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Embedding Hamiltonian paths in augmented cubes with a required vertex in a fixed position

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ABSTRACT

It is proved that there exists a path $P_l(\mathbf{x},\mathbf{y})$ of length l if $d_{AQ_n}(\mathbf{x},\mathbf{y}) \le l \le 2^n - 1$ between any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$ of AQ_n. Obviously, we expect that such a path $P_l(\mathbf{x},\mathbf{y})$ can be further extended by including the vertices not in $P_l(\mathbf{x},\mathbf{y})$ into a hamiltonian path from $\mathbf{x}$ to a fixed vertex $\mathbf{z}$ or a hamiltonian cycle. In this paper, we prove that there exists a hamiltonian path $R(\mathbf{x},\mathbf{y},\mathbf{z}; l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y}) = l$ for any three distinct vertices $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ of AQ_n with $n \ge 2$ and for any $d_{AQ_n}(\mathbf{x},\mathbf{y}) \le l \le 2^n - 1 - d_{AQ_n}(\mathbf{y},\mathbf{z})$. Furthermore, there exists a hamiltonian cycle $S(\mathbf{x},\mathbf{y}; l)$ such that $d_{S(\mathbf{x},\mathbf{y};l)}(\mathbf{x},\mathbf{y}) = l$ for any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$ and for any $d_{AQ_n}(\mathbf{x},\mathbf{y}) \le l \le 2^{n-1}$.

1. Introduction

In this paper, a network is represented as a loopless undirected graph. For the graph definitions and notation, we follow [1]. Let $G = (V, E)$ be a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. We use $Nbd_G(u)$ to denote the set $\{v \mid (u, v) \in E(G)\}$. The *degree* of a vertex u in G, denoted by $\deg_G(u)$, is $|Nbd_G(u)|$. We use $\delta(G)$ to denote $\min\{\deg_G(u) \mid u \in V(G)\}$. A graph is *k-regular* if $\deg_G(u) = k$ for every vertex u in G. A path is a sequence of adjacent vertices, written as $\langle v_0, v_1, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except that possibly $v_0 = v_m$. We also write the path $\langle v_0, P, v_m\rangle$, where $P = \langle v_0, v_1, \ldots, v_m\rangle$. The *length* of a path P, denoted by $l(P)$, is the number of edges in P. Let u and v be two vertices of G. The *distance* between u and v denoted by $d_G(u, v)$ is the length of the shortest path of G joining u and v. The *diameter* of a graph G, denoted by $D(G)$, is $\max\{d_G(u, v) \mid u, v \in V(G)\}$. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *hamiltonian cycle* is a cycle of length $V(G)$. A *hamiltonian path* is a path of length $V(G) - 1$.

Interconnection networks play an important role in parallel computing/communication systems. The graph embedding problem is a central issue in evaluating a network. The graph embedding problem asked if the guest graph is a subgraph of a host graph, and an important benefit of the graph embeddings is that we can apply existing algorithm for guest graphs to host graphs. This problem has attracted numerous studies in recent years. Cycle networks and path networks are suitable for designing simple algorithms with low communication costs. The cycle embedding problem, which deals with all possible lengths of the cycles in a given graph, is investigated in a lot of interconnection networks [2–6]. The path embedding problem, which deals with all possible lengths of the paths between given two vertices in a given graph, is investigated in a lot of interconnection networks [5–12].

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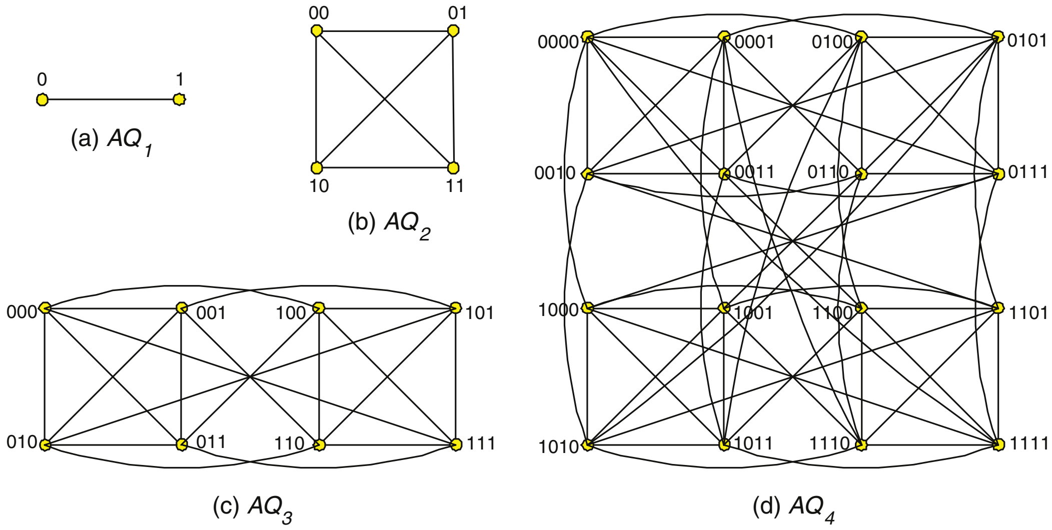

Fig. 1. The augmented cubes AQ_1, AQ_2, AQ_3 and AQ_4.

The hypercube Q_n is one of the most popular interconnection networks for parallel computer/communication system [13]. This is partly due to its attractive properties, such as regularity, recursive structure, vertex and edge symmetry, maximum connectivity, as well as effective routing and broadcasting algorithm. The augmented cube AQ_n is a variation of Q_n, proposed by Choudum and Sunitha [14], and not only retains some favorable properties of Q_n but also processes some embedding properties that Q_n does not [14–17,6]. For example, AQ_n contains cycles of all lengths from 3 to 2^n, but Q_n contains only even cycles.

For the path embedding problem on the augmented cube, Ma et al. [6] proved that between any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$ of AQ_n, there exists a path $P_l(\mathbf{x}, \mathbf{y})$ of length l with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \le l \le 2^n - 1$. Obviously, we expect that such a path $P_l(\mathbf{x}, \mathbf{y})$ can be further extended by including the vertices not in $P_l(\mathbf{x}, \mathbf{y})$ into a hamiltonian path from $\mathbf{x}$ to a fixed vertex $\mathbf{z}$ or a hamiltonian cycle. For this reason, we prove that for any three distinct vertices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of AQ_n, and for any $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \le l \le 2^n - 1 - d_{AQ_n}(\mathbf{y}, \mathbf{z})$ there exists a hamiltonian path $R(\mathbf{x}, \mathbf{y}, \mathbf{z}; l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x}, \mathbf{y}) = l$. As a corollary, we prove that for any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$, and for any $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \le l \le 2^{n-1}$, there exists a hamiltonian cycle $S(\mathbf{x}, \mathbf{y}; l)$ such that $d_{S(\mathbf{x},\mathbf{y};l)}(\mathbf{x}, \mathbf{y}) = l$.

In Section 2, we introduce the definition and some properties of the augmented cubes. In particular, we introduce another property, called 2RP, for augmented cubes. In Section 3, we prove that any AQ_n satisfies the 2RP-property if $n \ge 2$. Then we apply the 2RP-property to prove the aforementioned properties in Section 4.

2. Properties of augmented cubes

Assume that $n \ge 1$ is an integer. The graph of the *n-dimensional augmented cube*, denoted by AQ_n, has 2^n vertices, each labeled by an n-bit binary string $V(AQ_n) = \{u_1u_2 \ldots u_n \mid u_i \in \{0, 1\}\}$. For $n = 1$, AQ_1 is the graph K_2 with vertex set $\{0, 1\}$. For $n \ge 2$, AQ_n can be recursively constructed by two copies of AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, and by adding 2^n edges between AQ^0_{n-1} and AQ^1_{n-1} as follows:

Let $V(AQ^0_{n-1}) = \{0u_2u_3 \ldots u_n \mid u_i = 0 \text{ or } 1 \text{ for } 2 \le i \le n\}$ and $V(AQ^1_{n-1}) = \{1v_2v_3 \ldots v_n \mid v_i = 0 \text{ or } 1 \text{ for } 2 \le i \le n\}$. A vertex $\mathbf{u} = 0u_2u_3 \ldots u_n$ of AQ^0_{n-1} is adjacent to a vertex $\mathbf{v} = 1v_2v_3 \ldots v_n$ of AQ^1_{n-1} if and only if one of the following cases holds.

(i) $u_i = v_i$, for $2 \le i \le n$. In this case, $(\mathbf{u}, \mathbf{v})$ is called a *hypercube edge*. We set $\mathbf{v} = \mathbf{u}^h$.
(ii) $u_i = \bar{v}_i$, for $2 \le i \le n$. In this case, $(\mathbf{u}, \mathbf{v})$ is called a *complement edge*. We set $\mathbf{v} = \mathbf{u}^c$.

The augmented cubes AQ_1, AQ_2, AQ_3 and AQ_4 are illustrated in Fig. 1. It is proved in [14] that AQ_n is a vertex transitive, $(2n-1)$-regular, and $(2n-1)$-connected graph with 2^n vertices for any positive integer n. Let i be any index with $1 \le i \le n$ and $\mathbf{u} = u_1u_2u_3 \ldots u_n$ be a vertex of AQ_n. We use $\mathbf{u}^i$ to denote the vertex $\mathbf{v} = v_1v_2v_3 \ldots v_n$ such that $u_j = v_j$ with $1 \le j \ne i \le n$ and $u_i = \bar{v}_i$. Moreover, we use $\mathbf{u}^{i*}$ to denote the vertex $\mathbf{v} = v_1v_2v_3 \ldots v_n$ such that $u_j = v_j$ for $j < i$ and $u_j = \bar{v}_j$ for $i \le j \le n$. Obviously, $\mathbf{u}^n = \mathbf{u}^{n*}$, $\mathbf{u}^1 = \mathbf{u}^h$, $\mathbf{u}^c = \mathbf{u}^{1*}$, and $Nbd_{AQ_n}(\mathbf{u}) = \{\mathbf{u}^i \mid 1 \le i \le n\} \cup \{\mathbf{u}^{i*} \mid 1 \le i < n\}$.

Lemma 1. *Assume that $n \ge 2$. Then $|Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})| \ge 2$ if $(\mathbf{u}, \mathbf{v}) \in E(G)$.*

Proof. We prove this lemma by induction. Since AQ_2 is isomorphic to the complete graph K_4, the lemma holds for $n = 2$. Assume the lemma holds for $2 \le k < n$. Suppose that $\{\mathbf{u}, \mathbf{v}\} \subset V(AQ^i_{n-1})$ for some $i \in \{0, 1\}$. By induction, $|Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})| \ge 2$. Thus, consider the case that either $\mathbf{v} = \mathbf{u}^h$ or $\mathbf{v} = \mathbf{u}^c$. Obviously, $\{\mathbf{u}^{2*}, \mathbf{u}^c\} \subset Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})$ if $\mathbf{v} = \mathbf{u}^h$; and $\{\mathbf{u}^{2*}, \mathbf{u}^h\} \subset Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})$ if $\mathbf{v} = \mathbf{u}^c$. Then the statement holds. □

The following lemma can easily be obtained from the definition of AQ_n.

Lemma 2. *Assume that $n \ge 3$. For any two different vertices $\mathbf{u}$ and $\mathbf{v}$ of AQ_n, there exists two other vertices $\mathbf{x}$ and $\mathbf{y}$ of AQ_n such that the subgraph of $\{\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}\}$ containing a four cycle.*

Lemma 3 ([16]). *Let F be a subset of $V(AQ_n)$. Then there exists a hamiltonian path between any two vertices of $V(AQ_n) - F$ if $|F| \le 2n - 4$ for $n \ge 4$ and $|F| \le 1$ for $n = 3$.*

Lemma 4 ([14]). *Let $\mathbf{u}$ and $\mathbf{v}$ be any two vertices in AQ_n with $n \ge 2$. Suppose that both $\mathbf{u}$ and $\mathbf{v}$ are in AQ^i_{n-1} for $i = 0, 1$. Then $d_{AQ_n}(\mathbf{u}, \mathbf{v}) = d_{AQ^i_{n-1}}(\mathbf{u}, \mathbf{v})$. Suppose that $\mathbf{u}$ is a vertex in AQ^i_{n-1} and $\mathbf{v}$ is a vertex in AQ^{1-i}_{n-1}. Then there exist two shortest paths P_1 and P_2 of AQ_n joining $\mathbf{u}$ to $\mathbf{v}$ such that $(V(P_1) - \{\mathbf{v}\}) \subset V(AQ^i_{n-1})$ and $(V(P_2) - \{\mathbf{u}\}) \subset V(AQ^{1-i}_{n-1})$.*

With Lemma 4, we have Corollary 1.

Corollary 1. *Assume that $n \ge 3$. Let $\mathbf{x}$ and $\mathbf{y}$ be two vertices of AQ_n with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \ge 2$. Then, there are two vertices $\mathbf{p}$ and $\mathbf{q}$ in $Nbd_{AQ_n}(\mathbf{x})$ with $d_{AQ_n}(\mathbf{p}, \mathbf{y}) = d_{AQ_n}(\mathbf{q}, \mathbf{y}) = d_{AQ_n}(\mathbf{x}, \mathbf{y}) - 1$.*

Lemma 5 ([16]). *Let $\{\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}\}$ be any four distinct vertices of AQ_n with $n \ge 2$. Then there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ and $\mathbf{v}$, (2) P_2 is a path joining $\mathbf{x}$ and $\mathbf{y}$, and (3) $P_1 \cup P_2$ spans AQ_n.*

We refer to Lemma 5 as 2P-property of the augmented cube. This property is used for many applications of the augmented cubes [15,16]. Obviously, $l(P_1) \ge d_{AQ_n}(\mathbf{u}, \mathbf{v})$ and $l(P_2) \ge d_{AQ_n}(\mathbf{x}, \mathbf{y})$, and $l(P_1) + l(P_2) = 2^n - 2$. We expect that $l(P_1)$, hence, $l(P_2)$ can be an arbitrarily integer with the above constraint. However, such expectation is almost true. Let us consider AQ_3. Suppose that $\mathbf{u} = 001$, $\mathbf{v} = 110$, $\mathbf{x} = 101$, and $\mathbf{y} = 010$. Thus, $d_{AQ_3}(\mathbf{u}, \mathbf{v}) = 1$ and $d_{AQ_3}(\mathbf{x}, \mathbf{y}) = 1$. We can find P_1 and P_2 with $l(P_1) \in \{1, 3, 5\}$. Note that $\{\mathbf{x}, \mathbf{y}\} = Nbd_{AQ_3}(\mathbf{u}) \cap Nbd_{AQ_3}(\mathbf{v})$. We cannot find P_1 with $l(P_1) = 2$. Again, $\{\mathbf{u}, \mathbf{v}\} = Nbd_{AQ_3}(\mathbf{x}) \cap Nbd_{AQ_3}(\mathbf{y})$. We cannot find P_2 with $l(P_2) = 2$. Hence, we cannot find P_1 with $l(P_1) = 4$. Similarly, we consider AQ_4. Suppose that $\mathbf{u} = 0000$, $\mathbf{v} = 1001$, $\mathbf{x} = 0001$ and $\mathbf{y} = 1000$. Thus, $d_{AQ_4}(\mathbf{u}, \mathbf{v}) = 2$ and $d_{AQ_4}(\mathbf{x}, \mathbf{y}) = 2$. We can find P_1 and P_2 with $l(P_1) \in \{3, 4, \ldots, 11\}$. Note that $\{\mathbf{x}, \mathbf{y}\} = Nbd_{AQ_4}(\mathbf{u}) \cap Nbd_{AQ_4}(\mathbf{v})$. We cannot find P_1 with $l(P_1) = 2$. Again, $\{\mathbf{u}, \mathbf{v}\} = Nbd_{AQ_4}(\mathbf{x}) \cap Nbd_{AQ_4}(\mathbf{y})$. We cannot find P_2 with $l(P_2) = 2$.

Now, we propose the 2RP-property of AQ_n with $n \ge 2$: *Let $\{\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}\}$ be any four distinct vertices of AQ_n. Let l_1 and l_2 be two integers with $l_1 \ge d_{AQ_n}(\mathbf{u}, \mathbf{v})$, $l_2 \ge d_{AQ_n}(\mathbf{x}, \mathbf{y})$, and $l_1 + l_2 = 2^n - 2$. Then there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ and $\mathbf{v}$ with $l(P_1) = l_1$, (2) P_2 is a path joining $\mathbf{x}$ and $\mathbf{y}$ with $l(P_2) = l_2$, and (3) $P_1 \cup P_2$ spans AQ_n except for the following cases:* (a) *$l_1 = 2$ with $d_{AQ_n}(\mathbf{u}, \mathbf{v}) = 1$ such that $\{\mathbf{x}, \mathbf{y}\} = Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})$;* (b) *$l_2 = 2$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ such that $\{\mathbf{u}, \mathbf{v}\} = Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$;* (c) *$l_1 = 2$ with $d_{AQ_n}(\mathbf{u}, \mathbf{v}) = 2$ such that $\{\mathbf{x}, \mathbf{y}\} = Nbd_{AQ_n}(\mathbf{u}) \cap Nbd_{AQ_n}(\mathbf{v})$; and* (d) *$l_2 = 2$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 2$ such that $\{\mathbf{u}, \mathbf{v}\} = Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$.*

3. The 2RP-property of augmented cubes

Theorem 1. *Assume that n is a positive integer with $n \ge 2$. Then AQ_n satisfies 2RP-property.*

Proof. We prove this theorem by induction. By brute force, we check the theorem holds for $n = 2, 3, 4$. Assume the theorem holds for any AQ_k with $4 \le k < n$. Without loss of generality, we can assume that $l_1 \ge l_2$. Thus, $l_2 \le 2^{n-1} - 1$. By the symmetric property of AQ_n, we can assume that at least one of $\mathbf{u}$ and $\mathbf{v}$, say $\mathbf{u}$, is in $V(AQ^0_{n-1})$. Thus, we have the following cases:

Case 1: $\mathbf{v} \in V(AQ^0_{n-1})$ and $\{\mathbf{x}, \mathbf{y}\} \subset V(AQ^1_{n-1})$.

Subcase 1.1: $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \le l_2 \le 2^{n-1} - 3$ except that (1) $l_2 = 2^{n-1} - 4$ and (2) $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \ne Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. By Lemma 3, there exists a hamiltonian path R of AQ^0_{n-1} joining $\mathbf{u}$ to $\mathbf{v}$. Since $l(R) = 2^{n-1} - 1$, we can write R as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{q}, R_2, \mathbf{v} \rangle$ for some vertices $\mathbf{p}$ and $\mathbf{q}$ such that $\{\mathbf{p}^h, \mathbf{q}^h\} \cap \{\mathbf{x}, \mathbf{y}\} = \emptyset$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}^h$ to $\mathbf{q}^h$ with $l(S_1) = 2^{n-1} - l_2 - 2$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = l_2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}, \mathbf{p}^h, S_1, \mathbf{q}^h, \mathbf{q}, R_2, \mathbf{v} \rangle$ and set P_2 as S_2. Obviously, P_1 and P_2 are the required paths.

Subcase 1.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \ne Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Obviously, there exists a path P_2 of length 2 in $AQ_n - \{\mathbf{u}, \mathbf{v}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Subcase 1.3: $l_2 = 2^{n-1} - 4$. Obviously, there exists a vertex $\mathbf{p}$ in $V(AQ^1_{n-1}) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}^h, \mathbf{v}^h\}$, a vertex $\mathbf{q}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{p}) - \{\mathbf{x}, \mathbf{y}\}$, and a vertex $\mathbf{r}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{q}) - \{\mathbf{x}, \mathbf{y}, \mathbf{p}\}$. Suppose that $\mathbf{r}^h \notin \{\mathbf{u}, \mathbf{v}\}$. By induction, there exist two disjoint paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{u}$ to $\mathbf{p}^h$, (2) Q_2 is a path joining $\mathbf{r}^h$ to $\mathbf{v}$, and (3) $Q_1 \cup Q_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path P_2 of $AQ^1_{n-1} - \{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. We set P_1 as $\langle \mathbf{u}, Q_1, \mathbf{p}^h, \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{r}^h, Q_2, \mathbf{v}\rangle$. Suppose that $\mathbf{r}^h \in \{\mathbf{u}, \mathbf{v}\}$. Without loss of generality, we assume that $\mathbf{r}^h = \mathbf{v}$. By Lemma 3, there exists a hamiltonian path R of $AQ^0_{n-1} - \{\mathbf{v}\}$ joining $\mathbf{u}$ to $\mathbf{p}^h$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{p}^h, \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{r}^h = \mathbf{v}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 1.4: $l_2 = 2^{n-1} - 2$. Obviously, there exist a vertex $\mathbf{p} \in V(AQ^1_{n-1}) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}^h, \mathbf{u}^c, \mathbf{v}^h, \mathbf{v}^c\}$. By Lemma 5, there exists two disjoint paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{u}$ and $\mathbf{p}^h$, (2) Q_2 is a path joining $\mathbf{p}^c$ and $\mathbf{v}$, and (3) $Q_1 \cup Q_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path P_2 of $AQ^0_{n-1} - \{\mathbf{p}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. We set P_1 as $\langle \mathbf{u}, Q_1, \mathbf{p}^h, \mathbf{p}, \mathbf{p}^c, Q_2, \mathbf{v}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 1.5: $l_2 = 2^{n-1} - 1$. By Lemma 3, there exists a hamiltonian path P_1 of AQ^0_{n-1} joining $\mathbf{u}$ and $\mathbf{v}$ and there exists a hamiltonian path P_2 of AQ^1_{n-1} joining $\mathbf{x}$ to $\mathbf{y}$. Obviously, P_1 and P_2 are the required paths.

Case 2: $\mathbf{v} \in V(AQ^0_{n-1})$ and exactly one of $\mathbf{x}$ and $\mathbf{y}$ is in $V(AQ^0_{n-1})$. Without loss of generality, we assume that $\mathbf{x} \in V(AQ^0_{n-1})$.

Subcase 2.1: $l_2 = 1$. Obviously, $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$. We set P_2 as $\langle \mathbf{x}, \mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{y}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Subcase 2.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. The proof is the same to Subcase 1.2.

Subcase 2.3: $l_2 = 3$. Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$. There exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u}, \mathbf{v}\}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{p}^h\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x}, \mathbf{p}, \mathbf{p}^h, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 2$. By Lemma 4, there exists a path $\langle \mathbf{x}, \mathbf{p}, \mathbf{y}\rangle$ from $\mathbf{x}$ to $\mathbf{y}$ such that $\mathbf{p} \in V(AQ^1_{n-1})$. By Lemma 1, there exists a vertex $\mathbf{q} \in Nbd_{AQ^1_{n-1}}(\mathbf{p}) \cap Nbd_{AQ^1_{n-1}}(\mathbf{y})$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x}, \mathbf{p}, \mathbf{q}, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 3$. By Lemma 4, there exists a path P_2 from $\mathbf{x}$ to $\mathbf{y}$ such that $(V(P_2) - \{\mathbf{x}\}) \subset V(AQ^1_{n-1})$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Subcase 2.4: $4 \leq l_2 \leq 2^{n-1} - 2$ except that $l_2 = 2^{n-1} - 3$.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2. We first claim that there exists a vertex $\mathbf{p}$ in $Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Assume that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$. Obviously, either $\mathbf{y} = \mathbf{x}^h$ or $\mathbf{y} = \mathbf{x}^c$. We set $\mathbf{p} = \mathbf{x}^c$ if $\mathbf{y} = \mathbf{x}^h$; and we set $\mathbf{p} = \mathbf{x}^h$ if $\mathbf{y} = \mathbf{x}^c$. Assume that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 2$. By Lemma 4, there exists a path $\langle \mathbf{x}, \mathbf{p}, \mathbf{y}\rangle$ from $\mathbf{x}$ to $\mathbf{y}$ such that $\mathbf{p} \in V(AQ^1_{n-1})$. Obviously, $\mathbf{p}$ satisfies our claim. By Lemma 3, there exists a hamiltonian path R of $AQ^0_{n-1} - \{\mathbf{x}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. Since $l(R) = 2^{n-1} - 3$, we can write R as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{t}, R_2, \mathbf{v}\rangle$ such that $\{\mathbf{s}^h, \mathbf{t}^h\} \cap \{\mathbf{p}, \mathbf{y}\} = \emptyset$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^h$ to $\mathbf{t}^h$ with $l(S_1) = 2^{n-1} - 1 - l_2$, (2) S_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(S_2) = l_2 - 1$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S_1, \mathbf{t}^h, \mathbf{t}, R_2, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{p}, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \geq 3$. By Lemma 4, there exists a vertex $\mathbf{p}$ in $V(AQ^1_{n-1})$ such that $d_{AQ_n}(\mathbf{p}, \mathbf{y}) = d_{AQ_n}(\mathbf{x}, \mathbf{y}) - 1$. By Lemma 3, there exists a hamiltonian path R of $AQ^0_{n-1} - \{\mathbf{x}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We can write R as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{t}, R_2, \mathbf{v}\rangle$ such that $\{\mathbf{s}^h, \mathbf{t}^h\} \cap \{\mathbf{p}, \mathbf{y}\} = \emptyset$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^h$ to $\mathbf{t}^h$ with $l(S_1) = 2^{n-1} - 1 - l_2$, (2) S_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(S_2) = l_2 - 1$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S_1, \mathbf{t}^h, \mathbf{t}, R_2, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{p}, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 2.5: $l_2 = 2^{n-1} - 3$ or $l_2 = 2^{n-1} - 1$. Let $k = 3$ if $l_2 = 2^{n-1} - 3$ and $k = 1$ if $l_2 = 2^{n-1} - 1$. There exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u}, \mathbf{v}, \mathbf{y}^n\}$. By Lemma 3, there exists a hamiltonian path R of $AQ^0_{n-1} - \{\mathbf{x}, \mathbf{p}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We can write R as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{t}, R_2, \mathbf{v}\rangle$ such that $\{\mathbf{s}, \mathbf{t}\} \cap \{\mathbf{p}, \mathbf{y}^n\} = \emptyset$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^n$ to $\mathbf{t}^n$ with $l(S_1) = k$, (2) S_2 is a path joining $\mathbf{p}^n$ to $\mathbf{y}$ with $l(S_2) = 2^{n-1} - k - 2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^n, S_1, \mathbf{t}^n, \mathbf{t}, R_2, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{p}, \mathbf{p}^n, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Case 3: $\{\mathbf{v}, \mathbf{x}, \mathbf{y}\} \subset V(Q^0_{n-1})$.

Subcase 3.1: $l_2 = 1$. The proof is the same as Subcase 2.1.

Subcase 3.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \neq Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. The proof is the same as Subcase 1.2.

Subcase 3.3: $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \leq l_2 \leq 2^{n-2} - 1$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-1} - l_2 - 2$, (2) R_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(R_2) = l_2$, (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. We can write R_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{q}, R_4, \mathbf{v}\rangle$. By Lemma 3, there exists a hamiltonian path S of AQ^1_{n-1} joining $\mathbf{p}^h$ to $\mathbf{q}^h$. We set P_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{p}^h, S, \mathbf{q}^h, \mathbf{q}, R_4, \mathbf{v}\rangle$ and P_2 as R_2. Obviously, P_1 and P_2 are the required paths.

Subcase 3.4: $2^{n-2} + 1 \leq l_2 \leq 2^{n-1} - 1$ except that $l_2 = 2^{n-2} + 2$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-2} - 1$, (2) R_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(R_2) = 2^{n-2} - 1$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. We can write R_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{q}, R_4, \mathbf{v}\rangle$ and write R_2 as $\langle \mathbf{x}, R_5, \mathbf{s}, \mathbf{t}, R_6, \mathbf{y}\rangle$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}^h$ to $\mathbf{q}^h$ with $l(S_1) = 2^{n-1} - l_2 + 2^{n-2} - 2$, (2) S_2 is a path joining $\mathbf{s}^h$ to $\mathbf{t}^h$ with

$l(S_2) = l_2 - 2^{n-2}$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{p}^h, S_1, \mathbf{q}^h, \mathbf{q}, R_4, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_5, \mathbf{s}, \mathbf{s}^h, S_2, \mathbf{t}^h, \mathbf{t}, R_6, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 3.5: $l_2 = 2^{n-2}$ or $2^{n-2} + 2$. Let $k = 0$ if $l_2 = 2^{n-2}$ and $k = 2$ if $l_2 = 2^{n-2} + 2$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-2} - k$, (2) R_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(R_2) = 2^{n-2} + k - 2$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. We can write R_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{q}, R_4, \mathbf{v}\rangle$ and write R_2 as $\langle \mathbf{x}, R_5, \mathbf{s}, \mathbf{t}, R_6, \mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path S of $AQ^1_{n-1} - \{\mathbf{s}^n, \mathbf{t}^n\}$ joining $\mathbf{p}^n$ to $\mathbf{q}^n$. We set P_1 as $\langle \mathbf{u}, R_3, \mathbf{p}, \mathbf{p}^n, S, \mathbf{q}^n, \mathbf{q}, R_4, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_5, \mathbf{s}, \mathbf{s}^n, \mathbf{t}^n, \mathbf{t}, R_6, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Case 4: $\{\mathbf{x}, \mathbf{v}, \mathbf{y}\} \subset V(AQ^1_{n-1})$.

Subcase 4.1: $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \le l_2 \le 2^{n-1} - 3$ except that (1) $l_2 = 2^{n-1} - 4$ and (2) $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \ne Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{v}) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}^h\}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(S_1) = l_1 - 2^{n-1}$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = l_2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. By Lemma 3, there exists a hamiltonian path R of AQ^0_{n-1} joining $\mathbf{u}$ and $\mathbf{p}^h$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{p}^h, \mathbf{p}, S_1, \mathbf{v}\rangle$ and we set P_2 as S_2. Obviously, P_1 and P_2 are the required paths.

Subcase 4.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \ne Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. The proof is the same to Subcase 1.2.

Subcase 4.3: $l_2 = 2^{n-1} - 4$. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{v}) - \{\mathbf{x}, \mathbf{y}\}$, and there exists a vertex $\mathbf{q}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{p}) - \{\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{u}^h\}$. By Lemma 3, there exists a hamiltonian path R of AQ^0_{n-1} joining $\mathbf{u}$ to $\mathbf{q}^h$, and there exists a hamiltonian path P_2 of $AQ^1_{n-1} - \{\mathbf{v}, \mathbf{p}, \mathbf{q}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{q}^h, \mathbf{q}, \mathbf{p}, \mathbf{v}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 4.4: $l_2 = 2^{n-1} - 2$. Let $\mathbf{v}'$ be an element in $\{\mathbf{v}^h, \mathbf{v}^c\} - \{\mathbf{u}\}$. By Lemma 3, there exists a hamiltonian path R of AQ^0_{n-1} joining $\mathbf{u}$ to $\mathbf{v}'$, and there exists a hamiltonian path P_2 of $AQ^1_{n-1} - \{\mathbf{v}\}$ joining $\mathbf{x}$ to $\mathbf{y}$. We set P_1 as $\langle \mathbf{u}, R, \mathbf{v}', \mathbf{v}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 4.5: $l_2 = 2^{n-1} - 1$. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{v}) - \{\mathbf{x}, \mathbf{y}\}$. By induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(S_1) = 1$, (2) S_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_2) = 2^{n-1} - 3$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. Obviously, we can write S_2 as $\langle \mathbf{x}, S_2^1, \mathbf{r}, \mathbf{s}, S_2^2, \mathbf{y}\rangle$ for some vertex $\mathbf{r}$ and $\mathbf{s}$ such that $\mathbf{u} \notin \{\mathbf{r}^h, \mathbf{s}^h\}$. Again by induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{p}^h$ with $l(R_1) = 2^{n-1} - 3$, (2) R_2 is a path joining $\mathbf{r}^h$ to $\mathbf{s}^h$ with $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{p}^h, \mathbf{p}, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, S_2^1, \mathbf{r}, \mathbf{r}^h, \mathbf{s}^h, \mathbf{s}, S_2^2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Case 5: $\mathbf{v} \in V(AQ^1_{n-1})$ and $|\{x, y\} \cap V(AQ^0_{n-1})| = 1$. Without loss of generality, we assume that $\mathbf{x} \in V(AQ^0_{n-1})$.

Subcase 5.1: $l_2 = 1$. The proof is the same to Subcase 2.1.

Subcase 5.2: $l_2 = 2$ if $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$ or 2 with $\{\mathbf{u}, \mathbf{v}\} \ne Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. The proof is the same to Subcase 1.2.

Subcase 5.3: $l_2 = 3$. Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u}, \mathbf{v}^h\}$. We set P_2 as $\langle \mathbf{x}, \mathbf{p}, \mathbf{p}^h, \mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 2$. Assume that $\{\mathbf{u}, \mathbf{v}\} = Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Thus, we have either $\mathbf{v} = \mathbf{x}^h$ or $\mathbf{v} = \mathbf{x}^c$. Moreover, $\mathbf{u} = \mathbf{x}^\alpha$, and $\mathbf{y} = \mathbf{v}^\alpha$ for some $\alpha \in \{i \mid 2 \le i \le n\} \cup \{i{*} \mid 2 \le i \le n-1\}$. We set P_2 as $\langle \mathbf{x}, \mathbf{x}^{h*}, (\mathbf{x}^{h*})^\alpha, ((\mathbf{x}^h)^\alpha) = \mathbf{y}\rangle$ in the case of $\mathbf{v} = \mathbf{x}^h$. Otherwise, we set P_2 as $\langle \mathbf{x}, \mathbf{x}^h, (\mathbf{x}^h)^\alpha, ((\mathbf{x}^{h*})^\alpha) = \mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths. Now, assume that $\{\mathbf{u}, \mathbf{v}\} \ne Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. By Lemma 1, there exists a vertex $\mathbf{p}$ in $(Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})) - \{\mathbf{u}, \mathbf{v}\}$. Without loss of generality, we may assume that $\mathbf{p}$ is in AQ^0_{n-1}. By Lemma 1, there exists a vertex $\mathbf{q}$ in $(Nbd_{AQ^0_{n-1}}(\mathbf{p}) \cap Nbd_{AQ^0_{n-1}}(\mathbf{x})) - \{\mathbf{u}\}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{q}, \mathbf{p}, \mathbf{y}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x}, \mathbf{q}, \mathbf{p}, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 3$. By Lemma 4, there are two shortest paths R_1 and R_2 of AQ_n joining $\mathbf{x}$ to $\mathbf{y}$ such that R_1 can be written as $\langle \mathbf{x}, \mathbf{r_1}, \mathbf{r_2}, \mathbf{y}\rangle$ with $\{\mathbf{r_1}, \mathbf{r_2}\} \subset V(AQ^0_{n-1})$ and R_2 can be written as $\langle \mathbf{x}, \mathbf{s_1}, \mathbf{s_2}, \mathbf{y}\rangle$ with $\{\mathbf{s_1}, \mathbf{s_2}\} \subset V(AQ^1_{n-1})$. Suppose that $\mathbf{u} \ne \mathbf{r_2}$ or $\mathbf{v} \ne \mathbf{s_1}$. Without loss of generality, we assume that $\mathbf{u} \ne \mathbf{r_2}$. By Corollary 1, there exists a vertex $\mathbf{t} \in Nbd_{AQ^0_{n-1}}(\mathbf{x}) \cap Nbd_{AQ^0_{n-1}}(\mathbf{r_2}) - \{\mathbf{u}\}$. We set P_2 as $\langle \mathbf{x}, \mathbf{t}, \mathbf{r_2}, \mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths. Thus, we consider $\mathbf{u} = \mathbf{r_2}$ and $\mathbf{v} = \mathbf{s_1}$. By Corollary 1, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) \cap Nbd_{AQ^0_{n-1}}(\mathbf{u})$. Obviously, $d_{AQ_n}(\mathbf{p}, \mathbf{y}) = 2$. By Lemma 4, there exists a vertex $\mathbf{q}$ in $V(AQ^1_{n-1}) \cap Nbd_{AQ_n}(\mathbf{p}) \cap Nbd_{AQ_n}(\mathbf{y})$. Since $d_{AQ_n}(\mathbf{q}, \mathbf{y}) = 1$ and $d_{AQ_n}(\mathbf{v}, \mathbf{y}) = 2$, $\mathbf{q} \ne \mathbf{v}$. We set P_2 as $\langle \mathbf{x}, \mathbf{p}, \mathbf{q}, \mathbf{y}\rangle$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths.

Subcase 5.4: $4 \le l_2 \le 2^{n-1} - 1$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 1$. Suppose that $l_2 = 4$. Obviously, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u}, \mathbf{v}^h\}$. By Lemma 1, there exists a vertex $\mathbf{q}$ in $(Nbd_{AQ^0_{n-1}}(\mathbf{x}) \cap Nbd_{AQ^0_{n-1}}(\mathbf{p})) - \{\mathbf{u}\}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{p}^h, \mathbf{q}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x}, \mathbf{q}, \mathbf{p}, \mathbf{p}^h, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $5 \le l_2 \le 2^{n-1} - 1$ except that $l_2 = 2^{n-1} - 2$. Obviously, there exist a vertex $\mathbf{p}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{x}) - \{\mathbf{u}, \mathbf{v}^h, \mathbf{y}^h\}$ and a vertex $\mathbf{s}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{u}) - \{\mathbf{x}, \mathbf{p}, \mathbf{v}^h, \mathbf{y}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-1} - 2 - l_2$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{x}$ with $l(R_2) = l_2 - 2$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^1_{n-1} - \{\mathbf{y}, \mathbf{p}^h\}$ joining $\mathbf{s}^h$ to $\mathbf{v}$. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{p}, \mathbf{p}^h, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $l_2 = 2^{n-1} - 2$. Let $\mathbf{s}$ and $\mathbf{p}$ be two vertices in $V(AQ^0_{n-1}) - \{\mathbf{u}, \mathbf{x}, \mathbf{v}^h, \mathbf{y}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-2}$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{x}$ with $l(R_2) = 2^{n-2} - 2$, (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. Similarly, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^h$ to $\mathbf{v}$ with $l(S_1) = 2^{n-2} - 1$, (2) S_2 is a path joining $\mathbf{p}^h$ to $\mathbf{y}$ with $l(S_2) = 2^{n-2} - 1$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S_1, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{p}, \mathbf{p}^h, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 5.5: $4 \le l_2 \le 2^{n-1} - 1$ except $l_2 = 2^{n-1} - 3$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \ge 2$. Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 2$ with $\{\mathbf{u}, \mathbf{v}\} = Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Thus, we have either $\mathbf{v} = \mathbf{x}^h$ or $\mathbf{v} = \mathbf{x}^c$. Moreover, $\mathbf{u} = \mathbf{x}^\alpha$ and $\mathbf{y} = (\mathbf{x}^h)^\alpha$ for some $\alpha \in \{i \mid 2 \le i \le n\} \cup \{i* \mid 2 \le i \le n-1\}$. Obviously, there exists a vertex $\mathbf{t}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{v}) - \{\mathbf{x}^h, \mathbf{y}, \mathbf{x}^c, \mathbf{u}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{t}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-1} - 1 - l_2$, (2) R_2 is a path joining $\mathbf{x}^c$ to $\mathbf{y}$ with $l(R_2) = l_2 - 1$ in the case of $\mathbf{v} = \mathbf{x}^h$; otherwise R_2 is a path joining $\mathbf{x}^h$ to $\mathbf{y}$ with $l(R_2) = l_2 - 1$, and (3) $R_1 \cup R_2$ spans AQ^1_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^0_{n-1} - \{\mathbf{x}\}$ joining $\mathbf{t^h}$ to $\mathbf{u}$. We set P_1 as $\langle \mathbf{u}, S, \mathbf{t}^h, \mathbf{t}, R_1, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{x}^c, R_2, \mathbf{y}\rangle$ in the case of $\mathbf{v} = \mathbf{x}^h$; otherwise, we set P_2 as $\langle \mathbf{x}, \mathbf{x}^h, R_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 2$ with $\{\mathbf{u}, \mathbf{v}\} \ne Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Then, there exists a vertex $\mathbf{p}$ in $(Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})) - \{\mathbf{u}, \mathbf{v}\}$. Without loss of generality, we may assume that $\mathbf{p} \in V(AQ^1_{n-1})$. Obviously, there exists a vertex $\mathbf{t}$ in $Nbd_{AQ^1_{n-1}}(\mathbf{v}) - \{\mathbf{y}, \mathbf{p}, \mathbf{u}^h, \mathbf{x}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{t}$ to $\mathbf{v}$ with $l(R_1) = 2^{n-1} - 1 - l_2$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(R_2) = l_2 - 1$, and (3) $R_1 \cup R_2$ spans AQ^1_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^0_{n-1} - \{\mathbf{x}\}$ joining $\mathbf{t}^h$ to $\mathbf{u}$. We set P_1 as $\langle \mathbf{u}, S, \mathbf{t}^h, \mathbf{t}, R_1, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, \mathbf{p}, R_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Suppose that $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = k \ge 3$. By Lemma 4, there are two shortest paths S_1 and S_2 of AQ_n joining $\mathbf{x}$ to $\mathbf{y}$ such that S_1 can be written as $\langle \mathbf{x} = \mathbf{r_0}, \mathbf{r_1}, \mathbf{r_2}, \dots, \mathbf{r_{k-1}}, \mathbf{y}\rangle$ with $(V(S_1) - \{\mathbf{y}\}) \subset V(AQ^0_{n-1})$ and S_2 can be written as $\langle \mathbf{x}, \mathbf{s_1}, \mathbf{s_2}, \dots, \mathbf{s_{k-1}}, \mathbf{y}\rangle$ with $(V(S_2) - \{\mathbf{x}\}) \subset V(AQ^1_{n-1})$. Suppose that $\mathbf{u} \ne \mathbf{r_{k-1}}$. We set $\mathbf{p} = \mathbf{r_{k-1}}$. Again, there exists a vertex $\mathbf{s}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{u}) - \{\mathbf{x}, \mathbf{p}, \mathbf{y}^h, \mathbf{v}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-1} - 1 - l_2$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{x}$ with $l(R_2) = l_2 - 1$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^1_{n-1} - \{\mathbf{y}\}$ joining $\mathbf{s}^h$ to $\mathbf{v}$. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{p}, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Now we assume that $\mathbf{r_{k-1}} = \mathbf{u}$ and $\mathbf{s_1} = \mathbf{v}$. Since $d_{AQ_n}(\mathbf{r_{k-2}}, \mathbf{y}) = 2$, by Lemma 4, there exists a vertex $\mathbf{p} \in Nbd_{AQ_n}(\mathbf{r_{k-2}})$ in $V(AQ^1_{n-1})$ such that $d_{AQ_n}(\mathbf{p}, \mathbf{y}) = 1$. Suppose that $l_2 = 4$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 3$. Thus, $\langle \mathbf{x}, \mathbf{r_1}, \mathbf{p}, \mathbf{y}\rangle$ is a shortest path joining $\mathbf{x}$ and $\mathbf{y}$. By Lemma 1, there exists a vertex $\mathbf{q} \in Nbd_{AQ^1_{n-1}}(\mathbf{p}) \cap Nbd_{AQ^1_{n-1}}(\mathbf{y}) - \{\mathbf{v}\}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{r_1}, \mathbf{p}, \mathbf{q}, \mathbf{y}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. We set P_2 as $\langle \mathbf{x}, \mathbf{r_1}, \mathbf{p}, \mathbf{q}, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths. Suppose that $l_2 = 4$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) = 4$. Thus, $P_2 = \langle \mathbf{x}, \mathbf{r_1}, \mathbf{r_2}, \mathbf{p}, \mathbf{y}\rangle$ is a shortest path joining $\mathbf{x}$ and $\mathbf{y}$. By Lemma 3, there exists a hamiltonian path P_1 of $AQ_n - \{\mathbf{x}, \mathbf{r_1}, \mathbf{r_2}, \mathbf{p}, \mathbf{y}\}$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 are the required paths. Suppose that $5 \le l_2 \le 2^{n-2}$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \ge 3$. Obviously, there exists a vertex $\mathbf{s}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{u}) - \{\mathbf{x}, \mathbf{r_{k-2}}, \mathbf{y}^h, \mathbf{v}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-1} - l_2$, (2) R_2 is a path joining $\mathbf{r_{k-2}}$ to $\mathbf{x}$ with $l(R_2) = l_2 - 2$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. By Lemma 3, there exists a hamiltonian path S of $AQ^1_{n-1} - \{\mathbf{p}, \mathbf{y}\}$ joining $\mathbf{s}^h$ to $\mathbf{v}$. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{r_{k-2}}, \mathbf{p}, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths. Suppose that $2^{n-2} + 1 \le l_2 < 2^{n-1} - 1$ except $2^{n-1} - 3$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \ge 3$. Obviously, there exists a vertex $\mathbf{s}$ in $Nbd_{AQ^0_{n-1}}(\mathbf{u}) - \{\mathbf{x}, \mathbf{r_{k-2}}, \mathbf{y}^h, \mathbf{v}^h\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-2} + 1$, (2) R_2 is a path joining $\mathbf{r_{k-2}}$ to $\mathbf{x}$ with $l(R_2) = 2^{n-2} - 3$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. Again by induction, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^h$ to $\mathbf{v}$ with $l(S_1) = 2^{n-1} - l_2 + 2^{n-2} - 4$, (2) S_2 is a path joining $\mathbf{p}$ to $\mathbf{y}$ with $l(S_2) = l_2 - 2^{n-2} + 2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^h, S_1, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{r_{k-2}}, \mathbf{p}, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Subcase 5.6: $l_2 = 2^{n-1} - 3$ or $l_2 = 2^{n-1} - 1$ with $d_{AQ_n}(\mathbf{x}, \mathbf{y}) \ge 2$. Let $t = 0$ if $l_2 = 2^{n-1} - 3$ and $t = 1$ if $l_2 = 2^{n-1} - 1$. Obviously, there exist two vertices $\mathbf{s}$ and $\mathbf{p}$ in $AQ^0_{n-1} - \{\mathbf{u}, \mathbf{x}, \mathbf{v}^n, \mathbf{y}^n\}$. By induction, there exist two disjoint paths R_1 and R_2 such that (1) R_1 is a path joining $\mathbf{u}$ to $\mathbf{s}$ with $l(R_1) = 2^{n-2} - t$, (2) R_2 is a path joining $\mathbf{p}$ to $\mathbf{x}$ with $l(R_2) = 2^{n-2} + t - 2$, and (3) $R_1 \cup R_2$ spans AQ^0_{n-1}. Similarly, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{s}^n$ to $\mathbf{v}$ with $l(S_1) = 2^{n-2} - t$, (2) S_2 is a path joining $\mathbf{p}^n$ to $\mathbf{y}$ with $l(S_2) = 2^{n-2} + t - 2$, and (3) $S_1 \cup S_2$ spans AQ^1_{n-1}. We set P_1 as $\langle \mathbf{u}, R_1, \mathbf{s}, \mathbf{s}^n, S_1, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{p}, \mathbf{p}^n, S_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 are the required paths.

Thus, Theorem 1 is proved. □

4. The applications of the 2RP-property

Theorem 2. *Assume that n is a positive integer with $n \geq 2$. For any three distinct vertices* $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{z}$ *of* AQ_n *and for any* $d_{AQ_n}(\mathbf{x},\mathbf{y}) \leq l \leq 2^n - 1 - d_{AQ_n}(\mathbf{y},\mathbf{z})$, *there exists a hamiltonian path* $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ *from* $\mathbf{x}$ *to* $\mathbf{z}$ *such that* $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y}) = l$.

Proof. Obviously, the theorem holds for $n = 2$. Thus, we consider that $n \geq 3$. We have the following cases:

Case 1: $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 1$ and $d_{AQ_n}(\mathbf{y},\mathbf{z}) = 1$. By Lemma 1, there exists a vertex $\mathbf{w}$ in $(Nbd_{AQ_n}(\mathbf{y}) \cap Nbd_{AQ_n}(\mathbf{z})) - \{\mathbf{x}\}$. Similarly, there exists a vertex $\mathbf{p}$ in $(Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})) - \{\mathbf{z}\}$. Suppose that $l = 2$. By Theorem 1, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{x}$ to $\mathbf{p}$ with $l(S_1) = 1$, (2) S_2 is a path joining $\mathbf{y}$ to $\mathbf{z}$ with $l(S_2) = 2^n - 3$, and (3) $S_1 \cup S_2$ spans AQ_n. We set R as $\langle \mathbf{x}, \mathbf{p}, \mathbf{y}, S_2, \mathbf{z} \rangle$. Obviously, R forms a hamiltonian path from $\mathbf{x}$ to $\mathbf{z}$ such that $d_R(\mathbf{x},\mathbf{y}) = l$. Suppose that $l = 2^n - 3$. By Theorem 1, there exist two disjoint paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(Q_1) = 2^n - 3$, (2) Q_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(Q_2) = 1$, and (3) $Q_1 \cup Q_2$ spans AQ_n. We set R as $\langle \mathbf{x}, Q_1, \mathbf{y}, \mathbf{w}, \mathbf{z} \rangle$. Obviously, R forms a hamiltonian path from $\mathbf{x}$ to $\mathbf{z}$ such that $d_R(\mathbf{x},\mathbf{y}) = l$. Suppose that $1 \leq l \leq 2^n - 2$ with $l \notin \{2, 2^n - 3\}$. By Theorem 1, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_1) = l$, (2) P_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(P_2) = 2^n - 2 - l$, and (3) $P_1 \cup P_2$ spans AQ_n. We set R as $\langle \mathbf{x}, P_1, \mathbf{y}, \mathbf{w}, P_2, \mathbf{z} \rangle$. Obviously, R forms a hamiltonian path from $\mathbf{x}$ to $\mathbf{z}$ such that $d_R(\mathbf{x},\mathbf{y}) = l$.

Case 2: $d_{AQ_n}(\mathbf{x},\mathbf{y}) = 1$ and $d_{AQ_n}(\mathbf{y},\mathbf{z}) \neq 1$. By Lemma 1, there exists a vertex $\mathbf{p}$ in $Nbd_{AQ_n}(\mathbf{x}) \cap Nbd_{AQ_n}(\mathbf{y})$. Suppose that $l = 2$. By Theorem 1, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{x}$ to $\mathbf{p}$ with $l(S_1) = 1$, (2) S_2 is a path joining $\mathbf{y}$ to $\mathbf{z}$ with $l(S_2) = 2^n - 3$, and (3) $S_1 \cup S_2$ spans AQ_n. We set R as $\langle \mathbf{x}, \mathbf{p}, \mathbf{y}, S_2, \mathbf{z} \rangle$. Obviously, R forms a hamiltonian path from $\mathbf{x}$ to $\mathbf{z}$ such that $d_R(\mathbf{x},\mathbf{y}) = l$. Suppose that $1 \leq l \leq 2^n - 1 - d_{AQ_n}(\mathbf{y},\mathbf{z})$ with $l \neq 2$. By Corollary 1, there exists a vertex $\mathbf{w}$ in $Nbd_{AQ_n}(\mathbf{y}) - \{\mathbf{x}\}$ such that $d_{AQ_n}(\mathbf{w},\mathbf{z}) = d_{AQ_n}(\mathbf{y},\mathbf{z}) - 1$. By Theorem 1, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_1) = l$, (2) S_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(P_2) = 2^n - 2 - l$, and (3) $P_1 \cup P_2$ spans AQ_n. We set R as $\langle \mathbf{x}, P_1, \mathbf{y}, \mathbf{w}, P_2, \mathbf{z} \rangle$. Obviously, R forms a hamiltonian path from $\mathbf{x}$ to $\mathbf{z}$ such that $d_R(\mathbf{x},\mathbf{y}) = l$.

Case 3: $d_{AQ_n}(\mathbf{x},\mathbf{y}) \neq 1$ and $d_{AQ_n}(\mathbf{y},\mathbf{z}) = 1$. This case is similar as Case 2 by interchanging the roles of $\mathbf{x}$ and $\mathbf{z}$.

Case 4: $d_{AQ_n}(\mathbf{x},\mathbf{y}) \neq 1$ and $d_{AQ_n}(\mathbf{y},\mathbf{z}) \neq 1$. Let l be any integer with $d_{AQ_n}(\mathbf{x},\mathbf{y}) \leq l \leq 2^n - 1 - d_{AQ_n}(\mathbf{y},\mathbf{z})$. Let $\mathbf{w}$ be a vertex in $Nbd_{AQ_n}(\mathbf{y})$. By Theorem 1, there exist two disjoint paths S_1 and S_2 such that (1) S_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(S_1) = l$, (2) S_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(S_2) = 2^n - 2 - l$, and (3) $S_1 \cup S_2$ spans AQ_n. We set R as $\langle \mathbf{x}, S_1, \mathbf{y}, \mathbf{w}, S_2, \mathbf{z} \rangle$. Obviously, R forms a hamiltonian path from $\mathbf{x}$ to $\mathbf{z}$ such that $d_R(\mathbf{x},\mathbf{y}) = l$.

The theorem is proved. □

Corollary 2. *Assume that n is a positive integer with $n \geq 2$. For any two distinct vertices* $\mathbf{x}$ *and* $\mathbf{y}$ *and for any* $d_{AQ_n}(\mathbf{x},\mathbf{y}) \leq l \leq 2^{n-1}$, *there exists a hamiltonian cycle* $S(\mathbf{x},\mathbf{y};l)$ *such that* $d_{S(\mathbf{x},\mathbf{y};l)}(\mathbf{x},\mathbf{y}) = l$.

Proof. Let $\mathbf{z}$ be a vertex in $Nbd_{AQ_n}(\mathbf{x}) - \{\mathbf{y}\}$. By Theorem 2, there exists a hamiltonian path R joining $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y}) = l$. We set S as $\langle \mathbf{x}, R, \mathbf{z}, \mathbf{x} \rangle$. Obviously, S forms the required hamiltonian cycle. □

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