alpha}_{kj}\,\}\omega_k \\ &\quad -\lambda(1+<x,g>)\sum h^{\alpha}_{ik}\omega_{kj} - \lambda(1+<x,g>)\sum h^{\alpha}_{kj}\omega_{ki} \\ &\quad -\lambda(1+<x,g>)\sum h^{\beta}_{ij}\omega_{\beta\alpha} - \lambda\sum<e_{\beta},g>\delta_{ij}\omega_{\beta\alpha}\end{aligned}$$

and

$$
\begin{aligned}
d\bar{h}^{\alpha}_{ij} &= d\{\ \lambda[(1+<x,g>)h^{\alpha}_{ij}+<e_{\alpha},g>\delta_{ij}]\ \} \\
&= \lambda\sum <x,g>_k\ \omega_k h^{\alpha}_{ij} + \lambda(1+<x,g>)\sum dh^{\alpha}_{ij} + \lambda <de_{\alpha},g>\delta_{ij} \\
&= \lambda\sum <e_k,g>\omega_k h^{\alpha}_{ij} \\
&\quad +\lambda(1+<x,g>)\sum(h^{\alpha}_{ijk}\omega_k - h^{\alpha}_{ki}\omega_{kj} - h^{\alpha}_{kj}\omega_{ki} - h^{\beta}_{ij}\omega_{\beta\alpha}) \\
&\quad -\lambda\sum h^{\alpha}_{kl}<e_l,g>\omega_k\delta_{ij} + \lambda\sum <e_{\beta},g>\delta_{ij}\omega_{\alpha\beta} \\
&= \sum\{\ \lambda(1+<x,g>)h^{\alpha}_{ijk} + \lambda<e_k,g>h^{\alpha}_{ij} - \lambda<e_l,g>h^{\alpha}_{kl}\delta_{ij}\ \}\omega_k \\
&\quad -\lambda(1+<x,g>)\sum h^{\alpha}_{ki}\omega_{kj} - \lambda(1+<x,g>)\sum h^{\alpha}_{kj}\omega_{ki} \\
&\quad -\lambda(1+<x,g>)\sum h^{\beta}_{ij}\omega_{\beta\alpha} - \lambda\sum <e_{\beta},g>\delta_{ij}\omega_{\beta\alpha},
\end{aligned}
$$

hence

$$
\begin{aligned}
\bar{h}^{\alpha}_{ijk} &= \lambda^2(1+<x,g>)[(1+<x,g>)h^{\alpha}_{ijk} + h^{\alpha}_{ij}<e_k,g> + h^{\alpha}_{kj}<e_i,g> \\
&\quad + h^{\alpha}_{ik}<e_j,g> - \sum h^{\alpha}_{kl}<e_l,g>\delta_{ij} - \sum h^{\alpha}_{lj}<e_l,g>\delta_{ik} - \sum h^{\alpha}_{il}<e_l,g>\delta_{jk}.
\end{aligned}
$$

Now, we replace h^{α}_{ij} by $\phi^{\alpha}_{ij} + \frac{H^{\alpha}}{2}\delta_{ij}$.

$$
\begin{aligned}
\bar{h}^{\alpha}_{ijk} &= \bar{\phi}^{\alpha}_{ijk} + \frac{\bar{H}^{\alpha}_k}{2}\delta_{ij} \\
&= \bar{\phi}^{\alpha}_{ijk} + \frac{\lambda^2(1+<x,g>)}{2}\delta_{ij}[\ (1+<x,g>)H^{\alpha}_k - 2\sum\phi^{\alpha}_{kl}<e_l,g>\ ] \\
&= \bar{\phi}^{\alpha}_{ijk} + \frac{\lambda^2(1+<x,g>)^2}{2}\delta_{ij}H^{\alpha}_k - \lambda^2(1+<x,g>)\sum\phi^{\alpha}_{kl}<e_l,g>\delta_{ij}.
\end{aligned}
$$

On the other hand, we can get

$$
\begin{aligned}
\bar{h}^{\alpha}_{ijk} &= \lambda^2(1+<x,g>)[\ (1+<x,g>)\phi^{\alpha}_{ijk} + \frac{(1+<x,g>)H^{\alpha}_k}{2}\delta_{ij} + \phi^{\alpha}_{ij}<e_k,g> \\
&\quad +\frac{H^{\alpha}}{2}<e_k,g>\delta_{ij} + \phi^{\alpha}_{kj}<e_i,g> + \frac{H^{\alpha}}{2}<e_i,g>\delta_{kj} + \phi^{\alpha}_{ik}<e_j,g> \\
&\quad +\frac{H^{\alpha}}{2}<e_j,g>\delta_{ik} - \sum\phi^{\alpha}_{kl}<e_l,g>\delta_{ij} - \frac{H^{\alpha}}{2}<e_k,g>\delta_{ij} - \sum\phi^{\alpha}_{lj}<e_l,g>\delta_{ik} \\
&\quad -\frac{H^{\alpha}}{2}<e_j,g>\delta_{ki} - \sum\phi^{\alpha}_{il}<e_l,g>\delta_{jk} - \frac{H^{\alpha}}{2}<e_i,g>\delta_{kj}\ ] \\
&= \lambda^2(1+<x,g>)[\ (1+<x,g>)\phi^{\alpha}_{ijk} + \frac{(1+<x,g>)H^{\alpha}_k}{2}\delta_{ij} + \phi^{\alpha}_{ij}<e_k,g> \\
&\quad +\phi^{\alpha}_{kj}<e_i,g> + \phi^{\alpha}_{ik}<e_j,g> - \sum\phi^{\alpha}_{kl}<e_l,g>\delta_{ij} \\
&\quad -\sum\phi^{\alpha}_{lj}<e_l,g>\delta_{ik} - \sum\phi^{\alpha}_{il}<e_l,g>\delta_{jk}.
\end{aligned}
$$

Hence

$$\begin{aligned}
\bar{\phi}^{\alpha}_{ijk} &= \lambda^2(1+<x,g>)[(1+<x,g>)\phi^{\alpha}_{ijk}+\phi^{\alpha}_{ij}<e_k,g>+\phi^{\alpha}_{jk}<e_i,g>\\
&\quad +\phi^{\alpha}_{ki}<e_j,g>-\sum\phi^{\alpha}_{lj}<e_l,g>\delta_{ki}-\sum\phi^{\alpha}_{il}<e_l,g>\delta_{jk}].
\end{aligned}$$

□

For any given constant vector $g\in\mathbb{R}^{n+1}$, let $F^{\alpha}(x)=(1+<x,g>)H^{\alpha}+2<e_{\alpha},g>$. Then F^{α} satisfies the following equation

Lemma 2.2.8. $\Delta F^{\alpha}+\sum\phi^{\alpha}_{ij}\phi^{\beta}_{ij}F^{\beta}=0$.

Proof. It follows from the structure equations that

$$\begin{aligned}
<x,g>_i &= <e_i,g>,\\
<x,g>_{ij} &= \sum\phi^{\alpha}_{ij}<e_{\alpha},g>+\delta_{ij}\sum\frac{H^{\alpha}}{2}<e_{\alpha},g>-\delta_{ij}<x,g>,\\
<e_{\alpha},g>_i &= -\sum\phi^{\alpha}_{ij}<e_j,g>-\frac{H^{\alpha}}{2}<e_i,g>,\\
\Delta<e_{\alpha},g> &= -\sum H^{\alpha}_i<e_i,g>-\sum\phi^{\alpha}_{ij}\phi^{\beta}_{ij}<e_{\beta},g>-\sum\frac{H^{\alpha}H^{\beta}}{2}<e_{\beta},g>\\
&\quad +H^{\alpha}<x,g>.
\end{aligned}$$

We then have

$$F^{\alpha}_i=(1+<x,g>)H^{\alpha}_i-2\sum\phi^{\alpha}_{ij}<e_j,g>,$$

and

$$\begin{aligned}
\Delta F^{\alpha} &= H^{\alpha}\Delta<x,g>+2\sum<e_i,g>H^{\alpha}_i+(1+<x,g>)\Delta H^{\alpha}\\
&\quad +2\Delta<e_{\alpha},g>\\
&= \sum H^{\alpha}H^{\beta}<e_{\beta},g>-2H^{\alpha}<x,g>+2\sum<e_i,g>H^{\alpha}_i\\
&\quad -(1+<x,g>)\sum\phi^{\alpha}_{ij}\phi^{\beta}_{ij}H^{\beta}-2\sum H^{\alpha}_i<e_i,g>\\
&\quad -2\sum\phi^{\alpha}_{ij}\phi^{\beta}_{ij}<e_{\beta},g>-\sum H^{\alpha}H^{\beta}<e_{\beta},g>+2H^{\alpha}<x,g>\\
&= -\sum[\,(1+<x,g>)H^{\beta}+2<e_{\beta},g>\,]\,\phi^{\alpha}_{ij}\phi^{\beta}_{ij}\\
&= -\sum\phi^{\alpha}_{ij}\phi^{\beta}_{ij}F^{\beta}.
\end{aligned}$$

□

Finally, for any given constant vector $g \in \mathbb{R}^{n+1}$, let

$$\begin{aligned}\psi^\alpha_{ijk} &= (1+ < x, g >)\phi^\alpha_{ijk} + \phi^\alpha_{ij} < e_k, g > +\phi^\alpha_{jk} < e_i, g > +\phi^\alpha_{ki} < e_j, g > \\ &\quad -\sum \phi^\alpha_{lj} < e_l, g > \delta_{ki} - \sum \phi^\alpha_{il} < e_l, g > \delta_{jk},\end{aligned}$$

for all α, i, j, k. We will use the following properties.

Lemma 2.2.9. ψ^α_{ijk} *satisfies the following equations:*

1. $\psi^\alpha_{ijk} = \psi^\alpha_{jik}$, *for all* α, i, j, k.
2. $\Sigma\psi^\alpha_{jji} = 0$, *for all* i.
3. $\Sigma\psi^\alpha_{ijj} = \frac{F^\alpha_i}{2}$, *for all* α, i.

Proof. (1)By a direct computation, we have

$$\begin{aligned}\psi^\alpha_{ijk} &= (1+ < x, g >)\phi^\alpha_{ijk} + \phi^\alpha_{ij} < e_k, g > +\phi^\alpha_{jk} < e_i, g > +\phi^\alpha_{ki} < e_j, g > \\ &\quad -\sum \phi^\alpha_{lj} < e_l, g > \delta_{ki} - \sum \phi^\alpha_{il} < e_l, g > \delta_{jk} \\ &= (1+ < x, g >)\phi^\alpha_{jik} + \phi^\alpha_{ji} < e_k, g > +\phi^\alpha_{jk} < e_i, g > +\phi^\alpha_{ki} < e_j, g > \\ &\quad -\sum \phi^\alpha_{lj} < e_l, g > \delta_{ki} - \sum \phi^\alpha_{il} < e_l, g > \delta_{jk} \\ &= \psi^\alpha_{jik}.\end{aligned}$$

(2)It is an immediate consequence of the fact $\sum \phi^\alpha_{ii} = 0$.

$$\begin{aligned}\sum \psi^\alpha_{jji} &= (1+ < x, g >)\sum \phi^\alpha_{jji} + \sum \phi^\alpha_{jj} < e_i, g > + \sum \phi^\alpha_{ji} < e_j, g > \\ &\quad + \sum \phi^\alpha_{ij} < e_j, g > - \sum \phi^\alpha_{lj} < e_l, g > \delta_{ij} - \sum \phi^\alpha_{jl} < e_l, g > \delta_{ji} \\ &= 2\sum \phi^\alpha_{ji} < e_j, g > -2\sum \phi^\alpha_{li} < e_l, g > \\ &= 0.\end{aligned}$$

(3)Since $\sum \phi^\alpha_{ii} = 0$ and

$$\phi^\alpha_{ijk} = \phi^\alpha_{ikj} + \frac{H^\alpha_j}{2}\delta_{ik} - \frac{H^\alpha_k}{2}\delta_{ij},$$

we have that

$$\begin{aligned}
\sum \psi^{\alpha}_{ijj} &= (1+ < x, g >) \sum \phi^{\alpha}_{ijj} + \sum \phi^{\alpha}_{ij} < e_j, g > + \sum \phi^{\alpha}_{jj} < e_i, g > \\
&\quad + \sum \phi^{\alpha}_{ji} < e_j, g > - \sum \phi^{\alpha}_{lj} < e_l, g > \delta_{ji} - \sum \phi^{\alpha}_{il} < e_l, g > \delta_{jj} \\
&= (1+ < x, g >) \sum (\phi^{\alpha}_{jji} + \frac{H^{\alpha}_i}{2}\delta_{jj} - \frac{H^{\alpha}_j}{2}\delta_{ij}) \\
&\quad +2 \sum \phi^{\alpha}_{ji} < e_j, g > - \sum \phi^{\alpha}_{li} < e_l, g > -2 \sum \phi^{\alpha}_{li} < e_l, g > \\
&= (1+ < x, g >) \frac{H^{\alpha}_i}{2} - \sum \phi^{\alpha}_{ij} < e_j, g > \\
&= \frac{F^{\alpha}_i}{2}.
\end{aligned}$$

□

Chapter 3

Willmore surfaces in S^3

In this chapter we shall consider the Willmore surfaces in the unit 3-sphere, and establish an integral inequality for Φ and H. Based on this integral inequality, we characterize the totally umbilical spheres and the Clifford torus by a certain pinching condition. We then introduce a conformal invariant quantity which is formulated in terms of Φ and H, and prove that if this quantity is bounded above by that value of the Clifford torus then $x(M)$ is either a totally umbilical sphere or a conformal Clifford torus.

3.1 A pinching theorem of Willmore surfaces in the unit 3-sphere

Our pinching theorem of compact Willmore surfaces in S^3 is the following:

Theorem A. Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere S^3. Then

$$\int_M \Phi(2 + \frac{H^2}{4} - \Phi) \leq 0.$$

In particular, if

$$0 \leq \Phi \leq 2 + \frac{H^2}{4},$$

then either $\Phi = 0$ and M is totally umbilical, or $\Phi = 2 + \frac{H^2}{4}$ and M is the Clifford torus.

Proof. Integrating both sides of the Lemma 2.2.1 over M, we have

$$\begin{aligned} 0 &= \int_M \Big(\sum \phi_{ijk}^2 + \sum \phi_{ij} H_{ij} + \Phi(2 + \frac{H^2}{2} - \Phi) \Big) \\ &= \int_M \Big(\sum \phi_{ijk}^2 - \sum \phi_{ijj} H_i + \Phi(2 + \frac{H^2}{2} - \Phi) \Big). \end{aligned}$$

It follows from Lemmas 2.2.2 and 2.2.3 that

$$\begin{aligned} 0 &\geq \int_M \Big(\frac{1}{4}|\nabla H|^2 - \frac{1}{2}|\nabla H|^2 + \Phi(2 + \frac{H^2}{2} - \Phi) \Big) \\ &= \int_M \Big(-\frac{1}{4}|\nabla H|^2 + \Phi(2 + \frac{H^2}{2} - \Phi) \Big). \end{aligned}$$

We obtain from Lemma 2.2.4 that

$$\begin{aligned} 0 &\geq \int_M \Big(-\frac{1}{4}\Phi H^2 + \Phi(2 + \frac{H^2}{2} - \Phi) \Big) \\ &= \int_M \Phi(2 + \frac{H^2}{4} - \Phi). \end{aligned}$$

If $0 \leq \Phi \leq 2 + \frac{H^2}{4}$, then either $\Phi = 0$ and M is totally umbilical, or $\Phi = 2 + \frac{H^2}{4}$. In the latter case, all the integral inequalities become equalities. Assuming that $\Phi = 2 + \frac{H^2}{4}$, it follows from Lemmas 2.2.1 and 2.2.5 that

$$\begin{aligned} \int_M (2 + \frac{H^2}{2} - \Phi) &= \int_M \Big(\frac{1}{2}\frac{\Delta\Phi}{\Phi} - \frac{\sum \phi_{ijk}^2}{\Phi} - \frac{\sum \phi_{ij} H_{ij}}{\Phi}\Big) \\ &= \int_M \Big(\frac{1}{2}\frac{\Delta\Phi}{\Phi} - \frac{|\nabla\Phi|^2}{2\Phi^2} - \frac{|\nabla H|^2}{2\Phi} + \frac{\sum \phi_{ij} H_i \Phi_j}{\Phi^2} - \frac{\sum \phi_{ij} H_{ij}}{\Phi}\Big) \\ &= \int_M \Big(\frac{1}{2}\Delta \log \Phi - \frac{|\nabla H|^2}{2\Phi} + \frac{\sum \phi_{ij} H_i \Phi_j}{\Phi^2} - \frac{\sum \phi_{ij} H_{ij}}{\Phi}\Big) \\ &= \int_M \Big(-\frac{|\nabla H|^2}{2\Phi} + \frac{\sum \phi_{ij} H_i \Phi_j}{\Phi^2} + \sum (\frac{\phi_{ij}}{\Phi})_j H_i\Big) \\ &= \int_M \Big(-\frac{|\nabla H|^2}{2\Phi} + \frac{\sum \phi_{ij} H_i \Phi_j}{\Phi^2} + \sum \frac{\Phi\phi_{ijj} - \phi_{ij}\Phi_j}{\Phi^2} H_i\Big) \\ &= \int_M \Big(-\frac{|\nabla H|^2}{2\Phi} + \frac{|\nabla H|^2}{2\Phi}\Big) \\ &= 0. \end{aligned}$$

This implies that

$$0 = \int_M (2 + \frac{H^2}{2} - \Phi) = \int_M \frac{H^2}{4}.$$

Thus M is a minimal surface of S^3 with $S = 2$, we conclude that M is the Clifford torus (see [CDK]). This completes the proof of Theorem A.

□

3.2 A pinching theorem for conformal classes of Willmore surfaces in the unit 3-sphere

Our pinching theorem for conformal classes of Willmore Surfaces in S^3 is the following:

Theorem B. Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere S^3. If

$$inf_{g\in G} max_{g\circ x(M)}(\Phi_g - \frac{1}{4}H_g^2\ ) \leq 2,$$

where G is the conformal group of the ambient space S^3, Φ_g and H_g are the square of the length of the trace free part of the second fundamental form and the mean curvature of the immersion $g \circ x$ respectively, then $x(M)$ is either a totally umbilical sphere or a conformal Clifford torus.

Proof. By the hypothesis $inf_{g\in G} max_{g\circ x(M)}(\Phi_g - \frac{1}{4}H_g^2) \leq 2$, there is a sequence $g_m \in G$ such that $\Phi_m - \frac{1}{4}H_m^2 \leq 2 + \frac{1}{m}$ on M, for all $m = 1, 2, \cdots$, where Φ_m and H_m are the square of the length of the trace free part of the second fundamental form and the mean curvature of the immersion $g_m \circ x$, respectively. Without loss of generality, we may assume that $g_m \in D_4$. The closure of D_4 in R^4 being compact, there exists a convergent subsequence of g_m. We may assume that g_m converges to g_0 for some g_0 in the closed unit disk. If $g_0 \in D_4$, then $\Phi_m \to \Phi_{g_0}$ and $H_m \to H_{g_0}$ as $m \to \infty$. We find that $\Phi_{g_0} - \frac{1}{4}H_{g_0}^2 \leq 2$ on M, and the desired conclusion follows from Theorem A. So we need only consider the case that g_0 is a constant unit vector. In this case we shall show below that M is totally umbilical.

Suppose, to get a contradiction, that Φ is positive somewhere on M. To avoid ambiguity, we shall now use the notations da and da_m for the area measures of x and $g_m \circ x$, respectively. Since $g_m \circ x$ are Willmore surfaces, the integral inequality of

Theorem A gives

$$2\int_M \Phi_m \, da_m \le \int_M \Phi_m \,(\,\Phi_m - \frac{H_m^2}{4})\, da_m \le (2+\frac{1}{m})\int_M \Phi_m \, da_m.$$

It follows from Lemma 2.2.7 that, since Willmore functional is invariant under conformal transformations of S^3,

$$\begin{aligned} & 2(1-|g_m|^2)\int_M \Phi \, da \\ \le \ & \int_M \Phi\,[\,(1+<x,g_m>)^2\Phi - \frac{1}{4}((1+<x,g_m>)H + 2<e_3,g_m>)^2]\, da \\ \le \ & (2+\frac{1}{m})(1-|g_m|^2)\int_M \Phi \, da. \end{aligned}$$

Letting $m \longrightarrow \infty$, we find that

$$\int_M \Phi\,[\,(1+<x,g_0>)^2\Phi - \frac{1}{4}((1+<x,g_0>)H + 2<e_3,g_0>)^2]\, da = 0.$$

On the other hand, since $\Phi_m - \frac{1}{4}H_m^2 \le 2 + \frac{1}{m}$ on M, Lemma 2.2.7 gives

$$(1+<x,g_m>)^2\Phi - \frac{1}{4}((1+<x,g_m>)H + 2<e_3,g_m>)^2 \le (2+\frac{1}{m})(1-|g_m|^2).$$

When m tends to infinity, we find that $(1+<x,g_0>)^2\Phi - \frac{1}{4}((1+<x,g_0>)H + 2< e_3,g_0>)^2$ is nonpositive on M.

We then conclude that $(1+<x,g_0>)^2\Phi = \frac{1}{4}((1+<x,g_0>)H + 2<e_3,g_0>)^2$ or $\Phi = 0$ on M, and hence $(1+<x,g_0>)^2\Phi = \frac{1}{4}F^2$ provided $\Phi > 0$, where F is given as in Lemma 2.2.8 corresponding to the constant unit vector g_0. This implies that either $F = 2(1+<x,g_0>)\sqrt{\Phi}$ or $F = -2(1+<x,g_0>)\sqrt{\Phi}$ on each of the connected components of the set of points where $\Phi > 0$.

For each fixed m, let $\bar{x} = g_m \circ x$. Since $g_m \circ x$ is a Willmore immersion, we have again

$$\begin{aligned} 0 &= \int_M [\,\sum \bar{\phi}_{ijk}^2 + \sum \bar{\phi}_{ij}\bar{H}_{ij} + \bar{\Phi}(2+\frac{\bar{H}^2}{2} - \bar{\Phi})]\, d\bar{a} \\ &= \int_M [\,\sum \bar{\phi}_{ijk}^2 - \sum \bar{\phi}_{ijj}\bar{H}_i + \bar{\Phi}(2+\frac{\bar{H}^2}{2} - \bar{\Phi})\,]\, d\bar{a} \\ &= \int_M [\,\sum \bar{\phi}_{ijk}^2 - \frac{|\bar{\nabla}\bar{H}|^2}{2} + \bar{\Phi}(2+\frac{\bar{H}^2}{2} - \bar{\Phi})\,]\, d\bar{a}. \end{aligned}$$

When m tends to infinity, it follows from Lemmas 2.2.7 and 2.2.8 that

$$\begin{aligned}
0 &= \int_M \psi_{ijk}^2 - \frac{1}{2}|\nabla F|^2 + \Phi[\frac{1}{2}F^2 - (1+ < x, g_0 >)^2\Phi]\, da \\
&= \int_M \psi_{ijk}^2 - \frac{1}{2}|\nabla F|^2 + \frac{1}{4}\Phi F^2\, da \\
&= \int_M 2(\psi_{111}^2 + \psi_{122}^2 + \psi_{211}^2 + \psi_{222}^2) - \frac{1}{2}|\nabla F|^2 + \frac{1}{4}\Phi F^2\, da \\
&\geq \int_M (\psi_{111} + \psi_{122})^2 + (\psi_{211} + \psi_{222})^2 - \frac{1}{2}|\nabla F|^2 + \frac{1}{4}\Phi F^2\, da \\
&= \int_M -\frac{1}{4}|\nabla F|^2 + \frac{1}{4}\Phi F^2\, da. \\
&= 0,
\end{aligned}$$

here we use the identity $(1+ < x, g_0 >)^2\Phi^2 = \frac{1}{4}\Phi F^2$. Therefore we have $\psi_{111} = \psi_{122}$ and $\psi_{211} = \psi_{222}$. Combining the last two equations with Lemma 2.2.9 and simplifying, we can express ψ_{ijk} in terms of F_1 and F_2,

$$\psi_{111} = \psi_{122} = \psi_{212} = -\psi_{221} = \frac{1}{4}F_1$$

and

$$\psi_{121} = \psi_{211} = \psi_{222} = -\psi_{112} = \frac{1}{4}F_2.$$

Let $U = 2(1+ < x, g_0 >)\sqrt{\Phi}$, and let Ω be a connected component of the set of points where $\Phi > 0$. Then

$$\begin{aligned}
U_1 &= 2\sqrt{\Phi} < e_1, g_0 > +4\frac{\phi_{11}}{\sqrt{\Phi}}(1+ < x, g_0 >)\phi_{111} + 4\frac{\phi_{12}}{\sqrt{\Phi}}(1+ < x, g_0 >)\phi_{121}, \\
U_2 &= 2\sqrt{\Phi} < e_2, g_0 > +4\frac{\phi_{11}}{\sqrt{\Phi}}(1+ < x, g_0 >)\phi_{112} + 4\frac{\phi_{12}}{\sqrt{\Phi}}(1+ < x, g_0 >)\phi_{122}
\end{aligned}$$

on Ω. Since ψ_{ijk} can be expressed in terms of F_1 and F_2, we then obtain that for all i,

$$U_i = \sum \frac{\phi_{ij}}{\sqrt{\Phi}}F_j$$

on Ω. Therefore we have

$$|\nabla U|^2 = \frac{1}{\Phi}[(\phi_{11}F_1 + \phi_{12}F_2)^2 + (\phi_{21}F_1 + \phi_{22}F_2)^2] = \frac{1}{\Phi}(\phi_{11}^2 + \phi_{12}^2)|\nabla F|^2 = \frac{1}{2}|\nabla F|^2$$

on Ω. On the other hand, we know that $U = \pm F$ on Ω, $|\nabla U|^2 = |\nabla F|^2$ on Ω. We then conclude that $|\nabla F|$ vanishes on Ω, and hence F is a constant on Ω. Since every immersion is locally an embedding, $1+ < x, g_0 >$ vanishes only at most finite points on M, and $(1+ < x, g_0 >)^2\Phi^2 = \frac{1}{4}\Phi F^2$ on M, this constant must be nonzero by the continuity of Φ. Since F is a nonzero constant satisfying the equation $\Delta F + \Phi F = 0$, Φ vanishes on Ω, we get a contradiction. This contradiction shows that Φ vanishes identically, and M is totally umbilical. This completes the proof of Theorem B. □

From Theorem B, we obtain immediately the following.

Corollary 1. Let M be a compact immersed Willmore surface in the 3-dimensional unit sphere S^3. If

$$inf_{g\in G} max_{g\circ x(M)}(\Phi_g) \leq 2,$$

then M is either a totally umbilical sphere or a conformal Clifford torus.

Chapter 4

Willmore surfaces in S^n

In this chapter, as for the case $n = 3$, we also characterize the totally umbilical spheres and the Veronese surface by a pinching condition for the case $n \geq 4$. Analogous to the case $n = 3$, we then introduce a conformal invariant quantity, and prove that if this quantity is bounded above by that value of the Veronese surface then $x(M)$ is either a totally umbilical sphere or a conformal Veronese surface.

4.1 A pinching theorem of Willmore Surfaces in S^n

Our pinching theorem of Willmore Surfaces in S^n is the following:

Theorem C. Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n, $n \geq 4$. If

$$0 \leq \Phi \leq \frac{2}{3} + \frac{1}{8}H^2 + \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4},$$

then either $\Phi = 0$ and M is totally umbilical or $\Phi = \frac{2}{3} + \frac{1}{8}H^2 + \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4}$. In the latter case, $n = 4$ and M is the Veronese surface.

Proof. For simplicity, from now on in this section, let $r(H) = \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4}$. First, we wish to show that Φ is equal to either 0 or $\frac{2}{3} + \frac{H^2}{8} + r(H)$.

Integrating both sides of the Lemma 2.2.1 over M, we have

$$\begin{aligned} 0 &= \int_M [\, \sum (\phi^\alpha_{ijk})^2 + \sum \phi^\alpha_{ij} H^\alpha_{ij} + \Phi(2 + \frac{H^2}{2} - \Phi) - \sum R^2_{\alpha\beta 12}\,] \\ &= \int_M [\, \sum (\phi^\alpha_{ijk})^2 - \sum \phi^\alpha_{ijj} H^\alpha_{i} + \Phi(2 + \frac{H^2}{2} - \Phi) - \sum R^2_{\alpha\beta 12}\,]. \end{aligned}$$

It follows from Lemmas 2.2.2 and 2.2.3 that

$$0 \geq \int_M [\, -\frac{1}{4} \sum |\nabla H^\alpha|^2 + \Phi(2 + \frac{H^2}{2} - \Phi) - \sum R^2_{\alpha\beta 12}\,].$$

Since

$$\begin{aligned} \sum (R_{\alpha\beta 12})^2 &= 4 \sum (\phi^\alpha_{11}\phi^\beta_{12} - \phi^\beta_{11}\phi^\alpha_{12})^2 \\ &= 8 \sum (\phi^\alpha_{11})^2 \sum (\phi^\alpha_{12})^2 - 8 (\sum \phi^\alpha_{11}\phi^\alpha_{12})^2, \end{aligned}$$

by Lemmas 2.2.4 and 2.2.6 with $c = 1$, we get

$$\begin{aligned} 0 &\geq \int_M [\, -\frac{1}{4} \sum_{ij} (\sum_\alpha \phi^\alpha_{ij} H^\alpha)^2 - 8 \sum (\phi^\alpha_{11})^2 \sum (\phi^\alpha_{12})^2 + 8 (\sum \phi^\alpha_{11}\phi^\alpha_{12})^2 \\ &\qquad + \Phi(2 + \frac{H^2}{2} - \Phi)\,] \\ &= \int_M \{\, -\frac{1}{2} [\, (\sum \phi^\alpha_{11} H^\alpha)^2 + (\sum \phi^\alpha_{12} H^\alpha)^2 + 16 \sum (\phi^\alpha_{11})^2 \sum (\phi^\alpha_{12})^2 \\ &\qquad - 16 (\sum \phi^\alpha_{11}\phi^\alpha_{12})^2\,] + \Phi(2 + \frac{H^2}{2} - \Phi)\, \} \\ &\geq \int_M u(\Phi, H), \end{aligned}$$

where u is the continuous function given by $u(\Phi, H) = -\frac{3}{2}[\, \Phi^2 - (\frac{4}{3} + \frac{H^2}{4})\Phi + \frac{H^4}{192}\,]$, if $\Phi > \frac{H^2}{8}$; $u(\Phi, H) = \Phi(2 + \frac{H^2}{4} - \Phi)$, if $\Phi \leq \frac{H^2}{8}$.

Notice that u is nonnegative. In fact, if $\frac{2}{3} + \frac{H^2}{8} + r(H) \geq \Phi > \frac{H^2}{8}$, then

$$u(\Phi, H) \geq -\frac{3}{2}[\, \Phi - (\, \frac{2}{3} + \frac{H^2}{8} + r(H)\,)\,][-\frac{2}{3} + r(H)\,] \geq 0,$$

and if $\Phi \leq \frac{H^2}{8}$, then

$$u(\Phi, H) \geq \Phi(2 + \frac{H^2}{8}) \geq 0.$$

The preceding integral inequality then implies that if $0 \leq \Phi \leq \frac{2}{3} + \frac{H^2}{8} + r(H)$, then either $\Phi = 0$ and M is totally umbilical, or $\Phi = \frac{2}{3} + \frac{H^2}{8} + r(H)$. In the latter case we show below that M is minimal.

Now we shall simply assume that $\Phi = \frac{2}{3} + \frac{H^2}{8} + r(H)$. In this case, all the integral inequalities of previous argument become equalities. The proof of M is minimal is broken up into four steps.

Step 1. We establish the following two equations for later use:

$$|\nabla\Phi|^2 = \sum \phi_{ij}^{\alpha}\Phi_j H_i^{\alpha}$$

and

$$\int_M \frac{\sum |\nabla H^{\alpha}|^2}{4\Phi} = \int_M \frac{r(H)}{r(H) + \frac{2}{3} + \frac{H^2}{12}} \frac{|\nabla\Phi|^2}{\Phi^2} + \int_M \frac{1}{4\Phi} \sum(\sum \phi_{ij}^{\alpha} H^{\alpha})^2.$$

Because $\Phi = \frac{2}{3} + \frac{H^2}{8} + r(H)$, by Lemma 2.2.3, $\phi_{111}^{\alpha} = \phi_{122}^{\alpha} = \phi_{212}^{\alpha} = -\phi_{221}^{\alpha} = \frac{H_1^{\alpha}}{4}$ and $\phi_{211}^{\alpha} = \phi_{222}^{\alpha} = \phi_{121}^{\alpha} = -\phi_{112}^{\alpha} = \frac{H_2^{\alpha}}{4}$, it follows from a straight computation that

$$|\nabla\Phi|^2 = \sum \phi_{ij}^{\alpha}\Phi_j H_i^{\alpha} = (\sum \phi_{11}^{\alpha} H_1^{\alpha} + \sum \phi_{12}^{\alpha} H_2^{\alpha})^2 + (\sum \phi_{12}^{\alpha} H_1^{\alpha} + \sum \phi_{22}^{\alpha} H_2^{\alpha})^2.$$

We obtain the first equation.

Since $\Phi = \frac{2}{3} + \frac{H^2}{8} + r(H)$, we have

$$\Phi_i = (\frac{1}{4} + \frac{\frac{1}{6} + \frac{H^2}{48}}{r(H)}) H^{\alpha} H_i^{\alpha}.$$

Hence

$$H^{\alpha} H_i^{\alpha} \Phi_i = \frac{r(H)|\nabla\Phi|^2}{\frac{r(H)}{4} + \frac{1}{6} + \frac{H^2}{48}}.$$

Multiplying by H^{α}, dividing by Φ and integrating over M, the equation $\Delta H^{\alpha} + \sum \phi_{ij}^{\alpha}\phi_{ij}^{\beta} H^{\beta} = 0$ implies that

$$\begin{aligned}
0 &= \int_M \sum(\frac{H^{\alpha}\Delta H^{\alpha}}{\Phi} + \sum \frac{\phi_{ij}^{\alpha}\phi_{ij}^{\beta} H^{\alpha} H^{\beta}}{\Phi}) \\
&= \int_M [\, -\sum(\frac{H^{\alpha}}{\Phi})_i H_i^{\alpha} + \frac{1}{\Phi}\sum(\sum \phi_{ij}^{\alpha} H^{\alpha})^2 \,] \\
&= \int_M [\, -\sum(\frac{|\nabla H^{\alpha}|^2}{\Phi} + \frac{\Phi_i H^{\alpha} H_i^{\alpha}}{\Phi^2}) + \frac{1}{\Phi}\sum(\sum \phi_{ij}^{\alpha} H^{\alpha})^2 \,] \\
&= \int_M [\, -\sum \frac{|\nabla H^{\alpha}|^2}{\Phi} + \frac{r(H)}{\frac{r(H)}{4} + \frac{1}{6} + \frac{H^2}{48}} \frac{|\nabla\Phi|^2}{\Phi^2} + \frac{1}{\Phi}\sum(\sum \phi_{ij}^{\alpha} H^{\alpha})^2 \,].
\end{aligned}$$

This gives the second equation.

Step 2. We shall show that H^2 and Φ are constants. Dividing the equation of Lemma 2.2.1 by Φ and integrating over M, we get

$$\int_M \frac{\Delta\Phi}{2\Phi} = \int_M [\ \frac{\sum(\phi^\alpha_{ijk})^2}{\Phi} + \frac{\sum \phi^\alpha_{ij} H^\alpha_{ij}}{\Phi} + (2 + \frac{H^2}{2} - \Phi) - \frac{\sum R^2_{\alpha\beta 12}}{\Phi}\ ].$$

By applying Stokes' theorem, we obtain

$$\begin{aligned}\int_M \frac{|\nabla\Phi|^2}{2\Phi^2} &= \int_M [\ \frac{\sum|\nabla H^\alpha|^2}{4\Phi} - \sum \frac{\Phi\phi^\alpha_{ijj} - \phi^\alpha_{ij}\Phi_j}{\Phi^2} H^\alpha_i + (2 + \frac{H^2}{2} - \Phi) - \frac{\sum R^2_{\alpha\beta 12}}{\Phi}\ ] \\ &= \int_M [\ \frac{\sum|\nabla H^\alpha|^2}{4\Phi} - \frac{\sum|\nabla H^\alpha|^2}{2\Phi} + \frac{\sum \phi^\alpha_{ij}\Phi_j H^\alpha_i}{\Phi^2} + (2 + \frac{H^2}{2} - \Phi) - \frac{\sum R^2_{\alpha\beta 12}}{\Phi}\ ],\end{aligned}$$

where we have used $\sum(\phi^\alpha_{ijk})^2 = \frac{1}{4}\sum|\nabla H^\alpha|^2$ and $\sum \phi^\alpha_{ijj} = \frac{H^\alpha_i}{2}$ for all i. Consequently, we obtain from the equations of step 1 that

$$\begin{aligned} 0 &= \int_M [\ -\frac{|\nabla\Phi|^2}{2\Phi^2} - \frac{\sum|\nabla H^\alpha|^2}{4\Phi} + \frac{\sum \phi^\alpha_{ij}\Phi_j H^\alpha_i}{\Phi^2} + (2 + \frac{H^2}{2} - \Phi) - \frac{\sum R^2_{\alpha\beta 12}}{\Phi}\ ] \\ &= \int_M [\ -\frac{|\nabla\Phi|^2}{2\Phi^2} - \frac{r(H)}{r(H) + \frac{2}{3} + \frac{H^2}{12}} \frac{|\nabla\Phi|^2}{\Phi^2} - \frac{1}{4\Phi}\sum(\sum \phi^\alpha_{ij} H^\alpha)^2 + \frac{|\nabla\Phi|^2}{\Phi^2} \\ &\quad + (2 + \frac{H^2}{2} - \Phi) - \frac{\sum R^2_{\alpha\beta 12}}{\Phi}\ ] \\ &= \int_M [\ \frac{|\nabla\Phi|^2}{2\Phi^2}(1 - \frac{2r(H)}{r(H) + \frac{2}{3} + \frac{H^2}{12}}) - \frac{1}{4\Phi}\sum(\sum \phi^\alpha_{ij} H^\alpha)^2 + (2 + \frac{H^2}{2} - \Phi) \\ &\quad - \frac{8}{\Phi}\sum(\phi^\alpha_{11})^2 \sum(\phi^\alpha_{12})^2 + \frac{8}{\Phi}(\sum \phi^\alpha_{11}\phi^\alpha_{12})^2\ ] \\ &= \int_M \{\ \frac{|\nabla\Phi|^2}{2\Phi^2}(1 - \frac{2r(H)}{r(H) + \frac{2}{3} + \frac{H^2}{12}}) + \frac{1}{\Phi}[\Phi(2 + \frac{H^2}{2} - \Phi) - \frac{1}{2}((\sum \phi^\alpha_{11} H^\alpha)^2 \\ &\quad + (\sum \phi^\alpha_{12} H^\alpha)^2 + 16\sum(\phi^\alpha_{11})^2 \sum(\phi^\alpha_{12})^2 - 16(\sum \phi^\alpha_{11}\phi^\alpha_{12})^2)]\ \} \\ &= \int_M \{\ \frac{|\nabla\Phi|^2}{2\Phi^2}(1 - \frac{2r(H)}{r(H) + \frac{2}{3} + \frac{H^2}{12}}) + \frac{1}{\Phi}[\Phi(2 + \frac{H^2}{2} - \Phi) - \frac{1}{2}(\Phi + \frac{H^2}{8})^2]\ \}.\end{aligned}$$

Since the last term of the integrand vanishes,

$$\Phi(2 + \frac{H^2}{2} - \Phi) - \frac{1}{2}(\Phi + \frac{H^2}{8})^2 = -\frac{3}{2}[\ \Phi^2 - (\frac{4}{3} + \frac{H^2}{4})\Phi + \frac{H^4}{192}\ ] = 0,$$

we have

$$\int_M \frac{|\nabla\Phi|^2}{2\Phi^2}(1 - \frac{2r(H)}{r(H) + \frac{2}{3} + \frac{H^2}{12}}) = 0.$$

We note that the integrand is non-positive. In fact, let

$$f(x) = \frac{1}{2} + \frac{\frac{1}{3} + \frac{x}{24}}{\sqrt{\frac{4}{9} + \frac{1}{6}x + \frac{1}{96}x^2}}.$$

Then

$$f'(x) = -\frac{1}{108(\frac{4}{9} + \frac{1}{6}x + \frac{1}{96}x^2)^{\frac{3}{2}}} < 0$$

for all $x > 0$, f is decreasing for all $x \geq 0$, and $f(x) < f(0) = 1$ for all $x > 0$.

We then have $|\nabla\Phi| = 0$ or $H = 0$, thus Φ is constant on each connected component of the set where $H \neq 0$. Since H^2 satisfies the quadratic equation $\Phi^2 - (\frac{4}{3} + \frac{H^2}{4})\Phi + \frac{H^4}{192} = 0$, H^2 is also constant on each connected component of the set where $H \neq 0$. We conclude that, whether H is zero or not, H^2 and Φ are constants.

Step 3. Assume that H^2 is a positive constant. We establish the following five equations:

$$\Delta H^\alpha + \frac{1}{2}(\Phi + \frac{H^2}{8})H^\alpha = 0,$$

$$\sum |\nabla H^\alpha|^2 = \frac{1}{2}(\Phi + \frac{H^2}{8})H^2,$$

$$\sum \phi_{11}^\alpha H_1^\alpha = \sum \phi_{12}^\alpha H_1^\alpha = \sum \phi_{11}^\alpha H_2^\alpha = \sum \phi_{12}^\alpha H_2^\alpha = 0,$$

$$\sum (H_1^\alpha)^2 - (H_2^\alpha)^2 = 2(\Phi + \frac{H^2}{8}) \sum \phi_{11}^\alpha H^\alpha$$

and

$$\sum H_1^\alpha H_2^\alpha = (\Phi + \frac{H^2}{8}) \sum \phi_{12}^\alpha H^\alpha.$$

Since the equality in Lemma 2.2.6 with $c = 1$ holds, applying

$$H^\alpha = \frac{4}{\Phi + \frac{H^2}{8}}(\sum \phi_{11}^\beta H^\beta \phi_{11}^\alpha + \sum \phi_{12}^\beta H^\beta \phi_{12}^\alpha)$$

twice, we have

$$
\begin{aligned}
\phi^\alpha_{ij}\phi^\beta_{ij}H^\beta &= \frac{8}{\Phi+\frac{H^2}{8}}[\,(\sum(\phi^\beta_{11})^2\sum\phi^\beta_{11}H^\beta+\sum\phi^\beta_{11}\phi^\beta_{12}\sum\phi^\beta_{12}H^\beta)\phi^\alpha_{11} \\
&\qquad +(\sum\phi^\beta_{11}\phi^\beta_{12}\sum\phi^\beta_{11}H^\beta+\sum(\phi^\beta_{12})^2\sum\phi^\beta_{12}H^\beta)\phi^\alpha_{12}\,] \\
&= \frac{8}{\Phi+\frac{H^2}{8}}[\,\frac{1}{4}(\Phi+\frac{H^2}{8})\sum\phi^\beta_{11}H^\beta\phi^\alpha_{11}+\frac{1}{4}(\Phi+\frac{H^2}{8})\sum\phi^\beta_{12}H^\beta\phi^\alpha_{12}\,] \\
&= \frac{1}{2}(\Phi+\frac{H^2}{8})H^\alpha.
\end{aligned}
$$

Thus

$$\Delta H^\alpha+\frac{1}{2}(\Phi+\frac{H^2}{8})H^\alpha=0,$$

as desired. We obtain the first equation.

Since H^2 is a constant, the first equation gives

$$
\begin{aligned}
0 &= \frac{1}{2}\Delta H^2 \\
&= \sum|\nabla H^\alpha|^2+\sum H^\alpha\Delta H^\alpha \\
&= \sum|\nabla H^\alpha|^2-\frac{1}{2}(\Phi+\frac{H^2}{8})H^2.
\end{aligned}
$$

This is the second equation.

Now we show the third equation. Because the equality in Lemma 2.2.6 with $c=1$ holds, we have

$$
\begin{aligned}
A^2+B^2 &= \frac{H^2}{4}(\Phi+\frac{H^2}{8}), \\
A^2-B^2 &= 4(\Phi+\frac{H^2}{8})[\,\sum(\phi^\alpha_{11})^2-\sum(\phi^\alpha_{12})^2\,], \\
AB &= 4(\Phi+\frac{H^2}{8})\sum\phi^\alpha_{11}\phi^\alpha_{12},
\end{aligned}
$$

where $A=\sum\phi^\alpha_{11}H^\alpha$ and $B=\sum\phi^\alpha_{12}H^\alpha$.

Since A^2+B^2 and H^2 are constants,

$$
\begin{aligned}
0 &= 2A(\sum\phi^\alpha_{111}H^\alpha+\sum\phi^\alpha_{11}H^\alpha_1)+2B(\sum\phi^\alpha_{121}H^\alpha+\sum\phi^\alpha_{12}H^\alpha_1) \\
&= 2A\sum\phi^\alpha_{11}H^\alpha_1+2B\sum\phi^\alpha_{12}H^\alpha_1,
\end{aligned}
$$

we have

$$A\sum \phi_{11}^{\alpha} H_1^{\alpha} + B\sum \phi_{12}^{\alpha} H_1^{\alpha} = 0,$$

we make use here of the facts that $\phi_{111}^{\alpha} = \frac{H_1^{\alpha}}{4}$ and $\phi_{121} = \frac{H_2^{\alpha}}{4}$. Similarly, we also have

$$A\sum \phi_{11}^{\beta} H_2^{\beta} + B\sum \phi_{12}^{\beta} H_2^{\beta} = 0.$$

Since A^2+B^2 is a positive constant, $\sum \phi_{11}^{\alpha} H_1^{\alpha} = -tB$, $\sum \phi_{12}^{\alpha} H_1^{\alpha} = tA$, $\sum \phi_{11}^{\alpha} H_2^{\alpha} = -sB$ and $\sum \phi_{12}^{\alpha} H_2^{\alpha} = sA$, for some functions t and s.

Taking differentiation of equations $A^2 - B^2 = 4(\Phi + \frac{H^2}{8})[\ \sum(\phi_{11}^{\alpha})^2 - \sum(\phi_{12}^{\alpha})^2\ ]$ and $AB = 4(\Phi + \frac{H^2}{8})\sum \phi_{11}^{\alpha}\phi_{12}^{\alpha}$, and then substituting $\sum \phi_{11}^{\alpha} H_1^{\alpha} = -tB$, $\sum \phi_{12}^{\alpha} H_1^{\alpha} = tA$, $\sum \phi_{11}^{\alpha} H_2^{\alpha} = -sB$ and $\sum \phi_{12}^{\alpha} H_2^{\alpha} = sA$, we get

$$\begin{aligned}
2tAB &= (\Phi + \frac{H^2}{8})(sA + tB), \\
2sAB &= (\Phi + \frac{H^2}{8})(tA - sB), \\
t(A^2 - B^2) &= (\Phi + \frac{H^2}{8})(tA - sB), \\
s(A^2 - B^2) &= (\Phi + \frac{H^2}{8})(-sA - tB).
\end{aligned}$$

In particular, $t(A^2 - B^2) = 2sAB$, $s(A^2 - B^2) = -2tAB$, and $s^2AB = -t^2AB$. Since at least one of A and B is nonzero, there are three cases. If $A = 0$, then $-tB^2 = 0$, $-sB^2 = 0$, so that $t = s = 0$. Likewise, if $B = 0$, then $t = s = 0$. If A and B are nonzero, then $s^2 = -t^2$, and hence $t = s = 0$. In each case, $t = s = 0$. Therefore we have the third equation.

Taking differentiation of the third equation, and substituting $\phi_{111}^{\alpha} = \phi_{122}^{\alpha} = \phi_{212}^{\alpha} = -\phi_{221}^{\alpha} = \frac{H_1^{\alpha}}{4}$ and $\phi_{211}^{\alpha} = \phi_{222}^{\alpha} = \phi_{121}^{\alpha} = -\phi_{112}^{\alpha} = \frac{H_2^{\alpha}}{4}$, we find that

$$\begin{aligned}
\frac{1}{4}\sum[(H_1^{\alpha})^2 - (H_2^{\alpha})^2] + \sum \phi_{11}^{\alpha}\Delta H^{\alpha} &= 0, \\
-\frac{1}{2}\sum H_1^{\alpha}H_2^{\alpha} + \sum \phi_{11}^{\alpha}(H_{12}^{\alpha} - H_{21}^{\alpha}) &= 0, \\
\frac{1}{2}\sum H_1^{\alpha}H_2^{\alpha} + \sum \phi_{12}^{\alpha}\Delta H^{\alpha} &= 0, \\
\frac{1}{4}\sum[(H_1^{\alpha})^2 - (H_2^{\alpha})^2] + \sum \phi_{12}^{\alpha}(H_{12}^{\alpha} - H_{21}^{\alpha}) &= 0.
\end{aligned}$$

The equations four and five then follow from $\Delta H^\alpha + \frac{1}{2}(\Phi + \frac{H^2}{8})H^\alpha = 0$ and

$$H_{12}^\alpha - H_{21}^\alpha = \sum H^\beta R_{\beta\alpha 12} = 2\sum H^\beta(\phi_{12}^\alpha \phi_{11}^\beta - \phi_{11}^\alpha \phi_{12}^\beta).$$

Step 4. The hard part is to show that M is minimal. Suppose, to get a contradiction, that H^2 is a positive constant. The following computation is straightforward,

$$H_i^\alpha H_j^\alpha R_{ikjk} = |\nabla H^\alpha|^2 R_{1212} = (1 + \frac{H^2}{4} - \frac{\Phi}{2})|\nabla H^\alpha|^2.$$

Applying the third equation of step 3, we obtain

$$\sum H_i^\alpha H_j^\beta R_{\beta\alpha ij} = -2(H_1^\alpha H_2^\beta - H_2^\alpha H_1^\beta)(\phi_{11}^\alpha \phi_{12}^\beta - \phi_{12}^\alpha \phi_{11}^\beta) = 0.$$

Because $\phi_{111}^\alpha = \phi_{122}^\alpha = \phi_{212}^\alpha = -\phi_{221}^\alpha = \frac{H_1^\alpha}{4}$ and $\phi_{211}^\alpha = \phi_{222}^\alpha = \phi_{121}^\alpha = -\phi_{112}^\alpha = \frac{H_2^\alpha}{4}$,

$$\sum H_i^\alpha H^\beta R_{\beta\alpha ij,j} = \frac{1}{2}\sum[\,(H_1^\alpha)^2 - (H_2^\alpha)^2\,]\sum \phi_{11}^\alpha H^\alpha + \sum H_1^\alpha H_2^\alpha \sum \phi_{12}^\alpha H^\alpha.$$

Applying the fourth and fifth equations of step 3, we obtain

$$\sum H_i^\alpha H^\beta R_{\beta\alpha ij,j} = \frac{1}{4}(\Phi + \frac{H^2}{8})^2 H^2.$$

Because H^2 and Φ are constants, $\sum |\nabla H^\alpha|^2$ is also a constant, combining the above equations, we have

$$\begin{aligned}
0 &= \frac{1}{2}\Delta \sum |\nabla H^\alpha|^2 = (H_{ij}^\alpha)^2 + H_i^\alpha H_{ijj}^\alpha \\
&= \sum (H_{ij}^\alpha)^2 + H_i^\alpha(H_{jji}^\alpha + H_k^\alpha R_{kjij} + 2H_j^\beta R_{\beta\alpha ij} + H^\beta R_{\beta\alpha ij,j}) \\
&= \sum (H_{ij}^\alpha)^2 + H_i^\alpha(\Delta H^\alpha)_i + H_i^\alpha H_j^\alpha R_{ikjk} + 2H_i^\alpha H_j^\beta R_{\beta\alpha ij} + H_i^\alpha H^\beta R_{\beta\alpha ij,j} \\
&= \sum (H_{ij}^\alpha)^2 - \frac{1}{2}(\Phi + \frac{H^2}{8})|\nabla H^\alpha|^2 + (1 + \frac{H^2}{4} - \frac{\Phi}{2})|\nabla H^\alpha|^2 + \sum H_i^\alpha H^\beta R_{\beta\alpha ij,j} \\
&\geq \frac{1}{2}(\sum H_{ii}^\alpha)^2 - \frac{1}{2}(\Phi + \frac{H^2}{8})|\nabla H^\alpha|^2 + (1 + \frac{H^2}{4} - \frac{\Phi}{2})|\nabla H^\alpha|^2 + \sum H_i^\alpha H^\beta R_{\beta\alpha ij,j} \\
&= \frac{1}{8}(\Phi + \frac{H^2}{8})H^2(\frac{10}{3} + H^2 - r(H)) > 0.
\end{aligned}$$

We then have a contradiction. This contradiction shows that $H = 0$. Then we conclude that M is a minimal surface with $\Phi = \frac{4}{3}$, so that M is the Veronese surface (see [6]). This completes the proof of the Theorem C. □

From Theorem C, we obtain immediately the following.

Corollary 2. Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n, $n \geq 4$. If

$$0 \leq \Phi \leq \frac{4}{3} + \frac{1}{6}H^2 ,$$

then either $\Phi = 0$ and M is totally umbilical or $\Phi = \frac{4}{3} + \frac{1}{6}H^2$. In the latter case, $n = 4$ and M is the Veronese surface.

4.2 A pinching theorem for conformal classes of Willmore Surfaces in S^n

Our pinching theorem for conformal classes of Willmore Surfaces in S^n is the following:

Theorem D. Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n, $n \geq 4$. If

$$inf_{g\in G} max_{g\circ x(M)}(\Phi_g - \frac{1}{8}H_g^2 - \sqrt{\frac{4}{9} + \frac{1}{6}H_g^2 + \frac{1}{96}H_g^4}\ ) \leq \frac{2}{3},$$

where G is the conformal group of the ambient space S^n, Φ_g and H_g are the square of the length of the trace free part of the second fundamental form and the mean curvature of the immersion $g \circ x$ respectively, then $x(M)$ is either a totally umbilical sphere or a conformal Veronese surface.

Proof. The idea of the proof is to consider a minimizing sequence g_m of the conformal group G, such that the sequence g_m converges to an element g_0 of the closure of G. If $g_0 \in G$, then the result follows immediately from Theorem C. Otherwise we shall show that M is totally umbilical.

By the hypothesis of Theorem D, there is a sequence $g_m \in G$ such that $\Phi_m - \frac{1}{8}H_m^2 - r(H_m) \le \frac{2}{3} + \frac{1}{m}$ on M, for all m, where $r(H) = \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4}$, Φ_m and H_m are the square of the length of the trace free part of the second fundamental form and the mean curvature of the immersion $g_m \circ x$, respectively. Without loss of generality, we may assume that $g_m \in D_{n+1}$. Since the closure of D_{n+1} in R^{n+1} is compact, there is a subsequence, still denoted by g_m, which converges to g_0, for some g_0 in the closed unit disk. If $g_0 \in D_{n+1}$, then Φ_m tends to Φ_0, and H_m^2 tends to H_0^2 as m tends to infinity. In this case, we obtain that $\Phi_0 - \frac{1}{8}H_0^2 - r(H_0) \le \frac{2}{3}$ on M, and the desired conclusion follows from Theorem C. Thus from now on, we may assume that g_0 is a unit vector. In this case we shall show below that M is totally umbilical. There are four steps we want to do at this point.

Step 1. We want to show that $\Phi = 0$ or $(1+ < x, g_0 >)^2\Phi = \frac{3+\sqrt{6}}{24}F^2$. The proof is an adaptation of the proof of Theorem C. To avoid ambiguity, for each fixed m, let $\bar{x} = g_m \circ x$, and we shall now use the notations da and $d\bar{a}$ for the area measures of x and $\bar{x}$, respectively. We have to modify our integral inequality in the proof of Theorem C as follows

$$\begin{aligned}
0 &= \int_M [\, \sum (\bar{\phi}_{ijk}^\alpha)^2 + \sum \bar{\phi}_{ij}^\alpha \bar{H}_{ij}^\alpha + \bar{\Phi}(2 + \frac{\bar{H}^2}{2} - \bar{\Phi}) - \sum \bar{R}_{\alpha\beta 12}^2 \,]\, d\bar{a} \\
&= \int_M [\, \sum (\bar{\phi}_{ijk}^\alpha)^2 - \sum \bar{\phi}_{ijj}^\alpha \bar{H}_i^\alpha + \bar{\Phi}(2 + \frac{\bar{H}^2}{2} - \bar{\Phi}) - \sum \bar{R}_{\alpha\beta 12}^2 \,]\, d\bar{a} \\
&\ge \int_M [\, -\frac{1}{4} \sum |\bar{\nabla} \bar{H}^\alpha|^2 + \bar{\Phi}(2 + \frac{\bar{H}^2}{2} - \bar{\Phi}) - \sum \bar{R}_{\alpha\beta 12}^2 \,]\, d\bar{a} \\
&\ge \int_M [\, -\frac{1}{2} f(\bar{\Phi}, \bar{H}) + \bar{\Phi}(2 + \frac{\bar{H}^2}{2} - \bar{\Phi}) \,] d\bar{a} \\
&\ge \int_M \bar{\Phi} v(\bar{\Phi}, \bar{H}) d\bar{a}, \\
&= \int_M \Phi v(\bar{\Phi}, \bar{H})\, da,
\end{aligned}$$

where v is the continuous function defined on M, $v(\Phi, H) = -\frac{3}{2}[\, \Phi - (\frac{2}{3} + \frac{H^2}{8} + r(H))\,]$, if $\Phi > \frac{2}{3} + \frac{H^2}{8} + r(H)$; $v(\Phi, H) = -\frac{\sqrt{6}}{2}[\, \Phi - (\frac{2}{3} + \frac{H^2}{8} + r(H))\,]$, if $\frac{H^2}{8} \le \Phi \le \frac{2}{3} + \frac{H^2}{8} + r(H)$; $v(\Phi, H) = \frac{\sqrt{6}}{3} + \frac{H^2}{8} + \frac{\sqrt{6}}{2} r(H) - \Phi$, if $\Phi < \frac{H^2}{8}$.

Dividing the integral inequality by $\lambda_m^2 = \frac{1}{1-|g_m|^2}$ and letting $m \longrightarrow \infty$, Lemma

2.2.7 gives

$$0 \;\geq\; \int_M \Phi L(\Phi, F)\, da,$$

where $\mathbb{F} = \sum F^\alpha e_\alpha$, $F = |\mathbb{F}|$, was defined at Lemma 2.2.8 and L is the continuous function given by $L(\Phi, F) = -\frac{3}{2}[\,(1+<x, g_0>)^2\Phi - \frac{3+\sqrt{6}}{24}F^2\,]$, if $(1+<x, g_0>)^2\Phi \geq \frac{3+\sqrt{6}}{24}F^2$; $L(\Phi, F) = -\frac{\sqrt{6}}{2}[\,(1+<x, g_0>)^2\Phi - \frac{3+\sqrt{6}}{24}F^2\,]$, if $\frac{F^2}{8} \leq (1+<x, g_0>)^2\Phi \leq \frac{3+\sqrt{6}}{24}F^2$; $L(\Phi, F) = \frac{F^2}{4} - (1+<x, g_0>)^2\Phi$, if $(1+<x, g_0>)^2\Phi \leq \frac{F^2}{8}$.

On the other hand, since $\Phi_m - \frac{1}{8}H_m^2 - \sqrt{\frac{4}{9} + \frac{1}{6}H_m^2 + \frac{1}{96}H_m^4} \leq \frac{2}{3} + \frac{1}{m}$ on M, taking limits $m \longrightarrow \infty$, we see that

$$(1+<x, g_0>)^2\Phi - \frac{3+\sqrt{6}}{24}F^2 \leq 0,$$

and thus the integrand ΦL is nonnegative. We conclude that $\Phi = 0$ or $L = 0$, and hence $\Phi = 0$ or $(1+<x, g_0>)^2\Phi = \frac{3+\sqrt{6}}{24}F^2$. We note that all inequalities become equalities in the procedure for limits, and, in particular, $\psi_{ijj}^\alpha = \frac{F_i^\alpha}{4}$ for all α, i, j.

Step 2. We want to show that either M is totally umbilical or $(1+<x, g_0>)^2\Phi$ and F^2 are positive constants. Multiplying both sides of the equation for $\bar{\Phi}$ in Lemma 2.2.1 by $\bar{\Phi}$, integrating over M and applying pointwise estimates of Step 1, we obtain

$$\begin{aligned}
0 &= \int_M [\,\frac{1}{2}|\bar{\nabla}\bar{\Phi}|^2 + \frac{1}{2}\bar{\Phi}\bar{\Delta}\bar{\Phi}\,]\, d\bar{a} \\
&= \int_M \frac{1}{2}|\bar{\nabla}\bar{\Phi}|^2 + \bar{\Phi}[\,\sum(\bar{\phi}_{ijk}^\alpha)^2 + \sum \bar{\phi}_{ij}^\alpha \bar{H}_{ij}^\alpha + \bar{\Phi}(2 + \frac{\bar{H}^2}{2} - \bar{\Phi}) - \sum \bar{R}_{\alpha\beta 12}^2\,]\, d\bar{a} \\
&\geq \int_M \frac{1}{2}|\bar{\nabla}\bar{\Phi}|^2 - \frac{1}{4}\bar{\Phi}\sum|\bar{\nabla}\bar{H}^\alpha|^2 - \sum \bar{\phi}_{ij}^\alpha \bar{H}_i^\alpha \bar{\Phi}_j + \bar{\Phi}[\,\bar{\Phi}(2 + \frac{\bar{H}^2}{2} - \bar{\Phi}) - \sum \bar{R}_{\alpha\beta 12}^2\,]\, d\bar{a} \\
&= \int_M \frac{1}{2}|\bar{\nabla}\bar{\Phi}|^2 + \frac{1}{4}\sum \bar{\Phi}_i \bar{H}^\alpha \bar{H}_i^\alpha - \sum \bar{\phi}_{ij}^\alpha \bar{H}_i^\alpha \bar{\Phi}_j \\
&\quad + \bar{\Phi}[\,-\frac{1}{4}\sum(\sum \bar{\phi}_{ij}^\alpha \bar{H}^\alpha)^2 + \bar{\Phi}(2 + \frac{\bar{H}^2}{2} - \bar{\Phi}) - \sum \bar{R}_{\alpha\beta 12}^2\,]\, d\bar{a},
\end{aligned}$$

where in the last step we have used the identity

$$\int_M \bar{\Phi}\sum|\bar{\nabla}\bar{H}^\alpha|^2\, d\bar{a} = \int_M [\,-\sum \bar{\Phi}_i \bar{H}^\alpha \bar{H}_i^\alpha + \bar{\Phi}\sum(\sum \bar{\phi}_{ij}^\alpha \bar{H}^\alpha)^2\,]\, d\bar{a}.$$

In fact, this identity comes from multiplying the equation $\bar{\Delta}\bar{H}^\alpha + \sum \bar{\phi}_{ij}^\alpha \bar{\phi}_{ij}^\beta \bar{H}^\beta = 0$ by $\bar{\Phi}\bar{H}^\alpha$ and then integrating over M.

By using Lemma 2.2.6 again, we have

$$\begin{aligned} 0 \;\geq\; & \int_M [\, \frac{1}{2}|\bar{\nabla}\bar{\Phi}|^2 + \frac{1}{4}\sum \bar{\Phi}_i \bar{H}^\alpha \bar{H}_i^\alpha - \sum \bar{\phi}_{ij}^\alpha \bar{H}_i^\alpha \bar{\Phi}_j \,]\, d\bar{a} \\ & + \int_M \bar{\Phi}[\, -\frac{1}{2} f(\bar{\Phi}, \bar{H}) + \bar{\Phi}(2 + \frac{\bar{H}^2}{2} - \bar{\Phi}) \,]\, d\bar{a} \\ \geq\; & \int_M [\, \frac{1}{2}|\bar{\nabla}\bar{\Phi}|^2 + \frac{1}{4}\sum \bar{\Phi}_i \bar{H}^\alpha \bar{H}_i^\alpha - \sum \bar{\phi}_{ij}^\alpha \bar{H}_i^\alpha \bar{\Phi}_j + \int_M \bar{\Phi}^2 v(\bar{\Phi}, \bar{H}) \,]\, d\bar{a}, \end{aligned}$$

where v was given at Step 1. Substituting the relationships of Lemma 2.2.7 into this last integral, we get

$$\begin{aligned} 0 \;\geq\; & \int_M [\, 2\lambda_m^6 (1+ < x, g_m >)^4 \sum (\phi_{kl}^\alpha \psi_{kli}^\alpha)^2 - 2\lambda_m^6 (1+ < x, g_m >)^4 \sum \phi_{kl}^\alpha \psi_{kli}^\alpha \sum \phi_{ij}^\alpha F_j^\alpha \\ & + \frac{1}{2}\lambda_m^6 (1+ < x, g_m >)^3 \sum \phi_{kl}^\alpha \psi_{kli}^\alpha \sum F^\alpha F_i^\alpha \\ & + \lambda_m^4 (1+ < x, g_m >)^4 \Phi^2 v(\lambda_m^2 (1+ < x, g_m >)^2 \Phi, \lambda_m F) \,]\, \frac{1}{\lambda_m^2 (1+ < x, g_m >)^2}\, da. \end{aligned}$$

Dividing the integral inequality by λ_m^4 and letting $m \longrightarrow \infty$, we find that

$$\begin{aligned} 0 \;\geq\; & \int_M [\, 2(1+ < x, g_0 >)^2 \sum (\phi_{kl}^\alpha \psi_{kli}^\alpha)^2 - 2(1+ < x, g_0 >)^2 \sum \phi_{kl}^\alpha \psi_{kli}^\alpha \sum \phi_{ij}^\alpha F_j^\alpha \\ & + \frac{1}{2}(1+ < x, g_0 >) \sum \phi_{kl}^\alpha \psi_{kli}^\alpha \sum F^\alpha F_i^\alpha \,]\, da, \end{aligned}$$

this we can do because $\Phi = 0$ or $L = 0$. We assert that the integrand is nonnegative. Let Ω be a connected component of the set of points where $\Phi > 0$, and let $U = c(1+ < x, g_0 >)\sqrt{\Phi}$ defined on Ω, where $\frac{1}{c^2} = \frac{3+\sqrt{6}}{24}$. Then

$$U_i = c\sqrt{\Phi} < e_i, g_0 > + 2c \sum \frac{\phi_{11}^\alpha}{\sqrt{\Phi}} (1+ < x, g_0 >) \phi_{11i}^\alpha + 2c \sum \frac{\phi_{12}^\alpha}{\sqrt{\Phi}} (1+ < x, g_0 >) \phi_{12i}^\alpha,$$

for all i. Substituting $(1+ < x, g_0 >)\phi_{ijk}^\alpha$ in terms of ψ_{ijk}^α, Lemma 2.2.9 gives

$$U_i = \frac{c}{2\sqrt{\Phi}} \sum \phi_{ij}^\alpha F_j^\alpha = \frac{c}{\sqrt{\Phi}} \sum \phi_{kl}^\alpha \psi_{kli}^\alpha,$$

for all i, here we have used the fact that $\psi_{ijj}^\alpha = \frac{F_i^\alpha}{4}$ for all α, i, j. Since $F^2 = U^2$, we find that the integrand is equal to $(1+ < x, g_0 >)^2 \Phi(\frac{1}{2} - \frac{2}{c^2})|\nabla U|^2$ on Ω. When $\Phi = 0$ the integrand vanishes, when $\Phi > 0$, because $\frac{1}{2} - \frac{2}{c^2} = \frac{3-\sqrt{6}}{12} > 0$, the integrand is also nonnegative, as desired.

Since every immersion is locally an embedding, $1+ < x, g_0 >$ vanishes only at most finite points on M, thus $|\nabla U|^2 = 0$, if $\Phi > 0$. Therefore U is constant on each connected component of the set where $\Phi \neq 0$. A consequence of this is that either M is totally umbilical or $(1+ < x, g_0 >)^2\Phi$ and F^2 are constants.

Step 3. Assume that $(1+ < x, g_0 >)^2\Phi$ and F^2 are positive constants. It is important now to derive the following four equations which will require in Step 4:

$$F^\alpha = \frac{4}{\Phi + \frac{F^2}{8(1+<x,g_0>)^2}}(\sum \phi_{11}^\beta F^\beta \phi_{11}^\alpha + \sum \phi_{12}^\beta F^\beta \phi_{12}^\alpha),$$

$$\sum \phi_{11}^\alpha F_1^\alpha = \sum \phi_{12}^\alpha F_1^\alpha = \sum \phi_{11}^\alpha F_2^\alpha = \sum \phi_{12}^\alpha F_2^\alpha = 0,$$

$$(1+ < x, g_0 >)^2 \sum [(F_1^\alpha)^2 - (F_2^\alpha)^2] = 2[\ (1+ < x, g_0 >)^2\Phi + \frac{F^2}{8}\ ] \sum \phi_{11}^\alpha F^\alpha$$

and

$$(1+ < x, g_0 >)^2 \sum F_1^\alpha F_2^\alpha = [\ (1+ < x, g_0 >)^2\Phi + \frac{F^2}{8}\ ] \sum \phi_{12}^\alpha F^\alpha.$$

The way of proof is proceeding as the procedure of Step 1, but reverses the order of taking limits and applying Lemma 2.2.6. Since $g_m \circ x$ is a Willmore immersion, Lemma 2.2.7 gives

$$\begin{aligned}
0 &= \int_M [\ \sum (\bar\phi_{ijk}^\alpha)^2 + \sum \bar\phi_{ij}^\alpha \bar H_{ij}^\alpha + \bar\Phi(2 + \frac{\bar H^2}{2} - \bar\Phi) - \sum \bar R_{\alpha\beta 12}^2\ ]\, d\bar a \\
&= \int_M [\ \sum (\bar\phi_{ijk}^\alpha)^2 - \sum \bar\phi_{ijj}^\alpha \bar H_i^\alpha + \bar\Phi(2 + \frac{\bar H^2}{2} - \bar\Phi) - \sum \bar R_{\alpha\beta 12}^2\ ]\, d\bar a \\
&\geq \int_M [\ -\frac{1}{4} \sum |\bar\nabla \bar H^\alpha|^2 + \bar\Phi(2 + \frac{\bar H^2}{2} - \bar\Phi) - \sum \bar R_{\alpha\beta 12}^2\ ]\, d\bar a \\
&\geq \int_M \{\ -\frac{1}{2}[\ (\sum \bar\phi_{11}^\alpha \bar H^\alpha)^2 + (\sum \bar\phi_{12}^\alpha \bar H^\alpha)^2 + 16 \sum (\bar\phi_{11}^\alpha)^2 \sum (\bar\phi_{12}^\alpha)^2 \\
&\qquad -16(\sum \bar\phi_{11}^\alpha \bar\phi_{12}^\alpha)^2\ ] + \bar\Phi(2 + \frac{\bar H^2}{2} - \bar\Phi)\ \}\, d\bar a \\
&= \int_M \{\ -\frac{1}{2}\lambda_m^2 [\ (\sum \phi_{11}^\alpha F_m^\alpha)^2 + (\sum \phi_{12}^\alpha F_m^\alpha)^2 \\
&\qquad +16(1+ < x, g_m >)^2 \sum (\phi_{11}^\alpha)^2 \sum (\phi_{12}^\alpha)^2 - 16(1+ < x, g_m >)^2 (\sum \phi_{11}^\alpha \phi_{12}^\alpha)^2\ ] \\
&\qquad +\Phi(2 + \frac{\lambda_m^2 F_m^2}{2} - \lambda_m^2 (1+ < x, g_m >)^2 \Phi)\ \}\, da,
\end{aligned}$$

where $\lambda_m = \frac{1}{1-|g_m|^2}$, and $F_m^2 = \sum(F_m^\alpha)^2$ was defined at Lemma 2.2.8 with $g = g_m$. Dividing the integral inequality by λ_m^2 and letting $m \longrightarrow \infty$, we get

$$\begin{aligned} 0 \ \geq \ & \int_M \{ -\frac{1}{2}[(\sum \phi_{11}^\alpha F^\alpha)^2 + (\sum \phi_{12}^\alpha F^\alpha)^2 \\ & +16(1+<x,g_0>)^2 \sum(\phi_{11}^\alpha)^2 \sum(\phi_{12}^\alpha)^2 - 16(1+<x,g_0>)^2(\sum \phi_{11}^\alpha \phi_{12}^\alpha)^2] \\ & +\Phi(\frac{F^2}{2} - (1+<x,g_0>)^2\Phi) \} \, da, \end{aligned}$$

where F denote the function related to g_0.

Now, we apply Lemma 2.2.6 with $c = (1+<x,g_0>)^2$ to the first term of the integrand. Since $(1+<x,g_0>)^2\Phi$ is a positive constant, $1+<x,g_0>$ never vanishes and $(1+<x,g_0>)^2\Phi = \frac{3+\sqrt{6}}{24}F^2$, Lemma 2.2.6 gives

$$\begin{aligned} 0 \ \geq \ & \int_M \{ -\frac{1}{2}(1+<x,g_0>)^2[\ \Phi + \frac{F^2}{8(1+<x,g_0>)^2} \ ]^2 \\ & +\Phi[\ \frac{F^2}{2} - (1+<x,g_0>)^2\Phi \ ] \ \} \\ = \ & \int_M -\frac{3}{2}[\ (1+<x,g_0>)^2\Phi^2 - \frac{\Phi F^2}{4} + \frac{F^4}{192(1+<x,g_0>)^2} \ ] \\ = \ & 0. \end{aligned}$$

It follows that all the inequalities in the preceding process become equalities. In particular, the equality in Lemma 2.2.6 with $c = (1+<x,g_0>)^2$ holds, and hence the first equation follows immediately.

Applying the first equation twice, we have

$$\begin{aligned} \sum \phi_{ij}^\alpha \phi_{ij}^\beta F^\beta \ = \ & \frac{8}{\Phi + \frac{F^2}{8(1+<x,g>)^2}}[\ (\sum(\phi_{11}^\beta)^2 \sum \phi_{11}^\beta F^\beta + \sum \phi_{11}^\beta \phi_{12}^\beta \sum \phi_{12}^\beta F^\beta)\phi_{11}^\alpha \\ & +(\sum \phi_{11}^\beta \phi_{12}^\beta \sum \phi_{11}^\beta F^\beta + \sum(\phi_{12}^\beta)^2 \sum \phi_{12}^\beta F^\beta)\phi_{12}^\alpha \ ] \\ = \ & \frac{8}{\Phi + \frac{F^2}{8(1+<x,g>)^2}}[\ \frac{1}{4}(\Phi + \frac{F^2}{8(1+<x,g>)^2}) \sum \phi_{11}^\beta F^\beta \phi_{11}^\alpha \\ & +\frac{1}{4}(\Phi + \frac{F^2}{8(1+<x,g>)^2}) \sum \phi_{12}^\beta F^\beta \phi_{12}^\alpha \ ] \\ = \ & \frac{1}{2}[\ \Phi + \frac{F^2}{8(1+<x,g>)^2} \ ]F^\alpha, \end{aligned}$$

for all α. Thus F^α satisfies the following equation

$$\Delta F^\alpha + \frac{1}{2}[\ \Phi + \frac{F^2}{8(1+<x,g>)^2} \ ]F^\alpha = 0.$$

The scheme of showing others are similar to that of Step 3 in the proof of Theorem C. We made a brief sketch here for clarity and completeness. Let $\varphi_{ij}^{\alpha} = (1+ < x, g_0 >)\phi_{ij}^{\alpha}$ for all α, i, j. Because $\psi_{ijj}^{\alpha} = \frac{F_i^{\alpha}}{4}$, for all α, i, j, Lemma 2.2.9 gives

$$\varphi_{111}^{\alpha} = \frac{F_1^{\alpha}}{4} + 2 < e_2, g_0 > \phi_{12}^{\alpha},$$

$$\varphi_{112}^{\alpha} = -\frac{F_2^{\alpha}}{4} - 2 < e_1, g_0 > \phi_{12}^{\alpha},$$

$$\varphi_{121}^{\alpha} = \frac{F_2^{\alpha}}{4} - 2 < e_2, g_0 > \phi_{11}^{\alpha}$$

and

$$\varphi_{122}^{\alpha} = \frac{F_1^{\alpha}}{4} + 2 < e_1, g_0 > \phi_{11}^{\alpha}.$$

Because the equality in Lemma 2.2.6 with $c = (1+ < x, g >)^2$ holds, we have

$$\begin{aligned} A^2 + B^2 &= \frac{1}{2} C F^2, \\ A^2 - B^2 &= 8C[\, \sum (\phi_{11}^{\alpha})^2 - \sum (\phi_{12}^{\alpha})^2 \,], \\ AB &= 8C \sum \phi_{11}^{\alpha} \phi_{12}^{\alpha}, \end{aligned}$$

where $A = \sum \varphi_{11}^{\alpha} F^{\alpha}$, $B = \sum \varphi_{12}^{\alpha} F^{\alpha}$ and $C = \frac{1}{2}((1+ < x, g_0 >)^2 \Phi + \frac{F^2}{8})$.

Since $A^2 + B^2$ and F^2 are constants, differentiating $A^2 + B^2$ and substituting φ_{ijk}^{α} in terms of F_i^{α} and ϕ_{ij}^{α}, we obtain

$$A \sum \varphi_{11}^{\alpha} F_1^{\alpha} + B \sum \varphi_{12}^{\alpha} F_1^{\alpha} = 0,$$

$$A \sum \varphi_{11}^{\alpha} F_2^{\alpha} + B \sum \varphi_{12}^{\alpha} F_2^{\alpha} = 0.$$

Since $A^2 + B^2$ is a positive constant, $\sum \varphi_{11}^{\alpha} F_1^{\alpha} = -tB$, $\sum \varphi_{12}^{\alpha} F_1^{\alpha} = tA$, $\sum \varphi_{11}^{\alpha} F_2^{\alpha} = -sB$ and $\sum \varphi_{12}^{\alpha} F_2^{\alpha} = sA$, for some functions t and s.

Next, we differentiate the equations involved $A^2 - B^2$ and AB, obtaining

$$\begin{aligned} tAB &= C(sA + tB), \\ sAB &= C(tA - sB), \\ t(A^2 - B^2) &= 2C(tA - sB), \\ s(A^2 - B^2) &= 2C(-sA - tB). \end{aligned}$$

As before, this implies $s = t = 0$, and we get the second equation.

Differentiating the second equation, the proof of remaining part uses exactly the same argument as Theorem C, one just replaces H^α by F^α throughout.

Step 4. Finally, we assert that M is totally umbilical. Suppose that, to get a contradiction, M is not totally umbilical. It will then follow from Step 2 that both $(1+ < x, g_0 >)^2\Phi$ and F^2 are positive constants.

Setting $C = \frac{1}{2}[\ (1+ < x, g_0 >)^2\Phi + \frac{F^2}{8})\ ]$, since F^2 is a constant function, we have

$$\begin{aligned} 0 &= \frac{1}{2}(1+ < x, g_0 >)^2 \Delta F^2 \\ &= (1+ < x, g_0 >)^2 \sum |\nabla F^\alpha|^2 + (1+ < x, g_0 >)^2 \sum F^\alpha \Delta F^\alpha \\ &= (1+ < x, g_0 >)^2 \sum |\nabla F^\alpha|^2 - CF^2, \end{aligned}$$

and hence

$$(1+ < x, g_0 >)^2 \sum |\nabla F^\alpha|^2 = CF^2.$$

This means that $(1+ < x, g_0 >)^2 \sum |\nabla F^\alpha|^2$ is also a constant function. Both first derivatives being equal to zeros, we get

$$(1+ < x, g >)^2 \sum F^\alpha_j F^\alpha_{ji} < e_i, g_0 >= -(1+ < x, g_0 >) \sum |\nabla F^\alpha|^2 < e_i, g_0 >^2 .$$

Once again we use the fact that $(1+ < x, g_0 >)^2 \sum |\nabla F^\alpha|^2$ is a constant, we have

$$\begin{aligned} 0 &= \frac{1}{2}(1+ < x, g_0 >)^2 \Delta[\ (1+ < x, g_0 >)^2 \sum |\nabla F^\alpha|^2\ ] \\ &= \frac{1}{2}(1+ < x, g_0 >)^2 \sum |\nabla F^\alpha|^2 \Delta (1+ < x, g_0 >)^2 + \frac{1}{2}(1+ < x, g_0 >)^4 \Delta \sum |\nabla F^\alpha|^2 \\ &\quad +(1+ < x, g_0 >)^2 \nabla (1+ < x, g_0 >)^2 \cdot \nabla \sum |\nabla F^\alpha|^2 \\ &= CF^2[\ -3 \sum < e_i, g_0 >^2 + (1+ < x, g_0 >)(\sum H^\alpha < e_\alpha, g_0 > -2 < x, g_0 >)\ ] \\ &\quad + \frac{1}{2}(1+ < x, g_0 >)^4 \Delta \sum |\nabla F^\alpha|^2, \end{aligned}$$

here we have used the fact that $\Delta < x, g_0 >= \sum H^\alpha < e_\alpha, g_0 > -2 < x, g_0 >$. We need to adjust the last term,

$$\begin{aligned}\frac{1}{2}(1+<x,g_0>)^4\Delta\sum|\nabla F^\alpha|^2 &= (1+<x,g_0>)^4[\sum(F^\alpha_{ij})^2+\sum F^\alpha_i F^\alpha_{ijj}]\\ &= (1+<x,g_0>)^4[\sum(F^\alpha_{ij})^2+\sum F^\alpha_i(\Delta F^\alpha)_i\\ &\quad +\sum F^\alpha_i F^\alpha_j R_{ikjk}+2\sum F^\alpha_i F^\beta_j R_{\beta\alpha ij}\\ &\quad +\sum F^\alpha_i F^\beta R_{\beta\alpha ij,j}].\end{aligned}$$

Now we take care of these terms containing curvature. First, it is straightforward that

$$\sum F^\alpha_i F^\alpha_j R_{ikjk}=R_{1212}\sum|\nabla F^\alpha|^2=(1+\frac{H^2}{4}-\frac{\Phi}{2})\sum|\nabla F^\alpha|^2.$$

Next, applying the second equation of Step 3, we obtain

$$\sum F^\alpha_i F^\beta_j R_{\beta\alpha ij}=-2(F^\alpha_1F^\beta_2-F^\alpha_2F^\beta_1)(\phi^\alpha_{11}\phi^\beta_{12}-\phi^\alpha_{11}\phi^\beta_{12})=0.$$

Finally, substituting φ^α_{ijk} in terms of F^α_i and ϕ^α_{ij}, the second equation of Step 3 gives

$$\begin{aligned}(1+<x,g_0>)^2 &\sum F^\alpha_i F^\beta R_{\beta\alpha ij,j}\\ &= \frac{1}{2}\sum\varphi^\alpha_{11}F^\alpha\sum[(F^\alpha_1)^2-(F^\alpha_2)^2]+\sum\varphi^\alpha_{12}F^\alpha\sum F^\alpha_1F^\alpha_2.\end{aligned}$$

Then applying the third and fourth equations of Step 3, we have

$$\sum F^\alpha_i F^\beta R_{\beta\alpha ij,j}=\frac{F^2}{4}[\Phi+\frac{F^2}{8(1+<x,g_0>)^2}]^2.$$

Together these equations imply that

$$\begin{aligned}\frac{1}{2}(1+<x,g_0>)^4 \;&\Delta\;\sum|\nabla F^\alpha|^2\\ &= (1+<x,g_0>)^4\sum(F^\alpha_{ij})^2+CF^2(1+<x,g_0>)^2(1+\frac{H^2}{4}-\frac{\Phi}{2}).\end{aligned}$$

Substituting this into the original equation, it follows that

$$\begin{aligned}0 &= (1+<x,g_0>)^4\sum(F^\alpha_{ij})^2+CF^2[-3\sum<e_i,g_0>^2\\ &+ (1+<x,g_0>)(\sum H^\alpha<e_\alpha,g_0>-2<x,g_0>)+(1+<x,g_0>)^2(1+\frac{H^2}{4}-\frac{\Phi}{2})].\end{aligned}$$

To estimate the first term, let

$$\begin{aligned}\tilde{F}^\alpha_{ij} &= (1+<x,g_0>)^2F^\alpha_{ij}\\ &\quad +(1+<x,g_0>)[F^\alpha_i<e_j,g_0>+F^\alpha_j<e_i,g_0>-\sum F^\alpha_k<e_k,g_0>\delta_{ij}],\end{aligned}$$

for all α, i, j. Then

$$\sum \tilde{F}^{\alpha}_{ii} = (1+ < x, g_0 >)^2 \sum F^{\alpha}_{ii} = -CF^{\alpha},$$

and

$$\begin{aligned}
\sum (\tilde{F}^{\alpha}_{ij})^2 &= 2(1+ < x, g_0 >)^3 [\sum F^{\alpha}_{ij} F^{\alpha}_i < e_j, g_0 > + \sum F^{\alpha}_{ij} F^{\alpha}_j < e_i, g_0 > \\
&\qquad - \sum F^{\alpha}_{ii} F^{\alpha}_k < e_k, g_0 >] \\
&\quad + (1+ < x, g_0 >)^4 \sum (F^{\alpha}_{ij})^2 + 2(1+ < x, g_0 >)^2 \sum |\nabla F^{\alpha}|^2 < e_i, g_0 >^2 \\
&= 2(1+ < x, g_0 >)^3 (2 \sum F^{\alpha}_{ij} F^{\alpha}_i < e_j, g_0 > + \sum (F^{\alpha}_{ij} - F^{\alpha}_{ji}) F^{\alpha}_j < e_i, g_0 >) \\
&\quad + (1+ < x, g_0 >)^4 \sum (F^{\alpha}_{ij})^2 + 2(1+ < x, g_0 >) C \sum F^{\alpha} F^{\alpha}_k < e_k, g_0 > \\
&\quad + 2(1+ < x, g_0 >)^2 \sum |\nabla F^{\alpha}|^2 < e_i, g_0 >^2 \\
&= 2(1+ < x, g_0 >)^3 (2 \sum F^{\alpha}_{ij} F^{\alpha}_i < e_j, g_0 > + \sum F^{\beta} R_{\beta\alpha ij} F^{\alpha}_j < e_i, g_0 >) \\
&\quad + (1+ < x, g_0 >)^4 \sum (F^{\alpha}_{ij})^2 + 2(1+ < x, g_0 >)^2 \sum |\nabla F^{\alpha}|^2 < e_i, g_0 >^2 \\
&= -2(1+ < x, g_0 >)^2 \sum |\nabla F^{\alpha}| < e_i, g_0 >^2 + (1+ < x, g_0 >)^4 \sum (F^{\alpha}_{ij})^2.
\end{aligned}$$

Thus the first term can estimate from below by

$$\begin{aligned}
(1+ < x, g_0 >)^4 \sum (F^{\alpha}_{ij})^2 &= \sum (\tilde{F}^{\alpha}_{ij})^2 + 2CF^2 \sum < e_i, g_0 >^2 \\
&\geq \sum (\tilde{F}^{\alpha}_{ii})^2 + 2CF^2 \sum < e_i, g_0 >^2 \\
&\geq \frac{1}{2} (\sum \tilde{F}^{\alpha}_{ii})^2 + 2CF^2 \sum < e_i, g_0 >^2 \\
&= \frac{1}{2} C^2 F^2 + 2CF^2 \sum < e_i, g_0 >^2 .
\end{aligned}$$

Because $1 = < x, g_0 >^2 + \sum < e_i, g_0 >^2 + \sum < e_{\alpha}, g_0 >^2$, we conclude that

$$\begin{aligned}
0 &\geq CF^2 [\, 1 - \sum < e_i, g_0 >^2 - < x, g_0 >^2 + \frac{1}{4} (1+ < x, g_0 >)^2 H^2 \\
&\quad + (1+ < x, g_0 >) H^{\alpha} < e_{\alpha}, g_0 > + \frac{1}{32} F^2 - \frac{1}{4} (1+ < x, g_0 >)^2 \Phi \,] \\
&= CF^2 [\, \frac{9}{32} F^2 - \frac{1}{4} (1+ < x, g_0 >)^2 \Phi \,] \\
&= \frac{24 - \sqrt{6}}{96} CF^4 > 0.
\end{aligned}$$

This contradiction shows that M is totally umbilical. This completes the proof of Theorem D. □

As an immediate consequence of Theorem D, the pinching condition can be simplified as follows.

Corollary 3. Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n, $n \geq 4$. If

$$inf_{g\in G} max_{g\circ x(M)}(\Phi_g - \frac{1}{6}H_g^2) \leq \frac{4}{3},$$

then $x(M)$ is either a totally umbilical sphere or a conformal Veronese surface.

Chapter 5

Examples

The point of the following examples is that it shows our results about upper estimate for Φ, Theorems A and C may fail to be true if we make a slight change in the pinching condition.

Example 1. Let $x : S^1 \times S^1 \to S^3$ be the Clifford torus,

$$x(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta,\ \sin\theta,\ \cos\varphi,\ \sin\varphi).$$

Consider the Willmore surface $x_\epsilon = g \circ x$, where $g = (a, 0, a, 0)$ with $a = \frac{\epsilon}{\sqrt{2}(4+\epsilon)}$. Since the Clifford torus is a minimal surface with $\Phi = 2$, we have

$$\Phi_\epsilon - \frac{1}{4}H_\epsilon^2 = \frac{2}{1-2a^2}[\ (\frac{a}{\sqrt{2}}\cos\theta + \frac{a}{\sqrt{2}}\cos\varphi + 1)^2 - \frac{1}{2}(-\frac{a}{\sqrt{2}}\cos\theta + \frac{a}{\sqrt{2}}\cos\varphi)^2\ ].$$

The maximal value of $\Phi_\epsilon - \frac{1}{4}H_\epsilon^2$ over $S^1 \times S^1$ is

$$2\frac{1+\sqrt{2}a}{1-\sqrt{2}a} = 2 + \epsilon.$$

Thus for every $\epsilon > 0$, there is a compact Willmore surface M^2 in S^3, it is not the Clifford torus, with $0 < \Phi \leq 2 + \frac{H^2}{4} + \epsilon$.

Example 2. Let $x : S^2(\sqrt{3}) \to S^4$ be the Veronese surface,

$$\begin{aligned} x(\theta,\varphi) \ &= \ (\sqrt{3}\cos\theta\sin\theta\sin\varphi,\ \sqrt{3}\cos\theta\sin\theta\cos\varphi,\ \sqrt{3}\cos^2\theta\cos\varphi\sin\varphi, \\ &\quad \frac{\sqrt{3}}{2}\cos^2\theta(\cos^2\varphi - \sin^2\varphi),\ \frac{1}{2}\cos^2\theta - \sin^2\theta). \end{aligned}$$

Consider the Willmore surface $x_\epsilon = g \circ x$, where $g = (a, a, 0, 0, 0)$ with $a = \frac{-\sqrt{6}+\sqrt{6+3\epsilon(\frac{7}{2}+\frac{3\epsilon}{2})}}{7+3\epsilon}$. Since the Veronese surface is a minimal surface with $\Phi = \frac{4}{3}$, we must have

$$\begin{aligned}\Phi_\epsilon &- \frac{1}{8}H_\epsilon^2 - \sqrt{\frac{4}{9} + \frac{1}{6}H_\epsilon^2 + \frac{1}{96}H_\epsilon^4} \\ &= \frac{1}{1-2a^2}\{\ [\ a(\sin\varphi + \cos\varphi)\sin 2\theta + \frac{2}{\sqrt{3}}\ ]^2 - \frac{2a^2}{3}(\cos\varphi - \sin\varphi)^2\cos^2\theta\ \} \\ &\quad - \sqrt{\frac{4}{9} + \frac{2a^2}{3(1-2a^2)}(\cos\varphi - \sin\varphi)^2\cos^2\theta - \frac{a^4}{6(1-2a^2)^2}(\cos\varphi - \sin\varphi)^4\cos^4\theta}.\end{aligned}$$

The maximal value of $\Phi_\epsilon - \frac{1}{8}H_\epsilon^2 - \sqrt{\frac{4}{9} + \frac{1}{6}H_\epsilon^2 + \frac{1}{96}H_\epsilon^4}$ over $S^2(\sqrt{3})$ is

$$\frac{1}{1-2a^2}(a + \frac{2}{\sqrt{3}})^2) - \frac{2}{3} = \frac{2}{3} + \epsilon.$$

Thus for every $\epsilon > 0$, there is a compact Willmore surface M^2 in S^4, it is not the Veronese surface, with $0 < \Phi \leq \frac{2}{3} + \frac{H^2}{8} + \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4} + \epsilon$.

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