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The bipanpositionable bipancyclic property of the hypercube[☆,☆☆]

Yuan-Kang Shih [a], Cheng-Kuan Lin [a], Jimmy J.M. Tan [a,*], Lih-Hsing Hsu [b]

[a] Department of Computer Science, National Chiao Tung University, Hsinchu, 30010, Taiwan, ROC

[b] Department of Computer Science and Information Engineering, Providence University, Taichung, 43301, Taiwan, ROC

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ABSTRACT

A bipartite graph is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. A hamiltonian bipartite graph G is *bipanpositionable* if, for any two different vertices x and y, there exists a hamiltonian cycle C of G such that $d_C(x, y) = k$ for any integer k with $d_G(x, y) \leq k \leq |V(G)|/2$ and $(k - d_G(x, y))$ being even. A bipartite graph G is *k-cycle bipanpositionable* if, for any two different vertices x and y, there exists a cycle of G with $d_C(x, y) = l$ and $|V(C)| = k$ for any integer l with $d_G(x, y) \leq l \leq \frac{k}{2}$ and $(l - d_G(x, y))$ being even. A bipartite graph G is *bipanpositionable bipancyclic* if G is k-cycle bipanpositionable for every even integer k, $4 \leq k \leq |V(G)|$. We prove that the hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$.

1. Introduction

For the graph definitions and notations we follow Bondy and Murty [1]. Let $G = (V, E)$ be a *graph*, where V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* is represented by $\langle v_0, v_1, v_2, \dots, v_k\rangle$, where all vertices are distinct except possibly $v_0 = v_k$. The *length* of a path Q is the number of edges in Q. We also write the path $\langle v_0, v_1, v_2, \dots, v_k\rangle$ as $\langle v_0, Q_1, v_i, v_{i+1} \dots, v_j, Q_2, v_t, \dots, v_k\rangle$, where Q_1 is the path $\langle v_0, v_1, \dots, v_{i-1}, v_i\rangle$ and Q_2 is the path $\langle v_j, v_{j+1}, \dots, v_{t-1}, v_t\rangle$. We use $d_G(u, v)$ to denote the distance between u and v in G, i.e., the length of the shortest path joining u to v in G. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. We use $d_c(u, v)$ to denote the distance between u and v in a cycle C, i.e., the length of the shortest path joining u to v in C. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V(G) = V_0 \cup V_1$ and $E(G)$ is a subset of $\{(u, v) \mid u \in V_0$ and $v \in V_1\}$.

The *n-dimensional hypercube*, Q_n, consists of all n-bit binary strings as its vertices and two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if their binary labels are different in exactly one bit position. Let $\mathbf{u} = u_{n-1}u_{n-2}\dots u_1u_0$ and $\mathbf{v} = v_{n-1}v_{n-2}\dots v_1v_0$ be two n-bit binary strings. The Hamming distance $h(u, v)$ between two vertices u and v is the number of different bits in the corresponding strings of both vertices. Let Q_n^i be the subgraph of Q_n induced by $\{u_{n-1}u_{n-2}\cdots u_1u_0 \mid u_{n-1} = i\}$ for $i = 0, 1$. Therefore, Q_n can be constructed recursively by taking two copies of Q_{n-1}, Q_n^0 and Q_n^1, and adding a perfect matching between these two copies. For a vertex $\mathbf{u}$ in Q_n^0 (resp. Q_n^1), we use $\bar{\mathbf{u}}$ to denote the unique neighbor of $\mathbf{u}$ in Q_n^1 (resp. Q_n^0). The *hypercube* is a widely used topology in computer architecture, see Leighton [2].

A graph is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ inclusive. The concept of pancyclic graphs was proposed by Bondy [3]. Since there is no odd cycle in bipartite graph, the concept of a bipancyclic graph was proposed

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* Corresponding author.
E-mail addresses: ykshih@cs.nctu.edu.tw (Y.-K. Shih), cklin@cs.nctu.edu.tw (C.-K. Lin), jmtan@cs.nctu.edu.tw (J.J.M. Tan), lhhsu@pu.edu.tw (L.-H. Hsu).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
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by Mitchem and Schmeichel [4]. A bipartite graph is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. It is proved that the hypercube Q_n is bipancyclic if $n \geq 2$ [5,6]. A graph is *panconnected* if, for any two different vertices x and y, there exists a path of length l joining x and y for every l with $d_G(x, y) \leq l \leq |V(G)| - 1$. The concept of panconnected graphs was proposed by Alavi and Williamson [7]. It is easy to see that any bipartite graph with at least 3 vertices is not panconnected. Therefore, the concept of bipanconnected graphs is proposed. A bipartite graph is *bipanconnected* if, for any two different vertices x and y, there exists a path of length l joining x and y for every l with $d_G(x, y) \leq l \leq |V(G)| - 1$ and $(l - d_G(x, y))$ being even. It is proved that the hypercube is bipanconnected [5]. A hamiltonian graph G is *panpositionable* if for any two different vertices x and y of G and for any integer k with $d_G(x, y) \leq k \leq |V(G)|/2$, there exists a hamiltonian cycle C of G such that $d_C(x, y) = k$. A hamiltonian bipartite graph G is *bipanpositionable* if for any two different vertices x and y of G and for any integer k with $d_G(x, y) \leq k \leq |V(G)|/2$ and $(k - d_G(x, y))$ being even, there exists a hamiltonian cycle C of G such that $d_C(x, y) = k$. The concepts of panpositionable and bipanpositionable were proposed by Kao et al. [8]. They proved that the hypercube Q_n is bipanpositionable if $n \geq 2$ [8]. A bipartite graph G is *edge-bipancyclic* if for any edge in G, there is a cycle of every even length from 4 to $|V(G)|$ traversing through this edge. The concept of edge-bipancyclic was proposed by Alspach and Hare [9]. A bipartite graph G is *vertex-bipancyclic* if for any vertex in G, there is a cycle of every even length from 4 to $|V(G)|$ going through this vertex. The concept of vertex-bipancyclic was proposed by Hobbs [10]. Obviously, every edge-bipancyclic graph is vertex-bipancyclic. It is proved that the hypercube Q_n is edge-bipancyclic if $n \geq 2$ [5].

In this paper, we propose a more interesting property about hypercubes. A k-cycle is a cycle of length k. A bipartite graph G is *k-cycle bipanpositionable* if for every different vertices x and y of G and for any integer l with $d_G(x, y) \leq l \leq \frac{k}{2}$ and $(l - d_G(x, y))$ being even, there exists a k-cycle C of G such that $d_C(x, y) = l$. (Note that $d_C(x, y) \leq \frac{k}{2}$ for every cycle C of length k.) A bipartite graph G is *bipanpositionable bipancyclic* if G is k-cycle bipanpositionable for every even integer k with $4 \leq k \leq |V(G)|$. In this paper, we prove that the hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$. As a consequence of this result, we can see that many previous results on hypercubes follows directly from ours. For example, the hypercube is bipancyclic, bipanconnected, bipanpositionable, edge-bipancyclic and vertex-bipancyclic. Therefore, our result unifies these results in a general sense.

2. The bipanpositionable bipancyclic property

We prove our main result by induction as stated in Lemma 1 and Theorem 1 below.

Lemma 1. *The hypercube Q_3 is bipanpositionable bipancyclic.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be two different vertices in Q_3. Obviously, $d_{Q_3}(\mathbf{x}, \mathbf{y}) = 1$, 2 or 3. Since the hypercube is vertex symmetric, without loss of generality, we may assume that $\mathbf{x} = 000$.

Case 1: Suppose that $d_{Q_3}(\mathbf{x}, \mathbf{y}) = 1$. Since Q_3 is edge symmetric, we assume that $\mathbf{y} = 001$.

$\mathbf{y} = 001$	4-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 1$	$\langle 000, 001, 011, 010, 000\rangle$
	6-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 1$	$\langle 000, 001, 101, 111, 110, 100, 000\rangle$
		$d_C(\mathbf{x}, \mathbf{y}) = 3$	$\langle 000, 100, 101, 001, 011, 010, 000\rangle$
	8-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 1$	$\langle 000, 001, 101, 111, 011, 010, 110, 100, 000\rangle$
		$d_C(\mathbf{x}, \mathbf{y}) = 3$	$\langle 000, 100, 101, 001, 011, 111, 110, 010, 000\rangle$

Case 2: Suppose that $d_{Q_3}(\mathbf{x}, \mathbf{y}) = 2$. By symmetry, we assume that $\mathbf{y} = 011$.

$\mathbf{y} = 011$	4-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 2$	$\langle 000, 001, 011, 010, 000\rangle$
	6-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 2$	$\langle 000, 001, 011, 010, 110, 100, 000\rangle$
	8-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 2$	$\langle 000, 001, 011, 010, 110, 111, 101, 100, 000\rangle$
		$d_C(\mathbf{x}, \mathbf{y}) = 4$	$\langle 000, 001, 101, 111, 011, 010, 110, 100, 000\rangle$

Case 3: Suppose that $d_{Q_3}(\mathbf{x}, \mathbf{y}) = 3$. We have $\mathbf{y} = 111$.

$\mathbf{y} = 111$	6-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 3$	$\langle 000, 001, 011, 111, 110, 100, 000\rangle$
	8-cycle	$d_C(\mathbf{x}, \mathbf{y}) = 3$	$\langle 000, 001, 011, 111, 101, 100, 110, 010, 000\rangle$

Thus, Q_3 is bipanpositionable bipancyclic. $\square$

Theorem 1. *The hypercube Q_n is bipanpositionable bipancyclic for $n \geq 2$.*

Proof. We observe that Q_1 is not bipanpositionable bipancyclic. So we start with $n \geq 2$. We prove Q_n is bipanpositionable bipancyclic by induction on n. It is easy to see that Q_2 is bipanpositionable bipancyclic. By Lemma 1, this statement holds for $n = 3$. Suppose that Q_{n-1} is bipanpositionable bipancyclic for some $n \geq 4$. Let $\mathbf{x}$ and $\mathbf{y}$ be two distinct vertices in Q_n, and let k be an even integer with $k \geq \max\{4, 2d_{Q_n}(\mathbf{x}, \mathbf{y})\}$ and $k \leq 2^n$. For every integer l with $d_{Q_n}(\mathbf{x}, \mathbf{y}) \leq l \leq \frac{k}{2}$ and $(l - d_{Q_n}(\mathbf{x}, \mathbf{y}))$ being even, we need to construct a k-cycle C of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$.

Case 1: $d_{Q_n}(\mathbf{x}, \mathbf{y}) = 1$. Without loss of generality, we may assume that both $\mathbf{x}$ and $\mathbf{y}$ are in Q_n^0. $(l - d_{Q_n}(\mathbf{x}, \mathbf{y}))$ is even, so l is an odd number. Since Q_n^0 is isomorphic to Q_{n-1}, by introduction, there is a k-cycle of Q_n^0 with $d_C(\mathbf{x}, \mathbf{y}) = l$ for every $4 \leq k \leq 2^{n-1}$. Thus, we consider that $k \geq 2^{n-1} + 2$.

Case 1.1: $l = 1$. By induction, there is a (2^{n-1})-cycle $C' = \langle \mathbf{x}, P, \mathbf{z}, \mathbf{y}, \mathbf{x} \rangle$ of Q_n^0 where $d_P(\mathbf{x}, \mathbf{z}) = 2^{n-1} - 2$. Suppose that $k - 2^{n-1} = 2$. Then $C = \langle \mathbf{x}, P, \mathbf{z}, \bar{\mathbf{z}}, \bar{\mathbf{y}}, \mathbf{y}, \mathbf{x} \rangle$ forms a $(2^{n-1} + 2)$-cycle with $d_C(\mathbf{x}, \mathbf{y}) = 1$. Suppose that $k - 2^{n-1} \geq 4$. By induction, there is a $(k - 2^{n-1})$-cycle C'' of Q_n^1 such that $d_{C''}(\bar{\mathbf{z}}, \bar{\mathbf{y}}) = 1$. We write $C'' = \langle \bar{\mathbf{z}}, R, \bar{\mathbf{y}}, \bar{\mathbf{z}} \rangle$ with $d_R(\bar{\mathbf{z}}, \bar{\mathbf{y}}) = k - 2^{n-1} - 1$. Then $C = \langle \mathbf{x}, P, \mathbf{z}, \bar{\mathbf{z}}, R, \bar{\mathbf{y}}, \mathbf{y}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l = 1$.

Case 1.2: $l \geq 3$. Suppose that $k - l - 1 \leq 2^{n-1}$. By induction, there is an $(l + 1)$-cycle C' of Q_n^0 with $d_{C'}(\mathbf{x}, \mathbf{y}) = 1$. We write $C' = \langle \mathbf{x}, P, \mathbf{y}, \mathbf{x} \rangle$ where $d_P(\mathbf{x}, \mathbf{y}) = l$. By induction, there is a $(k - l - 1)$-cycle C'' of Q_n^1 with $d_{C''}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = 1$. We then write $C'' = \langle \bar{\mathbf{y}}, R, \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle$ such that $d_R(\bar{\mathbf{y}}, \bar{\mathbf{x}}) = k - l - 2$. Then $C = \langle \mathbf{x}, P, \mathbf{y}, \bar{\mathbf{y}}, R, \bar{\mathbf{x}}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$. Suppose that $k - l - 2 \geq 2^{n-1} + 1$. By induction, there is a $(k - 2^{n-1})$-cycle C' of Q_n^0 with $d_{C'}(\mathbf{x}, \mathbf{y}) = l$. We write $C' = \langle \mathbf{x}, P, \mathbf{y}, R, \mathbf{u}, \mathbf{x} \rangle$ with $d_P(\mathbf{x}, \mathbf{y}) = l$ and $d_R(\mathbf{y}, \mathbf{x}) = k - (2^{n-1} - 1) - l - 2$. By induction, there is a (2^{n-1})-cycle C'' of Q_n^1 with $d_{C''}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = 1$. We write $C'' = \langle \bar{\mathbf{x}}, \bar{\mathbf{u}}, S, \bar{\mathbf{x}} \rangle$ with $d_S(\bar{\mathbf{u}}, \bar{\mathbf{x}}) = 2^{n-1} - 1$. Then $C = \langle \mathbf{x}, P, \mathbf{y}, R, \mathbf{u}, \bar{\mathbf{u}}, S, \bar{\mathbf{x}}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$.

Case 2: $d_{Q_n}(\mathbf{x}, \mathbf{y}) \geq 2$ and $l = 2$. Since $d_{Q_n}(\mathbf{x}, \mathbf{y}) \leq l$ and $l = 2$, so $d_{Q_n}(\mathbf{x}, \mathbf{y}) = 2$. Without loss of generality, we may assume that $\mathbf{x}$ is in Q_n^0 and $\mathbf{y}$ is in Q_n^1. Then $d_{Q_n}(\bar{\mathbf{x}}, \mathbf{y}) = 1$ and $d_{Q_n}(\bar{\mathbf{y}}, \mathbf{x}) = 1$.

Suppose that $k = 4$. Then $C = \langle \mathbf{x}, \bar{\mathbf{x}}, \mathbf{y}, \bar{\mathbf{y}}, \mathbf{x} \rangle$ forms a 4-cycle of Q_n with $d_{Q_n}(\mathbf{x}, \mathbf{y}) = 2$. Suppose that $6 \leq k \leq 2^{n-1} + 2$. By induction, there is a $(k-2)$-cycle $C' = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{x} \rangle$ of Q_n^0 such that $d_P(\mathbf{x}, \bar{\mathbf{y}}) = k - 3$. Then $C = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{y}, \bar{\mathbf{x}}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = 2$. Suppose that $k \geq 2^{n-1} + 4$. By induction, there is a 2^{n-1}-cycle C' of Q_n^0 with $d_{C'}(\mathbf{x}, \bar{\mathbf{y}}) = 1$. We write $C' = \langle \mathbf{x}, P, \mathbf{z}, \bar{\mathbf{y}}, \mathbf{x} \rangle$ with $d_P(\mathbf{x}, \mathbf{z}) = 2^{n-1} - 2$. By induction, there is a $(k - 2^{n-1})$-cycle C'' of Q_n^1 with $d_{C''}(\mathbf{y}, \bar{\mathbf{z}}) = 1$. We write $C'' = \langle \mathbf{y}, \bar{\mathbf{z}}, R, \mathbf{y} \rangle$ with $d_R(\mathbf{y}, \bar{\mathbf{z}}) = k - 2^{n-1} - 1$. Then $C = \langle \mathbf{x}, P, \mathbf{z}, \bar{\mathbf{z}}, R, \mathbf{y}, \bar{\mathbf{y}}, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = 2$.

Case 3: $d_{Q_n}(\mathbf{x}, \mathbf{y}) \geq 2$ and $l \geq 3$. Without loss of generality, we may assume that $\mathbf{x}$ is in Q_n^0 and $\mathbf{y}$ is in Q_n^1. Suppose that $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2 \leq 2^{n-1}$. By induction, there is an $(l + d_{Q_n}(\mathbf{x}, \mathbf{y}) - 2)$-cycle $C' = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{u}, R, \mathbf{x} \rangle$ of Q_n^0 such that $d_P(\mathbf{x}, \bar{\mathbf{y}}) = l - 1$ and $d_R(\mathbf{u}, \mathbf{x}) = d_{Q_n}(\mathbf{x}, \mathbf{y}) - 2$. For $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2 \leq 2$, by induction, there is a $(k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2)$-cycle C'' of Q_n^1 with $d_{C''}(\mathbf{y}, \bar{\mathbf{u}}) = 1$. We write $C'' = \langle \mathbf{y}, S, \bar{\mathbf{u}}, \mathbf{y} \rangle$ with $d_S(\mathbf{y}, \bar{\mathbf{u}}) = k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 1$. We then set $C = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{y}, \bar{\mathbf{u}}, \mathbf{u}, R, \mathbf{x} \rangle$ if $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2 = 2$ or $C = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{y}, S, \bar{\mathbf{u}}, \mathbf{u}, R, \mathbf{x} \rangle$ if $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 2 \leq 4$. Then C forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$. Suppose that $k - l - d_{Q_n}(\mathbf{x}, \mathbf{y}) + 4 \geq 2^{n-1}$. By induction, there is a $(k - 2^{n-1})$-cycle $C' = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{u}, R, \mathbf{x} \rangle$ of Q_n^0 such that $d_P(\mathbf{x}, \bar{\mathbf{y}}) = l - 1$ and $d_R(\mathbf{u}, \mathbf{x}) = k - 2^{n-1} - l$. By induction, there is a 2^{n-1}-cycle C'' of Q_n^1 with $d_{C''}(\mathbf{y}, \bar{\mathbf{u}}) = 1$. We write $C'' = \langle \mathbf{y}, S, \bar{\mathbf{u}}, \mathbf{y} \rangle$ with $d_S(\mathbf{y}, \bar{\mathbf{u}}) = 2^{n-1} - 1$. Then $C = \langle \mathbf{x}, P, \bar{\mathbf{y}}, \mathbf{y}, S, \bar{\mathbf{u}}, \mathbf{u}, R, \mathbf{x} \rangle$ forms a k-cycle of Q_n with $d_C(\mathbf{x}, \mathbf{y}) = l$.

The theorem is proved. □

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