圖形的帶寬與輪廓
Bandwidth and Profiles of Graphs

研 究 生 ：曹友賓 | Student : Yu-Ping Tsao
指導教授 ：張鎮華 | Advisor : Gerard J. Chang

國 立 交 通 大 學
應 用 數 學 系

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學生：曹友賓　　　　指導教授：張鎮華

國立交通大學應用數學系博士班

摘　　要

圖形論之帶寬及輪廓在許多實際應用的領域裡是有用的參數，這兩個參數已經被廣泛地研究。許多文章對特殊或一般圖形的帶寬值及輪廓值均有著墨，本論文主要是在幾類合成圖上研究此二參數的作用方式。

首先，我們考慮帶寬問題，該問題尋求圖形 G 上的線性配置使能將邊的伸展達到最小化。明確地說，一個嵌射 $f: G \to \mathbf{N}$ 的帶寬是 $B_f(G) = \max_{uv \in E(G)} |f(u) - f(v)|$，而圖形 G 的帶寬 $B(G)$ 則是所有這些嵌射之帶寬的最小值。於此論文中，我們於三類距離圖 (包括 $G([n], D)$、$G(\mathbf{Z}_n, D)$ 及 $G(\mathbf{N}, D)$) 上考慮帶寬問題；也在與此三類距離圖作某些合成的圖類上考慮帶寬問題。我們在有些合成圖上導出了精確值，但對有些合成圖只能提出可達上界及下界。

接著，我們考慮輪廓問題，此問題與帶寬問題有著緊密的關係。目的在尋找一個嵌射，將圖形 G 的頂點集對應到正整數集，使得每一頂點的標號與它的閉鄰域中所有點之最小標號差的總和爲最小。本論文在兩類合成圖 (叉圖及構成圖) 上考慮輪廓問題，在叉圖的部分吾人僅獲得 $K_m \times K_n$、$(\overline{K_s} \vee G) \times K_n$ (其中 $|V(G)| = t \le s$ 且 $n \ge 4$) 及 $P_m \times K_n$ 的輪廓值；在構成圖的部分吾人則建立 $P(G[H])$ 之可達上界及下界，藉此也決定了當 G 爲區間圖或某些特定圖時 $P(G[H])$ 的精確值。

Bandwidths and Profiles of Graphs

Student : Yu-Ping Tsao **Advisor : Gerard J. Chang**

Department of Applied Mathematics
National Chiao Tung University

ABSTRACT

Bandwidth and profile in graph theory are useful parameters for many real applications. These two parameters have been extensively studied in the literature, This thesis emphasizes the study of these parameters on composite graphs.

We first discuss the bandwidth problem. The problem asks for a linear layout of a graph to minimize stretching of edges. More precisely, the bandwidth of an injection $f : V(G) \to \mathbf{N}$ is $B_f(G) = \max_{uv \in E(G)} |f(u) - f(v)|$. The bandwidth $B(G)$ of a graph G is $\min B_f(G)$ over all such injections f. In this thesis, we consider the problem for three kinds of distance graphs, including $G([n], D)$, $G(\mathbf{Z}_n, D)$ and $G(\mathbf{N}, D)$. We also consider several composites of them with arbitrary graphs. For some of these composites graphs we give exact values, and for some we only offer sharp bounds (both upper and lower ones).

We then study the profile problem. The problem is tightly related to the bandwidth problem. A profile of a proper numbering $f : V(G) \to [|V(G)|]$ is $P_f(G) = \sum_{v \in V(G)} \max_{x \in N[v]} (f(v) - f(x))$, where $N[v]$ means the closed neighborhood of v. The profile $P(G)$ of a graph G is $\min P_f(G)$ over all such proper numberings f. In this thesis, we consider the problem for product of graphs and composition of graphs. In the part of product of graphs, we barely obtain the profiles of $K_m \times K_n$, $(\overline{K_s} \vee G) \times K_n$ for $|V(G)| = t \leq s$

with $n \geq 4$ and $P_m \times K_n$. In the part of composition of graphs, we establish both sharp upper and lower bounds of the profile of $G[H]$, and proceed to determine the exact value when G is an interval graph as well as certain graphs.

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Chapter 1

Prologue: Introduction

In this chapter, we first describe motivations for studying the bandwidth and the profile problems. We then introduce some definitions needed in this thesis. Finally, we give an overview of our results.

1.1 Derivations of the problems

For an $n \times n$ symmetric matrix $[A] = [a_{i,j}]$, define the *row width* w_i of row i $(1 \leq i \leq n)$ as $w_i = \max\{0, i - \min\{j : a_{i,j} \neq 0\}\}$. The *bandwidth* $B(A)$ of the matrix A is $\max_{1 \leq i \leq n} w_i$, and the *profile* $P(A)$ of the matrix is $\sum_{1 \leq i \leq n} w_i$. For storage, the bandwidth represents the maximum length of a row that must be stored, and the profile represents the total amount of storage needed. In order to reduce both of them such that matrix operations can be performed faster, we need to permute the rows and columns of A simultaneously so that all nonzero entries of the resulting matrix lie near the diagonal and has the smallest bandwidth(profile). There is a direct one-to-one correspondence between symmetric $(0, 1)$ matrices with diagonal elements 0 and graphs. The position of the nonzero entries of an $n \times n$ symmetric matrix can define an adjacency matrix of a graph G on n vertices. So these problems can be reformulated in terms of graphs.

The two problems of graphs have a wide range of applications including solving linear equations, interconnection network, constraint satisfaction problem, data structure, coding theory and circuit layout of VLSI designs, which are introduced in [45]. Those problems become very important since the mid-1960s.

1.2 Basic definitions in graphs

This section gives some basic definitions and notation in graph theory. Other special definitions are mentioned later when they are used.

A *graph* G consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where each edge is an unordered pair $\{u, v\}$ of vertices called its *end-vertices*. For convenience, we write uv for an edge $\{u, v\}$. If $uv \in E(G)$, then u and v are *adjacent*. The cardinality of $V(G)$ is called the *order* of G, and the cardinality of $E(G)$ the *size*. The *degree* of a vertex v in a graph G, written $d_G(v)$, is the number of edges containing v. The *maximum degree* is denoted by $\Delta(G)$ and the *minimum degree* by $\delta(G)$. In a graph G, the *neighborhood* of a vertex v is $N_G(v) = \{x \in V(G) : xv \in E(G)\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N(v)$. If there is no ambiguity, we often use $N(v)$ for $N_G(v)$ and $N[v]$ for $N_G[v]$. A *clique* in a graph G is a set of pairwise adjacent vertices. A vertex v of a graph G is *simplicial* if $N(v)$ is a clique.

A *loop* is an edge whose end-vertices are equal. *Multiple edges* are edges having the same pairs of end-vertices. A *simple* graph is a graph having no loops or multiple edges. An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is isomorphic to H", written $G \cong H$, if there is an isomorphism from G to H. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset $S \subseteq V(G)$, the *subgraph induced by* S is the graph G_S with $V(G_S) = S$ and $E(G_S) = \{xy \in E(G) : x, y \in S\}$. The *complement* of a graph G, written $\overline{G}$, is a graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy \notin E(G) : x, y \in V(G)\}$. A *matching* in a graph G is a set of non-loop edges with no shared endpoints. The vertices incident to the edges of a matching M are *saturated* by M; the others are *unsaturated*. A *perfect matching* in a graph is a matching that saturates every vertex.

A *path* is an ordered list of distinct vertices $(v_0, v_1, \ldots, v_n)$ such that $v_{i-1}v_i$ is an edge for $1 \leq i \leq n$. The first and last vertices of a path are its *end-vertices*. A u, v-path is a path with end-vertices u and v. If a graph G has a u, v-path, then the *distance* from u to v, written $d(u, v)$, is the least length of a u, v-path; if G has no such path, then $d(u, v) = \infty$. The *diameter* $d(G)$ of a graph G is the maximum distance between two vertices in G. A graph G is *connected* if it has a u, v-path for each pair of vertices $u, v \in V(G)$. The *components* of a graph G are its maximal connected subgraphs. The *connectivity* of a graph G is the smallest number $\kappa(G)$ of vertices whose removal from G results a disconnected graph or a trivial graph. The *independence number* $\alpha(G)$ of G is the maximum size of a pairwise nonadjacent vertex set in G.

A *cycle* is an ordered list of distinct vertices $(v_0, v_1, \ldots, v_n)$, except $v_0 = v_n$ such that all $v_{i-1}v_i$ for $1 \leq i \leq n$ are edges. A graph is called *Hamiltonian* if it has a cycle containing all vertices of the graph. A graph with n vertices that is a path or a cycle is denoted by P_n or C_n, respectively. A *complete graph* of order n, written K_n, is a graph in which every pair of vertices is an edge. A *complete r-partite graph* is a graph whose vertex set can be partitioned into disjoint union of r nonempty parts, and two vertices are adjacent if and only if they are in different parts. We use $K_{n_1,n_2,\ldots,n_r}$ to denote the complete r-partite graph whose parts are of sizes $n_1, n_2, \ldots, n_r$, respectively.

The *join* of G and H, written $G \vee H$, is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G) \text{ and } y \in V(H)\}$. An *$n$-wheel* is a graph obtained from the join of C_{n-1} and an isolated vertex.

The *Cartesian product* of graphs G and H, written $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ such that (x, y) adjacent to (x', y') if and only if either $x = x'$ with $yy' \in E(H)$ or $y = y'$ with $xx' \in E(G)$.

The *product* (or *tensor product*) of two graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ such that (x, y) is adjacent to (x', y') in $G \times H$ if $xx' \in E(G)$ and $yy' \in E(H)$. Notice that $G \times H$ has $|V(G)||V(H)|$ vertices and $2|E(G)||E(H)|$ edges.

The *strong product* of two graphs G and H is the graph $G \boxtimes H$ with vertex set $V(G) \times V(H)$ such that (x, y) is adjacent to (x', y') in $G \boxtimes H$ if and only if $xx' \in E(G)$ with $yy' \in E(H)$, $x = x'$ with $yy' \in E(H)$, or $y = y'$ with $xx' \in E(G)$. Notice that $G \boxtimes H$ has $|V(G)||V(H)|$ vertices and $2|E(G)||E(H)| + |E(G)||V(H)| + |V(G)||E(H)|$ edges.

The *composition* of two graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$ such that (x, y) is adjacent to (x', y') in $G[H]$ if $xx' \in E(G)$ or $x = x'$ with $yy' \in E(H)$.

Notice that $G[H]$ has $|V(G)||V(H)|$ vertices and $|E(G)||V(H)|^2 + |V(G)||E(H)|$ edges.

The *corona* of two graphs G and H, denoted by $G \wedge H$, contains one copy of G and $|V(G)|$ copies H such that each vertex of G is joined to every vertex of corresponding copy of H.

For convenience, in either case of the Cartesian product, tensor product, strong product, and composition, suppose $V(G) = \{x_i : 1 \leq i \leq |V(G)|\}$ and $V(H) = \{y_j : 1 \leq j \leq |V(H)|\}$. We may write (x_i, y_j) as $v_{i,j}$. Let $R_i = \{v_{i,j} : 1 \leq j \leq |V(H)|\}$ represent the ith row (a copy of H) and $C_j = \{v_{i,j} : 1 \leq i \leq |V(G)|\}$ the jth column (a copy of G).

1.3 Survey of previous results

The first survey article on bandwidth was given by Chinn, Chvátalová, Dewdney, and Gibbs [4]. It provided many key concepts and inequalities upon which more later work is based. Some additional survey material was included in Chung [7]. Another excellent resource was written by Miller [55]. Lai and William gave a goodly new looking back on bandwidth, edge sum and profile before millennium [45].

A large number of approximation algorithms for bandwidth and profile had been extensively studied in the literature. Approximation algorithms for general graphs included those given in [3, 11, 14, 18, 20, 28, 31, 33, 41, 53, 59, 60, 63], and for trees or caterpillars in [19, 22, 56].

A considerable amount of work providing bounds on the bandwidth of graphs had been published, see [25, 26, 27, 47]. We now list some of the important bounds on bandwidth below.

For a connected graph G, Chung and Seymour [8] provided the *local density lower bound*,

$$B(G) \geq \max \frac{|V(G')| - 1}{d(G')},$$

where G' ranges over all connected subgraphs of G with $|V(G')| \geq 2$. They also showed that even in a tree with low local density the bandwidth can be arbitrarily large.

Lin [48] showed that for a connected graph G,

$$B(G) \geq \max_{1 \leq k \leq d(G)} \max_{S \in S_k} \frac{|S| - 1}{d(S)},$$

where $S_k = \{S \subseteq V(G) : S$ is a maximal subset with diameter $k\}$.

Let graph G be of order p and size q. It was conjectured by Chinn and then proved and published by Dutton and Brigham [13] that $B(G) \leq \frac{q+1}{2}$. Alavi, Liu, and McCanna [2] gave additional material to this inequality.

Lai and Williams [44] showed that, for connected graph G, $B(G) \geq p - r$, where $r = \max\{x \in \mathbf{Z} : x(x-1) \leq p(p-1) - 2q\}$. This bound is tight for all values of p and q. Alavi, Lam, Wand, and Yao [1] provided a related result as well as a number of bandwidth bounds that hold for special classes of graphs. Miller [55] had a similar lower bound

$$B(G) \geq p - \frac{1 + \sqrt{(2p-1)^2 - 8q}}{2}.$$

and showed the others such like

$$B(G) \geq \kappa(G),\ B(G) \geq \frac{p}{\alpha(G)} - 1.$$

Hare, Hare and Hedetniemi [24] showed that the bandwidth of a tree is bounded above by the width of the tree, where width is defined as the maximum number of vertices in any level of a level structure on the tree. De la Véga [12] gave some results on the bandwidth of random graphs.

Very few general bounds were known for the profile of a graph. Lin and Yuan [49] showed that $P(G) \geq q$ for any graph G. They also showed that

$$P(G) \geq \frac{\kappa(G)(2p - \kappa(G) - 1)}{2}.$$

Lai and Williams [44] provided an existence result for graphs with a given profile. Given integers $p \geq 2$ and $0 \leq q \leq \frac{n(n-1)}{2}$, there is a graph G of order p and size q such that $P(G) = q$.

The bandwidth and profile problems have been solved for a number of classes of graphs. [45] summarized the known exact results on many of the composite graphs.

1.4 Overview of the thesis

In this thesis, we study bandwidth and profile of graphs. We give a brief overview of the thesis.

In Chapter 1, we introduce basic definitions, terminologies and symbols in graphs. We also describe motivations of studying the bandwidth and the profile problems, and known results on these problems.

Chapter 2 is devoted to the bandwidth problem for some composites of three kinds of distance graphs (on $G([n], D), G(\mathbf{Z}_n, D), G(\mathbf{N}, D)$, respectively) with others, such as Cartesian product, tensor product, strong product, composition and corona.

Chapter 3 considers the profile problem. The formal half of this chapter is to present the profiles on products of complete graph with some other graphs. The latter half discusses the profiles on composition of two general graphs.

Chapter 4 makes a conclusion, in which we also give some open problems.

Chapter 2

Allegro: The Movement of Bandwidth

2.1 Preliminary for bandwidth

From Section 1.1 in Chapter 1, we know that the bandwidth problem can be defined in terms of graphs as follows.

A *proper numbering* of a graph G is a 1-1 mapping $f : V(G) \to \mathbf{N}$. The *bandwidth* of a proper numbering f of G is

$$B_f(G) = \max_{xy \in E(G)} |f(x) - f(y)|,$$

and the *bandwidth* of G is

$$B(G) = \min\{B_f(G) : f \text{ is a proper numbering of } G\}.$$

A proper numbering f is called a *bandwidth numbering* of G if $B_f(G) = B(G)$.

The bandwidth problem for a graph which asks for a linear layout to minimize stretching of edges (for VLSI circuit layout application) has been extensively studied during the past two decades.

From algorithmic points of view, the decision problem was shown to be NP-complete by Papadimitriou in [57]. Garey, Graham, Johnson, and Knuth [16] showed that the problem is NP-complete even for trees of maximum degree 3. Although many upper and lower bounds for bandwidths of graphs were developed in terms of various graph invariants, while the exact values or algorithms of bandwidths were only known for a few classes of graphs [10, 26, 30, 34, 38, 46, 48, 58], such like $B(P_n) = 1$, $B(C_n) = 2$,

$B(K_n) = n-1$, $B(K_{m,n}) = m + \lfloor \frac{n-1}{2} \rfloor$ for $m \le n$, and

$$B(K_{m_1,m_2,\ldots,m_k}) = \sum_{1\le i\le k} m_i - \left\lfloor \frac{\max\limits_{1\le i\le k} m_i + 1}{2} \right\rfloor,\ B(Q_n) = \sum_{0\le k\le n-1} \binom{k}{\lfloor \frac{k}{2} \rfloor},\ \text{...etc.}$$

Among the non-algorithmic results for bandwidths, researchers are more interested in graphs from graph operations. The classes of graphs in this line include Cartesian products, tensor products and strong products of certain graphs [9, 10, 23, 29, 42, 43, 64, 65, 67, 68], sum, composition and corona of certain graphs [5, 6, 40, 51, 52, 66]. The purpose of this chapter is to study bandwidths on three kinds of distance graphs and on the composites of the three kinds of distance graphs with others.

Let $\mathbf{N}$ denote the set of positive integers, $[n]$ be the set $\{1, 2, \ldots, n\}$ and $\mathbf{Z}_n$ means the set of integers modulo n. We use $\{1, 2, \ldots, n\}$ for $\mathbf{Z}_n$ if it is not vague. The first distance graph $G([n], D)$ we will handle is a graph with vertex set $[n]$ and edge set $\{ij : i, j \in [n]$ and $|i-j| \in D\}$, where D is a finite subset of $[n]$. The second one is $G(\mathbf{Z}_n, D)$ is a graph with vertex set $\mathbf{Z}_n$ and edge set $\{ij : i, j \in \mathbf{Z}_n$ and $i \equiv j \pm x \pmod{n}$ for some $x \in D\}$, where D is a subset of $\left[\lfloor \frac{n}{2} \rfloor\right]$. The last one is $G(\mathbf{N}, D)$ which is an infinite graph with vertex set $\mathbf{N}$ and edge set $\{ij : i, j \in \mathbf{N}$ and $|i-j| \in D\}$, where D is a finite subset of $\mathbf{N}$.

In this chapter, we always let $D = \{a_1, a_2, \ldots, a_k\}$ and denote $\max D$ by λ. Besides, H is assumed to be a connected graph of order m with $V(H) = \{y_j : 1 \le j \le m\}$ and we use $v_{i,j}$ to represent (i, y_j). In the following, we lead some properties which will be used later.

Proposition 2.1.1 *If G' is a subgraph of G, then $B(G') \le B(G)$.*

Proposition 2.1.2 *If a finite graph G with components G_i $(1 \le i \le m)$, then $B(G) = \max\limits_{1\le i\le m} B(G_i)$.*

For $S \subseteq V(G)$, ∂S denotes the set of vertices in S which are adjacent to some vertices in $V(G) \setminus S$. For a proper numbering f, let $S_t^f = \{v \in V(G) : f(v) \ge t+1\}$.

Proposition 2.1.3 *If f is a bandwidth numbering of a connected graph G, then $B(G) \ge |\partial S_t^f|$ for $t \in N$.*

Proof. Let $|\partial S_t^f| = k_t$. By the definition of S_t^f, $\max_{v\in\partial S_t^f} f(v) \geq k_t + t$. We then have

$$\begin{aligned} B(G) &= \max_{xy\in E(G)} |f(x) - f(y)| \\ &\geq \max\{f(x) - f(y) : x \in \partial S_t^f \text{ and } y \in N(x) \cap (V(G) \setminus S_t^f)\} \\ &\geq \max_{v\in\partial S_t^f} f(v) - t \\ &\geq k_t \\ &= |\partial S_t^f|. \end{aligned}$$

∎

Proposition 2.1.4 ([8]) *If G is a finite connected graph, then $B(G) \geq \max_{G'} \frac{|V(G')|-1}{d(G')}$, where G' ranges over all connected subgraphs of G with $|V(G')| \geq 2$.*

2.2 Bandwidths on $G([n], D)$ and the composites with others

2.2.1 Bandwidth on $G([n], D)$

Let us start with two lemmas related to our later assumption. They are easy to prove.

Lemma 2.2.1 *If $\gcd(n, D) = d$, then $G([n], D) \cong dG(\left[\frac{n}{d}\right], \frac{D}{d})$ and hence $B(G([n], D)) = B(G(\left[\frac{n}{d}\right], \frac{D}{d}))$.*

Lemma 2.2.2 *If $\gcd D = d$ and $\gcd(n, d) = 1$, then*

$$G([n], D) \cong (n - d\left\lfloor\frac{n}{d}\right\rfloor)G(\left[\left\lceil\frac{n}{d}\right\rceil\right], \frac{D}{d}) \cup (d - n + d\left\lfloor\frac{n}{d}\right\rfloor)G(\left[\left\lfloor\frac{n}{d}\right\rfloor\right], \frac{D}{d})$$

and hence

$$\begin{aligned} B(G([n], D)) &= \max\left\{B(G(\left[\left\lceil\frac{n}{d}\right\rceil\right], \frac{D}{d})), B(G(\left[\left\lfloor\frac{n}{d}\right\rfloor\right], \frac{D}{d}))\right\} \\ &= B(G(\left[\left\lceil\frac{n}{d}\right\rceil\right], \frac{D}{d})). \end{aligned}$$

By the lemmas above, we may assume later that D is co-prime. Let

$$\begin{aligned} X &= \left\{(u_i)_1^k : \sum_{1\leq i\leq k} a_i u_i = 1, u_i \in \mathbf{Z}\right\}, \\ c &= \min\left\{\frac{1}{2}\sum_{1\leq i\leq k} a_i(u_i + |u_i|) : (u_i)_1^k \in X\right\}, \text{ and } c_0 = \left\lceil\frac{c}{2}\right\rceil. \end{aligned}$$

Notice that $X \neq \emptyset$ since D is co-prime. There is one thing we need to mention is that c_0 depends on D.

We call $\langle p, q\rangle = \{i : p \leq i \leq q, i \in \mathbf{Z}\}$, for $p, q \in \mathbf{Z}$, a *discrete interval* on $\mathbf{Z}$.

Theorem 2.2.3 $B(G([n], D)) = \lambda$ *for* $n \geq 2c_0\lambda^2 - (2c_0 + 1)\lambda + 3$.

Proof. First, consider the numbering $g : V(G([n], D)) \to [n]$ defined by $g(i) = i$. Trivially, $B(G([n], D)) \leq B_g(G([n], D)) = \lambda$. Next, we must show $B(G([n], D)) \geq \lambda$ if n is larger than a certain number which is decided by D.

Let f be an optimal labeling and $\{f^{-1}(i) : 1 \leq i \leq t_0\} = \underset{1\leq i\leq m}{\cup}\langle p_i, q_i\rangle$, where $t_0 = c_0\lambda^2 - (c_0 + 1)\lambda + 2$ and $p_i \leq q_i < p_{i+1} \leq q_{i+1}$ with $p_{i+1} - q_i \geq 2$ for $1 \leq i \leq m - 1$.

In the case of $m \geq c_0\lambda$, for each $\ell \in [0, \lambda - 1] \cap \mathbf{Z}$, let

$$N^{(\ell)} = \left\{q_{1+\ell c_0} + \sum_{1\leq i\leq k} a_i u_i : q_{1+\ell c_0} + \sum_{1\leq i\leq k} a_i u_i \in \langle q_{1+\ell c_0} + 1, q_{1+\ell c_0} + c\rangle \cap S^f_{t_0}\right\}.$$

As $X \neq \emptyset$, $N^{(\ell)} \neq \emptyset$. Choose $i_\ell = q_{1+\ell c_0} + \sum_{1\leq i\leq k} a_i u_i^{(\ell)}$ with

$$\sum_{1\leq i\leq k} \left|u_i^{(\ell)}\right| = \min\left\{\sum_{1\leq i\leq k} |u_i| : q_{1+\ell c_0} + \sum_{1\leq i\leq k} a_i u_i \in N^{(\ell)}\right\}.$$

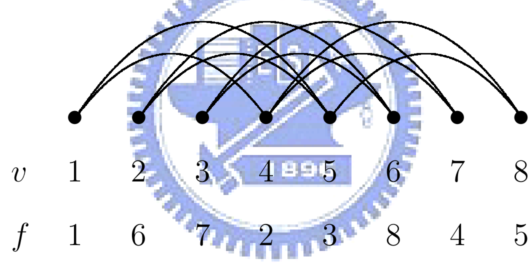
We claim that $i_\ell \in N(\underset{1\leq i\leq m}{\cup}\langle p_i, q_i\rangle)$, and thus we have $\left|\partial S^f_{t_0}\right| \geq \lambda$. For $1 \leq j \leq k$, let $i_{\ell,j} = q_{1+\ell c_0} + \sum_{1\leq i\neq j\leq k} a_i u_i^{(\ell)} + a_j(u_j^{(\ell)} - \mathrm{sgn}(u_j^{(\ell)}))$. By the meaning of i_ℓ, there is an $u_{j'}^{(\ell)} > 0$ with $i_{\ell,j'} \in \underset{1\leq i\leq m}{\cup}\langle p_i, q_i\rangle$, and so i_ℓ is incident to $i_{\ell,j'}$ through the definition of $G([n], D)$.

Thereupon, we consider the case of $m \leq c_0\lambda - 1$. Because $n \geq 2c_0\lambda^2 - (2c_0 + 1)\lambda + 3$, by the Pigeonhole's Principle, there is a discrete interval $\langle x, y\rangle$ in $S^f_{t_0}$ and a discrete interval $\langle z, w\rangle$ in $\overline{S^f_{t_0}}$ of order at least λ, respectively. Without loss of generality, we may assume $w < x$. For each i in $\langle w - \lambda + 1, w\rangle$, let $i + h_i\lambda = \min\left\{i + h\lambda : i + h\lambda \in S^f_{t_0}, h \in \mathbf{N}\right\}$. We claim that $i + h_i\lambda$ exists for each $i \in \langle w - \lambda + 1, w\rangle$. If so, since all $i + h_i\lambda$'s are trivially different, then $\left|\partial S^f_{t_0}\right| \geq \lambda$. We only need to show that for each $i \in \langle w - \lambda + 1, w\rangle$, there exists $h \in \mathbf{N}$ such that $i + h\lambda \in \langle x, y\rangle$. If there is a $i \in \langle w - \lambda + 1, w\rangle$ such that $i + h\lambda \leq x - 1$ or $i + h\lambda \geq y + 1$ for each $h \in \mathbf{N}$, suppose $i + h\lambda$ is the largest number such that $i + h\lambda \leq x - 1$, then $i + (h+1)\lambda \geq y + 1$. From $i + h\lambda \leq x - 1 < y + 1 \leq i + (h+1)\lambda$, we have $\lambda+1 \le (y+1)-(x-1) \le (i+(h+1)\lambda)-(i+h\lambda) = \lambda$, a contradiction. Otherwise, suppose $i+h\lambda$ is the smallest number such that $i+h\lambda \ge y+1$. This forces $i+(h-1)\lambda \le x-1$. From $i+h\lambda \ge y+1 > x-1 \ge i+(h-1)\lambda$, we have $\lambda+1 \le (y+1)-(x-1) \le (i+h\lambda)-(i+(h-1)\lambda) = \lambda$, a contradiction too. ∎

Corollary 2.2.4 *If $1 \in D$, then $B(G([n],D)) = \lambda$ for $n \ge 2\lambda^2 - 3\lambda + 3$.*

Proof. Clearly, $c_0 = 1$ if $1 \in D$. The result follows from By Theorem 2.2.3. ∎

We remark that if λ is large enough in proportional to n, the formulas in Theorem 2.2.3 and Corollary 2.2.4 are both not confidential. For example, $B(G([5],\{2,3\})) = 2 \ne \max\{2,3\}$ and $B(G([8],\{3,4\})) = 3 \ne \max\{3,4\}$. See Figure 2.1 for a bandwidth numbering of $G([8],\{3,4\})$.



$$B_f(G([8],\{3,4\})) = 3 \ne \max\{3,4\}$$

Figure 2.1: A bandwidth numbering of $G([8],\{3,4\})$.

2.2.2 Bandwidths on the composites of $G([n],D)$ with others

In the following, let $\varepsilon = \min\left\{\sum_{1\le i\le k} |u_i| : (u_i)_1^k \in X\right\}$, where we use precisely the same notation X as in Subsection 2.2.1. As it should be, ε also depends on D.

Theorem 2.2.5 *$B(G([n],D)\Box H) = m\lambda$ for $n \ge \varepsilon m\lambda^3 + (m^2-(\varepsilon+1)m-\varepsilon)\lambda^2 - (2m-\varepsilon-2)\lambda+2$.*

Proof. Obviously, the labeling g on $G([n],D)\Box H$ defined by $g(v_{i,j}) = (i-1)m+j$ makes $m\lambda$ the upper bound. Because $d(G([n],D)) \le \left\lfloor\frac{n-1}{\lambda}\right\rfloor + \varepsilon(\lambda-1)$, $d(H) \le m-1$, and

$d(G([n],D)\Box H) \le d(G([n],D)) + d(H)$, we have $\left\lceil \frac{nm-1}{d(G([n],D))+d(H)} \right\rceil \ge \left\lceil \frac{nm-1}{\lfloor \frac{n-1}{\lambda} \rfloor + \varepsilon(\lambda-1)+m-1} \right\rceil$. By solving the inequality

$$m\lambda - \frac{nm-1}{\frac{n-1}{\lambda} + \varepsilon(\lambda-1) + m - 1} < 1,$$

we know that if $n \ge \varepsilon m\lambda^3 + (m^2 - (\varepsilon+1)m - \varepsilon)\lambda^2 - (2m - \varepsilon - 2)\lambda + 2$, then

$$m\lambda - \frac{nm-1}{\lfloor \frac{n-1}{\lambda} \rfloor + \varepsilon(\lambda-1) + m - 1} < 1,$$

and therefore $\left\lceil \frac{nm-1}{\lfloor \frac{n-1}{\lambda} \rfloor + \varepsilon(\lambda-1)+m-1} \right\rceil = m\lambda$. From Proposition 2.1.4, we get

$$B(G([n],D)\Box H) \ge \left\lceil \frac{nm-1}{d(G([n],D)\Box H)} \right\rceil \ge \left\lceil \frac{nm-1}{d(G([n],D)) + d(H)} \right\rceil \ge m\lambda$$

for $n \ge \varepsilon m\lambda^3 + (m^2 - (\varepsilon+1)m - \varepsilon)\lambda^2 - (2m - \varepsilon - 2)\lambda + 2$. ∎

Remark: If $m = 1$ in Theorem 2.2.5, then we obtain another version of proof of Theorem 2.2.3. They have almost the same results. The slight and most important difference between them is the degrees of λ in the lower bounds restriction on n.

Corollary 2.2.6 *If* $1 \in D$, *then* $B(G([n],D)\Box H) = m\lambda$ *for* $n \ge m\lambda^3 + (m^2 - 2m - 1)\lambda^2 - (2m-3)\lambda + 2$.

Proof. Clearly, $\varepsilon = 1$ if $1 \in D$. The result then follows from Theorem 2.2.5. Notice that $m\lambda^3 + (m^2 - 2m - 1)\lambda^2 - (2m-3)\lambda + 2$ is almost independent of D. ∎

Theorem 2.2.7 $B(G([n],D) \wedge H) = (m+1)\lambda$ *for* $n \ge \varepsilon(m+1)\lambda^3 + (2m - m\varepsilon - 2\varepsilon + 2)\lambda^2 - (m - \varepsilon + 2)\lambda + 2$.

Proof. Let H_i be the copy of H corresponding to $i \in V(G([n],D))$. Define the labeling g on $G([n],D) \wedge H$ by numbering the vertices in $G([n],D)$ with $g(i) = (i-1)(m+1)+1$ for $1 \le i \le n$, and numbering the vertices j's $(1 \le j \le m)$, say j_i, in H_i with $g(j_i) = (i-1)(m+1)+1+j$ for $(i,j) \in [n] \times [m]$. Let two vertices u and v be adjacent in $G([n],D) \wedge H$. Then trivially u and v are in the same $V(H_i) \cup \{i\}$ or are adjacent in $G([n],D)$. In the former case, it is easy to see $|g(u) - g(v)| \le m$. In the latter case, we are certain that $|g(u)-g(v)| \leq (m+1)\lambda$ after checking carefully. These make sure that $(m+1)\lambda$ is an upper bound of $B(G([n],D)\wedge H)$. Next, to show that $(m+1)\lambda$ is also a lower bound. Because $d(G([n],D)) \leq \lfloor \frac{n-1}{\lambda} \rfloor + \varepsilon(\lambda-1)$, and $d(G([n],D)\wedge H) \leq d(G([n],D))+2$, we obtain $\left\lceil \frac{n(m+1)-1}{d(G(\mathbf{Z}_n,D))+2} \right\rceil \geq \left\lceil \frac{n(m+1)-1}{\lfloor \frac{n-1}{\lambda} \rfloor + \varepsilon(\lambda-1)+2} \right\rceil$. By solving the inequality

$$(m+1)\lambda - \frac{n(m+1)-1}{\frac{n-1}{\lambda} + \varepsilon(\lambda-1)+2} < 1,$$

we know that if $n \geq \varepsilon(m+1)\lambda^3 + (2m - m\varepsilon - 2\varepsilon + 2)\lambda^2 - (m-\varepsilon+2)\lambda + 2$, then

$$(m+1)\lambda - \frac{n(m+1)-1}{\lfloor \frac{n-1}{\lambda} \rfloor + \varepsilon(\lambda-1)+2} < 1$$

and therefore $\left\lceil \frac{n(m+1)-1}{\lfloor \frac{n-1}{\lambda} \rfloor + \varepsilon(\lambda-1)+2} \right\rceil = (m+1)\lambda$. From Proposition 2.1.4, we get

$$B(G([n],D)\wedge H) \geq \left\lceil \frac{n(m+1)-1}{d(G([n],D)\wedge H)} \right\rceil \geq \left\lceil \frac{n(m+1)-1}{d(G([n],D))+2} \right\rceil \geq (m+1)\lambda$$

for $n \geq \varepsilon(m+1)\lambda^3 + (2m - m\varepsilon - 2\varepsilon + 2)\lambda^2 - (m-\varepsilon+2)\lambda + 2$. ∎

Corollary 2.2.8 *If $1 \in D$, then $B(G([n],D)\wedge H) = (m+1)\lambda$ for $n \geq (m+1)\lambda^3 + m\lambda^2 - (m+1)\lambda + 2$.*

Proof. Clearly, $\varepsilon = 1$ if $1 \in D$. By Theorem 2.2.7, we have the result. ∎

2.3 Bandwidths on $G(\mathbf{Z}_n, D)$ and the composites with others

2.3.1 Bandwidth on $G(\mathbf{Z}_n, D)$

Since $G(\mathbf{Z}_n,D) \cong dG(\mathbf{Z}_{\frac{n}{d}}, \frac{1}{d}D)$, where $d = \gcd(n,D)$, by Proposition 2.1.2 we have $B(G(\mathbf{Z}_n,D)) = B(G(\mathbf{Z}_{\frac{n}{d}}, \frac{1}{d}D))$.

Lemma 2.3.1 *If $\gcd(n,D) = 1$ and $\gcd D = d$, then $G(\mathbf{Z}_n,D) \cong G(\mathbf{Z}_n, \frac{1}{d}D)$ and so $B(G(\mathbf{Z}_n,D)) = B(G(\mathbf{Z}_n, \frac{1}{d}D))$.*

Proof. Let $f : V(G(\mathbf{Z}_n, \frac{1}{d}D)) \to V(G(\mathbf{Z}_n,D))$ be defined by $f(i) = i'$ for $1 \leq i \leq n$, where $i' \equiv di \pmod{n}$. It is clear that f is well defined. Let $f(i) = f(j)$, $1 \leq i,j \leq n$.

The definition of f tells us $d(i-j) \equiv 0 \pmod{n}$. Since $\gcd(n,d)=1$, there are x and y in $\mathbf{Z}$ such that $nx+dy=1$. And hence $i-j=(i-j)(nx+dy)=(i-j)nx+(i-j)dy \equiv (i-j)\cdot 0+0\cdot y \equiv 0 \pmod{n}$. This ensures that f is a bijection. Next, suppose that i is adjacent to j in $G(\mathbf{Z}_n, \frac{1}{d}D)$, i.e. there is a $x \in \frac{1}{d}D$ such that $i \equiv j \pm x \pmod{n}$. Clearly, $f(i) \equiv di \equiv dj \pm dx \equiv f(j) \pm dx \pmod{n}$, where $dx \in d \cdot \frac{1}{d}D = D$. It means that $f(i)$ is adjacent to $f(j)$ in $G(\mathbf{Z}_n, D)$. From those, we know that $G(\mathbf{Z}_n, D) \cong G(\mathbf{Z}_n, \frac{1}{d}D)$, and so $B(G(\mathbf{Z}_n, D)) = B(G(\mathbf{Z}_n, \frac{1}{d}D))$. ∎

Henceforth, in the following we may assume $\gcd D = 1$. Let

$$\begin{aligned} X' &= \left\{ (u_i)_1^k : \sum_{1\le i\le k} a_i u_i = 1, u_i \in \mathbf{Z} \right\}, \\ c' &= \min \left\{ \frac{1}{2} \sum_{1\le i\le k} a_i(u_i+|u_i|) : (u_i)_1^k \in X' \right\}, \text{ and } c_0' = \left\lceil \frac{c'}{2} \right\rceil . \end{aligned}$$

Notice that $X' \neq \emptyset$, since $\gcd D = 1$. Likewise, we must highlight again that c_0' depends on D except $1 \in D$.

We call

$$\langle p,q\rangle_n = \begin{cases} \{i : p \le i \le q, i \in [n]\}, & \text{for } 1 \le p \le q \le n \\ \{i : 1 \le i \le q \text{ or } p \le i \le n, i \in [n]\}, & \text{for } 1 \le q \le p \le n \end{cases}$$

a *discrete interval modulus* n. Note that $\langle p,q\rangle_n = \langle p,n\rangle_n \cup \langle 1,q\rangle_n$ for $q \le p \le n$ under the meaning of modulus.

Theorem 2.3.2 *$B(G(\mathbf{Z}_n, D)) = 2\lambda$ for $n \ge 6c_0'\lambda^2 - (4c_0'+3)\lambda + 4$.*

Proof. First, we verify $B(G(\mathbf{Z}_n, D)) \le 2\lambda$. Consider the labeling $g : V(G(\mathbf{Z}_n, D)) \to [n]$ defined by

$$g(i) = \begin{cases} 2i-1, & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil; \\ 2(n+1-i), & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n. \end{cases}$$

It is not difficult to check that $B(G(\mathbf{Z}_n, D)) \le B_g(G(\mathbf{Z}_n, D)) = 2\lambda$. Afterwards, we need to corroborate $B(G(\mathbf{Z}_n, D)) \ge 2\lambda$.

Let f be an optimal labeling and $\{f^{-1}(i) : 1 \le i \le t_0\} = \bigcup_{1\le i\le m} \langle p_i, q_i\rangle_n$, where $t_0 = 2c_0'\lambda^2 - 2c_0'\lambda + 1$ with $p_i \le q_i < p_{i+1} \le q_{i+1} \le n$ and $p_{i+1} - q_i \ge 2$ for $1 \le i \le m-1$.

In the case of $m \geq 2c_0'\lambda + 1$, for each $\ell \in [0, 2\lambda - 1] \cap \mathbf{Z}$, let

$$N^{(\ell)} = \left\{ q_{1+\ell c_0'} + \sum_{1 \leq i \leq k} a_i u_i : q_{1+\ell c_0'} + \sum_{1 \leq i \leq k} a_i u_i \in \langle q_{1+\ell c_0'} + 1, q_{1+\ell c_0'} + c' \rangle_n \cap S_{t_0}^f \right\}.$$

As $X' \neq \emptyset$, $N^{(\ell)} \neq \emptyset$. Choose $i_\ell = q_{1+\ell c_0'} + \sum_{1 \leq i \leq k} a_i u_i^{(\ell)}$ with

$$\sum_{1 \leq i \leq k} \left| u_i^{(\ell)} \right| = \min \left\{ \sum_{1 \leq i \leq k} |u_i| : q_{1+\ell c_0'} + \sum_{1 \leq i \leq k} a_i u_i \in N^{(\ell)} \right\}.$$

We claim that $i_\ell \in N(\underset{1 \leq i \leq m}{\cup} \langle p_i, q_i \rangle_n)$ and therefore we get $\left| \partial S_{t_0}^f \right| \geq 2\lambda$. For $1 \leq j \leq k$, let $i_{\ell,j} = q_{1+\ell c_0'} + \sum_{1 \leq i \neq j \leq k} a_i u_i^{(\ell)} + a_j (u_j^{(\ell)} - \mathrm{sgn}(u_j^{(\ell)}))$. By the meaning of i_ℓ, it forces that there is a $u_{j'}^{(\ell)} > 0$ with $i_{\ell,j'} \in \underset{1 \leq i \leq m}{\cup} \langle p_i, q_i \rangle_n$, and so i_ℓ is incident to $i_{\ell,j'}$ through the definition of $G(\mathbf{Z}_n, D)$.

Thereupon, we consider the case of $m \leq 2c_0'\lambda$. Because $n \geq 6c_0'\lambda^2 - 4c_0'\lambda + 2$, by the Pigeonhole's Principle, there is a discrete interval $\langle x, y \rangle_n$ in $S_{t_0}^f$ and a discrete interval $\langle z, w \rangle_n$ in $\overline{S_{t_0}^f}$ of order at least 2λ and λ, respectively. Without loss of generality, we may assume $w < x$. For each i in $\langle w - \lambda + 1, w \rangle_n$, let $h_i = \min \left\{ h : i + h\lambda \in S_{t_0}^f, h \in \mathbf{N} \right\}$, and for each j in $\langle z, z + \lambda - 1 \rangle_n$, let $k_j = \min \left\{ k : j - k\lambda \text{ or } j - k\lambda + n \in S_{t_0}^f, k \in \mathbf{N} \right\}$. We claim that $i + h_i\lambda$ in $\mathbf{Z}_n$ exists for each $i \in \langle w - \lambda + 1, w \rangle_n$ and $j - k_j\lambda$ in $\mathbf{Z}_n$ exists for each $j \in \langle z, z + \lambda - 1 \rangle_n$. If so, since each $i + h_i\lambda$ and $j - k_j\lambda$ are trivially different, hence $\left| \partial S_{t_0}^f \right| \geq 2\lambda$. We only need to show that for each $i \in \langle w - \lambda + 1, w \rangle_n$, there exists $h \in \mathbf{N}$ such that $i + h\lambda \in \langle x, y \rangle_n$, and for each $j \in \langle z, z + \lambda - 1 \rangle_n$, there exists $k \in \mathbf{N}$ such that $j - k\lambda \in \langle x, y \rangle_n$. If there is an $i \in \langle w - \lambda + 1, w \rangle_n$ such that $i + h\lambda \leq x - 1$ or $i + h\lambda \geq y + 1$ for each $h \in \mathbf{N}$, suppose $i + h\lambda$ is the largest number such that $i + h\lambda \leq x - 1$. This forces $i + (h + 1)\lambda \geq y + 1$. From $i + h\lambda \leq x - 1 < y + 1 \leq i + (h + 1)\lambda$, we have $\lambda + 1 \leq (y + 1) - (x - 1) \leq (i + (h + 1)\lambda) - (i + h\lambda) = \lambda$, a contradiction. Otherwise, suppose $i + h\lambda$ is the smallest number such that $i + h\lambda \geq y + 1$. This forces $i + (h - 1)\lambda \leq x - 1$. From $i + h\lambda \geq y + 1 > x - 1 \geq i + (h - 1)\lambda$, we have $\lambda + 1 \leq (y + 1) - (x - 1) \leq (i + h\lambda) - (i + (h - 1)\lambda) = \lambda$, a contradiction too. If there is a $j \in \langle z, z + \lambda - 1 \rangle_n$ such that for each $k \in \mathbf{N}$, $j - k\lambda \leq x - 1$ or $j - k\lambda \geq y + 1$, a similar argument clarifies that it also wouldn't happen. ∎

Corollary 2.3.3 *If* $1 \in D$, *then* $B(G(\mathbf{Z}_n, D)) = 2\lambda$ *for* $n \geq 6\lambda^2 - 4\lambda + 2$.

Proof. Clearly, $c'_0 = 1$ if $1 \in D$. The result follows from Theorem 2.3.2. ∎

In the following, we attempt to explore the bandwidth of $G(\mathbf{Z}_n, D)$ when $\max D$ is close enough or equal to $\lfloor \frac{n}{2} \rfloor$.

Since $G(\mathbf{Z}_{2n}, \{k, n\}) \cong dG(\mathbf{Z}_{\frac{2n}{d}}, \{\frac{k}{d}, \frac{n}{d}\})$, where $d = \gcd(2n, k, n)$, by Proposition 2.1.2 we have $B(G(\mathbf{Z}_{2n}, \{k, n\})) = B(G(\mathbf{Z}_{\frac{2n}{d}}, \{\frac{k}{d}, \frac{n}{d}\}))$. We then may assume $\gcd(2n, k, n) = 1$ without loss of generality.

For the later usage, we introduce an operation on two graphs of the same orders.

Given $\sigma \in S_n$ and two graphs G and H with $V(G) = \{u_i : 1 \leq i \leq n\}$ and $V(H) = \{v_j : 1 \leq j \leq n\}$, respectively. We define a *permutation product* $G\sigma H$, which depends on the index order of $V(G)$ and $V(H)$, by

$$\begin{aligned} V(G\sigma H) &= V(G) \cup V(H), \\ E(G\sigma H) &= E(G) \cup E(H) \cup \{u_i v_{\sigma(i)} : 1 \leq i \leq n\}. \end{aligned}$$

Trivially, *Petersen graph* is a $C_5 \sigma C_5$ for some $\sigma \in S_n$.

Suppose i_d represents the identity permutation in S_n.

Lemma 2.3.4 $G(\mathbf{Z}_{2n}, \{k, n\}) \cong \begin{cases} G(\mathbf{Z}_{2n}, \{1, n\}), & \textit{for } \gcd(k,2) = 1 = \gcd(k,n); \\ C_n i_d C_n, & \textit{for } \gcd(k,2) \neq 1 = \gcd(k,n). \end{cases}$

Proof. In the case of $\gcd(k,2) = 1 = \gcd(k,n)$, let function $f : V(G(\mathbf{Z}_{2n}, \{1, n\})) \rightarrow V(G(\mathbf{Z}_{2n}, \{k, n\}))$ be defined by $f(i) = i'$ for $1 \leq i \leq 2n$, where $i' \equiv 1 + (i-1)k \pmod{2n}$ and $1 \leq j \leq 2n$. It is clear that f is well-defined. Let $f(i) = f(j)$, $1 \leq i, j \leq 2n$. The definition of f tells us $(i-j)k \equiv 0 \pmod{2n}$. Since $\gcd(k,2) = 1 = \gcd(k,n)$, there are x and y in $\mathbf{Z}$ such that $kx + 2ny = 1$. And hence $i - j = (i-j)(kx + 2ny) = (i-j)kx + (i-j)2ny \equiv 0 \cdot x + (i-j) \cdot 0 \equiv 0 \pmod{2n}$. This ensures that f is a bijection. Next, suppose that i is adjacent to j in $G(\mathbf{Z}_{2n}, \{1, n\})$, i.e. $i - j \equiv \pm 1$ or $\pm n \pmod{2n}$. Because $\gcd(k,2) = 1$, $f(i) - f(j) \equiv (1 + (i-1)k) - (1 + (j-1)k) = (i-j)k \equiv \pm k$ or $\pm nk \equiv \pm k$ or $\pm n \pmod{2n}$. It means that $f(i)$ is adjacent to $f(j)$ in $G(\mathbf{Z}_{2n}, \{k, n\})$. From these, we know that $G(\mathbf{Z}_{2n}, \{k, n\}) \cong G(\mathbf{Z}_{2n}, \{1, n\})$ if $\gcd(k,2) = 1 = \gcd(k,n)$.

In the other case $\gcd(k,2) \neq 1 = \gcd(k,n)$, let

$$\begin{aligned} A &= \{a \in V(G(\mathbf{Z}_{2n},\{k,n\})) : a \equiv 1 + i_a k \pmod{2n} \text{ for some } 0 \leq i_a \leq n-1\}, \\ B &= \{b \in V(G(\mathbf{Z}_{2n},\{k,n\})) : b \equiv (1+n) + j_b k \pmod{2n} \text{ for some } 0 \leq j_b \leq n-1\}. \end{aligned}$$

By $\gcd(k,n) = 1$, we have that the subgraph induced by A (so is B) is isomorphic to C_n. And by $\gcd(k,2) \neq 1$, it is easy to know that the two induced subgraphs are disjoint. Moreover, $N_B(1+ik) = \{(1+n)+ik\}$ for $0 \leq i \leq n-1$. Those imply $G(\mathbf{Z}_{2n},\{k,n\}) \cong C_n i_d C_n$ if $\gcd(k,2) \neq 1 = \gcd(k,n)$. ∎

Lemma 2.3.5 $B(G(\mathbf{Z}_{2n},\{1,n\})) = \begin{cases} 3, & \text{if } n = 2; \\ 4, & \text{if } n \neq 2. \end{cases}$

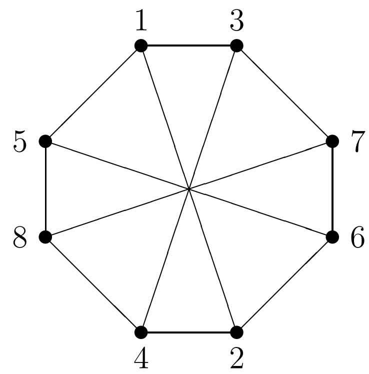
Proof. For $n = 2$, as $G(\mathbf{Z}_4,\{1,2\}) \cong K_4$, we have $B(G(\mathbf{Z}_4,\{1,2\})) = 3$.

For $n \geq 3$, to show at first that $B(G(\mathbf{Z}_{2n},\{1,n\})) \geq 4$. Suppose that f is an optimal labeling with $B_f(G(\mathbf{Z}_{2n},\{1,n\})) \leq 3$. Then, $N(f^{-1}(1)) = \{f^{-1}(2), f^{-1}(3), f^{-1}(4)\}$ by the fact that $G(\mathbf{Z}_{2n},\{1,n\})$ is 3-regular. We claim that $f^{-1}(3), f^{-1}(4) \notin N(f^{-1}(2))$. If not, then $n = 2$, contradicting to the hypothesis. And thus $N(f^{-1}(2)) = \{f^{-1}(1), f^{-1}(5)\}$. This conflicts with $\deg(f^{-1}(2)) = 3$. So we get $B(G(\mathbf{Z}_{2n},\{1,n\})) \geq 4$. Next, to verify $B(G(\mathbf{Z}_{2n},\{1,n\})) \leq 4$. Consider the labeling $g : V(G(\mathbf{Z}_{2n},\{1,n\})) \to [2n]$ defined by

$$g(i) = \begin{cases} 1 + 4(i-1), & \text{if } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil; \\ 4 + 4(n-i), & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n; \\ 2 + 4(i-n-1), & \text{if } n+1 \leq i \leq n + \left\lceil \frac{n}{2} \right\rceil; \\ 3 + 4(2n-i), & \text{if } n + \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq 2n. \end{cases}$$

It is not difficult to check that $B(G(\mathbf{Z}_{2n},\{1,n\})) \leq B_g(G(\mathbf{Z}_{2n},\{1,n\})) = 4$. ∎

Figure 2.2 shows a bandwidth numbering of $G(\mathbf{Z}_8,\{1,4\})$.



$B(G(\mathbf{Z}_8, \{1, 4\})) = 4$

Figure 2.2: A bandwidth numbering of $G(\mathbf{Z}_8, \{1, 4\})$.

Lemma 2.3.6 $B(C_n i_d C_n) = \begin{cases} 3, & \text{if } n = 3; \\ 4, & \text{if } n \neq 3. \end{cases}$

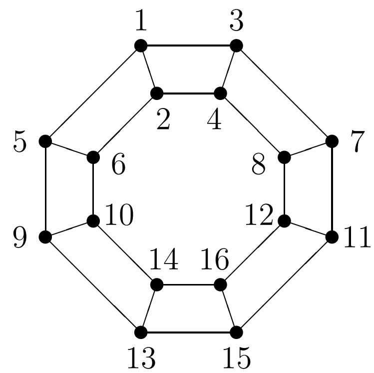
Proof. For the case of $n = 3$, since $C_3 i_d C_3$ is 3-regular, $B(C_3 i_d C_3) \geq 3$. Let $g : V(C_3 i_d C_3) \to [6]$ be a labeling such that the subgraphs induced by $\{g^{-1}(1), g^{-1}(2), g^{-1}(3)\}$, $\{g^{-1}(4), g^{-1}(5), g^{-1}(6)\}$ both are isomorphic to C_3, and edges $g^{-1}(1)g^{-1}(4), g^{-1}(2)g^{-1}(5)$, $g^{-1}(3)g^{-1}(6) \in E(C_3 i_d C_3)$, then $B(C_3 i_d C_3) \leq B_g(C_3 i_d C_3) = 3$. Notice that $C_3 i_d C_3 \cong G(\mathbf{Z}_6, \{2, 3\})$.

As to the case of $n \geq 4$, let f be an optimal labeling on $C_n i_d C_n$. Due to $\left|\partial S_4^f\right| \geq 4$, $B(C_n i_d C_n) \geq \left|\partial S_4^f\right| \geq 4$. To come here, we only need to explain why $B(C_n i_d C_n) \leq 4$. Suppose the outside n-cycle in $C_n i_d C_n$ is $v_1 v_2 v_3 \cdots v_{n-2} v_{n-1} v_n v_1$ and the inside n-cycle in $C_n i_d C_n$ is $v_{n+1} v_{n+2} v_{n+3} \cdots v_{2n-2} v_{2n-1} v_{2n} v_{n+1}$ such that v_i is adjacent to v_{n+i} for $1 \leq i \leq n$. Consider the labeling $g : V(C_n i_d C_n) \to [2n]$ defined by

$$g(v_i) = \begin{cases} 1 + 4(i-1), & \text{if } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil; \\ 3 + 4(n-i), & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n; \\ g(v_{i-n}) + 1, & \text{if } n+1 \leq i \leq 2n. \end{cases}$$

Clearly, $B(C_n i_d C_n) \leq B_g(C_n i_d C_n) = 4$. ∎

Figure 2.3 shows a bandwidth numbering of $C_8 i_d C_8$.



$B(C_8 i_d C_8) = 4$

Figure 2.3: A bandwidth numbering of $C_8 i_d C_8$.

Gathering up the above, we have the following theorem.

Theorem 2.3.7 $B(G(\mathbf{Z}_{2n}, \{k, n\})) = \begin{cases} 3, & \textit{if } (k,n) \in \{(1,2),(2,3)\}; \\ 4, & \textit{otherwise.} \end{cases}$

Proposition 2.3.8 $B(G(\mathbf{Z}_{2n}, \{1, n-1\})) = \begin{cases} 4, & \textit{if } n = 3; \\ 5, & \textit{if } n \geq 4. \end{cases}$

Proof. For the case of $n = 3$, because $G(\mathbf{Z}_6, \{1,2\})$ is 4-regular, $B(G(\mathbf{Z}_6, \{1,2\})) \geq 4$. Let $g : V(G(\mathbf{Z}_6, \{1,2\})) \to [6]$ be a labeling defined by

$$(g(1), g(2), g(3), g(4), g(5), g(6)) = (1, 2, 4, 6, 5, 3).$$

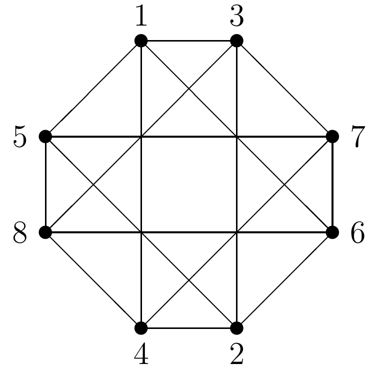
Obviously, $B(G(\mathbf{Z}_6, \{1,2\})) \leq B_g(G(\mathbf{Z}_6, \{1,2\})) = 4$.

For the case of $n \geq 4$, first of all we need to clarify $B_f(G(\mathbf{Z}_{2n}, \{1, n-1\})) \geq 5$ for each labeling f. Suppose f is a labeling with $B_f(G(\mathbf{Z}_{2n}, \{1, n-1\})) \leq 4$. If $f^{-1}(1)$ is not adjacent to $f^{-1}(2)$, as $G(\mathbf{Z}_{2n}, \{1, n-1\})$ is 4-regular, $\max\{i : f^{-1}(i) \in N(f^{-1}(1))\} \geq 6$. Hence $B_f(G(\mathbf{Z}_{2n}, \{1, n-1\})) \geq 5$, a contradiction to the assumption. This forces that $f^{-1}(1)$ must be adjacent to $f^{-1}(2)$. Even so, since $N(f^{-1}(1)) \cap N(f^{-1}(2)) = \emptyset$, $\left|\partial S_2^f\right| \geq 6$, and thus $B_f(G(\mathbf{Z}_{2n}, \{1, n-1\})) \geq \left|\partial S_2^f\right| \geq 6$. It also violates the assumption. Next, we give a labeling to certify $B(G(\mathbf{Z}_{2n}, \{1, n-1\})) \leq 5$. The labeling $g : V(G(\mathbf{Z}_{2n}, \{1, n-1\})) \to [2n]$ is given by

$$g(i) = \begin{cases} 1 + 4(i-1), & \text{if } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil; \\ 4 + 4(n-i), & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n; \\ 2 + 4(i-n-1), & \text{if } n+1 \leq i \leq n + \left\lceil \frac{n}{2} \right\rceil; \\ 3 + 4(2n-i), & \text{if } n + \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq 2n. \end{cases}$$

It is easy to check that $B(G(\mathbf{Z}_{2n},\{1,n-1\})) \le B_g(G(\mathbf{Z}_{2n},\{1,n-1\})) = 5$. ∎

Figure 2.4 shows the bandwidth numbering of $G(\mathbf{Z}_8,\{1,3\})$.



$B(G(\mathbf{Z}_8,\{1,3\})) = 5$

Figure 2.4: A bandwidth numbering of $G(\mathbf{Z}_8,\{1,3\})$.

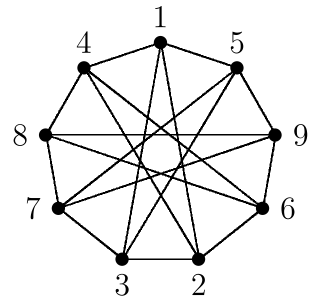
Proposition 2.3.9 $B(G(\mathbf{Z}_{2n+1},\{1,n\})) = 4$.

Proof. Suppose that f is an optimal labeling of $G(\mathbf{Z}_{2n+1},\{1,n\})$. Since $G(\mathbf{Z}_{2n+1},\{1,n\})$ is 4-regular, $\max\{i : f^{-1}(i) \in N(f^{-1}(1))\} \ge 5$, and hence $B(G(\mathbf{Z}_{2n+1},\{1,n\})) \ge 4$. Next, we give a labeling to show $B(G(\mathbf{Z}_{2n+1},\{1,n\})) \le 4$. The labeling $g : V(G(\mathbf{Z}_{2n+1},\{1,n\})) \to [2n+1]$ is defined by

$$g(i) = \begin{cases} 1+4(i-1), & \text{if } 1 \le i \le \lceil \frac{n}{2} \rceil; \\ 3+4(n-i), & \text{if } \lceil \frac{n}{2} \rceil + 1 \le i \le n; \\ 2+4(i-n-1), & \text{if } n+1 \le i \le n + \lceil \frac{n}{2} \rceil; \\ 2n+1, & \text{if } i = n + \lceil \frac{n}{2} \rceil + 1; \\ 4+4(2n+1-i), & \text{if } n + \lceil \frac{n}{2} \rceil + 2 \le i \le 2n+1. \end{cases}$$

It is easy to see that $B(G(\mathbf{Z}_{2n+1},\{1,n\})) \le B_g(G(\mathbf{Z}_{2n+1},\{1,n\})) = 4$. ∎

Figure 2.5 shows a bandwidth numbering of $G(\mathbf{Z}_9,\{1,4\})$.



$$B(G(\mathbf{Z}_9,\{1,4\}))=4$$

Figure 2.5: A bandwidth numbering of $G(\mathbf{Z}_9,\{1,4\})$.

Proposition 2.3.10 $B(G(\mathbf{Z}_{2n+1},\{1,n-1\}))=\begin{cases}4, & \text{if } n=3;\\ 5, & \text{if } n=4;\\ 6, & \text{if } n\geq 5.\end{cases}$

Proof. For the case of $n=3$, because $G(\mathbf{Z}_7,\{1,2\})$ is 4-regular, $B(G(\mathbf{Z}_7,\{1,2\}))\geq 4$. Let $g:V(G(\mathbf{Z}_7,\{1,2\}))\to[7]$ be the labeling defined by

$$(g(1),g(2),g(3),g(4),g(5),g(6),g(7))=(1,2,4,6,7,5,3).$$

Obviously, $B(G(\mathbf{Z}_7,\{1,2\}))\leq B_g(G(\mathbf{Z}_7,\{1,2\}))=4$. For the case of $n=4$, the labeling $g:V(G(\mathbf{Z}_8,\{1,3\}))\to[8]$ defined by

$$(g(1),g(2),g(3),g(4),g(5),g(6),g(7),g(8))=(1,5,2,4,9,7,3,8,6),$$

gives $B(G(\mathbf{Z}_8,\{1,3\}))\leq B_g(G(\mathbf{Z}_8,\{1,3\}))\leq 5$. Suppose f is an optimal labeling such that $B_f(G(\mathbf{Z}_8,\{1,3\}))\leq 4$. If $f^{-1}(2)\notin N(f^{-1}(1))$, then $\left|\partial S_2^f\right|=4$. As $G(\mathbf{Z}_8,\{1,3\})$ is 4-regular, $B_f(G(\mathbf{Z}_8,\{1,3\}))\geq 5$. This forces $f^{-1}(2)\in N(f^{-1}(1))$, thus we can see $\left|\partial S_2^f\right|=6$ and hence $B_f(G(\mathbf{Z}_8,\{1,3\}))\geq 6$, a contradiction. Therefore, $B(G(\mathbf{Z}_8,\{1,3\}))=B_f(G(\mathbf{Z}_8,\{1,3\}))\geq 5$ and so $B(G(\mathbf{Z}_8,\{1,3\}))=5$.

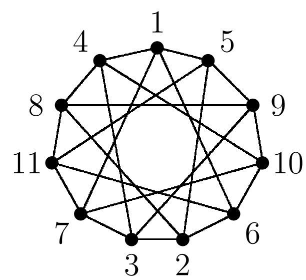
Next, let's settle the case of $n\geq 5$. First, we need to show $B_f(G(\mathbf{Z}_{2n+1},\{1,n-1\}))\geq 6$ for each labeling f. Suppose f is a labeling with $B_f(G(\mathbf{Z}_{2n+1},\{1,n-1\}))\leq 5$. If $f^{-1}(2)\in N(f^{-1}(1))$, then $\left|\partial S_2^f\right|\geq 6$ and hence $B_f(G(\mathbf{Z}_{2n+1},\{1,n-1\}))\geq\left|\partial S_2^f\right|\geq 6$, a contradiction to the assumption. This forces $f^{-1}(2)\notin N(f^{-1}(1))$. Inasmuch as $G(\mathbf{Z}_{2n+1},\{1,n-1\})$ is 4-regular and $|N(f^{-1}(1))\cap N(f^{-1}(2))|\leq 2$, $\left|\partial S_2^f\right|\geq 4+4-2=6$. It also conflicts with the assumption. Next, we gives a labeling to prove

$B(G(\mathbf{Z}_{2n+1},\{1,n-1\})) \le 6$. The labeling $g : V(G(\mathbf{Z}_{2n+1},\{1,n-1\})) \to [2n+1]$ is given by

$$g(i) = \begin{cases} 1+4(i-1), & \text{if } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1; \\ 2+4(n+1-i), & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n+1; \\ 3+4(i-n-2), & \text{if } n+2 \le i \le n + \left\lceil \frac{n}{2} \right\rceil + 1; \\ 4+4(2n+1-i), & \text{if } n + \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le 2n+1. \end{cases}$$

It is easy to check that $B(G(\mathbf{Z}_{2n+1},\{1,n-1\})) \le B_g(G(\mathbf{Z}_{2n+1},\{1,n-1\})) = 6$. ∎

Figure 2.6 shows a bandwidth numbering of $G(\mathbf{Z}_{11},\{1,4\})$.



$B(G(\mathbf{Z}_{11},\{1,4\})) = 6$

Figure 2.6: A bandwidth numbering of $G(\mathbf{Z}_{11},\{1,4\})$.

2.3.2 Bandwidths on the composites of $G(\mathbf{Z}_n, D)$ with others

In the following, let $\varepsilon = \min\left\{ \sum_{1\le i\le k} |u_i| : (u_i)_1^k \in X' \right\}$, where we use precisely the same notation X' as in Subsection 2.3.1. Naturally, ε depends on D except $1 \in D$.

Theorem 2.3.11 *$B(G(\mathbf{Z}_n, D)\Box H) = 2m\lambda$ for $n \ge 4\varepsilon m\lambda^3 + 2(2m^2 - 2(\varepsilon+1)m - \varepsilon)\lambda^2 - 2(2m-\varepsilon-2)\lambda + 2$.*

Proof. Obviously, the labeling g on $G(\mathbf{Z}_n, D)\Box H$ defined by

$$g(v_{i,j}) = \begin{cases} (2i-2)m + j, & \text{if } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil; \\ (2n-2i+1)m + j, & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, \end{cases}$$

gives the upper bound $2m\lambda$. Because $d(G(\mathbf{Z}_n, D)) \le \left\lfloor \frac{\lceil \frac{n}{2} \rceil - 1}{\lambda} \right\rfloor + \varepsilon(\lambda-1)$, $d(H) \le m-1$ and $d(G(\mathbf{Z}_n, D)\Box H) \le d(G(\mathbf{Z}_n, D)) + d(H)$, we have $\left\lceil \frac{nm-1}{d(G(\mathbf{Z}_n,D))+d(H)} \right\rceil \ge \left\lceil \frac{nm-1}{\left\lfloor \frac{\lceil \frac{n}{2} \rceil - 1}{\lambda} \right\rfloor + \varepsilon(\lambda-1) + m - 1} \right\rceil$. By solving the inequality

$$2m\lambda - \frac{nm-1}{\frac{n-1}{2\lambda} + \varepsilon(\lambda-1) + m - 1} < 1,$$

we know that if $n \geq 4\varepsilon m\lambda^3 + 2(2m^2 - 2(\varepsilon+1)m - \varepsilon)\lambda^2 - 2(2m - \varepsilon - 2)\lambda + 2)$, then

$$2m\lambda - \frac{nm-1}{\left\lfloor \frac{\lceil \frac{n}{2} \rceil - 1}{\lambda} \right\rfloor + \varepsilon(\lambda - 1) + m - 1} < 1,$$

and therefore $\left\lceil \frac{nm-1}{\left\lfloor \frac{\lceil \frac{n}{2} \rceil - 1}{\lambda} \right\rfloor + \varepsilon(\lambda-1)+m-1} \right\rceil = 2m\lambda$. From Proposition 2.1.4, we get

$$B(G(\mathbf{Z}_n, D)\Box H) \geq \left\lceil \frac{nm-1}{d(G(\mathbf{Z}_n, D)\Box H)} \right\rceil \geq \left\lceil \frac{nm-1}{d(G(\mathbf{Z}_n, D)) + d(H)} \right\rceil \geq 2m\lambda$$

for $n \geq 4\varepsilon m\lambda^3 + 2(2m^2 - 2(\varepsilon+1)m - \varepsilon)\lambda^2 - 2(2m - \varepsilon - 2)\lambda + 2$. ∎

Figure 2.7 shows a bandwidth numbering of $G(\mathbf{Z}_9, D)\Box H$ with $|V(H)| = 5$ in which the edges are not drawn for simplicity.

	• 6	• 7	• 8	• 9	• 10
	• 16	• 17	• 18	• 19	• 20
	• 26	• 27	• 28	• 29	• 30
	• 36	• 37	• 38	• 39	• 40
↑	• 41	• 42	• 43	• 44	• 45
$G(\mathbf{Z}_9, D)$	• 31	• 32	• 33	• 34	• 35
	• 21	• 22	• 23	• 24	• 25
	• 11	• 12	• 13	• 14	• 15
	• 1	• 2	• 3	• 4	• 5

$H \longrightarrow$

$B(G(\mathbf{Z}_9, D)\Box H) = 10\lambda$

Figure 2.7: A bandwidth numbering of $G(\mathbf{Z}_9, D)\Box H$ with $\max D = \lambda$ and $|V(H)| = 5$.

Remark: If $m = 1$ in Theorem 2.3.11, then we can obtain another proof of Theorem 2.3.2. They have almost the same results. The slight and most important difference between them is the degrees of λ in the lower bounds restriction on n.

Since $\varepsilon = 1$ if $1 \in D$, this immediately implies

Corollary 2.3.12 *If* $1 \in D$, *then* $B(G(\mathbf{Z}_n, D)\Box H) = 2m\lambda$ *for* $n \geq 4m\lambda^3 + 2(2m^2 - 4m - 1)\lambda^2 - 2(2m-3)\lambda + 2$.

Theorem 2.3.13 $B(G(\mathbf{Z}_n, D) \wedge H) = 2(m+1)\lambda$ *for* $n \geq 4\varepsilon(m+1)\lambda^3 + 2(4m - 2m\varepsilon - 3\varepsilon + 4)\lambda^2 + 2(\varepsilon - m - 2)\lambda + 2$.

Proof. Let H_i be the copy of H corresponding to $i \in V(G(\mathbf{Z}_n, D))$. Define the labeling g on $G(\mathbf{Z}_n, D) \wedge H$ by numbering the vertices in $G(\mathbf{Z}_n, D)$ with

$$g(i) = \begin{cases} (2i-2)(m+1)+1, & \text{if } 1 \leq i \leq \lceil \frac{n}{2} \rceil; \\ (2n-2i+1)(m+1)+1, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n, \end{cases}$$

and numbering the vertices j's ($1 \leq j \leq m$), say j_i, in H_i with

$$g(j_i) = \begin{cases} (2i-2)(m+1)+1+j, & \text{if } (i,j) \in \left[\lceil \frac{n}{2} \rceil\right] \times [m]; \\ (2n-2i+1)(m+1)+1+j, & \text{if } (i,j) \in \left([n] \setminus \left[\lceil \frac{n}{2} \rceil\right]\right) \times [m]. \end{cases}$$

Let u and v be two adjacent vertices in $G(\mathbf{Z}_n, D) \wedge H$. Then, trivially u and v are in the same $V(H_i) \cup \{i\}$ or are adjacent in $G(\mathbf{Z}_n, D)$. In the former case, it is easy to see $|g(u) - g(v)| \leq m$. In the latter case, we have that $|g(u) - g(v)| \leq 2(m+1)\lambda$ after checking carefully. These make sure that $2(m+1)\lambda$ is an upper bound of $B(G(\mathbf{Z}_n, D) \wedge H)$. Next, to show that $2(m+1)\lambda$ is also a lower bound. Because $d(G(\mathbf{Z}_n, D)) \leq \left\lfloor \frac{\lceil \frac{n}{2} \rceil - 1}{\lambda} \right\rfloor + \varepsilon(\lambda - 1)$ and $d(G(\mathbf{Z}_n, D) \wedge H) \leq d(G(\mathbf{Z}_n, D)) + 2$, we obtain $\left\lceil \frac{n(m+1)-1}{d(G(\mathbf{Z}_n,D))+2} \right\rceil \geq \left\lceil \frac{n(m+1)-1}{\left\lfloor \frac{\lceil \frac{n}{2} \rceil - 1}{\lambda} \right\rfloor + \varepsilon(\lambda-1)+2} \right\rceil$. By solving the inequality

$$2(m+1)\lambda - \frac{n(m+1)-1}{\frac{n-1}{2\lambda} + \varepsilon(\lambda - 1) + 2} < 1,$$

we know that if $n \geq 4\varepsilon(m+1)\lambda^3 + 2(4m - 2m\varepsilon - 3\varepsilon + 4)\lambda^2 + 2(\varepsilon - m - 2)\lambda + 2$, then

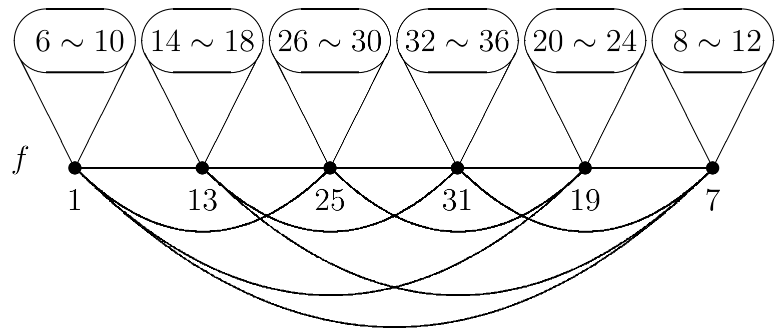
$$2(m+1)\lambda - \frac{n(m+1)-1}{\left\lfloor \frac{\lceil \frac{n}{2} \rceil - 1}{\lambda} \right\rfloor + \varepsilon(\lambda - 1) + 2} < 1,$$

and therefore $\left\lceil \frac{n(m+1)-1}{\left\lfloor \frac{\lceil \frac{n}{2} \rceil - 1}{\lambda} \right\rfloor + \varepsilon(\lambda-1)+2} \right\rceil = 2(m+1)\lambda$. From Proposition 2.1.4, we get

$$B(G(\mathbf{Z}_n, D) \wedge H) \geq \left\lceil \frac{n(m+1)-1}{d(G(\mathbf{Z}_n, D) \wedge H)} \right\rceil \geq \left\lceil \frac{n(m+1)-1}{d(G(\mathbf{Z}_n, D)) + 2} \right\rceil \geq 2(m+1)\lambda$$

for $n \geq 4\varepsilon(m+1)\lambda^3 + 2(4m - 2m\varepsilon - 3\varepsilon + 4)\lambda^2 + 2(\varepsilon - m - 2)\lambda + 2$. ∎

Figure 2.8 shows a bandwidth numbering of $G(\mathbf{Z}_6, \{1,2\}) \wedge H$ with $|V(H)| = 5$ in which the edges are not drawn for completely.



$B(G(\mathbf{Z}_6, \{1,2\}) \wedge H) = 24$

Figure 2.8: A bandwidth numbering of $G(\mathbf{Z}_6, \{1,2\}) \wedge H$ with $|V(H)| = 5$.

With the same reason of Corollary 2.3.12, we acquire

Corollary 2.3.14 *If* $1 \in D$, *then* $B(G(\mathbf{Z}_n, D) \wedge H) = 2(m+1)\lambda$ *for* $n \geq 4(m+1)\lambda^3 + 2(2m+1)\lambda^2 - 2(m+1)\lambda + 2$.

2.4 Bandwidths on $G(\mathbf{N}, D)$ and the composites with others

In this section, we use almost the same idea of the previous sections to establish the bandwidths of $G(\mathbf{N}, D)$.

2.4.1 Bandwidths on $G(\mathbf{N}, D)$, $G(\mathbf{N}, D)\Box H$, $G(\mathbf{N}, D)[H]$, and $G(\mathbf{N}, D) \wedge H$

Theorem 2.4.1 $B(G(\mathbf{N}, D)) = \lambda$.

Proof. First, consider the identity numbering i_d from $V(G(\mathbf{N}, D))$ to $\mathbf{N}$. Then we have $B(G(\mathbf{N}, D)) \leq B_{i_d}(G(\mathbf{N}, D)) = \lambda$. Next, we show that λ is the upper bound. Let f be a bandwidth numbering on $G(\mathbf{N}, D)$, and let $t = \max\{f(i) : 1 \leq i \leq \lambda\}$. Since t is finite, for $i \in [\lambda]$, there is $i + k_i\lambda = \min\{i + k\lambda : f(i + k\lambda) \geq t + 1, k \in \mathbf{N}\}$. It means that $i + k_i\lambda \in \partial S_t^f$ for $i \in [\lambda]$. Then by Proposition 2.1.3, we get $B(G(\mathbf{N}, D)) \geq \lambda$, as desired.∎

Theorem 2.4.2 $B(G(\mathbf{N}, D)\square H) = m\lambda$.

Proof. Consider the numbering g from $V(G(\mathbf{N}, D)\square H)$ to $\mathbf{N}$ defined by $g(v_{i,j}) = (i-1)m + j$. Obviously, $B(G(\mathbf{N}, D)\square H) \leq B_g(G(\mathbf{N}, D)\square H) = m\lambda$. Next, we need to show that $B(G(\mathbf{N}, D)\square H) \geq m\lambda$. Let f be a bandwidth numbering on $G(\mathbf{N}, D)\square H$, and let $t = \max\{f(v_{i,j}) : 1 \leq i \leq \lambda, 1 \leq j \leq m\}$. Since t is finite, for each $(i, j) \in [\lambda] \times [m]$, there is $i + k_{i,j}\lambda = \min\{i + k\lambda : f(v_{i+k\lambda,j}) \geq t + 1, k \in \mathbf{N}\}$. It means that $f(v_{i+(k_{i,j}-1)\lambda,j}) \leq t$, and so $v_{i+k_{i,j}\lambda,j} \in \partial S_t^f$ for each $(i, j) \in [\lambda] \times [m]$. For each j, let $i + k_{i,j}\lambda = i' + k_{i',j}\lambda$, where i, $i' \in [\lambda]$. As $0 \leq i - i' = (k_{i',j} - k_{i,j})\lambda < \lambda$, it forces $i = i'$ and $k_{i,j} = k_{i',j}$. And hence $\left|\partial S_t^f\right| \geq m\lambda$. By Proposition 2.1.3, we have $B(G(\mathbf{N}, D)\square H) \geq \left|\partial S_t^f\right| \geq m\lambda$. ∎

Theorem 2.4.3 $B(G(\mathbf{N}, D)[H]) = m\lambda + m - 1$.

Proof. Consider the numbering g from $V(G(\mathbf{N}, D)[H])$ to $\mathbf{N}$ defined by $g(v_{i,j}) = (i-1)m+j$. Then $B(G(\mathbf{N}, D)[H]) \leq B_g(G(\mathbf{N}, D)[H]) = m(\lambda+1)-1$. Next, we have to show that $B(G(\mathbf{N}, D)[H]) \geq m\lambda + m - 1$. Let f be a bandwidth numbering on $G(\mathbf{N}, D)[H]$, and let

$$
\begin{aligned}
t &= \max\{f(v_{i,j}) : 1 \leq i \leq \lambda, 1 \leq j \leq m\} \cup \{f(v_{\lambda+1,1})\}, \\
t_i &= \min\{f(v_{i,j}) : 1 \leq j \leq m\} \text{ for } i \in \mathbf{N}.
\end{aligned}
$$

Define $r(v_{i,j}) = i$ and let $\vartheta = \max\{r(f^{-1}(x)) : 1 \leq x \leq t\}$. Since t is finite, $\min\{t_i : t_i \geq t, i \geq \vartheta\}$ exists, say t' or $f(v_{i_0,j_0})$. Also, because t' is finite, for each $(i, j) \in [\lambda] \times [m]$, there is $i + k_{i,j}\lambda = \min\{i + k\lambda : f(v_{i+k\lambda,j}) \geq t' + 1, k \in \mathbf{N}\}$. Evidently, $i + k_{i,j}\lambda \leq i_0 + \lambda - 1$ for each $(i, j) \in [\lambda] \times [m]$ but $j = j_0$. Let $L = \{v_{i+k_{i,j}\lambda,j} : 1 \leq i \leq \lambda, 1 \leq j \leq m\}$, by the same argument as in Theorem 2.4.2, it is known that L has $m\lambda$ vertices in $\partial S_{t'}^f$. Besides, for $j \neq j_0$, $v_{i_0+\lambda,j} \in \partial S_{t'}^f \setminus L$ by the definition of t'. So $L \cup \{v_{i_0+\lambda,j} : j \neq j_0\} \subseteq \partial S_{t'}^f$, and hence $\left|\partial S_{t'}^f\right| \geq m\lambda + (m-1)$. ∎

Theorem 2.4.4 $B(G(\mathbf{N}, D) \wedge H) = (m+1)\lambda$.

Proof. Let $G' = G([n], D) \wedge H$. Since $D(G') \leq \lfloor \frac{n-1}{\lambda} \rfloor + \varepsilon(\lambda - 1) + 2$, from the Proposition 2.1.4, we have

$$\begin{aligned} B(G(\mathbf{N}, D) \wedge H) &\geq \frac{n(m+1)-1}{\lfloor \frac{n-1}{\lambda} \rfloor + \varepsilon(\lambda-1)+2} \\ &\geq \frac{n(m+1)-1}{\frac{n-1}{\lambda} + \varepsilon(\lambda-1)+2} \\ &= \frac{\lambda n(m+1)-\lambda}{n-1+\varepsilon\lambda(\lambda-1)+2\lambda} \\ &= \frac{(m+1)\lambda - \frac{\lambda}{n}}{1 + \frac{\varepsilon\lambda(\lambda-1)+2\lambda-1}{n}}. \end{aligned}$$

Take limitation on such n that $G([n], D) \wedge H$ is a connected subgraph of $G(\mathbf{N}, D) \wedge H$, and by Proposition 2.1.1, there is no doubt that $B(G(\mathbf{N}, D) \wedge H) \geq (m+1)\lambda$. Next, to show $(m+1)\lambda$ is an upper bound of $B(G(\mathbf{N}, D) \wedge H)$, we consider a numbering g of $G(\mathbf{N}, D) \wedge H$ by

$$\begin{cases} g(i) = (i-1)m + i, & \text{for } i \in \mathbf{N}; \\ (i-1)m + i + 1 \leq g(v) \leq (i-1)m + i + m, & \text{for } v \text{ is in the copy of } H \\ & \text{corresponding to } i. \end{cases}$$

Now if two vertices x and y are in the same component $\{i\} \vee H_i$, then we have $|g(x) - g(y)| \leq m$. The only other vertices adjacent in $G(\mathbf{N}, D) \wedge H$ are those which are adjacent in $G(\mathbf{N}, D)$. Assume i is adjacent to j in $G(\mathbf{N}, D)$. Then $|g(i) - g(j)| = |(i-j)(m+1)| \leq (m+1)\lambda$. These give $B(G(\mathbf{N}, D) \wedge H) \leq B_g(G(\mathbf{N}, D) \wedge H) = (m+1)\lambda$. (In fact, it is easy to prove that $B(G \wedge H) \leq B(G)|V(H)|$ for arbitrary graphs G and H.) ∎

2.4.2 Bandwidths on $G(\mathbf{N}, D) \times H$ and $G(\mathbf{N}, D) \boxtimes H$

Define two parameters as

$$\begin{aligned} &\underline{B_p}(H; k) = \min_{|A|=k} \left\{ \left| \bigcup_{v \in A} N(v) \right| - k : A \subseteq V(H) \right\}, \\ &\underline{B_p}(H) = \max_k \underline{B_p}(H; k). \end{aligned}$$

We may use them to express a lower bound of $B(G(\mathbf{N}, D) \times H)$ as follows.

Proposition 2.4.5 *Let H be a Hamiltonian graph or have a perfect matching, and let f be a bandwidth numbering on $G(\mathbf{N}, D) \times H$. Then there is a $t \in N$ such that $\left|\partial S_t^f\right| \geq m\lambda + \underline{B_p}(H)$, and therefore $B(G(\mathbf{N}, D) \times H) \geq m\lambda + \underline{B_p}(H)$.*

Proof.

Case 1. H is a Hamiltonian graph of order m with a spanning cycle $y_1y_2\cdots y_{m-1}y_my_1$. Let $A' = \{y_{j_s} : 1 \le s \le \ell\} \subseteq V(H)$ such that

$$\left|\bigcup_{v\in A'} N(v)\right| - \ell = \max_k \left[\min_{|A|=k}\left\{\left|\bigcup_{v\in A} N(v)\right| - k : A \subseteq V(H)\right\}\right].$$

And let

$$\begin{aligned}
\mu &= \max\{f(v_{i,j}) : 1 \le i \le \lambda, 1 \le j \le m\},\\
c(v_{i,j}) &= j,\\
A_{\rho,h} &= \left\{y_j \in V(H) : j \in \bigcup_{i\ge h} c(f^{-1}([\rho]) \cap R_i)\right\} \text{ for } \rho \ge \mu \text{ and } h \ge \lambda+1,\\
t &= \min\{\rho : |A_{\rho,h}| = \ell\},\\
h_t &= \min\{h : |A_{t,h}| = \ell\},\\
W &= \left\{v_{i(j),j} : j \in \bigcup_{i\ge h_t} c(f^{-1}([t]) \cap R_i), i(j) = \max r(f^{-1}([t]) \cap C_j)\right\}.
\end{aligned}$$

For $1 \le i \le \lambda$, $1 \le j \le m-1$, let

$$\begin{aligned}
i + 2a_{i,j}\lambda &= \min\{i + 2\theta\lambda : f(v_{i+2\theta\lambda,j}) \ge t+1, \theta \in N\},\\
i + (2b_{i,j}-1)\lambda &= \min\{i + (2\theta-1)\lambda : f(v_{i+(2\theta-1)\lambda,j+1}) \ge t+1, \theta \in N\},\\
\widetilde{v_{i,j}} &= \begin{cases} v_{i+2a_{i,j}\lambda,j}, & \text{for } 2a_{i,j} < 2b_{i,j}-1;\\ v_{i+(2b_{i,j}-1)\lambda,j+1}, & \text{for } 2a_{i,j} > 2b_{i,j}-1.\end{cases}
\end{aligned}$$

For $1 \le i \le \lambda$, $j = m$, let

$$\begin{aligned}
i + 2a_{i,m}\lambda &= \min\{i + 2\theta\lambda : f(v_{i+2\theta\lambda,m}) \ge t+1, \theta \in N\},\\
i + (2b_{i,m}-1)\lambda &= \min\{i + (2\theta-1)\lambda : f(v_{i+(2\theta-1)\lambda,1}) \ge t+1, \theta \in N\},\\
\widetilde{v_{i,m}} &= \begin{cases} v_{i+2a_{i,m}\lambda,m}, & \text{for } 2a_{i,m} < 2b_{i,m}-1;\\ v_{i+(2b_{i,m}-1)\lambda,1}, & \text{for } 2a_{i,m} > 2b_{i,m}-1.\end{cases}
\end{aligned}$$

And let $T = \{\widetilde{v_{i,j}} : 1 \le i \le \lambda, 1 \le j \le m\}$. Claim $|T| = m\lambda$. Suppose $\widetilde{v_{i,j}} = \widetilde{v_{k,l}}$.

(1) If $j, l \ne m$, then

$$\begin{cases} i + 2a_{i,j}\lambda = k + 2a_{k,l}\lambda\\ j = l\end{cases}, \quad\text{or}\quad \begin{cases} i + 2a_{i,j}\lambda = k + (2b_{k,l}-1)\lambda\\ j = l+1\end{cases}, \quad\text{or}$$

$$\begin{cases} i + (2b_{i,j}-1)\lambda = k + 2b_{k,l}\lambda\\ j+1 = l\end{cases}, \quad\text{or}\quad \begin{cases} i + (2b_{i,j}-1)\lambda = k + (2b_{k,l}-1)\lambda\\ j+1 = l+1\end{cases}.$$

Either of them forces inconsistencies or $i = k$, $j = l$.

(2) If $j = m$ or $l = m$ (by symmetry, we may assume $j = m$), then

$$\begin{cases} i + 2a_{i,m}\lambda = k + 2a_{k,l}\lambda \\ j = m = l \end{cases}, \quad \text{or} \begin{cases} i + 2a_{i,m}\lambda = k + (2b_{k,l} - 1)\lambda \\ j = m = l + 1 \end{cases}, \quad \text{or}$$

$$\begin{cases} i + (2b_{i,m} - 1)\lambda = k + 2a_{k,l}\lambda \\ 1 = l \end{cases}, \quad \text{or} \begin{cases} i + (2b_{i,m} - 1)\lambda = k + (2b_{k,l} - 1)\lambda \\ 1 = l + 1 \end{cases}.$$

Either of them also forces inconsistencies or $i = k$, $j = m = l$. In short, all $\widetilde{v_{i,j}}$'s in T are distinct, so $|T| = m\lambda$.

Additionally, let $T' = \left\{v_{i(j)+\lambda,n(j)} : v_{i(j),j} \in W, y_{n(j)} \in N_H(y_j)\right\}$. Trivially, for each $v_{i(j),j} \in W$, there is at most a $y_{n(j)} \in N_H(y_j)$ such that $v_{i(j)+\lambda,n(j)} \in T$ from the definition of T. And thus $|T \cap T'| \leq \ell$. Since $\partial S_t^f \supseteq T \cup T'$,

$$\begin{aligned} \left|\partial S_t^f\right| &\geq |T \cup T'| \\ &= |T| + |T'| - |T \cap T'| \\ &\geq m\lambda + \left|\bigcup_{v \in A_{t,h_t}} N(v)\right| - \ell \\ &\geq m\lambda + \left|\bigcup_{v \in A'} N(v)\right| - \ell \\ &= m\lambda + \max_k \left[\min_{|A|=k} \left\{\left|\bigcup_{v \in A} N(v)\right| - k : A \subseteq V(H)\right\}\right] \\ &= m\lambda + \underline{B_p}(H). \end{aligned}$$

Case 2. $M = \{y_{2j-1}y_{2j} : 1 \leq j \leq \frac{m}{2}\}$ is a perfect matching in H.

Let $A' = \{y_{j_s} : 1 \leq s \leq \ell\} \subseteq V(H)$ such that

$$\left|\bigcup_{v \in A'} N(v)\right| - \ell = \max_k \left[\min_{|A|=k} \left\{\left|\bigcup_{v \in A} N(v)\right| - k : A \subseteq V(H)\right\}\right].$$

And let

$$\begin{aligned}
\mu &= \max\{f(v_{i,j}) : 1 \le i \le \lambda, 1 \le j \le m\}, \\
c(v_{i,j}) &= j, \\
A_{\rho,h} &= \left\{ y_j \in V(H) : j \in \bigcup_{i \ge h} c(f^{-1}([\rho]) \cap R_i) \right\} \text{ for } \rho \ge \mu \text{ and } h \ge \lambda + 1, \\
t &= \min \{\rho : |A_{\rho,h}| = \ell\}, \\
h_t &= \min \{h : |A_{t,h}| = \ell\}, \\
W &= \left\{ v_{i(j),j} : j \in \bigcup_{i \ge h_t} c(f^{-1}([t]) \cap R_i), i(j) = \max r(f^{-1}([t]) \cap C_j) \right\}.
\end{aligned}$$

For $1 \le i \le \lambda$, $1 \le j \le m$, let

$$\begin{aligned}
i + 2a_{i,j}\lambda &= \min\{i + 2\theta\lambda : f(v_{i+2\theta\lambda,j}) \ge t+1, \theta \in N\}, \\
i + (2b_{i,j} - 1)\lambda &= \min\{i + (2\theta - 1)\lambda : f(v_{i+(2\theta-1)\lambda, j+(-1)^{j+1}}) \ge t+1, \theta \in N\}, \\
\widetilde{v_{i,j}} &= \begin{cases} v_{i+2a_{i,j}\lambda, j}, & \text{for } 2a_{i,j} < 2b_{i,j} - 1; \\ v_{i+(2b_{i,j}-1)\lambda, j+(-1)^{j+1}}, & \text{for } 2a_{i,j} > 2b_{i,j} - 1. \end{cases}
\end{aligned}$$

And let

$$\begin{aligned}
T &= \{\widetilde{v_{i,j}} : 1 \le i \le \lambda, 1 \le j \le m\}, \\
T' &= \left\{ v_{i(j)+\lambda, n(j)} : v_{i(j),j} \in W, y_{n(j)} \in N_H(y_j) \right\}.
\end{aligned}$$

With the similar argument of Case 1, we also get

$$\left|\partial S_t^f\right| \ge m\lambda + \max_k \left[\min_{|A|=k} \left\{ \left| \bigcup_{v \in A} N(v) \right| - k : A \subseteq V(H) \right\} \right] = m\lambda + \underline{B_p}(H).$$

∎

In more general, we have the following consequences by a careful application of Proposition 2.4.5.

Theorem 2.4.6 *If a graph H has a spanning subgraph which consists of a disjoint union of cycles or a matching, then $B(G(\mathbf{N}, D) \times H) \ge m\lambda + \underline{B_p}(H)$.*

We next give a weaker lower bound of $B(G(\mathbf{N}, D) \times H)$ which is easy to obtain from Theorem 2.4.6.

Corollary 2.4.7 *If a graph H has a spanning subgraph which consists of a disjoint union of cycles or a matching, then* $B(G(\mathbf{N}, D) \times H) \geq m\lambda + \delta(H) - 1$.

Proof. Taking $|A| = 1$ in Theorem 2.4.6, we then have this corollary. ∎

Lemma 2.4.8 $B(G(\mathbf{N}, D) \times H) \leq m\lambda + B(H)$ *for any finite graph H.*

Proof. Let f be a bandwidth numbering of H. Consider the numbering $g : V(G(\mathbf{N}, D) \times H) \to \mathbf{N}$ defined by $g(v_{i,j}) = (i-1)m + f(y_j)$. It is clear that $B(G(\mathbf{N}, D) \times H) \leq B_g(G(\mathbf{N}, D) \times H) = m\lambda + B(H)$. ∎

We give exact values of bandwidth for some $G(\mathbf{N}, D) \times H$'s below.

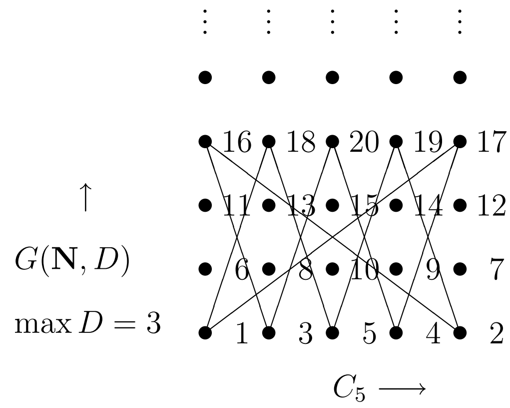
Example 2.4.9

(1)$B(G(\mathbf{N}, D) \times P_m) = m\lambda + 1$ *for* $m \in 2\mathbf{N} \setminus \{2\}$.

(2)$B(G(\mathbf{N}, D) \times C_m) = m\lambda + 2$ *for* $m \geq 4$.

Proof. It is trivial to get their upper bounds from Lemma 2.4.8. We know their lower bounds by taking $|A| = m - 1$ for (1) and $|A| = m - 2$ for (2) in Theorem 2.4.6. Thus the results hold. ∎

Figure 2.9 shows a bandwidth numbering of $G(\mathbf{N}, D) \times C_5$ with $\max D = 3$ in which the edges are not drawn completely.



$$\begin{aligned} B(G(\mathbf{N},D)\times C_5) &= 5\lambda+2 \\ &= 5\lambda + B(C_5) \end{aligned}$$

Figure 2.9: A bandwidth numbering of $G(\mathbf{N},D)\times C_5$ with $\max D = 3$.

With regard to $G(\mathbf{N},D)\times P_2$ and $G(\mathbf{N},D)\times C_3$, we have

$$\begin{cases} B(G(\mathbf{N},D)\times P_2) = 2\lambda \\ B(G(\mathbf{N},D)\times C_3) = 3\lambda+1 \end{cases}$$

by Example 2.4.12. In fact, $B(G(\mathbf{N},D)\times P_m) = m\lambda + 1$ for $m \in \mathbf{N}$.

For a graph H obtained from join, we give another substitutional bounds.

Theorem 2.4.10 *Let H_r be a graph of order m_r for $r \in [t]$ and $H = \bigvee_{1\le r\le t} H_r$ of order $m = \sum_{1\le r\le t} m_r$. If H has a spanning subgraph which consists of a disjoint union of cycles or a matching, then*

$$\max_k \left[\min_{1\le r\le t} \left(m - m_r + \underline{B_p}(H_r;k) \right) \right] \le B(G(\mathbf{N},D)\times H) - m\lambda \le \max_{1\le r\le t} (m - m_r + B(H_r)).$$

Proof. By Theorem 2.4.6, we know

$$B(G(\mathbf{N},D)\times H) \ge m\lambda + \max_k \left[\min_{|A|=k} \left\{ \left| \bigcup_{v\in A} N(v) \right| - k : A \subseteq V(H) \right\} \right].$$

If A with $|A| = k$ is contained in some $V(H_r)$, then

$$\left| \bigcup_{v\in A} N(v) \right| - k = m - m_r + \left| \bigcup_{v\in A} N_{H_r}(v) \right| - k \le m - k.$$

If not, then $\left|\underset{v\in A}{\cup} N(v)\right| - k = m - k$. This implies

$$\begin{aligned} & B(G(\mathbf{N}, D) \times H) \\ \geq \;& m\lambda + \max_k \left[\min_{|A|=k} \left\{\left|\underset{v\in A}{\cup} N(v)\right| - k : A \subseteq V(H)\right\}\right] \\ = \;& m\lambda + \max_k \left[\min_{1\leq r\leq t} \left(\min_{|A|=k} \left\{m - m_r + \left|\underset{v\in A}{\cup} N_{H_r}(v)\right| - k : A \subseteq V(H_r)\right\}\right)\right] \\ = \;& m\lambda + \max_k \left[\min_{1\leq r\leq t} \left(m - m_r + \underline{B_p}(H_r; k)\right)\right]. \end{aligned}$$

Next, we need to show that $m\lambda + \max\limits_{1\leq r\leq t}(m - m_r + B(H_r))$ is an upper bound of $B(G(\mathbf{N}, D) \times H)$. Let $V(H_1) = \{y_j : 1 \leq j \leq m_1\}$, $V(H_r) = \{y_j : \sum\limits_{1\leq s\leq r-1} m_s + 1 \leq j \leq \sum\limits_{1\leq s\leq r} m_s\}$ for $2 \leq r \leq k$, and f_r be the bandwidth numbering of H_r for $1 \leq r \leq t$. In addition, define $f(y_j) = f_r(y_j)$ for $\sum\limits_{1\leq s\leq r-1} m_s + 1 \leq j \leq \sum\limits_{1\leq s\leq r} m_s$. Suppose $i = \lambda a_i + b_i$ for each $i \in \mathbf{N}$, where $b_i \in [\lambda]$. Consider a numbering g from $V(G(\mathbf{N}, D) \times H)$ to $\mathbf{N}$ defined by

$$(i-1)m + 1 \leq g(v_{i,j}) \leq im, \text{ and } g(v_{i,j}) \equiv f(y_j) - \sum_{1\leq s\leq a_i} m_s \pmod m.$$

It is not hard to check that $B(G(\mathbf{N}, D) \times H) \leq B_g(G(\mathbf{N}, D) \times H) = m\lambda + \max\limits_{1\leq r\leq t}(m - m_r + B(H_r))$. ∎

Corollary 2.4.11 *If H_r is a graph of order m for all $r \in [t]$ and $H = \underset{1\leq r\leq t}{\vee} H_r$, then*

$$\max_k \left[\min_{1\leq r\leq t} \underline{B_p}(H_r; k)\right] \leq B(G(\mathbf{N}, D) \times H) - [tm\lambda + (t-1)m] \leq \max_{1\leq r\leq t} B(H_r).$$

Proof. Since H can be spanned by a disjoint union of some cycles and a matching, the corollary follows from Theorem 2.4.10. ∎

Example 2.4.12

(1)$B(G(\mathbf{N}, D) \times (\underset{1\leq r\leq t}{\vee} P_m)) = tm\lambda + (t-1)m + 1$ *for* $m \geq 3$.

(2)$B(G(\mathbf{N}, D) \times (\underset{1\leq r\leq t}{\vee} C_m)) = tm\lambda + (t-1)m + 2$ *for* $m \geq 4$.

Proof. Since $\max\limits_k \underline{B_p}(P_m; k) = 1 = B(P_m)$ for $m \geq 3$ and $\max\limits_k \underline{B_p}(C_m; k) = 2 = B(C_m)$ for $m \geq 4$, we have those consequences from the above corollary. ∎

Corollary 2.4.13 *Let H' be a graph of order $m' \leq m + \delta(H')$. If $H = \overline{K_m} \vee H'$ can be spanned by a disjoint union of some cycles and a matching, then $B(G(\mathbf{N}, D) \times H) = (m + m')\lambda + m' - 1$.*

Proof. By Theorem 2.4.10 and $m' \leq m + \delta(H')$, we know

$$\begin{aligned} B(G(\mathbf{N}, D) \times H) &\geq (m + m')\lambda + \max_k \left[\min_{1 \leq r \leq t} \left(m - m_r + \underline{B_p}(H_r; k) \right) \right] \\ &\geq (m + m')\lambda + \min \left\{ m' + \underline{B_p}(\overline{K_m}; 1), m + \underline{B_p}(H'; 1) \right\} \\ &= (m + m')\lambda + \min \{ m' + (-1), m + (\delta(H') - 1) \} \\ &= (m + m')\lambda + m' - 1. \end{aligned}$$

Next, we need to show that $(m + m')\lambda + m' - 1$ is an upper bound of $B(G(\mathbf{N}, D) \times H)$. Let $V(\overline{K_m}) = \{y_j : 1 \leq j \leq m\}$ and $V(H') = \{y_j : m + 1 \leq j \leq m + m'\}$. Consider a numbering g from $V(G(\mathbf{N}, D) \times H)$ to $\mathbf{N}$ defined by

$$g(v_{i,j}) = (i - 1)(m + m') + j \quad \text{for } (2k - 2)\lambda + 1 \leq i \leq (2k - 1)\lambda,$$

and for $(2k - 1)\lambda + 1 \leq i \leq 2k\lambda$

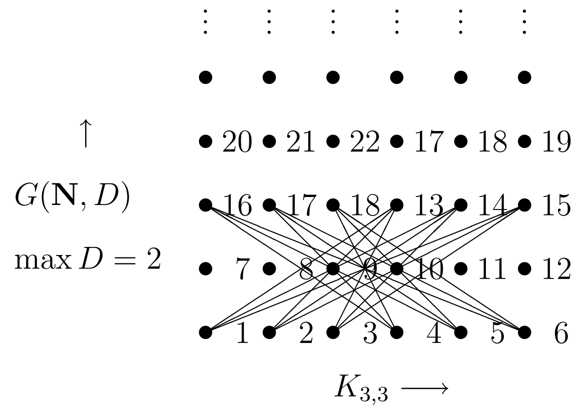
$$g(v_{i,j}) = \begin{cases} (i - 1)(m + m') + j + m', & \text{for } 1 \leq j \leq m; \\ (i - 1)(m + m') + j - m, & \text{for } m + 1 \leq j \leq m + m', \end{cases}$$

where $k \in \mathbf{N}$. It is not hard to check that $B(G(\mathbf{N}, D) \times H) \leq B_g(G(\mathbf{N}, D) \times H) = (m + m')\lambda + m' - 1$. ∎

Example 2.4.14 $B(G(\mathbf{N}, D) \times K_{t(m)}) = tm\lambda + (t - 1)m - 1$ *for* $t \geq 2$.

Proof. We obtain the result immediately by Corollary 2.4.13. Notice that $K_{m(1)}$ means K_m. This implies $B(G(\mathbf{N}, D) \times K_m) = m\lambda + m - 2$. ∎

Figure 2.10 shows a bandwidth of $G(\mathbf{N}, D) \times K_{3,3}$ with $\max D = 2$ in which the edges are not drawn completely.



$$
\begin{aligned}
B(G(\mathbf{N},D)\times K_{3,3}) &= 6\lambda+2\\
&< 6\lambda + B(K_{3,3})
\end{aligned}
$$

Figure 2.10: A bandwidth numbering of $G(\mathbf{N},D)\times K_{3,3}$ with $\max D = 2$.

In the following, we imitate the process of argument on $B(G(\mathbf{N},D)\times H)$ to give similar results of $B(G(\mathbf{N},D)\boxtimes H)$. Also, first of all, we define two parameters as

$$
\begin{aligned}
\underline{B_s}(H;k) &= \min_{|A|=k}\left\{\left|\bigcup_{v\in A} N[v]\right| - k : A\subseteq V(H)\right\},\\
\underline{B_s}(H) &= \max_k \underline{B_s}(H;k).
\end{aligned}
$$

And we have

Theorem 2.4.15 $\underline{B_s}(H) \le B(G(\mathbf{N},D)\boxtimes H) - m\lambda \le B(H)$.

Proof. The upper bound can be easily derived by the same argument as in the proof of Lemma 2.4.8. As to the lower bound, we discuss it in detail below. Suppose f is a bandwidth numbering on $G(\mathbf{N},D)\boxtimes H$. Let $A' = \{y_{j_s} : 1\le s\le \ell\}\subseteq V(H)$ such that

$$
\left|\bigcup_{v\in A'} N[v]\right| - \ell = \max_k \left[\min_{|A|=k}\left\{\left|\bigcup_{v\in A} N[v]\right| - k : A\subseteq V(H)\right\}\right].
$$

And let

$$\begin{aligned}
\mu &= \max\{f(v_{i,j}) : 1 \le i \le \lambda, 1 \le j \le m\},\\
c(v_{i,j}) &= j,\\
A_{\rho,h} &= \left\{ y_j \in V(H) : j \in \bigcup_{i\ge h} c(f^{-1}([\rho]) \cap R_i) \right\} \text{ for } \rho \ge \mu \text{ and } h \ge \lambda+1,\\
t &= \min\left\{\rho : |A_{\rho,h}| = \ell\right\},\\
h_t &= \min\left\{h : |A_{t,h}| = \ell\right\},\\
W &= \left\{ v_{i(j),j} : j \in \bigcup_{i \ge h_t} c(f^{-1}([t]) \cap R_i), i(j) = \max r(f^{-1}([t]) \cap C_j) \right\}.
\end{aligned}$$

Since t is finite, for each $(i,j) \in [\lambda] \times [m]$, there is $i + k_{i,j}\lambda = \min\{i + k\lambda : f(v_{i+k\lambda,j}) \ge t+1, k \in \mathbf{N}\}$. For $(i,j) \in [\lambda] \times [m]$, let $\widetilde{v_{i,j}} = v_{i+k_{i,j}\lambda,j}$ and let $T = \{\widetilde{v_{i,j}} : 1 \le i \le \lambda, 1 \le j \le m\}$. We claim $|T| = m\lambda$. For each j, let $i + k_{i,j}\lambda = i' + k_{i',j}\lambda$, where i, $i' \in [\lambda]$. As $0 \le i - i' = (k_{i',j} - k_{i,j})\lambda < \lambda$, it forces $i = i'$ and $k_{i,j} = k_{i',j}$. And hence $|T| = m\lambda$.

Moreover, let $T' = \left\{ v_{i(j)+\lambda,n(j)} : v_{i(j),j} \in W, y_{n(j)} \in N_H[y_j] \right\}$. Trivially, for each $v_{i(j),j} \in W$, there is at most a $y_{n(j)} \in N_H(y_j)$ such that $v_{i(j)+\lambda,n(j)} \in T$ from the definition of T. And thus $|T \cap T'| \le \ell$. Since $\partial S_t^f \supseteq T \cup T'$,

$$\begin{aligned}
\left|\partial S_t^f\right| &\ge |T \cup T'|\\
&= |T| + |T'| - |T \cap T'|\\
&\ge m\lambda + \left| \bigcup_{v \in A_{t,h_t}} N[v] \right| - \ell\\
&\ge m\lambda + \left| \bigcup_{v \in A'} N[v] \right| - \ell\\
&= m\lambda + \max_k \left[\min_{|A|=k} \left\{ \left| \bigcup_{v \in A} N[v] \right| - k : A \subseteq V(H) \right\} \right]\\
&= m\lambda + \underline{B_s}(H).
\end{aligned}$$

∎

We next also give a weaker lower bound of $B(G(\mathbf{N}, D) \boxtimes H)$ which is easy to obtain from Theorem 2.4.15.

Corollary 2.4.16 *$B(G(\mathbf{N}, D) \boxtimes H) \ge m\lambda + \delta(H)$.*

Proof. Taking $|A| = 1$ in Theorem 2.4.15, we then have the corollary. ∎

We also offer exact values of bandwidth for some $G(\mathbf{N}, D) \boxtimes H$'s in the underside.

Example 2.4.17

(1)$B(G(\mathbf{N}, D) \boxtimes P_m) = m\lambda + 1$.

(2)$B(G(\mathbf{N}, D) \boxtimes C_m) = m\lambda + 2$.

(3)$B(G(\mathbf{N}, D) \boxtimes K_m) = m\lambda + m - 1$.

Proof. The results follow from Theorem 2.4.15. ∎

For a graph H obtained from join, we still give another substitutional bounds.

Theorem 2.4.18 *If H_r is a graph of order m_r for $r \in [t]$ and $H = \bigvee_{1 \le r \le t} H_r$ is a graph of order $m = \sum_{1 \le r \le t} m_r$, then*

$$\max_k \left[\min_{1 \le r \le t} \left(m - m_r + \underline{B_s}(H; k)\right)\right] \le B(G(\mathbf{N}, D) \boxtimes H) - m\lambda \le \max_{1 \le r \le t} (m - m_r + B(H_r)).$$

Proof. By Theorem 2.4.15, we know

$$B(G(\mathbf{N}, D) \boxtimes H) \ge m\lambda + \max_k \left[\min_{|A|=k} \left\{\left|\bigcup_{v \in A} N[v]\right| - k : A \subseteq V(H)\right\}\right].$$

If A with $|A| = k$ is contained in some $V(H_r)$, then

$$\left|\bigcup_{v \in A} N[v]\right| - k = m - m_r + \left|\bigcup_{v \in A} N_{H_r}[v]\right| - k \le m - k.$$

If not, then $\left|\bigcup_{v \in A} N[v]\right| - k = m - k$. This implies

$$\begin{aligned} & B(G(\mathbf{N}, D) \boxtimes H) \\ \ge \;& m\lambda + \max_k \left[\min_{|A|=k} \left\{\left|\bigcup_{v \in A} N[v]\right| - k : A \subseteq V(H)\right\}\right] \\ = \;& m\lambda + \max_k \left[\min_{1 \le r \le t} \left(\min_{|A|=k} \left\{m - m_r + \left|\bigcup_{v \in A} N_{H_r}[v]\right| - k : A \subseteq V(H_r)\right\}\right)\right]. \end{aligned}$$

Next, we need to show that $m\lambda + \max_{1 \le r \le t} (m - m_r + B(H_r))$ is an upper bound of $B(G(\mathbf{N}, D) \boxtimes H)$. Let $V(H_1) = \{y_j : 1 \le j \le m_1\}$, $V(H_r) = \{y_j : \sum_{1 \le s \le r-1} m_s + 1 \le j \le \sum_{1 \le s \le r} m_s\}$ for $2 \le r \le k$, and f_r be the bandwidth numbering of H_r for $1 \le r \le t$. In addition, define

$f(y_j) = f_r(y_j)$ for $\sum_{1\le s\le r-1} m_s + 1 \le j \le \sum_{1\le s\le r} m_s$. Suppose $i = \lambda a_i + b_i$ for each $i \in \mathbf{N}$, where $b_i \in [\lambda]$. Consider a numbering g from $V(G(\mathbf{N}, D) \boxtimes H)$ to $\mathbf{N}$ defined by

$$(i-1)m + 1 \le g(v_{i,j}) \le im, \text{ and } g(v_{i,j}) \equiv f(y_j) - \sum_{1\le s\le a_i} m_s \pmod m.$$

It is not hard to check that $B(G(\mathbf{N}, D) \boxtimes H) \le B_g(G(\mathbf{N}, D) \boxtimes H) = m\lambda + \max_{1\le r\le t}(m - m_r + B(H_r))$. ∎

Corollary 2.4.19 *If H_r is a graph of order m for all $r \in [t]$ and $H = \bigvee_{1\le r\le t} H_r$, then*

$$\max_k \left[\min_{1\le r\le t} \underline{B_s}(H_r; k) \right] \le B(G(\mathbf{N}, D) \boxtimes H) - [tm\lambda + (t-1)m] \le \max_{1\le r\le t} B(H_r).$$

Proof. We may get this result directly from Theorem 2.4.18. ∎

Example 2.4.20

(1)$B(G(\mathbf{N}, D) \boxtimes (\bigvee_{1\le r\le t} P_m)) = tm\lambda + (t-1)m + 1$ *for* $m \ge 3$.

(2)$B(G(\mathbf{N}, D) \boxtimes (\bigvee_{1\le r\le t} C_m)) = tm\lambda + (t-1)m + 2$ *for* $m \ge 4$.

Proof. Since $\max_k \underline{B_s}(P_m; k) = 1 = B(P_m)$ for $m \ge 3$ and $\max_k \underline{B_s}(C_m; k) = 2 = B(C_m)$ for $m \ge 4$, we have these consequences from the above corollary. ∎

Corollary 2.4.21 *If H' is a graph of order $m' \le m + \delta(H')$ and $H = \overline{K_m} \vee H'$, then $B(G(\mathbf{N}, D) \boxtimes H) = (m + m')\lambda + m'$.*

Proof. By almost the same argument as in Corollary 2.4.13, we have the result. ∎

Example 2.4.22 *$B(G(\mathbf{N}, D) \boxtimes K_{t(m)}) = tm\lambda + (t-1)m$ for $t \ge 2$.*

Proof. We obtain it immediately by Corollary 2.4.21. Notice that $K_{m(1)}$ means K_m. This implies $B(G(\mathbf{N}; D) \boxtimes K_m) = m\lambda + m - 1$. ∎

Chapter 3

Andante: The Movement of Profile

3.1 Prerequisite for profile

The profile minimization problem arose from the study of sparse matrix technique. It can be defined in terms of graphs as follows. A *proper numbering* of a graph G of n vertices is a 1-1 mapping $f : V(G) \to \{1, 2, \ldots, n\}$. Given a proper numbering f, the *profile width* of a vertex v in G is

$$w_f(v) = \max_{x \in N[v]} (f(v) - f(x)),$$

where $N[v] = \{v\} \cup \{x \in V(G) : xv \in E(G)\}$. The *profile* of a proper numbering f of G is

$$P_f(G) = \sum_{v \in V(G)} w_f(v),$$

and the *profile* of G is

$$P(G) = \min\{P_f(G) : f \text{ is a proper numbering of } G\}.$$

A *profile numbering* of G is a proper numbering f such that $P_f(G) = P(G)$.

Lin and Yuan [49] proved $P(P_n) = n - 1$, $P(C_n) = 2n - 3$, $P(K_{m,n}) = mn + \binom{m}{2}$ for $m \leq n$ and indicated that the profile minimization problem of an arbitrary graph is equivalent to the interval graph completion problem, which was shown to be NP-complete by Garey and Johnson [17]. Kuo and Chang [36] provided a polynomial algorithm to achieve a profile numbering for an arbitrary tree. Aside from special classes of graphs in

[21, 36, 37, 38, 49], some people found the non-algorithmic results for profiles of composite graphs, see [37, 38, 39, 49, 50, 54].

In this chapter, we intend to probe the properties of the profile problem for product and composition of graphs, respectively. Most results of this chapter has been published in [61, 62].

The profile minimization problem is equivalent to the interval graph completion problem described as below. Recall that an *interval graph* is a graph whose vertices correspond to closed intervals in the real line, and two vertices are adjacent if and only if their corresponding intervals intersect. It is well-known that a graph G is an interval graph if and only if there exists an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$ such that $i < j < k$ and $v_i v_k \in E(G)$ imply $v_j v_k \in E(G)$. We call this ordering an *interval ordering* of G. This property can be re-stated as: A graph G of n vertices is an interval graph if and only if there is a proper numbering f such that

$$f(x) < f(y) < f(z) \text{ and } xz \in E(G) \text{ imply } yz \in E(G). \tag{3.1}$$

We call this property the *interval property*, which will be used frequently in this chapter. This property leads to the *perfect elimination property* which is also useful in this chapter:

$$f(x) < f(y) \text{ with } xy \in E(G) \text{ and } f(x) < f(z) \text{ with } xz \in E(G) \text{ imply } yz \in E(G). \tag{3.2}$$

The perfect elimination property in turn implies, in fact equivalent to, the *chordality property* which is also useful in this chapter:

$$\text{Every cycle of length greater than three has at least one chord.} \tag{3.3}$$

Having the interval property (3.1) in mind, it is then easy to see that for any proper numbering f of G, the graph G_f defined by the following is an interval super-graph of G with $|E(G_f)| = P_f(G)$:

$$V(G_f) = V(G) \text{ and } E(G_f) = \{yz : f(x) \leq f(y) < f(z), xz \in E(G)\}.$$

In other words, we have

Proposition 3.1.1 ([49]) *The profile minimization problem is the same as the interval graph completion problem. Namely,*

$$P(G) = \min\{|E(H)| : H \text{ is an interval super-graph of } G\}.$$

3.2 Profile minimization on product of graphs

3.2.1 Profile of $K_m \times K_n$

This subsection establishes the profile of $K_m \times K_n$.

Theorem 3.2.1 *If $m = 1$ or $n \geq \max\{m, 4\}$, then $P(K_m \times K_n) = \frac{1}{2}(m-1)(mn^2 + n^2 - n - 4)$.*

Proof. As the case of $m = 1$ is obvious, we may assume that $m \geq 2$ and $n \geq \max\{m, 4\}$.

First, consider a proper numbering g of $K_m \times K_n$ satisfying

$$g(v_{i,j}) = \begin{cases} j, & \text{for } i = 1 \text{ and } 1 \leq j \leq n-1; \\ mn, & \text{for } i = 1 \text{ and } j = n; \\ i + n - 2, & \text{for } 2 \leq i \leq m \text{ and } j = n, \end{cases}$$

while the other vertices are assigned numbers arbitrarily, see Figure 3.1 for g of $K_5 \times K_9$ in which the edges are not drawn for simplicity.

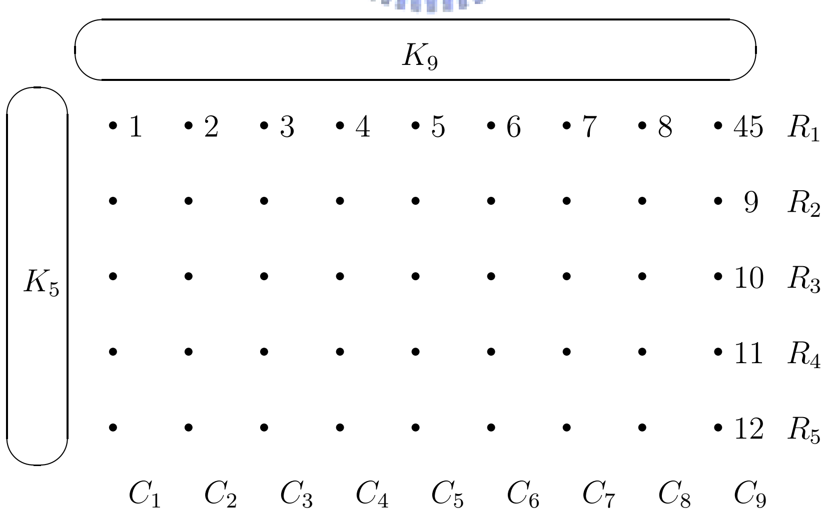


Figure 3.1: A proper numbering g of $K_5 \times K_9$.

The profile width of vertex $v_{i,j}$ is

$$w_g(v_{i,j}) = \begin{cases} 0, & \text{for } i = 1 \text{ and } 1 \leq j \leq n-1; \\ mn - n - m + 1, & \text{for } i = 1 \text{ and } j = n; \\ g(v_{i,j}) - 2, & \text{for } 2 \leq i \leq m \text{ and } j = 1; \\ g(v_{i,j}) - 1, & \text{for } 2 \leq i \leq m \text{ and } 2 \leq j \leq n. \end{cases}$$

Therefore,

$$\begin{aligned} P(K_m \times K_n) &\leq P_g(K_m \times K_n) \\ &= (mn - n - m + 1) + \sum_{k=n}^{mn-1} (k-1) - (m-1) \\ &= \frac{1}{2}(m-1)(mn^2 + n^2 - n - 4). \end{aligned}$$

Next, we shall prove that $P(K_m \times K_n) \geq \frac{1}{2}(m-1)(mn^2+n^2-n-4)$. Choose a profile numbering f of $K_m \times K_n$. Notice that $P(K_m \times K_n) = |E((K_m \times K_n)_f)|$. Without loss of generality, we may assume that $f(v_{1,1}) = 1$. For positive integers a and b, let $e_{a,b} = 2\binom{a}{2}\binom{b}{2} + (a-1)\binom{b}{2} + (b-2)\binom{a}{2} + 2\binom{a-1}{2}$. We consider the following three cases.

Case 1. $f^{-1}(2) \in R_1$, say $f(v_{1,j}) = j$ for $1 \leq j \leq r$ but $f(v_{s,t}) = r+1$ with $s \neq 1$ for some $r \geq 2$.

We shall count the number of edges in $(K_m \times K_n)_f$. Notice that besides the edges in $K_m \times K_n$, extra edges are due to the following cliques in $(K_m \times K_n)_f$ which are independent sets in $K_m \times K_n$.

Each row R_i with $2 \leq i \leq m$ is a clique in $(K_m \times K_n)_f$, since for $v_{i,p}, v_{i,q} \in R_i$ with $f(v_{i,p}) < f(v_{i,q})$, we can choose $k \in \{1,2\} - \{q\}$, such that $f(v_{1,k}) = k < f(v_{i,p}) < f(v_{i,q})$ and $v_{1,k}v_{i,q} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$, which imply $v_{i,p}v_{i,q} \in E((K_m \times K_n)_f)$. Notice that we use the interval property (3.1) in this implication. As the property will be used frequently, we shall not mention it every time.

Each column C_j with $2 \leq j \leq r$ is a clique in $(K_m \times K_n)_f$, since for $v_{p,j}, v_{q,j} \in C_j$ with $f(v_{p,j}) < f(v_{q,j})$, we have $q \geq 2$, and so $f(v_{1,1}) = 1 < f(v_{p,j}) < f(v_{q,j})$ and $v_{1,1}v_{q,j} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$, which imply $v_{p,j}v_{q,j} \in E((K_m \times K_n)_f)$.

For the case $r+1 \leq n$, any column C_j with $j \geq r+1$ but $j \neq t$ is a clique in $(K_m \times K_n)_f$, since for $v_{p,j}, v_{q,j} \in C_j$ with $f(v_{p,j}) < f(v_{q,j})$, we can choose $x = v_{1,1}$ (when $q \neq 1$) or $v_{s,t}$ (when $q = 1$), such that $f(x) < f(v_{p,j}) < f(v_{q,j})$ and $xv_{q,j} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$, which imply $v_{p,j}v_{q,j} \in E((K_m \times K_n)_f)$.

Similarly, $C_j - \{v_{1,j}\}$ is cliques in $(K_m \times K_n)_f$ for $1 \leq j \leq n$. In particular, this is true for $j = 1, t$.

Therefore, totally the graph $(K_m \times K_n)_f$ has at least $e_{m,n} = 2\binom{m}{2}\binom{n}{2} + (m-1)\binom{n}{2} + (n-2)\binom{m}{2} + 2\binom{m-1}{2} = \frac{1}{2}(m-1)(mn^2+n^2-n-4)$ edges, which gives that $P(K_m \times K_n) \geq \frac{1}{2}(m-1)(mn^2+n^2-n-4)$.

Case 2. $f^{-1}(2) \in C_1$.

Since $n \geq m$ and $n+m \geq 5$, we have $e_{n,m} - e_{m,n} = \binom{m}{2} - \binom{n}{2} + 2\binom{n-1}{2} - 2\binom{m-1}{2} = \frac{1}{2}(n+m-5)(n-m) \geq 0$. By an argument similar as Case 1, $P(K_m \times K_n) \geq e_{n,m} \geq e_{m,n} = \frac{1}{2}(m-1)(mn^2+n^2-n-4)$.

Case 3. $f^{-1}(2) \notin R_1 \cup C_1$, say $f(v_{2,2}) = 2$.

By an argument similar as Case 1, $R_1 - \{v_{1,1}, v_{1,2}\}$, $R_2 - \{v_{2,1}\}$, R_i for $3 \leq i \leq m$, $C_1 - \{v_{1,1}, v_{2,1}\}$, $C_2 - \{v_{1,2}\}$, C_j for $3 \leq j \leq n$ are all cliques in $(K_m \times K_n)_f$. Let $f^{-1}(3) = v_{s,t}$. Then, either $v_{s,t} \notin R_1 \cup C_2$ or $v_{s,t} \notin R_2 \cup C_1$. We may assume $v_{s,t} \notin R_1 \cup C_2$. Suppose $3 \leq q \leq n$. For the case $f(v_{1,2}) < f(v_{1,q})$, we have $f(v_{2,2}) = 2 < f(v_{1,2}) < f(v_{1,q})$ and $v_{2,2}v_{1,q} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$ implying $v_{1,2}v_{1,q} \in E((K_m \times K_n)_f)$. For the case $f(v_{1,2}) > f(v_{1,q})$, we have $f(v_{s,t}) = 3 < f(v_{1,q}) < f(v_{1,2})$ and $v_{s,t}v_{1,2} \in E(K_m \times K_n) \subseteq E((K_m \times K_n)_f)$ implying $v_{1,q}v_{1,2} \in E((K_m \times K_n)_f)$. So, in any case, $v_{1,2}v_{1,q} \in E((K_m \times K_n)_f)$. Similarly, $v_{1,2}v_{p,2} \in E((K_m \times K_n)_f)$ for $3 \leq p \leq m$. There are totally $n+m-4$ such edges. So $(K_m \times K_n)_f$ has at least $2\binom{m}{2}\binom{n}{2} + \binom{n-2}{2} + \binom{n-1}{2} + (m-2)\binom{n}{2} + \binom{m-2}{2} + \binom{m-1}{2} + (n-2)\binom{m}{2} + (n+m-4)$ edges. As $n \geq 4$, this number is greater than $e_{m,n}$ by $(n-1)(n-4)/2 \geq 0$ edges. Again, we have $P(K_m \times K_n) \geq \frac{1}{2}(m-1)(mn^2+n^2-n-4)$. ∎

The other cases remain are: $P(K_2 \times K_2) = 2$, $P(K_2 \times K_3) = 9$ and $P(K_3 \times K_3) = 28$. Figure 3.2 shows the profile numberings of $K_2 \times K_2$, $K_2 \times K_3$ and $K_3 \times K_3$, respectively.

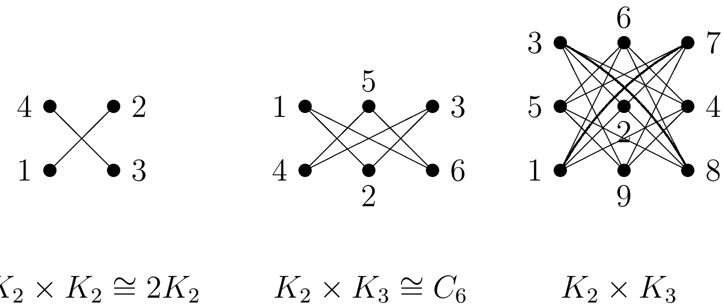


Figure 3.2: The profile numberings of $K_2 \times K_2$, $K_2 \times K_3$ and $K_3 \times K_3$, respectively.

3.2.2 Profile of $(\overline{K_s} \vee G) \times K_n$

This subsection determines the profile of $(\overline{K_s} \vee G) \times K_n$ with $|V(G)| = t \leq s$.

The notations we use in this subsection are the same as above except now we let $m = s + t$ and $V((\overline{K_s} \vee G)) = S \cup T$, where $S = \{x_1, x_2, \ldots, x_s\} = V(\overline{K_s})$ and $T = \{x_{s+1}, x_{s+2}, \ldots, x_{s+t}\} = V(G)$. We also let $S_j = \{v_{i,j} : x_i \in S\}$ and $T_j = \{v_{i,j} : x_i \in T\}$ for $1 \leq j \leq n$. Notice that $C_j = S_j \cup T_j$.

Theorem 3.2.2 *If G is a graph of order $t \leq s$ and $n \geq 4$, then* $P((\overline{K_s} \vee G) \times K_n) = \binom{nt}{2} + (n^2 - 2)st$.

Proof. To prove $P((\overline{K_s} \vee G) \times K_n) \leq \binom{nt}{2} + (n^2 - 2)st$, consider the proper numbering g of $(\overline{K_s} \vee G) \times K_n$ defined by

$$g(v_{i,j}) = \begin{cases} i + (j-1)s, & \text{for } 1 \leq i \leq s \text{ and } 1 \leq j \leq n-1; \\ i + (n-1)s + t, & \text{for } 1 \leq i \leq s \text{ and } j = n; \\ i + jt + (n-1)s, & \text{for } s+1 \leq i \leq s+t \text{ and } 1 \leq j \leq n-1; \\ i + (n-2)s, & \text{for } s+1 \leq i \leq s+t \text{ and } j = n. \end{cases}$$

See Figure 3.3 for g of $(\overline{K_4} \vee G) \times K_9$ where $|V(G)| = 3$ in which the edges are not drawn for simplicity.

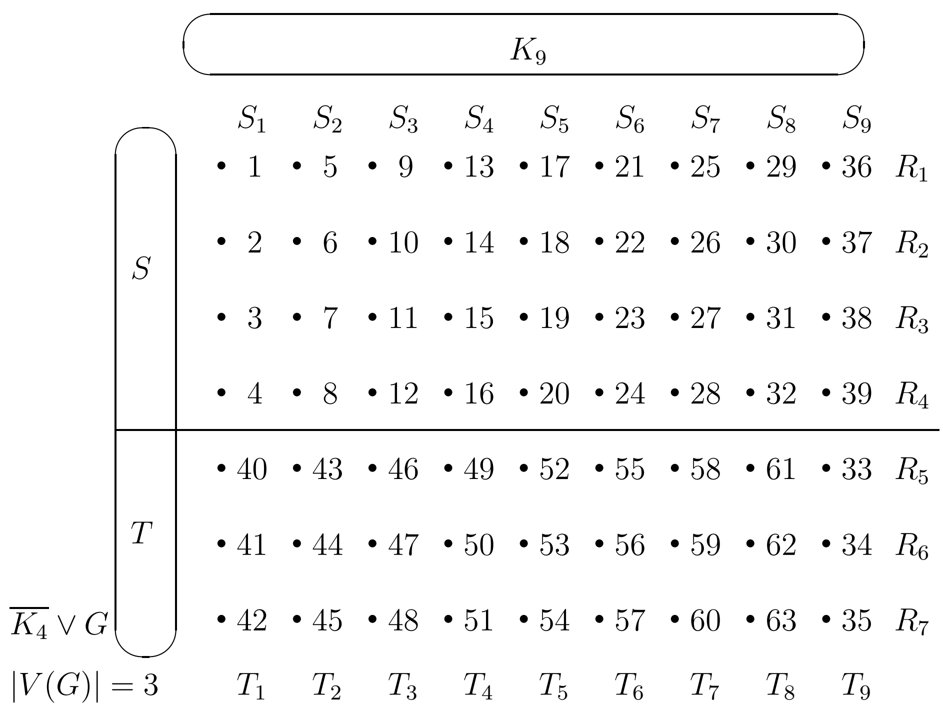


Figure 3.3: A proper numbering g of $(\overline{K_4} \vee G) \times K_9$.

Notice that two vertices $v_{i,j}, v_{i',j'}$ are adjacent in $(\overline{K_s} \vee G) \times K_n$ if and only if one is in S_j and the other in $T_{j'}$ for some $j \neq j'$ or one is in T_j and the other is in $T_{j'}$ with $x_i x_{i'} \in E(G)$. As no vertex in S_i is adjacent to a vertex with smaller numbering in $(\overline{K_s} \vee G) \times K_n$, $S \times V(K_n)$ is an independent set in $((\overline{K_s} \vee G) \times K_n)_g$.

For any two vertices $v_{i,j}$ and $v_{i',j'}$ in $T \times K_n$ with $g(v_{i,j}) < g(v_{i',j'})$, we may choose k from $\{1, 2\}$ such that $k \neq j'$. So, $g(v_{1,k}) < g(v_{i,j}) < g(v_{i',j'})$ and $v_{1,k}v_{i',j'} \in E((\overline{K_s} \vee G) \times K_n) \subseteq E(((\overline{K_s} \vee G) \times K_n)_g)$ imply that $v_{i,j}v_{i',j'} \in E(((\overline{K_s} \vee G) \times K_n)_g)$. This proves that $T \times V(K_n)$ is a clique in $((\overline{K_s} \vee G) \times K_n)_g$, which gives $\binom{nt}{2}$ edges.

For any $v_{i,j} \in S_j$ and $v_{i',j} \in T_j$ with $2 \leq j \leq n-1$, we have $g(v_{1,1}) < g(v_{i,j}) < g(v_{i',j})$ and $v_{1,1}v_{i',j} \in E((\overline{K_s} \vee G) \times K_n) \subseteq E(((\overline{K_s} \vee G) \times K_n)_g)$ implying that $v_{i,j}v_{i',j} \in E(((\overline{K_s} \vee G) \times K_n)_g)$. It is also the case that no vertex in S_j is adjacent to a vertex in T_j in $((\overline{K_s} \vee G) \times K_n)_g$ for $j = 1$ or n. So, vertices in S_j are adjacent to vertices in $T_{j'}$ in $((\overline{K_s} \vee G) \times K_n)_g$ for all j and j' except $j = j' \in \{1, n\}$. These give $(n^2 - 2)st$ edges.

Therefore, $P((\overline{K_s} \vee G) \times K_n) \leq |E(((\overline{K_s} \vee G) \times K_n)_g)| = \binom{nt}{2} + (n^2 - 2)st$.

Next, we shall prove that $P((\overline{K_s} \vee G) \times K_n) \geq \binom{nt}{2} + (n^2 - 2)st$. Choose a profile numbering f of $(\overline{K_s} \vee G) \times K_n$. For the first case, assume that $f(v_{1,1}) = 1$. Let $f(v_{a,b}) = \min\{f(v_{i,j}) : v_{i,j} \in T_2 \cup \ldots \cup T_n\}$.

For any vertices $v_{i,j} \in S_j$ and $v_{i',j'} \in T_{j'}$, by the definition, $v_{i,j}v_{i',j'} \in E((\overline{K_s} \vee G) \times K_n) \subseteq E(((\overline{K_s} \vee G) \times K_n)_f)$ if $j \neq j'$. Suppose $j = j' \notin \{1, b\}$. If $f(v_{i,j}) < f(v_{i',j'})$, then $f(v_{1,1}) < f(v_{i,j}) < f(v_{i',j'})$ and $v_{1,1}v_{i'j'} \in E(((\overline{K_s} \vee G) \times K_n)_f)$ imply that $v_{i,j}v_{i'j'} \in E(((\overline{K_s} \vee G) \times K_n)_f)$. If $f(v_{i,j}) > f(v_{i',j'})$, then $f(v_{a,b}) < f(v_{i',j'}) < f(v_{i,j})$ and $v_{a,b}v_{i,j} \in E(((\overline{K_s} \vee G) \times K_n)_f)$ imply that $v_{i,j}v_{i'j'} \in E(((\overline{K_s} \vee G) \times K_n)_f)$. So, vertices in S_j are adjacent to vertices in $T_{j'}$ for all j and j' except $j = j' \in \{1, b\}$. These give $(n^2 - 2)st$ edges.

Consider any two vertices $v_{i,j}$ and $v_{i',j'}$ in $T_1 \cup T_2 \cup \ldots \cup T_n$ such that $f(v_{i,j}) < f(v_{i',j'})$. For $j' \geq 2$, we have $f(v_{1,1}) < f(v_{i,j}) < f(v_{i',j'})$ and $v_{1,1}v_{i',j'} \in E(((\overline{K_s} \vee G) \times K_n)_f)$ implying $v_{i,j}v_{i',j'} \in E(((\overline{K_s} \vee G) \times K_n)_f)$. So, $T_2 \cup T_3 \cup \ldots \cup T_n$ is a clique in $((\overline{K_s} \vee G) \times K_n)_f$. This gives $\binom{(n-1)t}{2}$ edges. If $T_1 \cup T_2 \cup \ldots \cup T_n$ is a clique in $((\overline{K_s} \vee G) \times K_n)_f$, then these give $\binom{nt}{2}$ edges. Therefore, $P((\overline{K_s} \vee G) \times K_n) \geq \binom{nt}{2} + (n^2 - 2)st$. Now, we may assume that there are two non-adjacent vertices $v_{p,q}$ and $v_{p',q'}$ in $T_1 \cup T_2 \cup \ldots \cup T_n$ with $f(v_{p,q}) < f(v_{p',q'})$ and $q' = 1$.

For any two vertices $v_{i,j}$ and $v_{i',j'}$ in $S_2 \cup S_3 \cup \ldots \cup S_n$ such that $f(v_{i,j}) < f(v_{i',j'})$. If $f(v_{p,q}) > f(v_{i,j})$, then $f(v_{i,j}) < f(v_{p,q}) < f(v_{p',q'})$ and $v_{i,j}v_{p',q'} \in E(((\overline{K_s} \vee G) \times K_n)_f)$ imply $v_{p,q}v_{p',q'} \in E(((\overline{K_s} \vee G) \times K_n)_f)$, a contradiction. Therefore, it is always the case that $f(v_{p,q}) < f(v_{i,j}) < f(v_{i',j'})$. Except for the case when $q = j' = b$, we have $v_{p,q}v_{i',j'} \in E(((\overline{K_s} \vee G) \times K_n)_f)$, which together with the above inequalities gives that $v_{i,j}v_{i',j'} \in E(((\overline{K_s} \vee G) \times K_n)_f)$.

Now, if $q \neq b$, we have that $S_2 \cup S_3 \cup \ldots \cup S_n$ is a clique in $((\overline{K_s} \vee G) \times K_n)_f$. This gives $\binom{(n-1)s}{2}$ edges. And so $P((\overline{K_s} \vee G) \times K_n) \geq \binom{(n-1)s}{2} + \binom{(n-1)t}{2} + (n^2-2)st \geq 2\binom{(n-1)t}{2} + (n^2-2)st$ as $n \geq 4$. Hence we may assume that if $v_{p,q}$ and $v_{p',q'}$ are nonadjacent in $T_1 \cup T_2 \cup \ldots \cup T_n$ with $f(v_{p,q}) < f(v_{p',q'})$, then $q = b$ and $q' = 1$. In this case, $S_2 \cup S_3 \cup \ldots \cup S_{b-1} \cup S_{b+1} \cup S_{b+2} \cup \ldots \cup S_n$ is a clique and $T_1 \cup T_2 \cup \ldots \cup T_n$ is a clique in $((\overline{K_s} \vee G) \times K_n)_f$ except that vertices in T_1 are not necessarily adjacent to vertices in T_b. This gives $P((\overline{K_s} \vee G) \times K_n) \geq \binom{(n-2)s}{2} + \binom{nt}{2} - t^2 + (n^2-2)st \geq \binom{nt}{2} + (n^2-2)st$ as $n \geq 4$.

For the second case, assume that $f(v_{s+1,1}) = 1$. By symmetric argument of the first case, we also obtain $P((\overline{K_s} \vee G) \times K_n) \geq \binom{nt}{2} + (n^2-2)st$ as $n \geq 4$. ∎

3.2.3 Profile of $P_m \times K_n$

Finally in this section, we study the profile of $P_m \times K_n$.

The results in the previous subsections cover the case for $P_1 \times K_n = K_1 \times K_n$, $P_2 \times K_n = K_2 \times K_n = K_{1,1} \times K_n$ and $P_3 \times K_n = K_{1,2} \times K_n$. In the following we consider only for $m \geq 4$.

Theorem 3.2.3 *If $m, n \geq 4$, then $P(P_m \times K_n) = (m-2)\binom{n}{2} + (m-1)(n^2-1)$.*

Proof. For $P(P_m \times K_n) \leq (m-2)\binom{n}{2} + (m-1)(n^2-1)$, consider the proper numbering

g of $P_m \times K_n$ defined by

$$g(v_{i,j}) = \begin{cases} (i-1)n + j, & \text{for } 1 \le i \le m-2 \text{ and } 1 \le j \le n; \\ (m-1)n + j, & \text{for } i = m-1 \text{ and } 1 \le j \le n-1; \\ (m-1)n, & \text{for } i = m-1 \text{ and } j = n; \\ (m-2)n + j, & \text{for } i = m \text{ and } 1 \le j \le n-1; \\ mn, & \text{for } i = m \text{ and } j = n, \end{cases}$$

see Figure 3.4 for g of $P_5 \times K_9$ in which the edges are not drawn for simplicity.

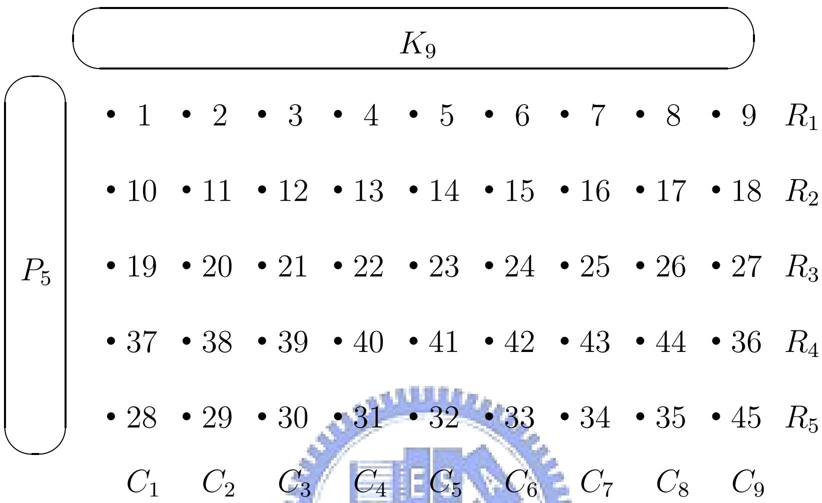


Figure 3.4: A proper numbering g of $P_5 \times K_9$.

The profile width of vertex $v_{i,j}$ is

$$w_g(v_{i,j}) = \begin{cases} 0, & \text{for } i = 1 \text{ and } 1 \le j \le n; \\ n-1, & \text{for } 2 \le i \le m-2 \text{ and } j = 1; \\ n-1+j, & \text{for } 2 \le i \le m-2 \text{ and } 2 \le j \le n; \\ 2n-1, & \text{for } i = m-1 \text{ and } j = 1; \\ 2n-1+j, & \text{for } i = m-1 \text{ and } 2 \le j \le n-1; \\ 2n-1, & \text{for } i = m-1 \text{ and } j = n; \\ 0, & \text{for } i = m \text{ and } 1 \le j \le n-1; \\ n-1, & \text{for } i = m \text{ and } j = n. \end{cases}$$

Therefore,

$$\sum_{j=1}^{n} w_g(v_{i,j}) = \begin{cases} 0, & \text{for } i = 1; \\ \binom{n}{2} + (n^2-1), & \text{for } 2 \le i \le m-2; \\ \binom{n}{2} + (2n^2-n-1), & \text{for } i = m-1; \\ n-1, & \text{for } i = m, \end{cases}$$

and so $P(P_m \times K_n) \le P_g(P_m \times K_n) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_g(v_{i,j}) = (m-2)\binom{n}{2} + (m-1)(n^2-1)$.

To prove that $P(P_m \times K_n) \ge (m-2)\binom{n}{2} + (m-1)(n^2-1)$, choose a profile numbering f of $P_m \times K_n$. We use the following notation:

Let

$$a_i = \min_{v_{i,j}\in R_i} f(v_{i,j}) \text{ and } f(v_{i,b_i}) = a_i \text{ for } 1 \le i \le m.$$

$$A = \{i : 2 \le i \le m-1 \text{ and } R_i \text{ is not a clique in } (P_m \times K_n)_f\} \text{ and } p = |A|.$$

$$B = \{i : 2 \le i \le m-1 \text{ and } a_i < \min\{a_{i-1}, a_{i+1}\}\} \text{ and } q = |B|.$$

$$\Lambda_{i,i'} = \{v_{i,j}v_{i',j'} \in E((P_m \times K_n)_f) : 1 \le j, j' \le n\}.$$

$$\lambda_{i,i'} = |\Lambda_{i,i'}| \text{ for } 1 \le i, i' \le m.$$

$$\Lambda^{=}_{i,i'} = \{v_{i,j}v_{i',j'} \in E((P_m \times K_n)_f) : 1 \le j = j' \le n\}.$$

$$\lambda^{=}_{i,i'} = |\Lambda^{=}_{i,i'}| \text{ for } 1 \le i, i' \le m.$$

$$\Lambda^{\le}_{i,i'} = \{v_{i,j}v_{i',j'} \in E((P_m \times K_n)_f) : 1 \le j \le j' \le n\}.$$

$$\lambda^{\le}_{i,i'} = |\Lambda^{\le}_{i,i'}| \text{ for } 1 \le i, i' \le m.$$

Claim 1. Suppose $|i-i'| = 1$. Then $\lambda^{=}_{i,i'} \ge n-2$ and so $\lambda_{i,i'} \ge n^2-2$. Furthermore, if $b_i = b_{i'}$, or $f(v_{i,b_{i'}}) < f(v_{i',b_{i'}})$, or R_i is a clique in $(P_m \times K_n)_f$ with $a_i < a_{i'}$, then $\lambda^{=}_{i,i'} \ge n-1$ and so $\lambda_{i,i'} \ge n^2-1$.

Proof of Claim 1. Consider any $j \notin \{b_i, b_{i'}\}$. If $f(v_{i,j}) < f(v_{i',j})$, then $f(v_{i,b_i}) < f(v_{i,j}) < f(v_{i',j})$ and $v_{i,b_i}v_{i',j} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$ imply $v_{i,j}v_{i',j} \in E((P_m \times K_n)_f)$. If $f(v_{i,j}) > f(v_{i',j})$, then $f(v_{i',b_{i'}}) < f(v_{i',j}) < f(v_{i,j})$ and $v_{i',b_{i'}}v_{i,j} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$ imply $v_{i',j}v_{i,j} \in E((P_m \times K_n)_f)$. In any case, $v_{i,j}v_{i',j} \in E((P_m \times K_n)_f)$ for $j \notin \{b_i, b_{i'}\}$, which give $\lambda^{=}_{i,i'} \ge n-2$. There are already other $n(n-1)$ edges between R_i and $R_{i'}$ in $E(P_m \times K_n)$, so we have $\lambda_{i,i'} \ge n^2-2$. For the case $b_i = b_{i'}$, there are at least $n-1$ edges $v_{i,j}v_{i',j} \in E((P_m \times K_n)_f)$ for $j \notin \{b_i, b_{i'}\}$. So, $\lambda^{=}_{i,i'} \ge n-1$ and $\lambda_{i,i'} \ge n^2-1$.

Now suppose $b_i \ne b_{i'}$. For the case $f(v_{i,b_{i'}}) < f(v_{i',b_{i'}})$, besides the $n-2$ edges $v_{i,j}v_{i',j}$ for $j \notin \{b_i, b_{i'}\}$, we also have the edge $v_{i,b_{i'}}v_{i',b_{i'}}$, since $f(v_{i,b_i}) < f(v_{i,b_{i'}}) < f(v_{i',b_{i'}})$ and $v_{i,b_i}v_{i',b_{i'}} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$ implying $v_{i,b_{i'}}v_{i',b_{i'}} \in E((P_m \times K_n)_f)$. For the case when $f(v_{i,b_{i'}}) > f(v_{i',b_{i'}})$ and R_i is a clique with $a_i < a_{i'}$, again $f(v_{i,b_i}) = a_i < a_{i'} = f(v_{i',b_{i'}}) < f(v_{i,b_{i'}})$ and $v_{i,b_i}v_{i,b_{i'}} \in E((P_m \times K_n)_f)$ imply $v_{i',b_{i'}}v_{i,b_{i'}} \in E((P_m \times K_n)_f)$. In any case, $v_{i,j}v_{i',j} \in E((P_m \times K_n)_f)$ for $j \ne b_i$, which gives $\lambda^{=}_{i,i'} \ge n-1$ and $\lambda_{i,i'} \ge n^2-1$. $\square$

Claim 2. If $i \in A$, then $\lambda^{\le}_{i-1,i+1} \ge \binom{n-1}{2} \ge 3$.

Proof of Claim 2. As R_i is not a clique in $(P_m \times K_n)_f$, we may choose $c \neq d$ such that $v_{i,c}v_{i,d} \notin E((P_m \times K_n)_f)$. Consider any $j, j' \notin \{c, d\}$ with $1 \leq j \leq j' \leq n$. In the 4-cycle $(v_{i,c}, v_{i-1,j}, v_{i,d}, v_{i+1,j'}, v_{i,c})$, we have $v_{i,c}v_{i,d} \notin E((P_m \times K_n)_f)$ implying $v_{i-1,j}v_{i+1,j'} \in E((P_m \times K_n)_f)$ by the chordality property (3.3). This gives that $\lambda^{\leq}_{i-1,i+1} \geq (1 + 2 + \ldots + (n-2)) = \binom{n-1}{2} \geq 3$. □

Claim 3. If $i \in B$, then $\lambda^{\leq}_{i-1,i+1} \geq \binom{n}{2} \geq 6$.

Proof of Claim 3. For any $j, j' \notin \{b_i\}$ with $1 \leq j \leq j' \leq n$, since $f(v_{i,b_i}) = a_i < a_{i-1} \leq f(v_{i-1,j})$ with $v_{i,b_i}v_{i-1,j} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$ and $f(v_{i,b_i}) = a_i < a_{i+1} \leq f(v_{i+1,j'})$ with $v_{i,b_i}v_{i+1,j'} \in E(P_m \times K_n) \subseteq E((P_m \times K_n)_f)$, by perfect elimination property (3.2), $v_{i-1,j}v_{i+1,j'} \in E((P_m \times K_n)_f)$. These give $\lambda^{\leq}_{i-1,i+1} \geq 1 + 2 + \ldots + (n-1) = \binom{n}{2} \geq 6$. □

Having these three claims in mind, we are ready to prove the theorem. As $n \geq 4$, there is a bijection from $\{\{j,k\} : 1 \leq j < k \leq n\}$ to itself such that $\{j,k\}$ is disjoint from its image $\{j',k'\}$. This can be done by setting $\{j',k'\} = \{(j+\delta) \bmod n, (k+\delta) \bmod n\}$, where $\delta = 2$ when j and k are consecutive under modulo n, and $\delta = 1$ otherwise. We may assume that $j' > k'$ for our convenience. Consider the following $(m-2)\binom{n}{2}$ disjoint sets:

$$S_{i,j,k} = \{v_{i,j}v_{i,k},\ v_{i-1,j'}v_{i+1,k'}\},$$

where $2 \leq i \leq m-2$ and $1 \leq j < k \leq n$. In the 4-cycle $(v_{i,j}, v_{i-1,j'}, v_{i,k}, v_{i+1,k'}, v_{i,j})$ (see Figure 3.5), at least one of the edge in $S_{i,j,k}$ must exist. These give totally at least $(m-2)\binom{n}{2}$ edges.

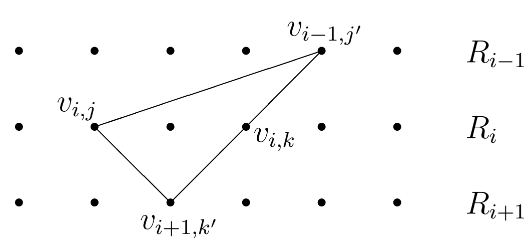


Figure 3.5: The 4-cycle $(v_{i,j}, v_{i-1,j'}, v_{i,k}, v_{i+1,k'}, v_{i,j})$.

Among the $m-2$ rows $R_2, R_3, \ldots, R_{m-1}$, there are p rows that are not cliques in $(P_m \times K_n)_f$ and the other $m-2-p$ rows are cliques. Among the $m-2-p$ clique rows, let there be p' consecutive pairs, that is, cliques R_i and $R_{i'}$ with $|i - i'| = 1$. By Claim 1, $\lambda_{i,i'} \geq n^2 - 1$ for these p' pairs and $\lambda_{i,i'} \geq n^2 - 2$ for the remaining $m-1-p'$ pairs of

i and i' with $|i-i'|=1$. These give totally at lease $p'(n^2-1)+(m-1-p')(n^2-2)=(m-1)(n^2-1)+(p'+1-m)$ edges.

By Claim 3, there are at least $6q$ extra edges from the sets $\Lambda^{\leq}_{i-1,i+1}$ for $i\in B$. By Claim 2, there are at least $3(p-q)$ extra edges from the sets $\Lambda^{\leq}_{i-1,i+1}$ for $i\in A\setminus B$. These give at least $3p+3q$ extra edges. So, we have

$$P(P_m\times K_n)\geq(m-2)\binom{n}{2}+(m-1)(n^2-1)+(p'+1-m+3p+3q).$$

In particular, $P(P_m\times K_n)\geq(m-2)\binom{n}{2}+(m-1)(n^2-1)$ when $p'+1-m+3p+3q\geq 0$. So, now assume that $p'+1-m+3p+3q\leq -1$ or $p'\leq m-3p-3q-2$.

Notice that there are p non-clique rows R_i with $2\leq i\leq m-1$. These rows separate the other rows into $p+1$ runs. Each run with α clique rows in $R_2,R_3,\ldots,R_{m-1}$ has $\max\{0,\alpha-1\}\geq\alpha-1$ consecutive pairs of cliques. Therefore, $p'\geq m-2-p-(p+1)=m-2p-3$ with equality holds if and only if $\alpha\geq 1$ for each run of clique rows. Or equivalently, any two rows in $A\cup\{R_1,R_m\}$ are not consecutive, which implies that $3\leq i\leq m-2$ for $i\in A$.

Now, $m-2p-3\leq p'\leq m-3p-3q-2$ imply that $p+3q\leq 1$. This is possible only when $p\leq 1$ and $q=0$. Suppose $p=1$, say $A=\{R_i\}$. Then, the above inequalities are in fact equalities, i.e., $m-2p-3=p'$ and so $3\leq i\leq m-2$. Therefore, R_{i-1} and R_{i+1} are clique rows. As $q=0$, we have $i\notin B$ and so either $a_{i-1}<a_i$ or $a_{i+1}<a_i$. By Claim 1, either $\lambda^{=}_{i-1,i}\geq n-1$ or $\lambda^{=}_{i,i+1}\geq n-1$. So in the above calculation, we in fact have $p'+1$, rather than p', consecutive pairs of i and i' with $\lambda_{i,i'}\geq n^2-1$. Thus,

$$P(P_m\times K_n)\geq(m-2)\binom{n}{2}+(m-1)(n^2-1)+(p'+2-m+3p+3q),$$

where $p'+2-m+3p+3q\geq(m-2p-3)+2-m+3p+3q=p+3q-1=0$ and so again $P(P_m\times K_n)\geq(m-2)\binom{n}{2}+(m-1)(n^2-1)$.

Now we may suppose that $p=q=0$. In other words, $R_2,R_3,\ldots,R_{m-1}$ are cliques and

$$a_1<a_2<\ldots<a_{r-1}<a_r \text{ and } a_r>a_{r+1}>a_{r+2}>\ldots>a_m \tag{3.4}$$

for some r. By Claim 1, we have

$$\lambda_{1,2}\geq n^2-2,\ \lambda_{i,i+1}\geq n^2-1 \text{ for } 2\leq i\leq m-2,\ \lambda_{m-1,m}\geq n^2-2.$$

These together with the $m-2$ clique rows gives at least $(m-2)\binom{n}{2}+(m-1)(n^2-1)-2$ edges. In the following, two extra edges, one with an end vertex in R_1 and the other with an end vertex in R_m, are to be found to make $P(P_m \times K_n) \geq (m-2)\binom{n}{2}+(m-1)(n^2-1)$. Assume, by symmetric, there is no such extra edge with a vertex in R_1 which we call an R_1-edge, we shall either get a contradiction or find two other extra edges.

First, we may assume that $b_1 \neq b_2$ and $a_1 < a_2$ and $f(v_{1,b_2}) > f(v_{2,b_2})$, for otherwise Claim 1 gives that $\lambda_{1,2} \geq n^2-1$ rather than only $\lambda_{1,2} \geq n^2-2$ which give an extra R_1-edge, a contradiction. Notice that the two non-edges between R_1 and R_2 are $v_{1,b_1}v_{2,b_1}$ and $v_{1,b_2}v_{2,b_2}$.

We claim that in fact $a_1 = 1$. Suppose to the contrary that $a_1 > 1$. By (3.4), we have $a_m = 1$. This together with $a_m < a_1 < a_2 \leq a_r$ implies that there is some i such that $a_r \geq a_{i-1} > a_1 > a_i \geq a_m = 1$. Then, for each $j \neq b_i$, we have $f(v_{i,b_i}) < f(v_{1,b_1}) < f(v_{i-1,j})$ and $v_{i,b_i}v_{i-1,j} \in E((P_m \times K_n)_f)$ implying $v_{1,b_1}v_{i-1,j} \in E((P_m \times K_n)_f)$, which gives $n-1$ extra R_1-edges, a contradiction. Thus, $a_1 = 1$.

As $a_1 = 1$ and $f(v_{1,b_2}) > a_2$, without loss of generality, we may assume that $f(v_{1,j}) = j$ for $1 \leq j \leq \ell-1$ but $f^{-1}(\ell) = v_{i^*,j^*} \notin R_1$, where $\ell \leq n$. Notice that we assume $b_1 = 1$ now. By the inequalities in (3.4), we have $\ell = a_m$ or $\ell = a_2$. For the case $\ell = a_m$, for any $j \neq 1$, we have $f(v_{1,1}) = 1 < \ell = a_m = f(v_{m,b_m}) < f(v_{2,j})$ and $v_{1,1}v_{2,j} \in E\left((P_m \times K_n)_f\right)$, implying $v_{m,b_m}v_{2,j'} \in E\left((P_m \times K_n)_f\right)$, which are $n-1 \geq 2$ extra edges as desired. For the case $\ell = a_2$, we may assume that $b_2 = n$. If $\ell < n$, then for any $j < n$, we have $f(v_{2,n}) < f(v_{1,\ell})$ with $v_{2,n}v_{1,\ell} \in E((P_m \times K_n)_f)$ and $f(v_{2,n}) < f(v_{3,j})$ with $v_{2,n}v_{3,j} \in E((P_m \times K_n)_f)$, implying $v_{1,\ell}v_{3,j} \in E((P_m \times K_n)_f)$ by the perfect elimination property (3.2). This gives $n-1 \geq 2$ extra edges as desired. So, we may assume that $\ell = n$.

Next, $f(v_{1,n}) > f(v_{3,1})$, for otherwise, $f(v_{1,n}) < f(v_{3,1})$ gives that $f(v_{2,n}) < f(v_{1,n}) < f(v_{3,1})$, this together with $v_{2,n}v_{3,1} \in E((P_m \times K_n)_f)$ implying $v_{1,n}v_{3,1} \in E((P_m \times K_n)_f)$, which is an extra R_1-edge, a contradiction. Similarly, for each j with $2 \leq j \leq n-1$ we have $f(v_{2,j}) > f(v_{3,1})$, for otherwise, $f(v_{2,j}) < f(v_{3,1})$ gives that $f(v_{2,j}) < f(v_{3,1}) < f(v_{1,n})$, this together with $v_{2,j}v_{1,n} \in E((P_m \times K_n)_f)$ implying $v_{3,1}v_{1,n} \in E((P_m \times K_n)_f)$, which is an extra R_1-edge, a contradiction. Also, $f(v_{4,2}) > f(v_{3,1})$, for otherwise, $f(v_{4,2}) < f(v_{3,1})$ gives that for each j with $2 \le j \le n-1$, we have $f(v_{1,1}) < f(v_{4,2}) < f(v_{3,1}) < f(v_{2,j})$, this together with $v_{1,1}v_{2,j} \in E((P_m \times K_n)_f)$ implying $v_{4,2}v_{2,j} \in E((P_m \times K_n)_f)$, which are $n-2 \ge 2$ extra edges as desired. Now, for each j with $2 \le j \le n-1$, we have $f(v_{3,1}) < f(v_{2,j})$ with $v_{3,1}v_{2,j} \in E((P_m \times K_n)_f)$, and $f(v_{3,1}) < f(v_{4,2})$ with $v_{3,1}v_{4,2} \in E((P_m \times K_n)_f)$, implying $v_{2,j}v_{4,2} \in E((P_m \times K_n)_f)$, which are $n-2 \ge 2$ extra edges as desired. ∎

3.3 Profile minimization on compositions of graphs

In this section we establish bounds for profiles $P(G[H])$ of compositions of graphs G and H. Also, exact value is determined when G is an interval graph as well as certain graphs.

3.3.1 Preliminary

A close related class of graphs to interval graphs are chordal graphs. A graph is *chordal* if every cycle of length greater than three has a chord. It is well-known that a graph G of n vertices is chordal if and only if it has a *perfect elimination ordering* which is an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$ such that

$$i < j < k,\ v_iv_j \in E(G) \text{ and } v_iv_k \in E(G) \text{ imply } v_jv_k \in E(G). \tag{3.5}$$

It is clear that an interval ordering is a perfect elimination ordering. Consequently, interval graphs are chordal. Notice that v_i is a simplicial vertex of the induced subgraph $G_{\{v_i,v_{i+1},\ldots,v_n\}}$ for $1 \le i \le n$.

Denote by $S(G)$ the set of all simplicial vertices of a graph G. It is clear by the definition that $S(G)$ induces a subgraph $G_{S(G)}$ in which every component is a clique. It is then the case that the number of components of $G_{S(G)}$ equals to the maximum number of an independent set in $G_{S(G)}$. We use $s(G)$ to denote this number.

Suppose now G is an interval graph, and $v_1, v_2, \ldots, v_n$ is an interval ordering of G. For

$1 \le i \le n$ and $x \in V(G)$, let

$$\begin{aligned} N_i(x) &= \{v_j \in N(x) : j \ge i\}, \\ N_i[x] &= \{v_j \in N[x] : j \ge i\}, \\ N^-(v_i) &= \{v_j \in N(v_i) : j < i\}. \end{aligned}$$

If necessary, we use $N^-(v_i; v_1, v_2, \ldots, v_n)$ for $N^-(v_i)$ to emphasize the ordering. We use $\sigma(G; v_1, v_2, \ldots, v_n)$ to denote the number of vertices v_i with $N^-(v_i) = \emptyset$. And let $\sigma(G) = \max \sigma(G; v_1, v_2, \ldots, v_n)$, where the maximum is taken over all interval orderings of G.

Lemma 3.3.1 *Suppose $v_1, v_2, \ldots, v_n$ is an interval ordering of an interval graph G. If $v_q \in N^-(v_p)$ and $N_q[v_p] \subseteq N_q[v_q]$, then the ordering $u_1, u_2, \ldots, u_n$ resulted from $v_1, v_2, \ldots, v_n$ by moving v_q to the position just after v_p is also an interval ordering of G.*

Proof. For $i < j < k$ with $u_i u_k \in E(G)$, we shall verify that $u_j u_k \in E(G)$ by considering three cases. Let $u_i = v_{i'}$, $u_j = v_{j'}$ and $u_k = v_{k'}$.

Case 1. $i' < j' < k'$. In this case, $v_{i'} v_{k'} = u_i u_k \in E(G)$ implies $v_{j'} v_{k'} \in E(G)$ and so $u_j u_k \in E(G)$.

Case 2. $q = k' < j' \le p$. In this case, $v_p v_q \in E(G)$ implies $v_{j'} \in N_q[v_p] \subseteq N_q[v_q]$ and so $u_j u_k = v_{j'} v_q \in E(G)$.

Case 3. $q = j' < i' \le p < k'$. In this case, $v_{i'} v_{k'} = u_i u_k \in E(G)$ implies $v_{k'} \in N_q[v_p] \subseteq N_q[v_q]$ and so $u_j u_k = v_q v_{k'} \in E(G)$. ∎

Proposition 3.3.2 *For any interval graph G, we have $\sigma(G) = s(G)$.*

Proof. Suppose $v_1, v_2, \ldots, v_n$ is an interval ordering of G with $\sigma(G; v_1, v_2, \ldots, v_n) = \sigma(G)$. By the definition of an interval ordering, any vertex v_i with $N^-(v_i) = \emptyset$ is simplicial. Also, $N^-(v_i) = N^-(v_j) = \emptyset$ imply that v_i and v_j are not adjacent. So, $\sigma(G) \le s(G)$.

Suppose $\sigma(G) < s(G)$. Then, by the definitions of $\sigma(G)$ and $s(G)$, the graph $G_{S(G)}$ has a component C containing no vertex v_i with $N^-(v_i) = \emptyset$. Let v_p be an arbitrarily vertex in C. For $v_{p-1} \in N^-(v_p)$, since v_p is simplicial, $N_{p-1}[v_p] \subseteq N_{p-1}[v_{p-1}]$. According to Lemma 3.3.1, we can move v_{p-1} to the position just after v_p to get a new interval ordering of G. Continue this process we shall get an interval ordering $u_1, u_2, \ldots, u_n$ with $N^-(v_p) = \emptyset$. More precisely, if $N^-(v_p; v_1, v_2, \ldots, v_n) = \{v_q, v_{q+1}, \ldots, v_{p-1}\}$, then in fact $u_1, u_2, \ldots, u_n$ is obtained from $v_1, v_2, \ldots, v_n$ by moving v_p into the position between v_{q-1} and v_q. So, $N^-(v_i; u_1, u_2, \ldots, u_n) = N^-(v_i; v_1, v_2, \ldots, v_n)$ for $i < q$ or $i > p$. Notice that by the definition of C and v_p, we have $N^-(v_i; v_1, v_2, \ldots, v_n) \neq \emptyset$ for $q \leq i \leq p$. Hence, $N^-(v_p; u_1, u_2, \ldots, u_n) = \emptyset$ implies that $\sigma(G; v_1, v_2, \ldots, v_n) < \sigma(G; u_1, u_2, \ldots, u_n)$, a contradiction. This proves the proposition. ∎

For a graph G, define $\widehat{s}(G) = \max\{s(\widehat{G}) : \widehat{G}$ is an interval completion of $G\}$ and $\widehat{\sigma}(G) = \max\{\sigma(\widehat{G}) : \widehat{G}$ is an interval completion of $G\}$. Obviously, $\widehat{\sigma}(G) = \widehat{s}(G)$ by using Proposition 3.3.2 directly.

Proposition 3.3.3 *If x is a simplicial vertex of a graph G, then x is also simplicial in any interval completion $\widehat{G}$ of G.*

Proof. Suppose to the contrary that x is not simplicial in $\widehat{G}$. Choose an interval ordering $v_1, v_2, \ldots, v_n$ of $\widehat{G}$ with $x = v_p$. We may assume that the interval ordering is chosen such that p is as small as possible. Then, there are $v_q, v_r \in N_{\widehat{G}}(v_p)$ such that $q < r$ and $v_q v_r \notin E(\widehat{G})$. We may assume that q is chosen as large as possible. I t is the case that $q < p$ by the interval ordering property. In fact, $q = p - 1$ for otherwise we have $N_{p-1}[v_p] \subseteq N_{p-1}[v_{p-1}]$. In this case, by Lemma 3.3.1, we may switch v_{p-1} and v_p to get a new interval ordering of G in which x has a smaller index than p, a contradiction.

Let s be the least index with $v_s \in N^-(v_p)$. It is easy to see that $v_1, v_2, \ldots, v_n$ is an interval ordering of $\widehat{G} - v_s v_p$. If $v_s v_p \notin E(G)$, then $\widehat{G} - v_s v_p$ is an interval super-graph of G with fewer edges than $\widehat{G}$, a contradiction. So, $v_s v_p \in E(G)$.

Since $v_r \in N_{\widehat{G}}(v_p) - N_{\widehat{G}}(v_{p-1})$, the least index t with $v_t \in N^-(v_r)$ is p. Again, $v_p v_r \in E(G)$ for otherwise $v_1, v_2, \ldots, v_n$ is an interval ordering of $\widehat{G} - v_p v_r$ which is an interval super-graph of G with fewer edges than $\widehat{G}$.

Since v_p is simplicial in G, both $v_s v_p, v_p v_r \in E(G)$ imply that $v_r v_s \in E(G) \subseteq E(\widehat{G})$. As $s \leq q < r$, by the interval ordering property, $v_q v_r \in E(\widehat{G})$, a contradiction. ∎

Proposition 3.3.4 *If I is an independent set of a graph G and $I \subseteq S(\widehat{G})$ for an interval completion $\widehat{G}$ of G, then I is also independent in $\widehat{G}$ and so $|I| \leq \widehat{\sigma}(G)$.*

Proof. Suppose to the contrary that $x, y \in I$ are such that $xy \notin E(G)$ but $xy \in E(\widehat{G})$. Choose an interval ordering $v_1, v_2, \ldots, v_n$ of $\widehat{G}$. Let $x = v_p$ and $y = v_{p'}$. Without loss of generality, we may assume that $p < p'$ and the interval ordering is chosen so that p is as small as possible. We then have $N^-(v_p) = \emptyset$, for otherwise there is some vertex $v_q \in N^-(v_p)$. Since v_p is simplicial in $\widehat{G}$, we have $N_q[v_p] \subseteq N_q[v_q]$. According to Lemma 3.3.1, we can move v_q to the position just after v_p to get a new interval ordering of $\widehat{G}$ in which x has a smaller index than p, a contradiction.

As $x = v_p$ and $y = v_{p'}$ are two adjacent simplicial vertices in $\widehat{G}$, we have $N_{\widehat{G}}[v_p] = N_{\widehat{G}}[v_{p'}]$. The fact that $N^-(v_p) = \emptyset$ then implies that the least index t with $v_t \in N^-(v_{p'})$ is p. It is then easy to see that $\widehat{G} - v_p v_{p'}$ is an interval super-graph of G, a contradiction. This proves the proposition. ∎

3.3.2 Bounds for profiles of compositions of graphs

This subsection establishes upper and lower bounds for the profiles $P(G[H])$ of compositions of graphs G and H. Exact value is also determined when G is an interval graph.

First, an upper bound.

Theorem 3.3.5 *If $\widehat{G}$ is an interval supper-graph of a graph G of order m and H is a graph of order n, then*

$$P(G[H]) \leq |E(\widehat{G})|n^2 + (m - \sigma(\widehat{G}))\binom{n}{2} + \sigma(\widehat{G})P(H).$$

Proof. Choose an interval completion $\widehat{H}$ of H. Then, $G[H]$ is a subgraph of $\widehat{G}[\widehat{H}]$ and so $P(G[H]) \leq P(\widehat{G}[\widehat{H}])$. Choose an interval ordering $x_1, x_2, \ldots, x_m$ of $\widehat{G}$ such that there are exactly $\sigma(\widehat{G})$ vertices x_i with $N^-(x_i) = \emptyset$. Also, choose an interval ordering $y_1, y_2, \ldots, y_n$ for $\widehat{H}$. Consider the ordering

$$v_{1,1}, v_{1,2}, \ldots, v_{1,n}, v_{2,1}, v_{2,2}, \ldots, v_{2,n}, \ldots, v_{m,1}, v_{m,2}, \ldots, v_{m,n}$$

using the lexicographical ordering. That is, $(i,j) < (i',j')$ if and only if $i < i'$ or $i = i'$ with $j < j'$. We shall check below that this is an interval ordering for the supper-graph Θ of $\widehat{G}[\widehat{H}]$ with $V(\Theta) = V(\widehat{G}[\widehat{H}])$ and $E(\Theta) = E(\widehat{G}[\widehat{H}]) \cup \{v_{i,j}v_{i,j'} : N^-(x_i) \neq \emptyset, 1 \leq j \neq j' \leq n\}$. Suppose $(i_1,j_1) < (i_2,j_2) < (i_3,j_3)$ with $v_{i_1,j_1}v_{i_3,j_3} \in E(\Theta)$.

Case 1. $i_1 \leq i_2 < i_3$.

In this case, $v_{i_1,j_1}v_{i_3,j_3} \in E(\Theta)$ implies that $x_{i_1}x_{i_3} \in E(\widehat{G})$. By the interval ordering property, $x_{i_2}x_{i_3} \in E(\widehat{G})$ and so $v_{i_2,j_2}v_{i_3,j_3} \in E(\widehat{G}[\widehat{H}]) \subseteq E(\Theta)$.

Case 2. $i_1 < i_2 = i_3$.

In this case, $v_{i_1,j_1}v_{i_3,j_3} \in E(\Theta)$ implies that $x_{i_1}x_{i_3} \in E(\widehat{G})$ and so $N^-(x_{i_3}) \neq \emptyset$. By the definition of Θ, we have $v_{i_2,j_2}v_{i_3,j_3} \in E(\Theta)$ since $i_2 = i_3$ and $j_2 \neq j_3$.

Case 3. $i_1 = i_2 = i_3$.

In this case, $j_1 < j_2 < j_3$. Suppose $v_{i_2,j_2}v_{i_3,j_3} \notin E(\Theta)$. By the definition of Θ, we have $N^-(x_{i_3}) = \emptyset$ and so $v_{i_1,j_1}v_{i_3,j_3} \in E(\widehat{G}[\widehat{H}])$. Then, $y_{j_1}y_{j_3} \in E(\widehat{H})$ and so $y_{j_2}y_{j_3} \in E(\widehat{H})$ which in turn implies that $v_{i_2,j_2}v_{i_3,j_3} \in E(\widehat{G}[\widehat{H}]) \subseteq E(\Theta)$.

Therefore, Θ is an interval super-graph of $\widehat{G}[\widehat{H}]$ with $|E(\widehat{G})|n^2 + (m - \sigma(\widehat{G}))\binom{n}{2} + \sigma(\widehat{G})P(H)$ edges. The theorem then follows. ∎

Corollary 3.3.6 *If G is a graph of order m and H is a graph of order n, then*

$$P(G[H]) \leq P(G)n^2 + (m - \widehat{\sigma}(G))\binom{n}{2} + \widehat{\sigma}(G)P(H).$$

Proof. The corollary follows from Theorem 3.3.5 by choosing an interval completion $\widehat{G}$ of G with $\widehat{\sigma}(G) = \sigma(\widehat{G})$. ∎

Next, we consider a lower bound.

Theorem 3.3.7 *If G is a $K_{2,3}$-free graph of order m and H is a graph of order n, then*

$$P(G[H]) \geq |E(G)|n^2 + (m - \widehat{\sigma}(G))\binom{n}{2} + \widehat{\sigma}(G)P(H).$$

Proof. Suppose K is an interval completion of $G[H]$. Notice that K is chordal.

Let

$$\begin{aligned} V(G) &= \{x_i : 1 \le i \le m\}, \\ V(H) &= \{y_j : 1 \le j \le n\}, \\ V(K) &= \{v_{i,j} = (x_i, y_j) : 1 \le i \le m, 1 \le j \le n\}, \\ R(K) &= \{x_i \in V(G) : K_{R_i} \text{ is not a clique in } K\} \text{ and } \eta = |R(K)|, \\ R'(K) &= \{x \in R(K) : x \text{ is not simplicial in } G\}. \end{aligned}$$

Claim 1. $R(K)$ is an independent set in G.

Proof of Claim 1. Suppose to the contrary that $x_p x_q \in E(G)$ for some $x_p, x_q \in R(K)$. By the definition of $R(K)$, there are four vertices $v_{p,a}, v_{p,b}, v_{q,c}, v_{q,d}$ in K such that $v_{p,a}v_{p,b} \notin E(K)$ and $v_{q,c}v_{q,d} \notin E(K)$. Since $x_p x_q \in E(G)$, we have $\{v_{p,a}v_{q,c}, v_{p,a}v_{q,d}, v_{p,b}v_{q,c}, v_{p,b}v_{q,d}\} \subseteq E(K)$ and hence $v_{p,a}v_{q,c}v_{p,b}v_{q,d}v_{p,a}$ is a chordless 4-cycle, a contradiction to the fact that K is chordal. □

Claim 2. If $x_i \in R(K)$ and $x_p \neq x_q$ are in $N_G(x_i)$, then $v_{p,a}v_{q,b} \in E(K)$ for $1 \le a,b \le n$.

Proof of Claim 2. By the definition of $R(K)$, $v_{i,j}v_{i,k} \notin E(K)$ for two distinct vertices $v_{i,j}$ and $v_{i,k}$. For $1 \le a, b \le n$, in the 4-cycle $v_{p,a}v_{i,j}v_{q,b}v_{i,k}v_{p,a}$, since $v_{i,j}v_{i,k} \notin E(K)$ we have $v_{p,a}v_{q,b} \in E(K)$. □

Claim 3. If $\widehat{\sigma}(G) < \eta$, then K has at least $(|E(G)| + \left\lceil \frac{\eta - \widehat{\sigma}(G)}{2} \right\rceil)n^2$ non-horizontal edges.

Proof of Claim 3. According to Claim 1, $R(K)$ is independent in G. Since $\widehat{\sigma}(G) < \eta$, by Proposition 3.3.4, in each interval completion $\widehat{G}$ of G, there are at least $r = \eta - \widehat{\sigma}(G) = \eta - \widehat{s}(G)$ vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ of $R(K)$ which are not simplicial in $\widehat{G}$. By Proposition 3.3.3, they are not simplicial in G and so are in $R'(K)$. For each x_{i_j} choose two neighbors $x_{p_j} \neq x_{q_j}$ with $x_{p_j}x_{q_j} \notin E(G)$. By Claim 2, there are n^2 non-horizontal edges $v_{p_j,a}v_{q_j,b}$ in K, where $1 \le a, b \le n$. As G contains no $K_{2,3}$ as an induced subgraph, each $\{x_{p_j}, x_{q_j}\}$ may equal to at most one $\{x_{p_{j'}}, x_{q_{j'}}\}$ with $j \neq j'$. Therefore, there are at least $\left\lceil \frac{\eta - \widehat{\sigma}(G)}{2} \right\rceil n^2$ non-horizontal edges other than those already in $G[H]$. □

We are now ready to prove the theorem. First, by the definition of $R(K)$, there are at least $(m-\eta)\binom{n}{2}+\eta P(H)$ horizontal edges in K.

If $\widehat{\sigma}(G)\geq\eta$, then

$$\begin{aligned} P(G[H]) &\geq |E(G)|\,n^2+(m-\eta)\binom{n}{2}+\eta P(H) \\ &\geq |E(G)|\,n^2+(m-\widehat{\sigma}(G))\binom{n}{2}+\widehat{\sigma}(G)P(H), \end{aligned}$$

since $P(H)\leq\binom{n}{2}$.

If $\widehat{\sigma}(G)<\eta$, then by Claim 3 we have

$$\begin{aligned} P(G[H]) &\geq (|E(G)|+\left\lceil\frac{\eta-\widehat{\sigma}(G)}{2}\right\rceil)n^2+(m-\eta)\binom{n}{2}+\eta P(H) \\ &\geq |E(G)|\,n^2+(m-\widehat{\sigma}(G))\binom{n}{2}+\widehat{\sigma}(G)P(H), \end{aligned}$$

since $\frac{n^2}{2}>\binom{n}{2}$ and $\eta>\widehat{\sigma}(G)$. The theorem then follows. ∎

Corollary 3.3.8 *If G is a chordal graph of order m and H is a graph of order n, then*

$$P(G[H])\geq|E(G)|n^2+(m-\widehat{\sigma}(G))\binom{n}{2}+\widehat{\sigma}(G)P(H).$$

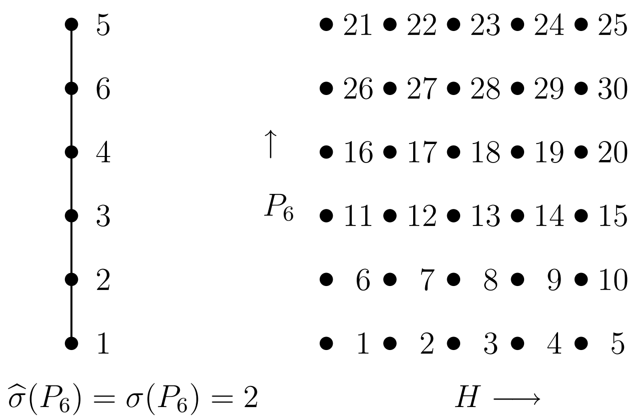
Proof. The corollary follows from that any chordal graph does not contain $K_{2,3}$ as an induced subgraph. ∎

Notice that the difference between the upper bound in Corollary 3.3.6 and the lower bound in Corollary 3.3.8 is at their first terms $P(G)n^2$ and $|E(G)|n^2$. For the case when the graph is interval, we have $P(G)=|E(G)|$ and so

Corollary 3.3.9 *If G is an interval graph of order m and H is a graph of order n, then*

$$P(G[H])=P(G)n^2+(m-\sigma(G))\binom{n}{2}+\sigma(G)P(H). \tag{3.6}$$

Figure 3.6 shows a profile numbering of $P_6[H]$ with $|V(H)|=5$ in which the edges are not drawn for simplicity.



$$\begin{aligned} P(P_m[H]) &= (6-1)5^2 + (6-2)\binom{5}{2} + 2P(H) \\ &= 165 + 2P(H) \end{aligned}$$

Figure 3.6: A profile numbering of $P_6[H]$ with $|V(H)| = 5$.

It is our interest to know for which graph G of order m equality (3.6) holds for any graph H of order n. For this purpose, let

$$\Omega = \left\{ G : P(G[H]) = P(G)n^2 + (m - \widehat{\sigma}(G))\binom{n}{2} + \widehat{\sigma}(G)P(H) \text{ for any graph } H \right\}$$

So, we have that Ω contains all interval graphs.

A slightly different lower bound is as follows.

Theorem 3.3.10 *If G is a graph of order m and H is a graph of order n, then either $G \in \Omega$ or*

$$\begin{aligned} P(G[H]) &\geq (P(G)+1)n^2 + (m-\eta)\binom{n}{2} + \eta P(H) \\ &\geq (P(G)+1)n^2 + (m-\alpha(G))\binom{n}{2} + \alpha(G)P(H) \end{aligned}$$

for some nonnegative integer $\eta \leq \alpha(G)$.

Proof. We use precisely the same notation $K, V(G), V(H), V(K), R(K), \eta, R'(K)$ as in the proof of Theorem 3.3.7. Notice that Claims 1 and 2 in Theorem 3.3.7 are still valid in this theorem.

Case 1. $\eta \leq \widehat{\sigma}(G)$.

For $j_1, j_2, \ldots, j_m \in \{1, 2, \ldots, n\}$, The subgraph $K_{\{v_{i,j_i}:1\leq i\leq m\}}$ is an interval supergraph of G and so has at least $P(G)$ edges. For each non-horizontal edge $v_{i',j'}v_{i'',j''}$ in K, there are n^{m-2} subgraphs $K_{\{v_{i,j_i}:1\le i\le m\}}$ contain this edge. Since there are n^m subgraphs $K_{\{v_{i,j_i}:1\le i\le m\}}$, there are at least $n^m P(G)/n^{m-2} = P(G)n^2$ non-horizontal edges in K. By the definition of η, we have

$$\begin{aligned} P(G[H]) &\ge P(G)n^2 + (m-\eta)\binom{n}{2} + \eta P(H) \\ &\ge P(G)n^2 + (m-\widehat{\sigma}(G))\binom{n}{2} + \widehat{\sigma}(G)P(H). \end{aligned}$$

This together with Corollary 3.3.6 gives that $G \in \Omega$.

Case 2. $\eta > \widehat{\sigma}(G)$.

In this case, we claim that each $K_{\{v_{i,j_i}:1\le i\le m\}}$ has at least $P(G)+1$ edges and hence the desired inequalities hold. Suppose to the contrary that there is some $K_{\{v_{i,j_i}:1\le i\le m\}}$ having just $P(G)$ edges. We may view v_{i,j_i} as x_i and then $K_{\{v_{i,j_i}:1\le i\le m\}}$ is an interval completion of G. By Claim 1, $R(K)$ is independent in G. By Claim 2, $R(K) \subseteq S(K_{\{v_{i,j_i}:1\le i\le m\}})$. Hence, by Proposition 3.3.4, $R(K)$ is also independent in $K_{\{v_{i,j_i}:1\le i\le m\}}$. And then $\eta = |R(K)| \le \widehat{\sigma}(G)$, a contradiction. ∎

Corollary 3.3.11 *If $\alpha(G) - \widehat{\sigma}(G) \le 2$, then $G \in \Omega$.*

Proof. Suppose to the contrary that $G \notin \Omega$. According to Corollary 3.3.6 and Theorem 3.3.10,

$$(P(G)+1)n^2 + (m-\alpha(G))\binom{n}{2} + \alpha(G)P(H) \le P(G)n^2 + (m-\widehat{\sigma}(G))\binom{n}{2} + \widehat{\sigma}(G)P(H)$$

for some graph H of n vertices. This gives $n^2 \le (\alpha(G)-\widehat{\sigma}(G))\left(\binom{n}{2} - P(H)\right) \le n(n-1) - 2P(H)$, which is impossible. Therefore, $G \in \Omega$. ∎

3.3.3 Gap between the upper and the lower bounds

There is a gap between the upper bound in Corollary 3.3.6 and the lower bound in Theorem 3.3.10. This subsection gives examples for which the upper or the lower bound are attainable. We also give conditions for which the upper bound attains.

Theorem 3.3.12 *If G_i is a graph of m_i vertices for $1 \le i \le k$ with $\sum_{1\le i\le k} m_i = m$, then*

$$P(\bigvee_{1\le i\le k} G_i) = \min_{1\le i\le k} \left\{ P(G_i) + m_i(m-m_i) + \binom{m-m_i}{2} \right\}.$$

Furthermore, if $P(\bigvee_{1\le i\le k} G_i) = P(G_j) + m_j(m-m_j) + \binom{m-m_j}{2}$, then $\widehat{\sigma}(\bigvee_{1\le i\le k} G_i) = \widehat{\sigma}(G_j)$.

Proof. The theorem follows from the fact that for any interval super-graph K of $\bigvee_{1\le i\le k} G_i$, at least one of $\{\bigcup_{1\le i\ne j\le k} V(G_i) : 1 \le j \le k\}$ is a clique in K. If not, there exists $x_p, y_p \in V(G_p)$ and $x_q, y_q \in V(G_q)$ such that $x_p y_p \notin E(K)$ and $x_q y_q \notin E(K)$. And then $x_p x_q y_p y_q x_p$ is a chordless 4-cycle in K which is impossible. ∎

Theorem 3.3.13 *If G_i is a graph of m_i vertices for $1 \le i \le k$ with $\sum_{1\le i\le k} m_i = m$, and H a graph of n vertices, then $(\bigvee_{1\le i\le k} G_i)[H] = \bigvee_{1\le i\le k} G_i[H]$ and so*

$$P((\bigvee_{1\le i\le k} G_i)[H]) = \min_{1\le i\le k} \left\{ P(G_i[H]) + m_i(m-m_i)n^2 + \binom{(m-m_i)n}{2} \right\}.$$

Proof. The first equality follows from definition. The second equality then follows from Theorem 3.3.12. ∎

Now, let G_1 be the path P_7 and G_2 the graph obtained from $K_{1,6}$ by adding a new edge. Notice that both G_1 and G_2 are interval graphs of 7 vertices; and G_1 has 6 edges while G_2 has 7 edges. Also, $\sigma(G_1) = 2$ and $\sigma(G_2) = 5$. Then, for any graph H of n

vertices, we have

$$\begin{aligned}
P(G_1 \vee G_2) &= 6 + 7 \cdot 7 + \binom{7}{2} = 76, \\
P(G_1[H]) &= 6n^2 + (7-2)\binom{n}{2} + 2P(H) = 8.5n^2 - 2.5n + 2P(H), \\
P(G_2[H]) &= 7n^2 + (7-5)\binom{n}{2} + 5P(H) = 8n^2 - n + 5P(H), \\
P\left((G_1 \vee G_2)[H]\right) &= \min\{P(G_1[H]), P(G_2[H])\} + \binom{7n}{2} + 7n \cdot 7n \\
&= \min\{P(G_1[H]), P(G_2[H])\} + 73.5n^2 - 3.5n, \\
\widehat{\sigma}(G_1 \vee G_2) &= 2, \\
\alpha(G_1 \vee G_2) &= 5, \\
\text{upper bound} &= 76n^2 + (14-2)\binom{n}{2} + 2P(H) = 82n^2 - 6n + 2P(H), \\
\text{lower bound} &= (76+1)n^2 + (14-5)\binom{n}{2} + 5P(H) = 81.5n^2 - 4.5n + 5P(H).
\end{aligned}$$

Depending on H, it is possible that $P(G_1[H]) < P(G_2[H])$ or $P(G_1[H]) \geq P(G_2[H])$. For the former case, $P\left((G_1 \vee G_2)[H]\right)$ is equal to the upper bound; for the latter case, $P\left((G_1 \vee G_2)[H]\right)$ is equal to the lower bound.

Theorem 3.3.14 *Suppose G_1, G_2 and H are graphs of order m_1, m_2 and n, respectively. If $G_1 \in \Omega$, $G_2 \notin \Omega$, $\binom{m_2}{2} - \binom{m_1}{2} \leq P(G_2) - P(G_1)$ and $\alpha(G_2) \leq \widehat{\sigma}(G_1) + 2$, then $G_1 \vee G_2 \in \Omega$ and*

$$P((G_1 \vee G_2)[H]) = (P(G_1) + m_1 m_2 + \binom{m_2}{2})n^2 + (m_1 + m_2 - \widehat{\sigma}(G_1))\binom{n}{2} + \widehat{\sigma}(G_1)P(H).$$

Proof. By the assumption $\binom{m_2}{2} - \binom{m_1}{2} \leq P(G_2) - P(G_1)$ and Theorem 3.3.12, we have

$$P(G_1 \vee G_2) = P(G_1) + m_1 m_2 + \binom{m_2}{2} \text{ and } \widehat{\sigma}(G_1 \vee G_2) = \widehat{\sigma}(G_1).$$

Now

$$
\begin{aligned}
& P(G_1[H]) + m_1m_2n^2 + \binom{m_2n}{2} \\
= \;& P(G_1)n^2 + (m_1 - \widehat{\sigma}(G_1))\binom{n}{2} + \widehat{\sigma}(G_1)P(H) + m_1m_2n^2 + \binom{m_2n}{2} \\
= \;& (P(G_1) + m_1m_2 + \binom{m_2}{2})n^2 + (m_1 + m_2 - \widehat{\sigma}(G_1))\binom{n}{2} + \widehat{\sigma}(G_1)P(H) \\
\leq \;& (P(G_2) + m_1m_2 + \binom{m_1}{2})n^2 + (m_1 + m_2 - \alpha(G_2))\binom{n}{2} + \alpha(G_2)P(H) + 2\binom{n}{2} \\
\leq \;& (P(G_2) + 1)\,n^2 + (m_2 - \alpha(G_2))\binom{n}{2} + \alpha(G_2)P(H) + m_1m_2n^2 + \binom{m_1n}{2} \\
\leq \;& P(G_2[H]) + m_1m_2n^2 + \binom{m_1n}{2}.
\end{aligned}
$$

Notice that in the above formulas, the first equality follows from that $G_1 \in \Omega$, the second equality from that $\binom{m_2n}{2} = \binom{m_2}{2}n^2 + m_2\binom{n}{2}$, the third inequality from that $\binom{m_2}{2} - \binom{m_1}{2} \leq P(G_2) - P(G_1)$ and $\alpha(G_2) \leq \widehat{\sigma}(G_1) + 2$, the forth inequality from that $\binom{m_1n}{2} = \binom{m_1}{2}n^2 + m_1\binom{n}{2}$ and $2\binom{n}{2} \leq n^2$, and the fifth inequality from Theorem 3.3.10. The theorem then follows from Theorem 3.3.13. ∎

Theorem 3.3.15 *Suppose G_1, G_2 and H are graphs of order m_1, m_2 and n, respectively. If $G_1, G_2 \in \Omega$, $\binom{m_2}{2} - \binom{m_1}{2} \leq P(G_2) - P(G_1)$ and $\widehat{\sigma}(G_2) \leq \widehat{\sigma}(G_1)$, then $G_1 \vee G_2 \in \Omega$ and*

$$P((G_1 \vee G_2)[H]) = (P(G_1) + m_1m_2 + \binom{m_2}{2})n^2 + (m_1 + m_2 - \widehat{\sigma}(G_1))\binom{n}{2} + \widehat{\sigma}(G_1)P(H).$$

Proof. The arguments are similar to those for the proof of Theorem 3.3.14. ∎

3.3.4 Exact values

By using the theorems in the previous subsections, we are able to get exact values for many $P(G[H])$ when G are given precisely. In this subsection, we give exact profiles of compositions of graphs by means of the results in Subsection 3.3.2.

We first consider the case when G is a caterpillar. A *caterpillar* is a tree from whom the removing of all leaves resulting a path(possibly empty). More precisely, suppose $m = r + \sum_{2 \leq i \leq r-1} s_i$, where $r \geq 2$ and each $s_i \geq 0$. A caterpillar with the parameter $(m; r; s_2, s_3, \ldots, s_{r-1})$ is the tree T with a vertex set $V(T) = \{x_i : 1 \leq i \leq$

$r\} \cup (\bigcup_{2\leq i\leq r-1}\{y_j^{(i)} : 1 \leq j \leq s_i\})$ and an edge set $E(T) = \{x_i x_{i+1} : 1 \leq i \leq r-1\} \cup (\bigcup_{2\leq i\leq r-1}\{x_i y_j^{(i)} : 1 \leq j \leq s_i\})$.

Theorem 3.3.16 *If T is a caterpillar with the parameter $(m; r; s_2, s_3, \ldots, s_{r-1})$ and H is a graph of n vertices, then*

$$P(T[H]) = \begin{cases} \frac{1}{2}n(3n-1) + P(H), & \textit{for } m = 2, \\ \frac{1}{2}n((2m+r-4)n - (r-2)) + (m-r+2)P(H), & \textit{for } m \geq 3. \end{cases}$$

Proof. At first, consider the function $f : V(T) \to \{1, 2, \ldots, m\}$ via $f(x_1) = 1$, $f(y_j^{(i)}) = f(x_{i-1}) + j$ for $2 \leq i \leq r-1$ and $1 \leq j \leq s_i$, $f(x_i) = f(x_{i-1}) + s_i + 1$ for $2 \leq i \leq r-2$, $f(x_{r-1}) = r + \sum_{2\leq i\leq r-1} s_i$, $f(x_r) = r + \sum_{2\leq i\leq r-1} s_i - 1$. Clearly, f is an interval ordering of T, and hence we have that T is an interval graph. It is trivial that $\sigma(T) = 1$ if $m = 2$. For $m \geq 3$, since the maximum independent subset of $S(T)$ is the set of all leaves in T, by computing the number of leaves in T, we have $\sigma(T) = s(T) = m - r + 2$ from Proposition 3.3.2. Apply Corollary 3.3.9, we obtain

$$P(T[H]) = \begin{cases} \frac{1}{2}n(3n-1) + P(H), & \text{for } m = 2, \\ \frac{1}{2}n((2m+r-4)n - (r-2)) + (m-r+2)P(H), & \text{for } m \geq 3. \end{cases}$$

∎

Theorem 3.3.17 *If $G = K_m - E(K_{m_1,m_2,\ldots,m_k})$ is a subgraph of K_m obtaining by deleting a set of all edges in an isomorphic complete multipartite subgraph $K_{m_1,m_2,\ldots,m_k}$ of K_m and H is a graph of n vertices, then*

$$P(G[H]) = (\binom{m}{2} - \sum_{1\leq i<j\leq k} m_i m_j)n^2 + (m-k)\binom{n}{2} + kP(H).$$

Proof. Let $V(K_{m_1,m_2,\ldots,m_k}) = \dot{\bigcup}_{1\leq i\leq k} X_i$ with $|X_i| = m_i$ and $V(G) = (\dot{\bigcup}_{1\leq i\leq k} X_i) \dot{\cup} Y$. Define a proper numbering $f : V(G) \to \{1, 2, \ldots, m\}$ with $1 \leq f(v) \leq m_1$ for $v \in X_1$ and $\sum_{1\leq j\leq i-1} m_j + 1 \leq f(v) \leq \sum_{1\leq j\leq i} m_j$ for $v \in X_i$ $(2 \leq i \leq k)$. Easily to check up that f is an interval ordering of G and hence G is an interval graph. Since each $X_i \dot{\cup} Y$ is a clique in

G and there are no edges between all X_i's, we may choose a vertex from each X_i at will to form an independent set of simplicial vertices with largest size k, by Corollary 3.3.9 we finally have

$$P(G[H]) = (\binom{m}{2} - \sum_{1 \le i < j \le k} m_i m_j) n^2 + (m-k)\binom{n}{2} + kP(H).$$

Notice that if $m_i = 0$ for each i, then $G = K_m$ is also an interval graph and $k = 1$. Hence

$$P(K_m[H]) = \binom{m}{2} n^2 + (m-1)\binom{n}{2} + P(H).$$

∎

Thereupon, we deal with some cases which are not interval graphs.

Theorem 3.3.18 *If* $\sum_{1 \le i \le k} m_i = m$ *with* $m' = \max_{1 \le i \le k} m_i$ *and* H *is a graph of* n *vertices, then*

$$P(K_{m_1,m_2,\ldots,m_k}[H]) = (m'(m-m') + \binom{m-m'}{2}) n^2 + (m-m')\binom{n}{2} + m'P(H).$$

Proof. Since $K_{m_1,m_2,\ldots,m_k} \cong \bigvee_{1 \le i \le k} \overline{K_{m_i}}$, by Theorem 3.3.13 we have it. ∎

Corollary 3.3.19 *If* $G = K_m - E(\dot{\bigcup}_{1 \le i \le k} K_{m_i})$ *(where* $\min_{1 \le i \le k} m_i \ge 2$ *and* $m' = \max_{1 \le i \le k} m_i$*) is a subgraph of* K_m *obtaining by deleting a set of all edges in an isomorphic subgraph* $\dot{\bigcup}_{1 \le i \le k} K_{m_i}$ *of* K_m *and* H *is a graph of* n *vertices, then*

$$P(G[H]) = (\binom{m}{2} - \binom{m'}{2}) n^2 + (m-m')\binom{n}{2} + m'P(H).$$

Proof. Let $r = m - \sum_{1 \le i \le k} m_i$, then $G \cong K_r \vee K_{m_1,m_2,\ldots,m_k}$. And the theorem follows from Theorem 3.3.13, Theorem 3.3.18 and Corollary 3.3.9 by a careful computing. ∎

Corollary 3.3.20 *If* $G = K_m - E(M_k)$ *is a subgraph of* K_m *obtaining by deleting a set of all edges in an isomorphic subgraph* M_t *of* K_m *(where* M_k *is a matching of* k *edges in* K_m*) and* H *is a graph of* n *vertices, then*

$$P(G[H]) = (\binom{m}{2} - 1) n^2 + (m-2)\binom{n}{2} + 2P(H).$$

Proof. Since $K_m - E(M_k) \cong K_m - E(\overset{\bullet}{\bigcup}_{1\le i\le k} K_2)$, by Corollary 3.3.19 we get it. ∎

Let X, Y be two graphs. We define the graph $B(X,Y)$ to be the union of X, Y and a bipartite graph B with bipartition $V(X), V(Y)$.

Theorem 3.3.21 *Let $G = B(K_m, K_{m'})$ be connected with $B \neq K_{m,m'}$ and H be a graph of order n, then $G \in \Omega$ and*

$$P(G[H]) = P(G)n^2 + (m + m' - 2)\binom{n}{2} + 2P(H).$$

Proof. Since $\alpha(G) = 2$, we have $G \in \Omega$ from Corollary 3.3.11. In the following, we first show $\widehat{G} \neq K_{m+m'}$. Let f be a proper numbering on $V(G)$ with $f^{-1}(1)f^{-1}(2) \notin E(G)$, then trivially $G_f \neq K_{m+m'}$. This implies $\widehat{G} \neq K_{m+m'}$ and hence $\widehat{\sigma}(G) = 2$. It leads

$$P(G[H]) = P(G)n^2 + (m + m' - 2)\binom{n}{2} + 2P(H).$$

∎

Theorem 3.3.22 *Let $G = B(K_m, \overline{K_{m'}})$ be connected and H be a graph of order n, then $G \in \Omega$ and*

$$P(G[H]) = \begin{cases} P(G)n^2 + m\binom{n}{2} + m'P(H), & \text{for } N_B(V(\overline{K_{m'}})) = V(K_m), \\ P(G)n^2 + (m-1)\binom{n}{2} + (m'+1)P(H), & \text{for otherwise.} \end{cases}$$

Proof.

Case 1. When $N_B(V(\overline{K_{m'}})) = V(K_m)$

At first we show $\widehat{\sigma}(G) = m'$. Since the vertices of $V(\overline{K_{m'}})$ are all simplicial in G, so are in $\widehat{G}$ by Proposition 3.3.3. Besides, $V(\overline{K_{m'}})$ is the only maximum independent set in G, so is in $\widehat{G}$ by Proposition 3.3.4. Hence $\widehat{\sigma}(G) = m'$. Using $\alpha(G) = m'$, by Corollary 3.3.11, we derive

$$P(G[H]) = P(G)n^2 + m\binom{n}{2} + m'P(H).$$

Case 2. When $N_B(V(\overline{K_{m'}})) \neq V(K_m)$ (i.e. There is an $x \in V(K_m) \setminus N_B(V(\overline{K_{m'}}))$)

In the beginning, we show $\widehat{\sigma}(G) = m' + 1$. Since the vertices of $V(\overline{K_{m'}}) \cup \{x\}$ are all simplicial in G, so are in $\widehat{G}$ by Proposition 3.3.3. Besides, $V(\overline{K_{m'}}) \cup \{x\}$ is the only maximum independent set in G, so is in $\widehat{G}$ by Proposition 3.3.4. Hence $\widehat{\sigma}(G) = m' + 1$. Using $\alpha(G) = m' + 1$, by Corollary 3.3.11, we acquire

$$P(G[H]) = P(G)n^2 + (m-1)\binom{n}{2} + (m'+1)P(H).$$

No matter what case, $G \in \Omega$. ∎

Corollary 3.3.23 *Suppose $\{p_i q_i\}_{i=1}^{k}$ is increasing. If $G = K_m - E(\dot{\bigcup}_{1\le i\le k} K_{p_i,q_i})$ is a subgraph of K_m obtaining by deleting a set of all edges in an isomorphic subgraph $\dot{\bigcup}_{1\le i\le k} K_{p_i,q_i}$ of K_m and H is a graph of n vertices, then*

$$P(G[H]) = (\binom{m}{2} - p_k q_k)n^2 + (m-2)\binom{n}{2} + 2P(H).$$

Proof. We may regard $K_m - E(\dot{\bigcup}_{1\le i\le k} K_{p_i,q_i})$ as $B(K_{m-t}, K_t)$ (where $t = \sum_{1\le i\le k} q_i$) for a certain bipartite graph $B \neq K_{m-t,t}$. Next, we show $\widehat{G} = K_m - E(K_{p_k,q_k})$. Let $V(K_{p_i,q_i}) = X_i, V(G) = (\dot{\bigcup}_{1\le i\le k} X_i) \dot{\cup} Y$ and f be a proper numbering on $V(G)$. If $f^{-1}(1) \in X_i$, then $|E(G_f)| \ge |E(K_m) - E(K_{p_i,q_i})|$. Trivially, the equality holds if and only if $f^{-1}(\ell) \in X_i$ for $1 \le \ell \le p_i + q_i$, and hence $G_f = K_m - E(K_{p_i,q_i})$. If $f^{-1}(1) \notin V(\dot{\bigcup}_{1\le i\le k} K_{p_i,q_i})$, then $G_f = K_m$. Thus, we conclude that $\widehat{G} = K_m - E(K_{p_k,q_k})$. Using Theorem 3.3.21, it forces

$$P(G[H]) = (\binom{m}{2} - p_k q_k)n^2 + (m-2)\binom{n}{2} + 2P(H).$$

∎

At the end of this subsection we consider one of the case that can not be deduced directly by the previous properties, namely for the case when $G = C_m$ with $m \ge 4$.

Lemma 3.3.24 *If $m \ge 4$ and C is a non-complete interval super-graph of C_m, then $|E(C)| \ge 2m - 5 + s(C)$.*

Proof. Since C is chordal, C contains at least $m-3$ chords of C_m and so $|E(C)| \ge 2m-3$. The lemma is clearly true for $s(C) \le 2$. We may now assume that $s(C) \ge 3$. It is then the case that $m \geq 6$. Choose an interval ordering $v_1, v_2, \ldots, v_m$ of C. Let $i < j < k$ and v_i, v_j, v_k are independent simplicial vertices of C. Choose a v_i-v_k path P in C_m not passing v_j. As $i < j < k$, in this path there are adjacent vertices $v_{i'}$ and $v_{k'}$ with $i' < j < k'$. By the interval ordering property, we have $v_j v_{k'} \in E(C)$. Let $v_{j'}, v_{j''}$ be the two neighbors of v_j in C_m. Then $v_{j'} v_{j''} \in E(C)$ as v_j is simplicial in C. So $C' = C - v_j$ is an interval super-graph of C_{m-1} with $s(C') \geq s(C) - 1 \geq 2$, which implies that C' is not a complete graph. By the induction hypothesis, $|E(C')| \geq 2(m-1) - 5 + s(C) - 1$. As the path P does not pass v_j, we have $v_{k'} \notin \{v_{j'}, v_{j''}\}$ and so $|E(C)| \geq |E(C')| + 3 \geq 2m - 5 + s(C)$. ∎

Theorem 3.3.25 *If $m \geq 4$ and H is a graph of order n, then*

$$P(C_m[H]) = (2m-3)n^2 + (m-2)\binom{n}{2} + 2P(H).$$

Consequently, $C_m \in \Omega$.

Proof. Let $G = C_m$ and we use the same notation $K, V(G), V(H), V(K), R(K), \eta, R'(K)$ as in the proof of Theorem 3.3.7. Notice that Claims 1 and 2 in Theorem 3.3.7 are still valid in this theorem. Consider the interval super-graph C' obtained from C_m by adding $m-3$ chords passing a fixed vertex. Then $|E(C')| = 2m-3$ and $\widehat{\sigma}(C') = s(C') = 2$. Suppose C'' is an interval completion of C_m with $\sigma(C'') = \widehat{\sigma}(C_m)$. It is clear that C'' is not a complete graph, and so $\widehat{\sigma}(C'') \geq 2$. By Lemma 3.3.24, $2m - 3 = |E(C')| \geq |E(C'')| \geq 2m - 5 + \sigma(C'') \geq 2m - 3$ and so in fact $P(C_m) = 2m - 3$ and $\widehat{\sigma}(C_m) = 2$.

By Corollary 3.3.6, $P(C_m[H]) \leq (2m-3)n^2 + (m-2)\binom{n}{2} + 2P(H)$. To see the other inequality, we consider two cases.

Case 1. $\eta \leq 2$.

For $j_i, j_2, \ldots, j_m \in \{1, 2, \ldots, n\}$, The subgraph $K_{\{v_{i,j_i}:1 \leq i \leq m\}}$ is an interval super-graph of C_m and so has at least $P(C_m) = 2m-3$ edges. For each non-horizontal edge $v_{i',j'} v_{i'',j''}$ in K, there are n^{m-2} subgraphs $K_{\{v_{i,j_i}:1 \leq i \leq m\}}$ contain this edge. Since there are n^m subgraphs $K_{\{v_{i,j_i}:1 \leq i \leq m\}}$, there are at least $n^m(2m-3)/n^{m-2} = (2m-3)n^2$ non-horizontal edges in

K. By the definition of η, we have

$$\begin{aligned} P(C_m[H]) &\geq (2m-3)n^2 + (m-\eta)\binom{n}{2} + \eta P(H) \\ &\geq (2m-3)n^2 + (m-2)\binom{n}{2} + 2P(H). \end{aligned}$$

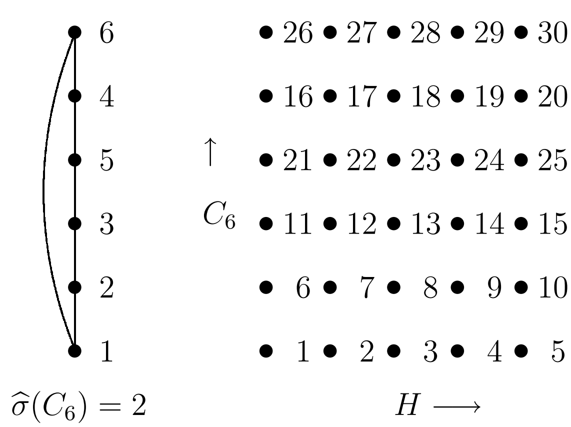
Case 2. $\eta > 2$.

In this case, we may view v_{i,j_i} as x_i and then $C = K_{\{v_{i,j_i}:1\leq i\leq m\}}$ is an interval supergraph of C_m. By Claim 1, $R(K)$ is independent in C_m. By Claim 2, $R(K) \subseteq S(C)$. Hence, $s(C) \geq \eta$ and so $|E(C)| \geq 2m-5+s(C) \geq 2m-5+\eta$ by Lemma 3.3.24. As in the proof of case 1, there are at least $((2m-5)+\eta)n^2$ non-horizontal edges. By the definition of η, we have

$$\begin{aligned} P(C_m[H]) &\geq (2m-5+\eta)n^2 + (m-\eta)\binom{n}{2} + \eta P(H) \\ &\geq (2m-3)n^2 + (m-2)\binom{n}{2} + 2P(H). \end{aligned}$$

∎

Figure 3.7 shows a profile numbering of $C_6[H]$ with $|V(H)| = 5$ in which the edges are not drawn for simplicity.



$$\begin{aligned} P(C_6[H]) &= (2\cdot 6-3)5^2 + (6-2)\tbinom{5}{2} + 2P(H) \\ &= 265 + 2P(H) \end{aligned}$$

Figure 3.7: A profile numbering of $C_6[H]$ with $|V(H)| = 5$.

Corollary 3.3.26 *If $m \geq 5$ and H is a graph of order n, then*

$$P(W_m[H]) = (3m-6)n^2 + (m-2)\binom{n}{2} + 2P(H).$$

Proof. Because $W_m = K_1 \vee C_{m-1}$, it is easy to obtain from Theorem 3.3.13 and Theorem 3.3.25. ∎

Chapter 4

Epilogue: Conclusions and Further Topics

This thesis studies two problems on graphs of operations: the bandwidth problem and the profile problem. Many of our results are solved by exact formulas or sharp bounds.

In the part of bandwidth problem, the following bandwidths have been determined:

Let $m = |V(H)|$, $\gcd D = 1$ and $\lambda = \max D$. Then

$$1.\begin{cases} (1)\ B(G([n],D)) = \lambda \\ (2)\ B(G([n],D)\Box H) = m\lambda \\ (3)\ B(G([n],D)\wedge H) = (m+1)\lambda \end{cases}, \text{ for } n \text{ larger than a certain number decided by } D.$$

$$2.\begin{cases} (1)\ B(G(\mathbf{Z}_n,D)) = 2\lambda \\ (2)\ B(G(\mathbf{Z}_n,D)\Box H) = 2m\lambda \\ (3)\ B(G(\mathbf{Z}_n,D)\wedge H) = 2(m+1)\lambda \end{cases}, \text{ for } n \text{ larger than a certain number decided by } D.$$

$$3.\begin{cases} (1)\ B(G(\mathbf{Z}_{2n},\{k,n\})) = \begin{cases} 3, & \text{if } (k,n) \in \{(1,2),(2,3)\}; \\ 4, & \text{otherwise.} \end{cases} \\ (2)\ B(G(\mathbf{Z}_{2n},\{1,n-1\})) = \begin{cases} 4, & \text{if } n = 3; \\ 5, & \text{if } n \geq 4. \end{cases} \\ (3)\ B(G(\mathbf{Z}_{2n+1},\{1,n\})) = 4. \\ (4)\ B(G(\mathbf{Z}_{2n+1},\{1,n-1\})) = \begin{cases} 4, & \text{if } n = 3; \\ 5, & \text{if } n = 4; \\ 6, & \text{if } n \geq 5. \end{cases} \end{cases}$$

Let $m = |V(H)|$, and $\lambda = \max D$. Then

$$4.\begin{cases} (1)\ B(G(\mathbf{N},D)) = \lambda. \\ (2)\ B(G(\mathbf{N},D)\Box H) = m\lambda. \\ (3)\ B(G(\mathbf{N},D)[H]) = m\lambda + m - 1. \\ (4)\ B(G(\mathbf{N},D)\wedge H) = (m+1)\lambda. \end{cases}$$

Define two parameters by

$$\underline{B_p}(H;k) = \min_{|A|=k} \left\{ \left| \underset{v\in A}{\cup} N(v) \right| - k : A \subseteq V(H) \right\}, \text{ and } \underline{B_p}(H) = \max_k \underline{B_p}(H;k);$$
$$\underline{B_s}(H;k) = \min_{|A|=k} \left\{ \left| \underset{v\in A}{\cup} N[v] \right| - k : A \subseteq V(H) \right\}, \text{ and } \underline{B_s}(H) = \max_k \underline{B_s}(H;k).$$

We have

5. If H is a graph spanned by a disjoint union of some cycles or a matching, then

$$\underline{B_p}(H) \le B(G(\mathbf{N}, D) \times H) - m\lambda \le B(H).$$

6. $\underline{B_s}(H) \le B(G(\mathbf{N}, D) \boxtimes H) - m\lambda \le B(H)$.

About the part of profile problem, we have presented the following profiles:

$$7. \begin{cases} (1)\ P(K_m \times K_n) = \frac{1}{2}(m-1)(mn^2+n^2-n-4) \text{ for } m=1 \text{ or } n \ge \max\{m,4\}. \\ (2)\ P((\overline{K_s} \vee G) \times K_n) = \binom{nt}{2} + (n^2-2)st \text{ for } |V(G)| = t \le s \text{ and } n \ge 4. \\ (3)\ P(P_m \times K_n) = (m-2)\binom{n}{2} + (m-1)(n^2-1) \text{ for } m,n \ge 4. \end{cases}$$

Let G and H be graphs of order m and n, respectively, then

$$8. \begin{cases} (1)\ P(G[H]) \le P(G)n^2 + (m - \widehat{\sigma}(G))\binom{n}{2} + \widehat{\sigma}(G)P(H). \\ (2)\ P(G[H]) \ge |E(G)|n^2 + (m - \widehat{\sigma}(G))\binom{n}{2} + \widehat{\sigma}(G)P(H) \text{ if } G \text{ is } K_{2,3}\text{-free}. \end{cases}$$

Define

$$\Omega = \left\{ G : P(G[H]) = P(G)n^2 + (m - \widehat{\sigma}(G))\binom{n}{2} + \widehat{\sigma}(G)P(H) \text{ for any graph } H \right\}.$$

We have

9. If G is an interval graph, then $G \in \Omega$.

10. Suppose G_1,G_2 are graphs of order m_1,m_2, respectively. If $G_1 \in \Omega$, $G_2 \notin \Omega$, $\binom{m_2}{2} - \binom{m_1}{2} \le P(G_2) - P(G_1)$ and $\alpha(G_2) \le \widehat{\sigma}(G_1) + 2$, then $G_1 \vee G_2 \in \Omega$ and

$$P((G_1 \vee G_2)[H]) = (P(G_1) + m_1m_2 + \binom{m_2}{2})n^2 + (m_1 + m_2 - \widehat{\sigma}(G_1))\binom{n}{2} + \widehat{\sigma}(G_1)P(H).$$

11. Suppose G_1,G_2 are graphs of order m_1,m_2, respectively. If $G_1, G_2 \in \Omega$, $\binom{m_2}{2} - \binom{m_1}{2} \le P(G_2) - P(G_1)$ and $\widehat{\sigma}(G_2) \le \widehat{\sigma}(G_1)$, then $G_1 \vee G_2 \in \Omega$ and

$$P((G_1 \vee G_2)[H]) = (P(G_1) + m_1m_2 + \binom{m_2}{2})n^2 + (m_1 + m_2 - \widehat{\sigma}(G_1))\binom{n}{2} + \widehat{\sigma}(G_1)P(H).$$

12. If $G \notin \Omega$, then $P(G[H]) \ge (P(G)+1)n^2 + (m - \alpha(G))\binom{n}{2} + \alpha(G))P(H)$.

Although some results of the above two problems are obtained, there are still many questions remain open. We describe below some of them that we concern most.

In Chapter 2, we use Proposition 2.1.3 and Proposition 2.1.4 frequently to get the bandwidths on three simple distance graphs and some composites of them with other arbitrary graphs. After this, we expect to solve the bandwidths of $G(\mathbf{X}, D)[H]$, $G(\mathbf{X}, D) \times H$, and $G(\mathbf{X}, D) \boxtimes H$ for $\mathbf{X} \in \{[n], \mathbf{Z}_n\}$ and H is an arbitrary finite graph. Moreover, we wish to make sure that $B(G(\mathbf{Z}_{2n}, \{1, n-k\}))$ (we guess it equals to $4k$) for $2 \leq k \ll n$, $B(G(\mathbf{Z}_{2n+s}, D))$ as $\max D$ close to n for $s \in \{0, 1\}$, $B(G([n], D) * H)$ as $\max D$ close to n and $B(G(\mathbf{Z}_n, D) * H)$ as $\max D$ close to $\lfloor \frac{n}{2} \rfloor$, for $* \in \{\Box, \times, \boxtimes, [\ ], \wedge\}$. We believe deeply that the lower bounds for $B(G(\mathbf{N}, D) \times H)$ and $B(G(\mathbf{N}, D) \boxtimes H)$ are valid for $B(G \times H)$ and $B(G \boxtimes H)$, respectively, where G is an infinite graph with finite bandwidth λ. It is interesting to characterize graphs H that result $\underline{B_p}(H) = B(H)$ and to give an efficient algorithm to find $\underline{B_p}(H)$. We are also interested in characterizing graphs H that result $\underline{B_s}(H) = B(H)$ and in finding to get an efficient algorithm to get $\underline{B_s}(H)$. Another nature question is that can we extend our results to more harder distance graphs, even to general Caley graphs.

Besides the relation between the *profile minimization problem* and the *interval graph completion problem*, the *interval property* and *perfect elimination property* play important roles in Chapter 3, we utilize them almost all the time.

For the profile minimization on product of graphs, we have given the profiles of $K_m \times K_n$, $(\overline{K_s} \vee G) \times K_n$ for $|V(G)| = t \leq s$ with $n \geq 4$ and $P_m \times K_n$. It is desirable to study $P(G \times H)$ for general graphs G and H, or at least $P(G \times K_n)$ for a general graph G. Before these, maybe we at first need to make clear the exact values of $P((\overline{K_s} \vee G) \times K_n)$ for $|V(G)| = t \geq s$ with $n \geq 4$ and of $P(C_m \times K_n)$.

For the profile minimization on composition of graphs, a sharp upper bound and a sharp lower bound of $P(G[H])$ are acquired. In addition, exact formula is set up when G is an interval graph. We also determine the exact values of $P(G[H])$ for some non-interval graphs G. It is our hope to find a speedy algorithm for $\widehat{\sigma}(G)$ and to characterize graphs G in Ω.

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