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Integral Equations and Operator Theory

Numerical Ranges of Radial Toeplitz Operators on Bergman Space

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Abstract. A Toeplitz operator T_ϕ with symbol ϕ in $L^\infty(\mathbb{D})$ on the Bergman space $A^2(\mathbb{D})$, where $\mathbb{D}$ denotes the open unit disc, is radial if $\phi(z) = \phi(|z|)$ a.e. on $\mathbb{D}$. In this paper, we consider the numerical ranges of such operators. It is shown that all finite line segments, convex hulls of analytic images of $\mathbb{D}$ and closed convex polygonal regions in the plane are the numerical ranges of radial Toeplitz operators. On the other hand, Toeplitz operators T_ϕ with ϕ harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ and radial Toeplitz operators are convexoid, but certain compact quasinilpotent Toeplitz operators are not.

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The *Bergman space* $A^2(\mathbb{D})$ of the open unit disc $\mathbb{D}$ in the plane consists of analytic functions $f : \mathbb{D} \to \mathbb{C}$ which are square-integrable with respect to the area measure dA. It is a Hilbert space under the inner product

$$\langle f, g\rangle = \iint_{\mathbb{D}} f(z)\overline{g(z)}dA(z) \quad \text{for } f, g \in A^2(\mathbb{D}),$$

and has the orthonormal basis $\{e_n\}_{n=0}^\infty$, where

$$e_n(z) = \sqrt{\frac{n+1}{\pi}}z^n \quad \text{for } z \in \mathbb{D}.$$

For any (essentially) bounded function ϕ on $\mathbb{D}$, the *Toeplitz operator* T_ϕ with symbol ϕ is the operator on $A^2(\mathbb{D})$ defined by

$$T_\phi f = P(\phi f) \quad \text{for } f \in A^2(\mathbb{D}),$$

where P denotes the (orthogonal) projection from $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$. T_ϕ or ϕ is said to be *radial* if $\phi(z) = \phi(|z|)$ for almost all z in $\mathbb{D}$. Such operators have been investigated intensively in recent years (cf. [8, 4, 9]). The purpose of this paper is to study their numerical ranges.

Recall that the *numerical range* of an operator A on the Hilbert space H is the set $W(A) = \{\langle Ax, x\rangle : x \in H,\ \|x\| = 1\}$, where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and its associated norm in H. The numerical range is always convex. For other properties of the numerical range, the reader may consult [6, Chapter 22] and [5].

We start with the general Toeplitz operators. The next proposition is all we can say about their numerical ranges at the present time.

Proposition 1. *If ϕ is a nonconstant function in $L^\infty(\mathbb{D})$, then $W(T_\phi)$ is contained in the relative interior of the convex hull of the essential range of ϕ.*

The *essential range* R_ϕ of a function ϕ in $L^\infty(\mathbb{D})$ is the set of complex numbers u for which $\{z \in \mathbb{D} : |\phi(z) - u| < \epsilon\}$ has (strictly) positive area measure for every $\epsilon > 0$, the *convex hull* $R^\wedge$ of a subset R of the plane is the smallest convex set containing R, and the *relative interior*, Rel Int $\triangle$, of a (nonempty nonsingleton) convex subset $\triangle$ is its interior relative to the affine subspace generated by it.

Note that, in Proposition 1, $W(T_\phi)$ is in general not equal to the asserted relative interior as the following example shows.

Example 2. If

$$\phi(z) = \begin{cases} 1 & \text{if } |z| \leq 1/2, \\ 0 & \text{if } 1/2 < |z| < 1, \end{cases}$$

then T_ϕ has the matrix representation diag $(1/4, 1/16, \ldots, 1/2^{2(n+1)}, \ldots)$ relative to the standard basis $\{e_n\}_{n=0}^\infty$ of $A^2(\mathbb{D})$. Hence $W(T_\phi) = (0, 1/4]$, which is not equal to the relative interior $(0, 1)$ of the convex hull of $R_\phi = \{0, 1\}$. Note also that the spectrum $\sigma(T_\phi)$ of T_ϕ is equal to $\{1/2^{2(n+1)} : n \geq 0\} \cup \{0\}$, which is not contained in R_ϕ.

Proof of Proposition 1. Let M_ϕ be the multiplication operator $M_\phi f = \phi f$ on $L^2(\mathbb{D})$. Since T_ϕ dilates to M_ϕ, we have

$$W(T_\phi) \subseteq \overline{W(M_\phi)} = R_\phi^\wedge$$

(cf. [6, Problems 81 and 216]). Assume that $W(T_\phi)$ is not contained in the relative interior of $R_\phi^\wedge$. Then we can find a real θ and a unit vector f in $A^2(\mathbb{D})$ such that

$$\langle T_{\mathrm{Re}\,(e^{i\theta}\phi)} f, f\rangle = \max R^\wedge_{\mathrm{Re}\,(e^{i\theta}\phi)} \equiv a.$$

Hence

$$\langle M_{\mathrm{Re}\,(e^{i\theta}\phi)} f, f\rangle = \max W(M_{\mathrm{Re}\,(e^{i\theta}\phi)}) = a,$$

from which we infer that (Re $(e^{i\theta}\phi))f = af$. The analyticity of the nonzero f implies that the set $\{z \in \mathbb{D} : \mathrm{Re}\ (e^{i\theta}\phi(z)) \neq a\}$ has area measure zero. Hence Re $(e^{i\theta}\phi) = a$ a.e. on $\mathbb{D}$. This says that the essential range of ϕ is contained in a line. Repeating the above arguments with Im $(e^{i\theta}\phi)$ replacing Re $(e^{i\theta}\phi)$ yields that ϕ is constant, contradicting our assumption. Thus we must have $W(T_\phi) \subseteq$ Rel Int $R_\phi^\wedge$. □

If the symbol ϕ of a Toeplitz operator T_ϕ on $A^2(\mathbb{D})$ is (complex-valued) harmonic, then $W(T_\phi)$ has been considered by Thukral [10]. The next result, though not stated explicitly, is essentially due to him.

Proposition 3. *If ϕ is a nonconstant harmonic function in $L^\infty(\mathbb{D})$, then $W(T_\phi)$ equals the relative interior of $R_\phi^\wedge$.*

Proof. By Proposition 1, $W(T_\phi)$ is contained in the relative interior of $R_\phi^\wedge$. If they are not equal, then $W(T_{\mathrm{Re}\,(e^{i\theta}(\phi+c))}) \subsetneqq \mathrm{Rel\ Int}\ R^\wedge_{\mathrm{Re}\,(e^{i\theta}(\phi+c))}$ for some real θ and complex c, which is in contradiction to [10, Lemma 1 and Theorem 2]. □

The assertion in the preceding proposition is analogous to the corresponding one for Toeplitz operators on the Hardy space (cf. [7]).

An operator A is said to be *convexoid* if $\overline{W(A)} = \sigma(A)^\wedge$. Note that it is unknown whether $\sigma(T_\phi) = R_\phi$ holds for harmonic ϕ in $L^\infty(\mathbb{D})$. If this is indeed the case, then we would have the convexoidity of Toeplitz operators with harmonic symbols. The following result is a partial confirmation of this.

Proposition 4. *If ϕ is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, then T_ϕ is convexoid.*

Proof. For a continuous ϕ on $\overline{\mathbb{D}}$, it is known that $\sigma_e(T_\phi)$, the essential spectrum of T_ϕ, equals $\phi(\partial\mathbb{D})$ (cf. [1, Corollary 10]). Hence $\phi(\partial\mathbb{D}) \subseteq \sigma(T_\phi)$. Next we show that every extreme point of $R_\phi^\wedge = \phi(\overline{\mathbb{D}})^\wedge$ is in $\phi(\partial\mathbb{D})$. Indeed, if z_0 is an extreme point of $\phi(\overline{\mathbb{D}})^\wedge$, then it is in $\partial\phi(\overline{\mathbb{D}})$. Let the real θ_0 and r_0 and the complex c_0 be such that $\phi_0 \equiv e^{i\theta_0}\phi + c_0$ satisfies $\phi_0(\overline{\mathbb{D}}) \subseteq \{z \in \mathbb{C} : |z| \le r_0\}$ and $|e^{i\theta_0}z_0 + c_0| = r_0$. Then ϕ_0 is harmonic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and $e^{i\theta_0}z_0 + c_0$ in $\partial\phi_0(\overline{\mathbb{D}})$ satisfies $|e^{i\theta_0}z_0 + c_0| = \max|\phi_0(\overline{\mathbb{D}})|$. The maximum modulus principle says that $e^{i\theta_0}z_0 + c_0 = \phi_0(u_0)$ for some u_0 in $\partial\mathbb{D}$. Hence $z_0 = \phi(u_0)$ is in $\phi(\partial\mathbb{D})$. Therefore, the Krein–Milman theorem implies that

$$R_\phi^\wedge = \phi(\overline{\mathbb{D}})^\wedge \subseteq \phi(\partial\mathbb{D})^\wedge \subseteq \sigma(T_\phi)^\wedge.$$

This, together with Proposition 1 or 3, yields $\overline{W(T_\phi)} \subseteq \sigma(T_\phi)^\wedge$. Since $\sigma(T_\phi)^\wedge \subseteq \overline{W(T_\phi)}$ always holds (cf. [6, Problem 214]), the convexoidity of T_ϕ follows. □

We now consider the main topic of this paper: radial Toeplitz operators. The following characterization of such operators is known in the literature (cf. [9, p. 631]).

Proposition 5. *Let ϕ be a function in $L^\infty(\mathbb{D})$. Then T_ϕ has a diagonal matrix representation relative to the standard basis $\{e_n\}_{n=0}^\infty$ of $A^2(\mathbb{D})$ if and only if ϕ is radial. In this case, the asserted matrix representation of T_ϕ is*

$$\mathrm{diag}\ \left(2(n+1)\int_0^1 r^{2n+1}\phi(r)\,\mathrm{d}r.\right.$$

The next corollary is an easy consequence.

Corollary 6. *Let ϕ be a radial function in $L^\infty(\mathbb{D})$. If $\phi(1^-) \equiv \lim_{r\to 1^-}\phi(r)$ exists, then T_ϕ is the sum of the scalar operator $\phi(1^-)I$ and a compact operator.*

By Proposition 4, certain Toeplitz operators with harmonic symbols are convexoid. The same is true for radial Toeplitz operators since they are normal and normal operators are convexoid (cf. [6, Problem 216]). The next theorem gives examples of nonconvexoid Toeplitz operators.

Theorem 7. *If ϕ is a radial continuous function on $\overline{\mathbb{D}}$ with $\phi(1) = 0$ and ψ is a function in H^∞ with $\psi(0) = 0$, then $T_{\phi\psi}$ is compact and quasinilpotent. If, in addition, $\phi\psi$ is nonzero, then $T_{\phi\psi}$ is not convexoid.*

An operator A is *quasinilpotent* if its spectrum $\sigma(A)$ is the singleton $\{0\}$. Note that the preceding theorem implies that a Toeplitz operator with continuous symbol may not be convexoid.

Proof of Theorem 7. Let $\psi(z) = \sum_{k=1}^{\infty} a_k z^k$ on $\mathbb{D}$. For $m, n \geq 0$, we have

$$\begin{aligned} b_{mn} \equiv& \langle T_{\phi\psi} e_n, e_m \rangle = \frac{\sqrt{(n+1)(m+1)}}{\pi} \iint_{\mathbb{D}} \phi(z)\psi(z) z^n \bar{z}^m dA(z) \\ =& \frac{\sqrt{(n+1)(m+1)}}{\pi} \sum_{k=1}^{\infty} a_k \iint_{\mathbb{D}} \phi(z) z^{n+k} \bar{z}^m dA(z) \\ =& \frac{\sqrt{(n+1)(m+1)}}{\pi} \sum_{k=1}^{\infty} a_k \left(\int_0^1 r^{n+k+m+1} \phi(r) dr \right) \left(\int_0^{2\pi} e^{i(n+k-m)\theta} d\theta \right) \\ =& \begin{cases} 2\sqrt{(n+1)(m+1)} a_{m-n} \displaystyle\int_0^1 r^{2m+1} \phi(r) dr & \text{if } m > n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus $A = [b_{mn}]_{m,n=0}^{\infty}$, the matrix representation of $T_{\phi\psi}$ relative to the standard basis $\{e_n\}_{n=0}^{\infty}$, is lower triangular with zero diagonals. For each $j \geq 0$, let A_j be the matrix obtained from A by replacing the b_{mn}'s with $m > j$ by 0. Since ϕ is radial with $\phi(1) = 0$, the Toeplitz operator T_ϕ is compact (cf. [8]). The same is true for $T_{\phi\psi} = T_\phi T_\psi$. Hence A_j converges to A in norm and $\sigma(A)$ is totally disconnected. It follows that $\sigma(A_j)$ converges to $\sigma(A)$ in the Hausdorff metric (cf. [3, Corollary 3.4]). Because $\sigma(A_j) = \{0\}$ for all j, we conclude that A is quasinilpotent and hence so is $T_{\phi\psi}$. If $\phi\psi$ is nonzero, then $\overline{W(T_{\phi\psi})} \neq \{0\} = \sigma(T_{\phi\psi})^\wedge$, that is, $T_{\phi\psi}$ is not convexoid. □

In the following, we show that many commonly seen convex subsets of the plane are numerical ranges of radial Toeplitz operators. This we start with intervals on the real line.

Proposition 8. *If ϕ is a real-valued radial function in $L^\infty(\mathbb{D})$, then*

$$\overline{W(T_\phi)} = \left[\inf_{n\geq 0} \lambda_n, \sup_{n\geq 0} \lambda_n \right], \quad \textit{where } \lambda_n = 2(n+1) \int_0^1 r^{2n+1} \phi(r) \, \mathrm{d}r \textit{ for } n \geq 0.$$

If, in addition, $\phi(r)$ is (almost) nonconstant and increasing (resp., decreasing) in r on $[0, 1)$, then

$$W(T_\phi) = \left[2\int_0^1 r\phi(r)\,\mathrm{d}r, \phi(1^-)\right) \quad \left(\textit{resp. } \left(\phi(1^-), 2\int_0^1 r\phi(r)\,\mathrm{d}r\right]\right).$$

An example of decreasing $\phi(r)$ is given in Example 2.

Proof of Proposition 8. The first assertion is an easy consequence of Proposition 5 and the fact that normal operators are convexoid [6, Problem 216].

Now assume that $\phi(r)$ is increasing in r. By the change of variable $s = r^{2n+2}$, we have

$$\lambda_n = \int_0^1 \phi(s^{1/(2n+2)})\,\mathrm{d}s \geq \int_0^1 \phi(s^{1/(2n)})\,\mathrm{d}s = \lambda_{n-1}$$

for $n \geq 1$. Moreover, if here the equality holds for some $n \geq 1$, then $\phi(s^{1/(2n+2)}) = \phi(s^{1/(2n)})$ a.e. or $\phi(r) = \phi(r^{(n+1)/n})$ a.e. on $[0, 1)$, which is impossible since $\phi(r)$ is nonconstant and increasing. Hence the λ_n's are strictly increasing in n. Our assertion for $W(T_\phi)$ follows immediately. Analogous arguments apply to decreasing $\phi(r)$. □

Some of our later results on the numerical ranges of radial Toeplitz operators are proved based on the construction for the essential spectrum due to Grudsky and Vasilevski [4]. These we summarize briefly below.

For any real $t \neq 0$, let

$$\phi_t(z) = \begin{cases} \dfrac{1}{\Gamma(1+it)}(\ln|z|^{-2})^{it} & \text{if } z \in \mathbb{D} \text{ and } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases} \tag{1}$$

where $\Gamma(\cdot)$ denotes the usual Gamma function. Then it was shown that the corresponding $\lambda_n \equiv 2(n+1)\int_0^1 r^{2n+1}\phi(r)dr$, $n \geq 0$, is given by $(n+1)^{-it}$ (cf. [4, Example 4]). Thus we can derive that $W(T_{\phi_t}) = \mathbb{D} \cup \{(n+1)^{-it} : n \geq 0\}$ for $t \neq 0$, $W(T_{\mathrm{Im}\,\phi_1}) = (-1, 1)$ and $W(T_{\mathrm{Im}\,\phi_1 + i\mathrm{Im}\,\phi_\pi}) = (-1, 1) \times (-1, 1)$.

Theorem 9. *Any finite line segment in the plane is the numerical range of some radial Toeplitz operator.*

Proof. We may assume that the finite line segment I is on the real line. If $I = [a, b)$ (resp., $(a, b]$), then it is the numerical range of T_ϕ, where

$$\phi(z) = \begin{cases} 4a - 3b & \text{if } |z| \leq 1/2, \\ b & \text{if } 1/2 < |z| < 1 \end{cases}$$

(resp.,

$$\phi(z) = \begin{cases} 4b - 3a & \text{if } |z| \leq 1/2, \\ a & \text{if } 1/2 < |z| < 1) \end{cases}$$

(cf. Example 2 or Proposition 8).

If $I = (a, b)$, then $I = W(T_\phi)$ for

$$\phi = -\frac{a-b}{2}\mathrm{Im}\,\phi_1 + \frac{a+b}{2},$$

where ϕ_1 is the radial function given in (1).

Finally, consider $I = [a, b]$. Let r_1 and r_2 be such that $1/\sqrt{2} < r_1 < 1/2^{1/4}$ and $2^{1/4}r_1 < r_2 < 1$. If

$$\phi(z) = \begin{cases} 1 & \text{if } 0 \le |z| \le r_1, \\ -1 & \text{if } r_1 < |z| \le r_2, \\ 0 & \text{if } r_2 < |z| < 1, \end{cases}$$

then $2\int_0^1 r\phi(r)dr = 2r_1^2 - r_2^2 > 0$, $4\int_0^1 r^3\phi(r)\,\mathrm{d}r = 2r_1^4 - r_2^4 < 0$, and

$$2(n+1)\int_0^1 r^{2n+1}\phi(r)\,\mathrm{d}r = 2r_1^{2n+2} - r_2^{2n+2} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus $W(T_\phi)$ is some closed interval $[c, d]$ with $c < 0 < d$. If

$$\psi = \frac{a-b}{c-d}\phi + \frac{bc-ad}{c-d},$$

then $W(T_\psi) = [a, b]$, completing the proof. □

For convex sets in the plane with nonempty interior, we make use of the radial functions ϕ_t in (1) to prove the following theorem.

Theorem 10. *For any function f analytic on an open set containing $\overline{\mathbb{D}}$, there is a radial function ϕ in $L^\infty(\mathbb{D})$ such that $\sigma_e(T_\phi) = f(\partial\mathbb{D})$ and $W(T_{\phi_m}) = f(\mathbb{D})^\wedge$, where $\phi_m(z) = |z|^{2m}\phi(z)$ for $z \in \mathbb{D}$, for all $m \ge 1$.*

Proof. Let $f(z) = \sum_{k=0}^\infty a_k z^k$ on $\mathbb{D}$. By our assumption on f, we have $\alpha \equiv \limsup_{k\to\infty} |a_k|^{1/k} < 1$. Let $0 < t < -(2/\pi)\ln\alpha$ and

$$\phi(z) = \begin{cases} \displaystyle\sum_{k=0}^\infty \frac{a_k}{\Gamma(1+itk)}(\ln|z|^{-2})^{itk} & \text{if } z \in \mathbb{D} \text{ and } z \ne 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Then

$$\begin{aligned}\limsup_{k\to\infty}\Big|\frac{a_k}{\Gamma(1+itk)}\Big|^{1/k} &= \limsup_{k\to\infty}|a_k|^{1/k}\lim_{k\to\infty}|\Gamma(1+itk)|^{-1/k} \\ &= \alpha\lim_{k\to\infty}(2\pi)^{-1/(2k)}|e^{-itk(-1/k)}||(itk)^{(itk+(1/2))(-1/k)}| \\ &= \alpha\lim_{k\to\infty}|e^{(-it-(1/(2k)))(\ln(tk)+i(\pi/2))}| \\ &= \alpha\lim_{k\to\infty}e^{-(1/(2k))\ln(tk)+t(\pi/2)} \\ &= \alpha e^{t(\pi/2)} < 1,\end{aligned}$$

where the second equality is a consequence of Stirling's formula

$$\lim_{\substack{|z|\to\infty \\ -\pi+\epsilon<\arg z<\pi-\epsilon}} \frac{\Gamma(1+z)}{\sqrt{2\pi}e^{-z}z^{z+(1/2)}} = 1, \quad \epsilon > 0$$

(cf. [2, p. 253, Section 34D]). This shows that the radius of convergence of the power series $\sum_k (a_k/\Gamma(1+itk))z^k$ is bigger than 1. Hence ϕ is a radial function in $L^\infty(\mathbb{D})$. For $n \geq 0$, we have

$$\begin{aligned}
2(n+1)\int_0^1 r^{2n+1}\phi(r)\,\mathrm{d}r &= 2(n+1)\int_0^1 r^{2n+1}\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(1+itk)}(\ln r^{-2})^{itk}\,\mathrm{d}r \\
&= \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(1+itk)}\int_0^1 (\ln s^{-1/(n+1)})^{itk}\,\mathrm{d}s \quad (\text{ letting } s = r^{2(n+1)}) \\
&= \sum_{k=0}^{\infty} a_k(n+1)^{-itk} = f((n+1)^{-it}).
\end{aligned}$$

Since the set $\{(n+1)^{-it} : n \geq 0\}$ is dense in $\partial\mathbb{D}$, we obtain $\sigma_e(T_\phi) = f(\partial\mathbb{D})$.

For the numerical range, we may assume that $f(0) = 0$. This is because if $\tilde{f}(z) = f(z) - a_0$ and $\tilde{f}(\mathbb{D})^\wedge = W(T_\psi)$ for some radial ψ in $L^\infty(\mathbb{D})$, then

$$f(\mathbb{D})^\wedge = \tilde{f}(\mathbb{D})^\wedge + a_0 = W(T_\psi) + a_0 = W(T_{\psi+a_0}).$$

A computation as above with ϕ replaced by ϕ_m yields that

$$2(n+1)\int_0^1 r^{2n+1}\phi_m(r)dr = \frac{n+1}{n+m+1}f((n+m+1)^{-it}), \quad n \geq 0.$$

Since 0 is in $f(\mathbb{D})$ and $\{(n+m+1)^{-it} : n \geq 0\}$ is dense in $\partial\mathbb{D}$, the convexity of $f(\mathbb{D})^\wedge$ implies that $W(T_{\phi_m}) = f(\mathbb{D})^\wedge$. □

Corollary 11. *Any open elliptic disc is the numerical range of some radial Toeplitz operator.*

Proof. If E is an open elliptic disc, then let

$$\psi(z) = (a\mathrm{Re}\, z + b\mathrm{Im}\, z + c) + i(u\mathrm{Re}\, z + v\mathrm{Im}\, z + w),$$

where a, b, c, u, v and w are real with $av \neq bu$, be an affine transformation which maps $\mathbb{D}$ onto E. Theorem 10 says that $\mathbb{D} = W(T_\phi)$ for some radial function ϕ in $L^\infty(\mathbb{D})$. If $\eta = \psi \circ \phi$, then η is radial in $L^\infty(\mathbb{D})$ and

$$W(T_\eta) = W(\psi(T_\phi)) = \psi(W(T_\phi)) = \psi(\mathbb{D}) = E.$$ □

The proof of Theorem 10 can be combined with the arguments for [4, Corollary 3.10] to yield the following proposition, whose proof we omit.

Proposition 12. *For any polynomial p (resp., trigonometric polynomial q), the convex set $p((-1,1)\times(-1,1))^\wedge$ (resp.,* Int *$q(\partial D)^\wedge$) is the numerical range of some radial Toeplitz operator.*

Note that if the assertion in Theorem 10 is true for any function analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, then, in view of the Riemann mapping theorem, every nonempty bounded open convex subset of the plane is the numerical range of some radial Toeplitz operator. Unable to prove this, we show that at least an asymptotic version of it is indeed true.

Proposition 13. *Let $\triangle$ be a nonempty bounded open convex subset of the plane. Then there is a sequence of radial functions ϕ_n in $L^\infty(\mathbb{D})$ such that $W(T_{\phi_n})$ is open for all n and increases to $\triangle$.*

Proof. By the Riemann mapping theorem, there is an analytic function $f(z) = \sum_{k=0}^\infty a_k z^k$ on $\mathbb{D}$ which maps $\mathbb{D}$ onto $\triangle$ injectively. For each $n \geq 0$, let $D_n = \{nz/(n+1) : z \in \mathbb{D}\}$ and $f_n(z) = \sum_{k=0}^n a_k z^k$. Since the boundaries $\partial f(D_n)$ are compact and pairwise disjoint, we have $d_n \equiv \operatorname{dist}(\partial f(D_n), \partial f(D_{n+1})) > 0$ for all n. Let $\{k_n\}_{n=1}^\infty$ be a (strictly) increasing sequence such that

$$\sup\{|f_{k_n}(z) - f(z)| : z \in D_n\} < \min\left\{\frac{d_{n-1}}{2}, \frac{d_n}{2}\right\}, \quad n \geq 1.$$

Since $f(D_n)$ increases to $\triangle$, from the construction of the k_n's, we derive that $f_{k_n}(D_n)$ also increases to $\triangle$. By Theorem 10, each $f_{k_n}(D_n)^\wedge$ is the numerical range of some radial Toeplitz operator T_{ϕ_n}. We conclude that $W(T_{\phi_n})$ is open and increases to $\triangle$. □

Finally, we come to closed polygonal regions.

Theorem 14. *Any compact convex polygonal region is the numerical range of some radial Toeplitz operator.*

To prove this, we need the following lemma.

Lemma 15. *For any $\epsilon > 0$ and complex numbers $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$, there is an integer $m \geq n-1$ and a function f of the form*

$$f(k) = a_0\left(\frac{1}{2}\right)^{k+1} + a_1\left(\frac{1}{4}\right)^{k+1} + \cdots + a_m\left(\frac{1}{2^{m+1}}\right)^{k+1}, \quad k \geq 0, \tag{2}$$

such that $f(k) = \lambda_k$ for $0 \leq k \leq n-1$ and $|f(k)| \leq \epsilon$ for all $k \geq n$.

Proof. Let A_m, $m \geq 1$, denote the $(m+1)$-by-$(m+1)$ Vandermonde-type matrix $[1/2^{i+j+1}]_{i,j=0}^m$. Since the determinant of A_m equals the nonzero

$$\left(\prod_{i=0}^m \frac{1}{2^{i+1}}\right)\left(\prod_{0\leq i<j\leq m}\left(\frac{1}{2^{j+1}} - \frac{1}{2^{i+1}}\right)\right),$$

A_m is invertible. Let $A_m^{-1} = [b_{ij}]_{i,j=0}^m$. Here the entries b_{ij} depend on m (for the sake of simplicity, we don't add further indices to them). For a large m (to be determined later), let $a_i = \sum_{j=0}^{n-1} b_{ij}\lambda_j$, $0 \le i \le m$, and let f be defined as in (2). If a and c denote the $(m+1)$-vectors

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda_0 \\ \vdots \\ \lambda_{n-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

respectively, then $a = A_m^{-1}c$. Hence $A_m a = c$, which is the same as

$$f(k) = \begin{cases} \lambda_k & \text{if } 0 \le k \le n-1, \\ 0 & \text{if } n \le k \le m. \end{cases}$$

We now check that $|f(k)|$ can be made arbitrarily small for any $k > m$. Indeed, we have

$$\begin{aligned} |f(k)| &\le \sum_{i=0}^{m} |a_i| \big(\frac{1}{2^{i+1}}\big)^{k+1} \\ &\le \sum_{i=0}^{m}\sum_{j=0}^{n-1} |b_{ij}||\lambda_j| \big(\frac{1}{2^{i+1}}\big)^{m+1} \\ &= \sum_{j=0}^{n-1} |\lambda_j| \bigg(\sum_{i=0}^{m} |b_{ij}| \big(\frac{1}{2^{i+1}}\big)^{m+1} \bigg). \end{aligned}$$

To proceed further, we show that

$$\sum_{i=0}^{m} |b_{ij}| \left(\frac{1}{2^{i+1}}\right)^{m+1} \longrightarrow 0 \quad \text{as } m \to \infty$$

for any j, $0 \le j \le n-1$. Let the (Vandermonde interpolation) polynomial

$$p_i(x) = \prod_{\substack{l=0 \\ l \ne i}}^{m} \frac{x - \frac{1}{2^{l+1}}}{\frac{1}{2^{i+1}} - \frac{1}{2^{l+1}}}, \quad 0 \le i \le m,$$

be expanded as $(1/2^{i+1}) \sum_{j=0}^{m} c_{ij} x^j$. Then

$$\frac{1}{2^{i+1}} \sum_{j=0}^{m} c_{ij} \left(\frac{1}{2^{l+1}}\right)^j = p_i\left(\frac{1}{2^{l+1}}\right) = \delta_{il}, \quad 0 \le l \le m,$$

which shows that $c_{ij} = b_{ij}$ for all i and j. Moreover, for each fixed i, the b_{ij}'s have alternating signs. This can be seen by computing the higher-order derivatives $p_i^{(j)}$

of p_i and noting that $b_{ij} = 2^{i+1}p_i^{(j)}(0)/j!$. Hence

$$\begin{aligned}
&\sum_{i=0}^{m} |b_{ij}| \Big(\frac{1}{2^{i+1}}\Big)^{m+1} \\
&\le \sum_{i=0}^{m} 2^j \Big| \sum_{j=0}^{m} b_{ij}\Big(-\frac{1}{2}\Big)^j \Big| \Big(\frac{1}{2^{i+1}}\Big)^{m+1} \\
&= 2^j \sum_{i=0}^{m} 2^{i+1} \Big| p_i\Big(-\frac{1}{2}\Big)\Big| \Big(\frac{1}{2^{i+1}}\Big)^{m+1} \\
&= 2^j \sum_{i=0}^{m} 2^{i+1} \Big| \prod_{\substack{l=0\\ l\ne i}}^{m} \frac{\frac{1}{2}+\frac{1}{2^{l+1}}}{\frac{1}{2^{i+1}}-\frac{1}{2^{l+1}}} \Big| \Big(\frac{1}{2^{i+1}}\Big)^{m+1} \\
&= 2^j \sum_{i=0}^{m} \Big| \prod_{\substack{l=0\\ l\ne i}}^{m} \frac{\frac{1}{2}+\frac{1}{2^{l+1}}}{1-2^{i-l}} \Big| \\
&\le 2^j \Big(\prod_{l=1}^{\infty}(1-2^{-l})\Big)^{-1} \sum_{i=0}^{m} \prod_{l=0}^{m-1} \Big(\frac{1}{2}+\frac{1}{2^{l+1}}\Big) \\
&\le 2^j \Big(\prod_{l=1}^{\infty}(1-2^{-l})\Big)^{-1} (m+1)\Big(\frac{1}{2}+\frac{1}{4}\Big)^{m-1} \longrightarrow 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Hence for a large $m \ge n-1$ we have $|f(k)| \le \epsilon$ for all $k \ge n$. □

We now proceed to prove Theorem 14.

Proof of Theorem 14. Let $\triangle$ be a compact convex polygonal region with n (≥ 3) vertices $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. We may assume that 0 is in its interior. Let $\epsilon > 0$ be such that the circular disc $\{z \in \mathbb{D} : |z| \le \epsilon\}$ is contained in $\triangle$, and let

$$\phi(z) = \sum_{i=0}^{m} a_i \chi_{[0,1/\sqrt{2^{i+1}}]}(|z|) \qquad \text{for } z \in \mathbb{D},$$

where m and the a_i's are as given in Lemma 15. Then ϕ is a radial function in $L^\infty(\mathbb{D})$ and

$$\begin{aligned}
2(k+1)\int_0^1 r^{2k+1}\phi(r)dr &= \sum_{i=0}^{m} a_i 2(k+1)\int_0^{1/\sqrt{2^{i+1}}} r^{2k+1}dr \\
&= \sum_{i=0}^{m} a_i \big(\frac{1}{2^{i+1}}\big)^{k+1} = f(k), \quad k \ge 0,
\end{aligned}$$

by (2). Since Lemma 15 says that $f(k) = \lambda_k$ for $0 \le k \le n-1$ and $|f(k)| \le \epsilon$ for all $k \ge n$, the convex hull of the $f(k)$'s equals $\triangle$, that is, $W(T_\phi) = \triangle$ as required. □

We conclude this paper by asking which (nonempty bounded convex) subset of the plane is the numerical range of a radial Toeplitz operator. One constraint is that it can have at most countably many extreme points as is the case for any normal operator (on a separable Hilbert space). In particular, is every nonempty bounded open convex subset the numerical range of some radial T_ϕ? This seems to be quite plausible although we don't know how to prove it at present.

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