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ZETA FUNCTIONS FOR HIGHER-DIMENSIONAL SHIFTS OF FINITE TYPE

WEN-GUEI HU* and SONG-SUN LIN†
Department of Applied Mathematics,
National Chiao Tung University,
Hsin-Chu 30010, Taiwan
**wwk.am94g@nctu.edu.tw*
†sslin@math.nctu.edu.tw

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This work investigates zeta functions for d-dimensional shifts of finite type, $d \geq 3$. First, the three-dimensional case is studied. The trace operator $\mathbf{T}_{a_1,a_2;b_{12}}$ and rotational matrices $R_{x;a_1,a_2;b_{12}}$ and $R_{y;a_1,a_2;b_{12}}$ are introduced to study $\begin{bmatrix} a_1 & b_{12} & b_{23} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}$-periodic patterns. The rotational symmetry of $\mathbf{T}_{a_1,a_2;b_{12}}$ induces the reduced trace operator $\tau_{a_1,a_2;b_{12}}$ and then the associated zeta function $\zeta_{a_1,a_2;b_{12}} = (\det(I - s^{a_1 a_2}\tau_{a_1,a_2;b_{12}}))^{-1}$. The zeta function ζ is then expressed as $\zeta = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} \prod_{b_{12}=0}^{a_1-1} \zeta_{a_1,a_2;b_{12}}$, a reciprocal of an infinite product of polynomials. The results hold for any inclined coordinates, determined by unimodular transformation in $GL_3(\mathbb{Z})$. Hence, a family of zeta functions exists with the same integer coefficients in their Taylor series expansions at the origin, and yields a family of identities in number theory. The methods used herein are also valid for d-dimensional cases, $d \geq 4$, and can be applied to thermodynamic zeta functions for the three-dimensional Ising model with finite range interactions.

Keywords: Zeta function; shift of finite type; patterns generation problem; phase-transition; Ising model; cellular neural networks.

1. Introduction

This study investigates the zeta functions for shifts of finite type on $\mathbb{Z}^d$, $d \geq 3$. Zeta functions are important subjects in the fields of number theory, geometry, dynamical systems and statistical physics. They have been extensively studied for many years. This work is an extension of our previous results on $\mathbb{Z}^2$ [Ban *et al.*, 2008a], following the works of Artin and Mazur [1965], Bowen and Lanford [1970], Ruelle [1978] and Lind [1996]. Let ϕ be an action of $\mathbb{Z}^d$ on $\mathbb{X}$. Lind [1996] defined a zeta function ζ_ϕ as

$$\zeta_\phi(s) = \exp\left(\sum_{L\in\mathcal{L}_d} \frac{\Gamma_L(\phi)}{[L]} s^{[L]}\right), \tag{1}$$

where $\mathcal{L}_d$ is the set of finite-index subgroups of $\mathbb{Z}^d$, $[L] = \text{index}[\mathbb{Z}^d/L]$ and $\Gamma_L(\phi)$ is the number of fixed points by $\phi^{\mathbf{n}}$ for all $\mathbf{n} \in L$.

Our two-dimensional work [Ban *et al.*, 2008a] is reviewed first. Based on Eq. (1), Ban *et al.* [2008a] studied the two-dimensional zeta functions $\zeta^0_{\mathcal{B}}$ of shifts of finite type, which are

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generated by admissible local patterns $\mathcal{B}$. $\zeta^0_{\mathcal{B}}$ is defined by

$$\zeta^0_{\mathcal{B}} = \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{nk}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k\end{bmatrix}\right)s^{nk}\right) \tag{2}$$

and the nth order zeta function $\zeta_{\mathcal{B};n}(s)$ is

$$\zeta_{\mathcal{B};n}(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k\end{bmatrix}\right)s^{nk}\right) \tag{3}$$

for any $n \geq 1$, where $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k\end{bmatrix}\right)$ is the number of $\begin{bmatrix} n & l \\ 0 & k\end{bmatrix}$-periodic patterns that can be generated by $\mathcal{B}$. The zeta function $\zeta_{\mathcal{B}}(s)$ is now given by

$$\zeta_{\mathcal{B}}(s) = \prod_{n=1}^{\infty}\zeta_{\mathcal{B};n}(s). \tag{4}$$

In deriving Eq. (3), the trace operator $\mathbf{T}_n(\mathcal{B})$ and rotational matrix R_n are introduced to accommodate the periodic patterns. Based on the rotational symmetry of the trace operator, the reduced trace operator $\tau_n(\mathcal{B})$ is defined. $\zeta_{\mathcal{B};n}$ and $\zeta_{\mathcal{B}}$ can be expressed as

$$\zeta_{\mathcal{B};n} = (\det(I - s^n\tau_n))^{-1} \tag{5}$$

and

$$\zeta_{\mathcal{B}} = \prod_{n=1}^{\infty}(\det(I - s^n\tau_n))^{-1}. \tag{6}$$

The latter is a reciprocal of an infinite product of polynomials. The results also hold when inclined coordinates are used for any unimodular transformation $\gamma \in GL_2(\mathbb{Z})$. Therefore, there exists a family of zeta functions with the same integer coefficients of their Taylor series expansions at $s = 0$ and the family of zeta functions yields a family of identities. The two-dimensional thermodynamic zeta functions for the Ising model with finite range interactions are also studied.

It is clear that in many situations the three-dimensional problems are more related to our real world phenomena. In this work, the zeta functions of d-dimensional shifts of finite type are studied for $d \geq 3$, and the previous results of $\mathbb{Z}^2$ are extended. For simplicity, only the zeta functions for three-dimensional shifts of finite type are introduced and the general case is studied in Sec. 5.

Let $\mathbb{Z}_{m\times m\times m}$ be the $m \times m \times m$ cubic lattice in $\mathbb{Z}^3$ and $\mathcal{S}$ be the finite set of symbols (alphabets or colors). $\mathcal{S}^{\mathbb{Z}_{m\times m\times m}}$ is the set of all local patterns on $\mathbb{Z}_{m\times m\times m}$. Denote $\mathcal{B} \subset \mathcal{S}^{\mathbb{Z}_{m\times m\times m}}$ as a basic set of admissible local patterns and $\mathcal{P}(\mathcal{B})$ the set of all periodic patterns that are generated by $\mathcal{B}$ on $\mathbb{Z}^3$.

As in two other works [Lind, 1996; Ban *et al.*, 2008a], the Hermite normal form [MacDuffie, 1956] can be used to parameterize $\mathcal{L}_3$ as

$$\mathcal{L}_3 = \left\{\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3\end{bmatrix}\mathbb{Z}^3 : a_i \geq 1, 1 \leq i \leq 3, \right.$$
$$\left. 0 \leq b_{ij} \leq a_i - 1, i+1 \leq j \leq 3\right\}.$$

Given a basic set $\mathcal{B}$, let $L = \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3\end{bmatrix}\mathbb{Z}^3 \in \mathcal{L}_3$, denote $\mathcal{P}_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3\end{bmatrix}\right)$ as the set of all L-periodic patterns that are generated by $\mathcal{B}$ on $\mathbb{Z}^3$ and $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3\end{bmatrix}\right)$ as the number of $\mathcal{P}_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3\end{bmatrix}\right)$. Then, the zeta function in Eq. (1) is

$$\zeta^0_{\mathcal{B}} = \exp\left(\sum_{i=1}^{3}\sum_{a_i=1}^{\infty}\sum_{j=i+1}^{3}\sum_{b_{ij}=0}^{a_i-1}\frac{1}{a_1a_2a_3}\Gamma_{\mathcal{B}}\right.$$
$$\left.\times\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3\end{bmatrix}\right)s^{a_1a_2a_3}\right). \tag{7}$$

Similar to Eqs. (3) and (4), the $(a_1, a_2; b_{12})$th zeta function is defined by

$$\zeta_{\mathcal{B};a_1,a_2;b_{12}}(s) = \exp\left(\frac{1}{a_1a_2}\sum_{a_3=1}^{\infty}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\frac{1}{a_3}\Gamma_{\mathcal{B}}\right.$$
$$\left.\times\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3\end{bmatrix}\right)s^{a_1a_2a_3}\right) \tag{8}$$

and the zeta function $\zeta_{\mathcal{B}}(s)$ is given by

$$\zeta_{\mathcal{B}}(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\zeta_{\mathcal{B};a_1,a_2;b_{12}}(s). \tag{9}$$

The trace operator $\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B})$ and rotational matrices $R_{x;a_1,a_2;b_{12}}$ and $R_{y;a_1,a_2;b_{12}}$ are introduced. After the rotational symmetry of $\mathbf{T}_{a_1,a_2;b_{12}}$ is demonstrated the reduced trace operator $\tau_{a_1,a_2;b_{12}}(\mathcal{B})$ can be defined. Finally, as in Eq. (5),

$\zeta_{\mathcal{B};a_1,a_2;b_{12}}(s)$ can be represented as a rational function:

$$\zeta_{\mathcal{B};a_1,a_2;b_{12}}(s) = (\det(I - s^{a_1a_2}\tau_{a_1,a_2;b_{12}}))^{-1}. \quad (10)$$

Hence,

$$\zeta_{\mathcal{B}}(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}(\det(I - s^{a_1a_2}\tau_{a_1,a_2;b_{12}}))^{-1} \quad (11)$$

is a reciprocal of an infinite product of polynomials. The proof of Eq. (10) in this paper is new and simpler than in an earlier wok [Ban *et al.*, 2008a], in which the proof is also valid for any $d \geq 3$.

Additionally, for any $\gamma \in GL_3(\mathbb{Z})$, the zeta function can also be represented in γ-coordinates. Therefore, a family of zeta functions exists that have the same integer coefficients in their Taylor series expansions at $s = 0$.

According to [Lind, 1996] and [Ruelle, 1978], the thermodynamic zeta function with weight function $\theta : \mathbb{X} \to (0,\infty)$ is defined as

$$\zeta^0_{\phi,\theta}(s) = \exp\left(\sum_{L\in\mathcal{L}_d}\left\{\sum_{x\in fix_L(\phi)}\prod_{\mathbf{k}\in\mathbb{Z}^d/L}\theta(\phi^{\mathbf{k}}x)\right\}\frac{s^{[L]}}{[L]}\right), \quad (12)$$

where $fix_L(\phi)$ is the set of points fixed by $\phi^{\mathbf{n}}$ for all $\mathbf{n} \in L$. Let ϕ be a shift of finite type given by $\mathcal{B}$. As in the two-dimensional case [Ban *et al.*, 2008a], the thermodynamic zeta function for the three-dimensional Ising model with finite range interactions can also be represented as a reciprocal of an infinite product of polynomials. The three-dimensional model can be applied to study three-dimensional phase-transition problems. Further results need to be investigated.

Various works relate to this study, including zeta functions and related topics [Artin & Mazur, 1965; Ban *et al.*, 2008a; Bowen & Lanford, 1970; Fried, 1987; Hardy & Ramanujan, 1918; Hardy & Wright, 1988; Kitchens & Schmidt, 1989; Ledrappier, 1978; Lind, 1996; Lind & Marcus, 1995; Lind *et al.*, 1990; Manning, 1971; Markley & Paul, 1979, 1981; Pollicott & Schmidt, 1996; Ruelle, 1978; Schmidt, 1995], patterns generation problems and lattice dynamical systems [Ban *et al.*, 2001a, 2002; Ban & Lin, 2005; Ban *et al.*, 2007, 2008b, 2001b; Chow & Mallet-Paret, 1995; Chow *et al.*, 1996a, 1996b; Chua *et al.*, 1995; Chua & Roska, 1993; Chua & Yang, 1988a, 1988b; Chua, 1998; Chua & Itoh, 2003, 2005; Juang & Lin, 2000; Juang *et al.*, 2000; Lin & Yang, 2000, 2002], and phase-transitions in statistical physics [Baxter, 1971, 1982; Lieb, 1967a, 1967b; Onsager, 1994].

The rest of this article is organized as follows. Section 2 discusses periodic patterns, the trace operators and rotational matrices. Section 3 shows the rotational symmetry of trace operator and introduces the reduced trace operator. The rationality of $\zeta_{a_1,a_2;b_{12}}$ is then obtained for $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$. Section 4 studies the zeta function in γ-coordinates for $\gamma \in GL_3(\mathbb{Z})$. Section 5 extends the previous result to d-dimensions, $d \geq 4$, and to more symbols on a larger lattice. The thermodynamic zeta function for the three-dimensional Ising model with finite range interactions is also investigated.

2. Periodic Patterns, Trace Operator and Rotational Matrices

This section studies the properties of the periodic patterns and derives trace operator and rotational matrices. Furthermore, $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)$ can be expressed in terms of the trace of the products of the trace operator and rotational matrices.

For clarity, two symbols on $2 \times 2 \times 2$ lattice $\mathbb{Z}_{2\times2\times2}$ are examined first. For given positive integers N_1, N_2 and N_3, the rectangular lattice $\mathbb{Z}_{N_1\times N_2\times N_3}$ is defined by

$$\mathbb{Z}_{N_1\times N_2\times N_3} = \{(n_1,n_2,n_3) : 0 \leq n_i \leq N_i - 1, \; 1 \leq i \leq 3\}.$$

In particular,

$$\mathbb{Z}_{2\times2\times2} = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1), (1,0,0),(1,0,1),(1,1,0),(1,1,1)\}.$$

Define the set of all global patterns on $\mathbb{Z}^3$ with two symbols $\{0,1\}$ by

$$\Sigma_2^3 = \{0,1\}^{\mathbb{Z}^3} = \left\{U | U : \mathbb{Z}^3 \to \{0,1\}\right\}.$$

Here, $\mathbb{Z}^3 = \{(n_1,n_2,n_3) : n_1,n_2,n_3 \in \mathbb{Z}\}$, the set of all three-dimensional lattice points (vertices). The set of all local patterns on $\mathbb{Z}_{N_1\times N_2\times N_3}$ is defined by

$$\Sigma_{N_1\times N_2\times N_3} = \{U|_{\mathbb{Z}_{N_1\times N_2\times N_3}} : U \in \Sigma_2^3\},$$

and a local pattern of a global pattern U on $\mathbb{Z}_{N_1\times N_2\times N_3}$ is denoted by

$$\begin{aligned}U_{N_1\times N_2\times N_3} &\equiv U|_{\mathbb{Z}_{N_1\times N_2\times N_3}}\\ &= (u_{\alpha_1,\alpha_2,\alpha_3})_{0\le\alpha_i\le N_i-1,1\le i\le 3},\end{aligned}$$

where $u_{\alpha_1,\alpha_2,\alpha_3} \in \{0,1\}$. To simplify the notation, the subscripts of $U_{N_1\times N_2\times N_3}$ and $(u_{\alpha_1,\alpha_2,\alpha_3})_{0\le\alpha_i\le N_i-1,1\le i\le 3}$ are omitted whenever such omission will not cause confusion.

Now, for any given $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$, $\mathcal{B}$ is called a basic set of admissible local patterns. In short, $\mathcal{B}$ is a basic set. A local pattern $U_{N_1\times N_2\times N_3} = (u_{\alpha_1,\alpha_2,\alpha_3})$ is called $\mathcal{B}$-admissible if for any vertex (lattice point) (n_1, n_2, n_3) with $0 \le n_i \le N_i - 2$, $1 \le i \le 3$, there exists a $2\times 2\times 2$ admissible local pattern $(\beta_{k_1,k_2,k_3})_{0\le k_1,k_2,k_3\le 1} \in \mathcal{B}$ such that

$$u_{n_1+k_1,n_2+k_2,n_3+k_3} = \beta_{k_1,k_2,k_3}$$

for $0 \le k_1, k_2, k_3 \le 1$.

Given a lattice $L \in \mathcal{L}_3$ with Hermite normal form,

$$L = \begin{bmatrix} a_1 & b_{12} & b_{13}\\ 0 & a_2 & b_{23}\\ 0 & 0 & a_3\end{bmatrix}\mathbb{Z}^3, \tag{13}$$

where $a_i \ge 1$ for $1 \le i \le 3$ and $0 \le b_{ij} \le a_i - 1$ for $i+1 \le j \le 3$. A global pattern $U = (u_{\alpha_1,\alpha_2,\alpha_3})_{\alpha_1,\alpha_2,\alpha_3\in\mathbb{Z}}$ is called L-periodic or $\left[\begin{smallmatrix} a_1 & b_{12} & b_{13}\\ 0 & a_2 & b_{23}\\ 0 & 0 & a_3\end{smallmatrix}\right]$-periodic if for every $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$

$$u_{\alpha_1+a_1p+b_{12}q+b_{13}r,\alpha_2+a_2q+b_{23}r,\alpha_3+a_3r} = u_{\alpha_1,\alpha_2,\alpha_3} \tag{14}$$

for all $p, q, r \in \mathbb{Z}$.

The periodicity of $\left[\begin{smallmatrix} a_1 & b_{12} & b_{13}\\ 0 & a_2 & b_{23}\\ 0 & 0 & a_3\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} a_1 & 0 & 0\\ 0 & a_2' & 0\\ 0 & 0 & a_3'\end{smallmatrix}\right]$ are closely related as follows.

Proposition 2.1. *For $a_i \ge 1$, $1 \le i \le 3$, $0 \le b_{ij} \le a_i - 1$, $i+1 \le j \le 3$, let*

$$s_1 = \frac{a_1}{(a_1, b_{12})} \quad \text{and} \quad s_2 = \left[\frac{a_1}{(a_1, b_{13})}, \frac{s_1 a_2}{(s_1 a_2, b_{23})}\right],$$

where (m, n) is the greatest common divisor of m and n and $[p, q]$ is the least common multiple of p and q. Then, $\left[\begin{smallmatrix} a_1 & b_{12} & b_{13}\\ 0 & a_2 & b_{23}\\ 0 & 0 & a_3\end{smallmatrix}\right]$-periodic patterns are $\left[\begin{smallmatrix} a_1 & 0 & 0\\ 0 & s_1a_2 & 0\\ 0 & 0 & s_2a_3\end{smallmatrix}\right]$-periodic.

Proof. By Eq. (14), the $\left[\begin{smallmatrix} a_1 & b_{12} & b_{13}\\ 0 & a_2 & b_{23}\\ 0 & 0 & a_3\end{smallmatrix}\right]$-periodic pattern is easily identified as $\left[\begin{smallmatrix} a_1 & m_1b_{12} & m_2b_{13}\\ 0 & m_1a_2 & m_2b_{23}\\ 0 & 0 & m_2a_3\end{smallmatrix}\right]$-periodic for all $m_1, m_2 \in \mathbb{N}$. By taking $m_1 = s_1$ and $m_2 = s_2$, the result holds. ■

Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$, defined on cubic lattice $\mathbb{Z}_{2\times 2\times 2}$, the L-periodic patterns that are $\mathcal{B}$-admissible must be verified on $\mathbb{Z}_{2\times 2\times 2}$. For $n_1, n_2, n_3 \in \mathbb{Z}$, let $\mathbb{Z}_{2\times 2\times 2}((n_1, n_2, n_3))$ be the cubic lattice with the smallest vertex (n_1, n_2, n_3):

$$\begin{aligned}&\mathbb{Z}_{2\times 2\times 2}((n_1, n_2, n_3))\\ &\quad = \{(n_1+k_1, n_2+k_2, n_3+k_3) :\\ &\qquad 0 \le k_1, k_2, k_3 \le 1\}.\end{aligned}$$

Now, the admissibility of L-periodic patterns is demonstrated to be verified on finite cubic lattices.

Proposition 2.2. *An L-periodic pattern U is B-admissible if and only if*

$$U|_{\mathbb{Z}_{2\times 2\times 2}((\alpha_1,\alpha_2,\alpha_3))} \in \mathcal{B}$$

for $0 \le \alpha_i \le a_i - 1$, $1 \le i \le 3$.

Proof. Sine $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$, it is sufficient to prove

$$\begin{aligned}&\{U|_{\mathbb{Z}_{2\times 2\times 2}((\alpha_1,\alpha_2,\alpha_3))} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\}\\ &\quad = \{U|_{\mathbb{Z}_{2\times 2\times 2}((\alpha_1,\alpha_2,\alpha_3))} : 0 \le \alpha_i \le a_i - 1,\\ &\qquad 1 \le i \le 3\}.\end{aligned}$$

The proof follows easily from Eq. (14). The details are left to the reader. ■

According to Proposition 2.2, the admissibility of an L-periodic pattern U is determined by $U|_{\mathbb{Z}_{(a_1+1)\times(a_2+1)\times(a_3+1)}} = (u_{\alpha_1,\alpha_2,\alpha_3})$ and $U|_{\mathbb{Z}_{(a_1+1)\times(a_2+1)\times(a_3+1)}}$ has the periodic property that is given by Eq. (14), which can be divided into two parts:

$$\begin{cases} u_{a_1,\alpha_2,\alpha_3} = u_{0,\alpha_2,\alpha_3}\\ u_{\alpha_1,a_2,\alpha_3} = u_{[\alpha_1-b_{12}]_{a_1},0,\alpha_3}\end{cases} \tag{15}$$

for $0 \le \alpha_i \le a_i$, $1 \le i \le 3$, where $[m]_n \equiv m \pmod{n}$;

$$u_{\alpha_1,\alpha_2,\alpha_3}=\begin{cases}u_{[\alpha_1-b_{12}-b_{13}]_{a_1},0,0} & \text{if } \alpha_2-b_{23}=a_2\\ u_{[\alpha_1-b_{13}]_{a_1},\alpha_2-b_{23},0} & \text{if } 0\le\alpha_2-b_{23}\le a_2-1\\ u_{[\alpha_1+b_{12}-b_{13}]_{a_1},\alpha_2-b_{23}+a_2,0} & \text{if } -a_2+1\le\alpha_2-b_{23}\le-1\end{cases}\tag{16}$$

for $0\le\alpha_1\le a_1$, $0\le\alpha_2\le a_2$.

Notably, $(u_{\alpha_1,\alpha_2,\alpha_3})_{0\le\alpha_1\le a_1,0\le\alpha_2\le a_2,\alpha_3}$ has the same structure Eq. (15) for all $0\le\alpha_3\le a_3$; this fact is useful in constructing the cylindrical ordering matrix. Then, the set of all local patterns in $\Sigma_{a_1+1,a_2+1,a_3+1}$ that satisfy the periodic property Eq. (15) is denoted by $\mathbb{P}_{a_1,a_2;b_{12};a_3+1}$. However, Eq. (16) is important in allowing patterns in $\mathbb{P}_{a_1,a_2;b_{12};a_3+1}$ to become L-periodic and it will be used to define the rotational matrices later.

Now, the counting function for $U_{n_1\times n_2\times n_3}=(u_{\alpha_1,\alpha_2,\alpha_3})$ in $\Sigma_{n_1\times n_2\times n_3}$, $n_1,n_2,n_3\ge 1$, is defined by

$$\psi(U_{n_1\times n_2\times n_3})=1+\sum_{\alpha_1=0}^{n_1-1}\sum_{\alpha_2=0}^{n_2-1}\sum_{\alpha_3=0}^{n_3-1}u_{\alpha_1,\alpha_2,\alpha_3}\times 2^{n_1n_2(n_3-1-\alpha_3)+n_1(n_2-1-\alpha_2)+n_1-1-\alpha_1}.\tag{17}$$

Similar to Eq. (17), the counting function $\overline{\psi}$ for patterns $\overline{U}$ in $\mathbb{P}_{n_1,n_2;l;1}$, $0\le l\le n_1-1$, is defined by

$$\overline{\psi}(\overline{U})\equiv\psi(\overline{U}|_{\mathbb{Z}_{n_1\times n_2\times 1}}).\tag{18}$$

Notably, $\overline{\psi}$ is bijective from $\mathbb{P}_{n_1,n_2;l;1}$ to $\{i|1\le i\le 2^{n_1n_2}\}$.

Given $n_1,n_2\ge 1$, $0\le l\le n_1-1$, $h\ge 1$, a local pattern $\overline{U}$ in $\mathbb{P}_{n_1,n_2;l;h}$ can be represented as

$$\overline{U}=\overline{U}_0\oplus_z\overline{U}_1\oplus_z\cdots\oplus_z\overline{U}_{h-1},\tag{19}$$

where $\overline{U}_i\in\mathbb{P}_{n_1,n_2;l;1}$, $0\le i\le h-1$, and $\overline{U}'\oplus_z\overline{U}''$ means that $\overline{U}''$ is put on the top (in the z-direction) of $\overline{U}'$. Therefore, the cylindrical ordering matrix $\mathbb{C}_{n_1,n_2;l;h}=[C_{n_1,n_2;l;h;i,j}]_{2^{n_1n_2}\times 2^{n_1n_2}}$ of patterns in $\mathbb{P}_{n_1,n_2;l;h}$ is defined by

$$C_{n_1,n_2;l;h;i,j}=\{\overline{U}_0\oplus_z\cdots\oplus_z\overline{U}_{h-1}|\overline{\psi}(\overline{U}_0)=i \text{ and } \overline{\psi}(\overline{U}_{h-1})=j\}.\tag{20}$$

In particular, for $h=2$, $\mathbb{C}_{n_1,n_2;l;2}$ can be applied to construct the associated trace operator. Notably, the set $\mathbb{C}_{n_1,n_2;l;2;i,j}$ contains exactly one pattern.

Now, given $\mathcal{B}\subset\Sigma_{2\times2\times2}$, the associated trace operator $\mathbf{T}_{n_1,n_2;l}(\mathcal{B})=[t_{n_1,n_2;l;i,j}]$, with $t_{n_1,n_2;l;i,j}\in\{0,1\}$, can be defined by

$$t_{n_1,n_2;l;i,j}=1 \quad \text{if and only if the pattern in } C_{n_1,n_2;l;2;i,j} \text{ is } \mathcal{B}\text{-admissible.}\tag{21}$$

Remark 2.3. Given $L'=\begin{bmatrix}a_1&b_{12}&0\\0&a_2&0\\0&0&a_3\end{bmatrix}\mathbb{Z}^3$, Eqs. (15) and (16) easily verify that

$$\{U|_{\mathbb{Z}_{a_1+1,a_2+1,a_3+1}}:U \text{ is } L'\text{-periodic}\}=\{\overline{U}=\overline{U}_0\oplus_z\cdots\oplus_z\overline{U}_{a_3}\in\mathbb{P}_{a_1,a_2;b_{12};a_3+1}:\overline{U}_0=\overline{U}_{a_3}\}.\tag{22}$$

Furthermore, given $\mathcal{B}\subset\Sigma_{2\times2\times2}$, from Proposition 2.2 and the construction of the transition matrix $\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B})$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix}a_1&b_{12}&0\\0&a_2&0\\0&0&a_3\end{bmatrix}\right)=\mathrm{tr}(\mathbf{T}^{a_3}_{a_1,a_2;b_{12}}(\mathcal{B})).\tag{23}$$

The shift maps and the related rotational matrices are considered below for general $L=\begin{bmatrix}a_1&b_{12}&b_{13}\\0&a_2&b_{23}\\0&0&a_3\end{bmatrix}\mathbb{Z}^3$.

Let $n_1,n_2\ge 1$, $0\le l\le n_1-1$; the shift (to the left) in the x-direction of any pattern $\overline{U}=(u_{\alpha_1,\alpha_2,0})$ in $\mathbb{P}_{n_1,n_2;l;1}$, $u_{\alpha_1,\alpha_2,0}\in\{0,1\}$, is defined by

$$\sigma_{x;n_1,n_2;l}((u_{\alpha_1,\alpha_2,0}))=(u^{(1)}_{\alpha_1,\alpha_2,0})_{0\le\alpha_1\le n_1,0\le\alpha_2\le n_2}$$

where

$$u^{(1)}_{\alpha_1,\alpha_2,0}=\begin{cases}u_{[\alpha_1+1-l]_{n_1},0,0} & \text{if } \alpha_2=n_2,\\ u_{[\alpha_1+1]_{n_1},\alpha_2,0} & \text{if } 0\le\alpha_2\le n_2-1.\end{cases}\tag{24}$$

Similarly, the shift (to the below) in the y-direction is defined by

$$\sigma_{y;n_1,n_2;l}((u_{\alpha_1,\alpha_2,0}))=(u^{(2)}_{\alpha_1,\alpha_2,0})_{0\le\alpha_1\le n_1,0\le\alpha_2\le n_2}$$

where

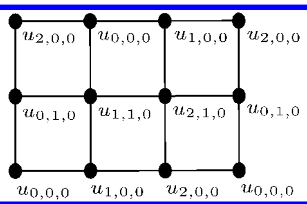
$$u^{(2)}_{\alpha_1,\alpha_2,0}=\begin{cases}u_{[\alpha_1-l]_{n_1},\alpha_2+1-n_2,0} & \text{if } \alpha_2+1\ge n_2,\\ u_{[\alpha_1]_{n_1},\alpha_2+1,0} & \text{if } 0\le\alpha_2+1\le n_2-1.\end{cases}\tag{25}$$

Notably, $\sigma_{x;n_1,n_2;l}$ and $\sigma_{y;n_1,n_2;l}$ are automorphisms on $\mathbb{P}_{n_1,n_2;l;1}$.

The following example illustrates $\sigma_{x;n_1,n_2;l}$ and $\sigma_{y;n_1,n_2;l}$.

Example 2.4. Let

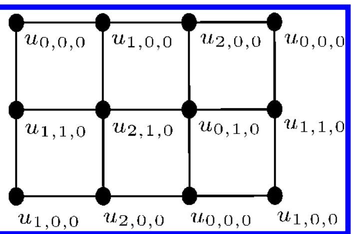
$\overline{U} = (u_{\alpha_1,\alpha_2,0}) \equiv$



$\in \mathbb{P}_{3,2;1;1}$

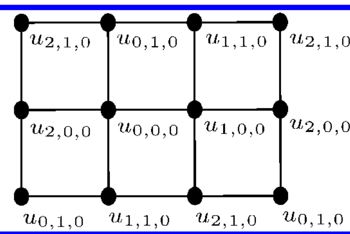
be a local pattern that lies on the plane $\{(z_1, z_2, 0) : z_1, z_2 \in \mathbb{Z}\}$. Now, consider $\sigma_{x;3,2;1}$ and $\sigma_{y;3,2;1}$ which are acting on $\overline{U}$. Then it is easy to see

$\sigma_{x;3,2;1}\left(\overline{U}\right) =$



and

$\sigma_{y;3,2;1}\left(\overline{U}\right) =$



.

Moreover, both $\sigma_{x;3,2;1}(\overline{U})$ and $\sigma_{y;3,2;1}(\overline{U})$ also belong to $\mathbb{P}_{3,2;1;1}$.

From Eqs. (24) and (25), for $0 \leq r_i \leq n_i - 1$, $i = 1, 2$, the following can be straightforwardly verified;

$$\sigma^{r_1}_{x;n_1,n_2;l}(\sigma^{r_2}_{y;n_1,n_2;l}((u_{\alpha_1,\alpha_2,0}))) = (u^{(3)}_{\alpha_1,\alpha_2,0})_{0\leq\alpha_1\leq n_1, 0\leq\alpha_2\leq n_2}$$

where

$$u^{(3)}_{\alpha_1,\alpha_2,0} = \begin{cases} u_{[\alpha_1+r_1-l]_{n_1},\alpha_2+r_2-n_2,0} & \text{if } n_2 \leq \alpha_2 + r_2 \leq 2n_2 - 1, \\ u_{[\alpha_1+r_1]_{n_1},\alpha_2+r_2,0} & \text{if } 0 \leq \alpha_2 + r_2 \leq n_2 - 1. \end{cases} \tag{26}$$

Furthermore,

$$\sigma_{y;n_1,n_2;l} \circ \sigma_{x;n_1,n_2;l} = \sigma_{x;n_1,n_2;l} \circ \sigma_{y;n_1,n_2;l} \tag{27}$$

and

$$\sigma^{n_1}_{x;n_1,n_2;l} = \sigma^{l}_{x;n_1,n_2;l}(\sigma^{n_2}_{y;n_1,n_2;l}) = \text{ identity map.} \tag{28}$$

Hence,

$$\sigma^{-1}_{x;n_1,n_2;l} \equiv \sigma^{n_1-1}_{x;n_1,n_2;l} \quad \text{and} \quad \sigma^{-1}_{y;n_1,n_2;l} \equiv \sigma^{l}_{x;n_1,n_2;l}(\sigma^{n_2-1}_{y;n_1,n_2;l}). \tag{29}$$

Therefore, for $0 \leq r_i \leq n_i - 1$, $i = 1, 2$,

$$\sigma^{-r_1}_{x;n_1,n_2;l}(\sigma^{-r_2}_{y;n_1,n_2;l}((u_{\alpha_1,\alpha_2,0}))) = (u^{(4)}_{\alpha_1,\alpha_2,0})_{0\leq\alpha_1\leq n_1, 0\leq\alpha_2\leq n_2}$$

where

$$u^{(4)}_{\alpha_1,\alpha_2,0} = \begin{cases} u_{[\alpha_1-r_1-l]_{n_1},0,0} & \text{if } \alpha_2 - r_2 = n_2, \\ u_{[\alpha_1-r_1]_{n_1},\alpha_2-r_2,0} & \text{if } 0 \leq \alpha_2 - r_2 \leq n_2 - 1, \\ u_{[\alpha_1-r_1+l]_{n_1},\alpha_2-r_2+n_2,0} & \text{if } -n_2 + 1 \leq \alpha_2 - r_2 \leq -1. \end{cases} \tag{30}$$

Now, the two rotational matrices $R_{x;n_1,n_2;l}$ and $R_{y;n_1,n_2;l}$ are defined as follows.

Definition 2.5. The $2^{n_1n_2} \times 2^{n_1n_2}$ x-rotational matrix $R_{x;n_1,n_2;l} = [R_{x;n_1,n_2;l;i,j}]$, $R_{x;n_1,n_2;l;i,j} \in \{0, 1\}$, is defined by

$$R_{x;n_1,n_2;l;i,j} = 1 \quad \text{if and only if } i = \overline{\psi}(\overline{U}) \quad \text{and} \quad j = \overline{\psi}(\sigma_{x;n_1,n_2;l}(\overline{U})), \tag{31}$$

where $\overline{U} \in \mathbb{P}_{n_1,n_2;l;1}$. From Eq. (31), for convenience, denote by

$$j = \sigma_x(i). \tag{32}$$

Similarly, the $2^{n_1n_2} \times 2^{n_1n_2}$ y-rotational matrix $R_{y;n_1,n_2;l} = [R_{y;n_1,n_2;l;i,j}]$, $R_{y;n_1,n_2;l;i,j} \in \{0,1\}$, is defined by

$$R_{y;n_1,n_2;l;i,j} = 1 \quad \text{if and only if } i = \overline{\psi}(\overline{U}) \quad \text{and} \quad j = \overline{\psi}(\sigma_{y;n_1,n_2;l}(\overline{U})), \tag{33}$$

where $\overline{U} \in \mathbb{P}_{n_1,n_2;l;1}$. From Eq. (33), for convenience, denote by

$$j = \sigma_y(i). \tag{34}$$

Obviously, $R_{x;n_1,n_2;l}$ and $R_{y;n_1,n_2;l}$ are permutation matrices. By Eq. (28), $R^{n_1}_{x;n_1,n_2;l} = R^{l}_{x;n_1,n_2;l}R^{n_2}_{y;n_1,n_2;l} = I_{2^{n_1n_2}}$, where I_n is the $n \times n$ identity matrix.

Example 2.6. Let $n_1 = 2$, $n_2 = 1$ and $l = 1$,

$$R_{x;2,1;1} = R_{y;2,1;1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$R^2_{x;2,1;1} = R_{x;2,1;1}R_{y;2,1;1} = I_4 \text{ but } R_{y;2,1;1} \neq I_4.$$

The following proposition shows the permutation characters of $R_{x;n_1,n_2;l}$ and $R_{y;n_1,n_2;l}$.

Proposition 2.7. *Let $\mathbb{M} = [M_{i,j}]_{2^{n_1n_2}\times 2^{n_1n_2}}$ be a matrix where $M_{i,j}$ denotes a number or a pattern or a set of patterns. Then*

$$(\mathbb{M}R_{x;n_1,n_2;l})_{i,j} = M_{i,\sigma_x^{-1}(j)} \quad \text{and} \quad (\mathbb{M}R_{y;n_1,n_2;l})_{i,j} = M_{i,\sigma_y^{-1}(j)}. \tag{35}$$

Furthermore, for any $r \geq 1$

$$(\mathbb{M}R^r_{x;n_1,n_2;l})_{i,j} = M_{i,\sigma_x^{-r}(j)} \quad \text{and} \quad (\mathbb{M}R^r_{y;n_1,n_2;l})_{i,j} = M_{i,\sigma_y^{-r}(j)}. \tag{36}$$

Proof. For any $1 \leq i,j \leq 2^{n_1n_2}$, by Eq. (32),

$$\begin{aligned}(\mathbb{M}R_{x;n_1,n_2;l})_{i,j} &= \sum_q M_{i,q}R_{x;n_1,n_2;l;q,j} \\ &= M_{i,\sigma_x^{-1}(j)}R_{x;n_1,n_2;l;\sigma_x^{-1}(j),j} \\ &= M_{i,\sigma_x^{-1}(j)}.\end{aligned}$$

Similarly,

$$\begin{aligned}(\mathbb{M}R_{y;n_1,n_2;l})_{i,j} &= \sum_q M_{i,q}R_{y;n_1,n_2;l;q,j} \\ &= M_{i,\sigma_y^{-1}(j)}R_{y;n_1,n_2;l;\sigma_y^{-1}(j),j} \\ &= M_{i,\sigma_y^{-1}(j)}.\end{aligned}$$

Applying Eq. (35) r times yields Eq. (36). The proof is complete. ∎

Now, the following lemma can be obtained.

Lemma 2.8. *Given $L = \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\mathbb{Z}^3$,*

$$\begin{aligned}&\{U|_{\mathbb{Z}_{(a_1+1)\times(a_2+1)\times(a_3+1)}} : U \text{ is } L\text{-periodic}\} \\ &\quad = \{\overline{U} = \overline{U}_0 \oplus_z \cdots \oplus_z \overline{U}_{a_3} \in \mathbb{P}_{a_1,a_2;b_{12};a_3+1} : \\ &\qquad \overline{U}_{a_3} = \sigma^{-b_{13}}_{x;a_1,a_2;b_{12}}(\sigma^{-b_{23}}_{y;a_1,a_2;b_{12}}(\overline{U}_0))\}\end{aligned} \tag{37}$$

Proof. From Eqs. (16) and (30),

$$\begin{aligned}&\{\overline{U} = \overline{U}_0 \oplus_z \cdots \oplus_z \overline{U}_{a_3} \in \mathbb{P}_{a_1,a_2;b_{12};a_3+1} : \\ &\quad \overline{U}_{a_3} = \sigma^{-b_{13}}_{x;a_1,a_2;b_{12}}(\sigma^{-b_{23}}_{y;a_1,a_2;b_{12}}(\overline{U}_0))\} \\ &\qquad = \{\overline{U} \in \mathbb{P}_{a_1,a_2;b_{12};a_3+1} : \overline{U} \text{ satisfies Eq. (16)}\}.\end{aligned}$$

Then, by the construction of $\mathbb{P}_{a_1,a_2;b_{12};a_3+1}$, the last set is equal to

$$\begin{aligned}&\{U \in \Sigma_{a_1+1,a_2+1,a_3+1} : U \text{ satisfies} \\ &\quad \text{Eqs. (15) and (16)}\} \\ &\qquad = \{U \in \Sigma_{a_1+1,a_2+1,a_3+1} : U \text{ satisfies Eq. (14)}\}.\end{aligned}$$

Therefore, Eq. (37) follows. The proof is complete. ∎

Propositions 2.2, 2.7 and Lemma 2.8 yield the following main results for $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right)$.

Theorem 2.9. *Given a basic set $\mathcal{B} \subset \Sigma_{2\times2\times2}$. For $a_i \geq 1, 1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq 3$,*

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) = \mathrm{tr}(\mathbf{T}^{a_3}_{a_1,a_2;b_{12}}(\mathcal{B})R^{b_{13}}_{x;a_1,a_2;b_{12}}R^{b_{23}}_{y;a_1,a_2;b_{12}}). \tag{38}$$

Furthermore,

$$\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) = tr(\mathbf{T}^{a_3}_{a_1,a_2;b_{12}}(\mathcal{B})\mathbf{R}_{a_1,a_2;b_{12}}), \tag{39}$$

where

$$\mathbf{R}_{a_1,a_2;b_{12}} = \sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1} R^{b_{13}}_{x;a_1,a_2;b_{12}}R^{b_{23}}_{y;a_1,a_2;b_{12}}. \tag{40}$$

Proof. From Proposition 2.2, Lemma 2.8 and the construction of $\mathbb{C}_{a_1,a_2;b_{12};a_3+1}$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) = \sum_{i=1}^{2^{a_1a_2}} \sharp\{\overline{U} \in C_{a_1,a_2;b_{12};a_3+1;i,j} : \overline{U} \text{ is } \mathcal{B}\text{-admissible and } j = \sigma_x^{-b_{13}}(\sigma_y^{-b_{23}}(i))\},$$

where $\sharp S$ is the cardinal number of set S.

Then, Proposition 2.7 and the construction of $\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B})$, $R_{x;a_1,a_2;b_{12}}$ and $R_{y;a_1,a_2;b_{12}}$ easily yield Eq. (38). Equation (39) holds from Eqs. (38) and (40). The proof is complete. ■

The $(a_1, a_2; b_{12})$th zeta function $\zeta_{a_1,a_2;b_{12}}(s)$ can now be obtained as follows.

Theorem 2.10. *Given a basic set $\mathcal{B} \subset \Sigma_{2\times2\times2}$. For $a_i \geq 1, 1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i + 1 \leq j \leq 3$,*

$$\zeta_{a_1,a_2;b_{12}}(s) = \exp\left(\frac{1}{a_1a_2}\sum_{a_3=1}^{\infty}\frac{1}{a_3}\mathrm{tr}(\mathbf{T}^{a_3}_{a_1,a_2;b_{12}}(\mathcal{B}) \times \mathbf{R}_{a_1,a_2;b_{12}})s^{a_1a_2a_3}\right). \quad (41)$$

Proof. The results follow from Theorem 2.9. ■

3. Rationality of $\zeta_{a_1,a_2;b_{12}}$

This section proves that $\zeta_{a_1,a_2;b_{12}}$ is a rational function. First, the rotational symmetry of $\mathbf{T}_{a_1,a_2;b_{12}}$ is introduced.

Theorem 3.1. *Given $\mathcal{B} \subset \Sigma_{2\times2\times2}$. Denote by $\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B}) = [t_{a_1,a_2;b_{12};i,j}]$. For $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$,*

$$t_{a_1,a_2;b_{12};\sigma_x^{-1}(i),\sigma_x^{-1}(j)} = t_{a_1,a_2;b_{12};i,j} \quad (42)$$

and

$$t_{a_1,a_2;b_{12};\sigma_y^{-1}(i),\sigma_y^{-1}(j)} = t_{a_1,a_2;b_{12};i,j} \quad (43)$$

for all $1 \leq i, j \leq 2^{a_1a_2}$. Furthermore,

$$t_{a_1,a_2;b_{12};\sigma_x^{-r_1}(\sigma_y^{-r_2}(i)),\sigma_x^{-r_1}(\sigma_y^{-r_2}(j))} = t_{a_1,a_2;b_{12};i,j} \quad (44)$$

for all $1 \leq i, j \leq 2^{a_1a_2}$, $-a_1 + 1 \leq r_1 \leq a_1 - 1$ and $-a_2 + 1 \leq r_2 \leq a_2 - 1$.

Proof. The proof of Eq. (43) is similar to that of Eq. (42) and is omitted. We now prove Eq. (42).

Given $1 \leq i, j \leq 2^{a_1a_2}$, $C_{a_1,a_2;b_{12};2;i,j}$ and $C_{a_1,a_2;b_{12};2;\sigma_x^{-1}(i),\sigma_x^{-1}(j)}$ contain only one pattern respectively. Let

$$\overline{U} = \overline{U}_0 \oplus_z \overline{U}_1 = (u_{\alpha_1,\alpha_2,\alpha_3}) \in C_{a_1,a_2;b_{12};2;i,j}$$

with $\overline{\psi}(\overline{U}_0) = i$ and $\overline{\psi}(\overline{U}_1) = j$, and

$$\overline{U}' = \overline{U}'_0 \oplus_z \overline{U}'_1 = (u'_{\alpha_1,\alpha_2,\alpha_3}) \in C_{a_1,a_2;b_{12};2;\sigma_x^{-1}(i),\sigma_x^{-1}(j)}$$

with $\overline{\psi}(\overline{U}'_0) = \sigma_x^{-1}(i)$ and $\overline{\psi}(\overline{U}'_1) = \sigma_x^{-1}(j)$. Since $\mathcal{B} \subset \Sigma_{2\times2\times2}$ and Eq. (21), to prove Eq. (42) is equal,

$$\{(u_{n_1+k_1,n_2+k_2,k_3})_{0\leq k_1,k_2,k_3\leq1} : 0 \leq n_1 \leq a_1 - 1, 0 \leq n_2 \leq a_2 - 1\} = \{(u'_{n_1+k_1,n_2+k_2,k_3})_{0\leq k_1,k_2,k_3\leq1} : 0 \leq n_1 \leq a_1 - 1, 0 \leq n_2 \leq a_2 - 1\}. \quad (45)$$

Since $\overline{\psi}(\overline{U}_0) = i$ and $\overline{\psi}(\overline{U}'_0) = \sigma_x^{-1}(i)$, by Eq. (30),

$$u'_{\alpha_1,\alpha_2,0} = \begin{cases} u_{[\alpha_1-1-b_{12}]_{a_1},0,0} & \text{if } \alpha_2 = a_2, \\ u_{[\alpha_1-1]_{a_1},\alpha_2,0} & \text{if } 0 \leq \alpha_2 \leq a_2 - 1. \end{cases}$$

Similarly, from $\overline{\psi}(\overline{U}_1) = j$ and $\overline{\psi}(\overline{U}'_1) = \sigma_x^{-1}(j)$,

$$u'_{\alpha_1,\alpha_2,1} = \begin{cases} u_{[\alpha_1-1-b_{12}]_{a_1},0,1} & \text{if } \alpha_2 = a_2, \\ u_{[\alpha_1-1]_{a_1},\alpha_2,1} & \text{if } 0 \leq \alpha_2 \leq a_2 - 1. \end{cases}$$

Then, Eq. (45) is directly obtained.

Therefore, Eqs. (42) and (43) hold. For $0 \leq r_1 \leq a_1 - 1$ and $0 \leq r_2 \leq a_2 - 1$, by applying Eq. (43) r_2 times and Eq. (42) r_1 times, Eq. (44) holds. From Eqs. (27)–(29), Eq. (44) follows. The proof is complete. ■

To study the rationality of $\zeta_{a_1,a_2;b_{12}}$, we need more definitions and properties about the two shifts in Eqs. (32) and (34) as follows.

Given $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$, for $1 \leq i \leq 2^{a_1a_2}$, the equivalent class $\mathcal{C}_{a_1,a_2;b_{12}}(i)$ of

i is defined by

$$\mathcal{C}_{a_1,a_2;b_{12}}(i) \equiv \{\sigma_x^{-r_1}(\sigma_y^{-r_2}(i)) : 0 \le r_1 \le a_1 - 1,\ 0 \le r_2 \le a_2 - 1\}. \tag{46}$$

Clearly,

$$\text{either } \mathcal{C}_{a_1,a_2;b_{12}}(i) = \mathcal{C}_{a_1,a_2;b_{12}}(j) \quad \text{or} \quad \mathcal{C}_{a_1,a_2;b_{12}}(i) \cap \mathcal{C}_{a_1,a_2;b_{12}}(j) = \emptyset. \tag{47}$$

The cardinal number of $\mathcal{C}_{a_1,a_2;b_{12}}(i)$ is denoted by $\omega_{a_1,a_2;b_{12};i}$. Let i be the smallest element in its equivalent class, and the index set $\mathcal{I}_{a_1,a_2;b_{12}}$ is defined by

$$\mathcal{I}_{a_1,a_2;b_{12}} = \{i : 1 \le i \le 2^{a_1 a_2}, i \le j \text{ for all } j \in \mathcal{C}_{a_1,a_2;b_{12}}(i)\}. \tag{48}$$

Therefore,

$$\{j : 1 \le j \le 2^{a_1 a_2}\} = \bigcup_{i \in \mathcal{I}_{a_1,a_2;b_{12}}} \mathcal{C}_{a_1,a_2;b_{12}}(i). \tag{49}$$

The cardinal number of $\mathcal{I}_{a_1,a_2;b_{12}}$ is denoted by $\chi_{a_1,a_2;b_{12}}$.

The following example illustrates $\mathcal{C}_{2,2;j}(i)$.

Example 3.2.

$$\begin{cases} \mathcal{C}_{2,2;0}(1) = \{1\} \\ \mathcal{C}_{2,2;0}(2) = \{2,3,5,9\} \\ \mathcal{C}_{2,2;0}(4) = \{4,13\} \\ \mathcal{C}_{2,2;0}(6) = \{6,11\} \\ \mathcal{C}_{2,2;0}(7) = \{7,10\} \\ \mathcal{C}_{2,2;0}(8) = \{8,12,14,15\} \\ \mathcal{C}_{2,2;0}(16) = \{16\} \\ \mathcal{I}_{2,2;0} = \{1,2,4,6,7,8,16\} \end{cases}$$

$$\begin{cases} \mathcal{C}_{2,2;1}(1) = \{1\} \\ \mathcal{C}_{2,2;1}(2) = \{2,3,5,9\} \\ \mathcal{C}_{2,2;1}(4) = \{4,13\} \\ \mathcal{C}_{2,2;1}(6) = \{6,7,10,11\} \\ \mathcal{C}_{2,2;1}(8) = \{8,12,14,15\} \\ \mathcal{C}_{2,2;1}(16) = \{16\} \\ \mathcal{I}_{2,2;1} = \{1,2,4,6,8,16\} \end{cases}$$

The equivalent classes are invariant under the two shift maps. Therefore, the following proposition is directly obtained and the proof is omitted.

Proposition 3.3. *Given $a_1, a_2 \ge 1$ and $0 \le b_{12} \le a_1 - 1$. Let $N \equiv 2^{a_1 a_2}$ and $V = (v_1, v_2, \ldots, v_N)^t$, for $1 \le i \le N$,*

$$\sum_{r_1=0}^{a_1-1} \sum_{r_2=0}^{a_2-1} v_{\sigma_x^{-r_1}(\sigma_y^{-r_2}(i))} = \frac{a_1 a_2}{\omega_{a_1,a_2;b_{12};i}} \sum_{j \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} v_j. \tag{50}$$

For the rationality of $\zeta_{a_1,a_2;b_{12}}$, the reduced trace operator $\tau_{a_1,a_2;b_{12}}$ of $\mathbf{T}_{a_1,a_2;b_{12}}$ is introduced as follows.

Definition 3.4. For $a_1, a_2 \ge 1$, $0 \le b_{12} \le a_1 - 1$, the reduced trace operator $\tau_{a_1,a_2;b_{12}} = [\tau_{a_1,a_2;b_{12};i,j}]$ of $\mathbf{T}_{a_1,a_2;b_{12}} = [t_{a_1,a_2;b_{12};i,j}]$ is a $\chi_{a_1,a_2;b_{12}} \times \chi_{a_1,a_2;b_{12}}$ matrix and is defined by

$$\tau_{a_1,a_2;b_{12};i,j} = \sum_{k \in \mathcal{C}_{a_1,a_2;b_{12}}(j)} t_{a_1,a_2;b_{12};i,k} \tag{51}$$

for each $i, j \in \mathcal{I}_{a_1,a_2;b_{12}}$.

The following theorem expresses the average of $\Gamma_{\mathcal{B}}$ in terms of the trace of the reduced trace operator τ and plays a crucial role in proving the rationality of $\zeta_{a_1,a_2;b_{12}}$. The proof here is simpler and more straightforward than the proofs in [Ban *et al.*, 2008a] for $d = 2$.

Theorem 3.5. *Given $\mathcal{B} \subset \Sigma_{2\times2\times2}$. For $a_i \ge 1$, $1 \le i \le 3$, $0 \le b_{ij} \le a_i - 1$, $i+1 \le j \le 3$,*

$$\frac{1}{a_1 a_2} \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) = \mathrm{tr}(\tau_{a_1,a_2;b_{12}}^{a_3}) = \sum_{\lambda \in \Sigma(\tau_{a_1,a_2;b_{12}})} \chi_{a_1,a_2;b_{12}}(\lambda) \lambda^{a_3}, \tag{52}$$

where $\Sigma(\tau_{a_1,a_2;b_{12}})$ is the spectrum of $\tau_{a_1,a_2;b_{12}}$ and $\chi_{a_1,a_2;b_{12}}(\lambda)$ is the algebraic multiplicity of $\tau_{a_1,a_2;b_{12}}$ with eigenvalue λ.

Proof. For simplicity, let $N = 2^{a_1 a_2}$ and $\mathbf{T}_{a_1,a_2;b_{12}} = [t_{i,j}]$. From Proposition 2.7 and Theorem 2.9,

$$\frac{1}{a_1 a_2} \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) = \frac{1}{a_1 a_2} \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \times \mathrm{tr}(\mathbf{T}_{a_1,a_2;b_{12}}^{a_3} R_{x;a_1,a_2;b_{12}}^{b_{13}} R_{y;a_1,a_2;b_{12}}^{b_{23}})$$

$$= \frac{1}{a_1 a_2} \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \sum_{i=1}^{N} \sum_{j=1}^{a_3-1} \sum_{k_j=1}^{N} \times t_{i,k_1} t_{k_1,k_2} \cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}(\sigma_y^{-b_{23}}(i))}.$$

Now, by Eq. (49), the last sum becomes

$$\frac{1}{a_1 a_2} \sum_{i \in \mathcal{I}_{a_1,a_2;b_{12}}} \sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \sum_{j=1}^{a_3-1} \sum_{k_j=1}^{N} \times t_{q,k_1} t_{k_1,k_2} \cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}(\sigma_y^{-b_{23}}(q))}. \quad (53)$$

Fixed $q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)$, there exist $0 \le r_1 \le a_1 - 1$ and $0 \le r_2 \le a_2 - 1$ such that $q = \sigma_x^{-r_1}(\sigma_y^{-r_2}(i))$. Then, by Theorem 3.1,

$$\begin{aligned}
&\sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \sum_{j=1}^{a_3-1} \sum_{k_j=1}^{N} \times t_{q,k_1} t_{k_1,k_2} \cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}(\sigma_y^{-b_{23}}(q))} \\
&= \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \sum_{j=1}^{a_3-1} \sum_{k_j=1}^{N} \times t_{\sigma_x^{r_1}(\sigma_y^{r_2}(q)),\sigma_x^{r_1}(\sigma_y^{r_2}(k_1))} t_{\sigma_x^{r_1}(\sigma_y^{r_2}(k_1)),\sigma_x^{r_1}(\sigma_y^{r_2}(k_2))} \\
&\quad \cdots t_{\sigma_x^{r_1}(\sigma_y^{r_2}(k_{a_3-1})),\sigma_x^{r_1}(\sigma_y^{r_2}(\sigma_x^{-b_{13}}(\sigma_y^{-b_{23}}(q))))} \\
&= \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \sum_{j=1}^{a_3-1} \sum_{k_j=1}^{N} \times t_{i,\sigma_x^{r_1}(\sigma_y^{r_2}(k_1))} t_{\sigma_x^{r_1}(\sigma_y^{r_2}(k_1)),\sigma_x^{r_1}(\sigma_y^{r_2}(k_2))} \\
&\quad \cdots t_{\sigma_x^{r_1}(\sigma_y^{r_2}(k_{a_3-1})),\sigma_x^{-b_{13}}(\sigma_y^{-b_{23}}(i))}
\end{aligned}$$

Since $\{\sigma_x^{r_1}(\sigma_y^{r_2}(m)) : 1 \le m \le N\} = \{m : 1 \le m \le N\}$, the last sum becomes

$$\sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \sum_{j=1}^{a_3-1} \sum_{k_j=1}^{N} \times t_{i,k_1} t_{k_1,k_2} \cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}(\sigma_y^{-b_{23}}(i))} \quad (54)$$

Therefore, Eq. (53) is equal to

$$\frac{1}{a_1 a_2} \sum_{i \in \mathcal{I}_{a_1,a_2;b_{12}}} \omega_{a_1,a_2;b_{12};i} \sum_{j=1}^{a_3-1} \sum_{k_j=1}^{N} \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \times t_{i,k_1} t_{k_1,k_2} \cdots t_{k_{a_3-1},\sigma_x^{-b_{13}}(\sigma_y^{-b_{23}}(i))}. \quad (55)$$

According to Proposition 3.3, Eq. (55) is equal to

$$\begin{aligned}
&\sum_{i \in \mathcal{I}_{a_1,a_2;b_{12}}} \sum_{j=1}^{a_3-1} \sum_{k_j=1}^{N} t_{i,k_1} \cdots t_{k_{a_3-2},k_{a_3-1}} \times \left(\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q} \right) \\
&= \sum_{i \in \mathcal{I}_{a_1,a_2;b_{12}}} \sum_{j=1}^{a_3-1} \sum_{k_j \in \mathcal{I}_{a_1,a_2;b_{12}}} \sum_{q_j \in \mathcal{C}_{a_1,a_2;b_{12}}(k_j)} \times t_{i,q_1} \cdots t_{q_{a_3-2},q_{a_3-1}} \left(\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{q_{a_3-1},q} \right).
\end{aligned} \quad (56)$$

For any $q_{a_3-1} \in \mathcal{C}_{a_1,a_2;b_{12}}(k_{a_3-1})$, there exist $0 \le r_1 \le a_1 - 1$ and $0 \le r_2 \le a_2 - 1$ such that

$$q_{a_3-1} = \sigma_x^{-r_1}(\sigma_y^{-r_2}(k_{a_3-1})).$$

Then, by Theorem 3.1,

$$\begin{aligned}
\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{q_{a_3-1},q} &= \sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{\sigma_x^{-r_1}(\sigma_y^{-r_2}(k_{a_3-1})),q} \\
&= \sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},\sigma_x^{r_1}(\sigma_y^{r_2}(q))} \\
&= \sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^{a_3-1} \sum_{q_j \in \mathcal{C}_{a_1,a_2;b_{12}}(k_j)} t_{i,q_1} \cdots t_{q_{a_3-2},q_{a_3-1}} \left(\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{q_{a_3-1},q} \right) \\
&= \sum_{j=1}^{a_3-2} \sum_{q_j \in \mathcal{C}_{a_1,a_2;b_{12}}(k_j)} t_{i,q_1} \cdots \left(\sum_{q_{a_3-1} \in \mathcal{C}_{a_1,a_2;b_{12}}(k_{a_3-1})} t_{q_{a_3-2},q_{a_3-1}} \right) \left(\sum_{q \in \mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q} \right)
\end{aligned}$$

$$= \sum_{j=1}^{a_3-2} \sum_{q_j\in\mathcal{C}_{a_1,a_2;b_{12}}(k_j)} t_{i,q_1}\cdots\left(\sum_{q_{a_3-1}\in\mathcal{C}_{a_1,a_2;b_{12}}(k_{a_3-1})} t_{k_{a_3-2},q_{a_3-1}}\right)\left(\sum_{q\in\mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q}\right)$$

$$\vdots$$

$$= \left(\sum_{q_1\in\mathcal{C}_{a_1,a_2;b_{12}}(k_1)} t_{i,q_1}\right)\left[\prod_{j=2}^{a_3-1}\left(\sum_{q_j\in\mathcal{C}_{a_1,a_2;b_{12}}(k_j)} t_{k_{j-1},q_j}\right)\right]\left(\sum_{q\in\mathcal{C}_{a_1,a_2;b_{12}}(i)} t_{k_{a_3-1},q}\right)$$

$$= \tau_{a_1,a_2;b_{12};i,k_1}\tau_{a_1,a_2;b_{12};k_1,k_2}\cdots\tau_{a_1,a_2;b_{12};k_{a_3-1},i}$$

Finally, Eq. (56) is equal to

$$\sum_{i\in\mathcal{I}_{a_1,a_2;b_{12}}} \sum_{j=1}^{a_3-1} \sum_{k_j\in\mathcal{I}_{a_1,a_2;b_{12}}} \times\tau_{a_1,a_2;b_{12};i,k_1}\tau_{a_1,a_2;b_{12};k_1,k_2}\cdots\tau_{a_1,a_2;b_{12};k_{a_3-1},i}$$
$$= \mathrm{tr}(\tau_{a_1,a_2;b_{12}}^{a_3})$$
$$= \sum_{\lambda\in\Sigma(\tau_{a_1,a_2;b_{12}})} \chi_{a_1,a_2;b_{12}}(\lambda)\lambda^{a_3}.$$

The proof is complete. ■

Therefore, the rationality of $\zeta_{a_1,a_2;b_{12}}$ and ζ can be obtained as follows.

Theorem 3.6. *For* $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$,

$$\zeta_{a_1,a_2;b_{12}}(s) = (\det(I - s^{a_1a_2}\tau_{a_1,a_2;b_{12}}))^{-1} = \prod_{\lambda\in\Sigma(\tau_{a_1,a_2;b_{12}})} (1-\lambda s^{a_1a_2})^{-\chi_{a_1,a_2;b_{12}}(\lambda)}, \tag{57}$$

and

$$\zeta(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1} (\det(I - s^{a_1a_2}\tau_{a_1,a_2;b_{12}}))^{-1} = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\prod_{\lambda\in\Sigma(\tau_{a_1,a_2;b_{12}})} \times(1-\lambda s^{a_1a_2})^{-\chi_{a_1,a_2;b_{12}}(\lambda)}. \tag{58}$$

Proof. By using the power series

$$-\log(1-t) = \sum_{n=1}^{\infty}\frac{t^n}{n}, \tag{59}$$

Eq. (57) follows from Eq. (8) and Theorem 3.5. Equation (58) follows from Eqs. (9) and (57). ■

The following example is used to demonstrate the application of the above result.

Example 3.7. Consider

$$\mathcal{B} = \{U_{2\times2\times2} = (u_{\alpha_1,\alpha_2,\alpha_3}) \in \Sigma_{2\times2\times2} : u_{0,0,j} = u_{1,0,j} = u_{0,1,j} = u_{1,1,j} \text{ for } j = 0,1\}.$$

Clearly, the set $\mathcal{P}(\mathcal{B})$ of all $\mathcal{B}$-admissible and periodic patterns is

$$\{U = (u_{\alpha_1,\alpha_2,\alpha_3}) \in \Sigma_2^3 : u_{i,j,k} = u_{0,0,k} \text{ for all } i,j,k \in \mathbb{Z}\}.$$

Then, it is easy to verify that

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}\right) = 2^{a_3}$$

for $a_i \geq 1$, $1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq 3$. Therefore,

$$\zeta_{a_1,a_2;b_{12}}(s) = (1-2s^{a_1a_2})^{-1} \tag{60}$$

and

$$\zeta(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}(1-2s^{a_1a_2})^{-a_1}. \tag{61}$$

However, Eqs. (60) and (61) can be obtained from Eqs. (57) and (58). The trace operator

$$\mathbf{T}_{a_1,a_2;b_{12}}(\mathcal{B}) = \mathbf{T}_{a_1,a_2;0}(\mathcal{B}) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{2^{a_1a_2}\times2^{a_1a_2}}.$$

Since $\mathcal{C}_{a_1,a_2;b_{12}}(1) = \{1\}$ and $\mathcal{C}_{a_1,a_2;b_{12}}(2^{a_1a_2}) = \{2^{a_1a_2}\}$, the reduced trace operator

$$\tau_{a_1,a_2;b_{12}}(\mathcal{B}) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{\chi_{a_1,a_2;b_{12}} \times \chi_{a_1,a_2;b_{12}}}.$$

Therefore,

$$\begin{aligned} \zeta_{a_1,a_2;b_{12}}(s) &= (\det(I - s^{a_1a_2}\tau_{a_1,a_2;b_{12}}))^{-1} \\ &= (1 - 2s^{a_1a_2})^{-1} \end{aligned}$$

and

$$\zeta(s) = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} (1 - 2s^{a_1a_2})^{-a_1}.$$

Equations (60) and (61) are recovered.

4. Zeta Functions in Inclined Coordinates

This section presents the zeta function with respect to inclined coordinates, determined by applying the unimodular transformations in $GL_3(\mathbb{Z})$. $\mathbb{Z}^3$ is known to be invariant under the unimodular transformation in $GL_3(\mathbb{Z})$. Indeed, Lind [1996] proved that the zeta function $\zeta_{\mathcal{B}}^0$ is independent of a choice of basis for $\mathbb{Z}^3$. Recall that

$$\begin{aligned} GL_d(\mathbb{Z}) = \{\gamma = [\gamma_{ij}]_{1\le i,j\le d} : \gamma_{ij} \in \mathbb{Z} \text{ for } 1 \le i, \\ j \le d \text{ and } |\det(\gamma)| = 1\}. \end{aligned}$$

This section presents the construction of the trace operator $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B})$ and the reduced trace operator $\tau_{\gamma;a_1,a_2;b_{12}}(\mathcal{B})$, and then determines $\zeta_{\gamma;a_1,a_2;b_{12}}$ and $\zeta_{\mathcal{B};\gamma}$. Finally, $\zeta_{\mathcal{B};\gamma}$ is obtained as

$$\begin{aligned} \zeta_{\mathcal{B};\gamma}(s) = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} \prod_{b_{12}=0}^{a_1-1} \\ \times (\det(I - s^{a_1a_2}\tau_{\gamma;a_1,a_2;b_{12}}))^{-1}. \end{aligned} \tag{62}$$

For simplicity, only $\mathcal{B} \subset \Sigma_{2\times2\times2}$ with two symbols are considered. The general cases can be treated analogously.

For a given $\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \in GL_3(\mathbb{Z})$, the lattice points in γ-coordinates are

$$\begin{aligned} (1,0,0)_\gamma &= (\gamma_{11}, \gamma_{12}, \gamma_{13}), \\ (0,1,0)_\gamma &= (\gamma_{21}, \gamma_{22}, \gamma_{23}) \quad \text{and} \\ (0,0,1)_\gamma &= (\gamma_{31}, \gamma_{32}, \gamma_{33}), \end{aligned}$$

and the unit vectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_\gamma = \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_\gamma = \begin{pmatrix} \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_\gamma = \begin{pmatrix} \gamma_{31} \\ \gamma_{32} \\ \gamma_{33} \end{pmatrix}.$$

Notably, when $\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, standard rectangular coordinates are used and the subscript γ is omitted.

The matrix M_γ is defined by

$$M_\gamma = \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma = \gamma^t \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}.$$

Let $L_\gamma = M_\gamma \mathbb{Z}^3$. Then,

$$L_\gamma = \gamma^t \begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix} \mathbb{Z}^3 \tag{63}$$

is easily verified.

A global pattern $U_\gamma = (u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})_{\alpha_1,\alpha_2,\alpha_3\in\mathbb{Z}}$ is called L_γ-periodic or $\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma$-periodic if for every $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$

$$u_{(\alpha_1+a_1p+b_{12}q+b_{13}r,\alpha_2+a_2q+b_{23}r,\alpha_3+a_3r)_\gamma} = u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma} \tag{64}$$

for all $p, q, r \in \mathbb{Z}$. Therefore, the $(a_1, a_2; b_{12})$th zeta function of $\zeta_{\mathcal{B}}^0(s)$ with respect to γ is defined by

$$\begin{aligned} \zeta_{\mathcal{B};\gamma;a_1,a_2;b_{12}}(s) \\ = \exp\Bigg(\frac{1}{a_1a_2} \sum_{a_3=1}^{\infty} \sum_{b_{13}=0}^{a_1-1} \sum_{b_{23}=0}^{a_2-1} \frac{1}{a_3} \Gamma_{\mathcal{B}} \\ \times \left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma \right) s^{a_1a_2a_3} \Bigg) \end{aligned} \tag{65}$$

and the zeta function $\zeta_{\mathcal{B};\gamma}$ with respect to γ is defined by

$$\zeta_{\mathcal{B};\gamma}(s) \equiv \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}\zeta_{\mathcal{B};\gamma;a_1,a_2;b_{12}}. \tag{66}$$

The following introduces the cylindrical ordering matrix, the trace operator and the rotational matrices. The proofs of the results as in previous sections are omitted.

Fix a $\gamma \in GL_3(\mathbb{Z})$. Let $\mathbb{Z}_{\gamma;n_1\times n_2\times n_3}$ be the $n_1 \times n_2 \times n_3$ lattice with the basis

$$\gamma_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}_\gamma = \begin{pmatrix}\gamma_{11}\\\gamma_{12}\\\gamma_{13}\end{pmatrix}, \quad \gamma_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}_\gamma = \begin{pmatrix}\gamma_{21}\\\gamma_{22}\\\gamma_{23}\end{pmatrix}$$
$$\text{and} \quad \gamma_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}_\gamma = \begin{pmatrix}\gamma_{31}\\\gamma_{32}\\\gamma_{33}\end{pmatrix}.$$

The total number of lattice points on $\mathbb{Z}_{\gamma;n_1\times n_2\times n_3}$ is $n_1 \cdot n_2 \cdot n_3$.

Since the basic set $\mathcal{B} \subset \Sigma_{2\times2\times2}$, the L_γ-periodic patterns that are $\mathcal{B}$-admissible must be verified on $\mathbb{Z}_{2\times2\times2}$. Let $(n_1, n_2, n_3)_\gamma = (m_1, m_2, m_3)$,

$$\begin{aligned}&\mathbb{Z}_{2\times2\times2}((n_1, n_2, n_3)_\gamma)\\&\quad = \{(m_1 + k_1, m_2 + k_2, m_3 + k_3) : 0\\&\qquad \le k_1, k_2, k_3 \le 1\}.\end{aligned}$$

Now, the admissibility is demonstrated to be verified on finite lattice as follows.

Proposition 4.1. *Given* $\gamma = \begin{bmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13}\\ \gamma_{21} & \gamma_{22} & \gamma_{23}\\ \gamma_{31} & \gamma_{32} & \gamma_{33}\end{bmatrix} \in GL_3(\mathbb{Z})$. *An L_γ-periodic pattern U is $\mathcal{B}$-admissible if and only if*

$$U|_{\mathbb{Z}_{2\times2\times2}}((\alpha_1, \alpha_2, \alpha_3)_\gamma) \in \mathcal{B}$$

for $0 \le \alpha_i \le a_i - 1, 1 \le i \le 3$.

For $a_1, a_2, a_3 \ge 1$, it is easy to verify that there exist positive integers $\widehat{a}_1(\gamma)$, $\widehat{a}_2(\gamma)$ and $\widehat{a}_3(\gamma)$ such that

$$\begin{aligned}&\bigcup_{i=1}^{3}\bigcup_{\alpha_i=0}^{a_i-1}\mathbb{Z}_{2\times2\times2}((\xi_1+\alpha_1, \xi_2+\alpha_2, \xi_3+\alpha_3)_\gamma)\\&\qquad \subseteq \mathbb{Z}_{\gamma;\widehat{a}_1\times\widehat{a}_2\times\widehat{a}_3}\end{aligned}$$

for some $\xi_1, \xi_2, \xi_3 \in \mathbb{Z}$.

According to Proposition 4.1, the admissibility of an L_γ-periodic pattern U is determined by $U|_{\mathbb{Z}_{\gamma;\widehat{a}_1\times\widehat{a}_2\times\widehat{a}_3}} = (u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})_{0\le\alpha_i\le\widehat{a}_i-1, 1\le i\le 3}$ and $U|_{\mathbb{Z}_{\gamma;\widehat{a}_1\times\widehat{a}_2\times\widehat{a}_3}}$ has the the periodic condition that is given by Eq. (64), which can be divided into two parts: (i) for $0 \le \alpha_i \le \widehat{a}_i - 1$, $1 \le i \le 3$ and $p, q \in \mathbb{Z}$, if $0 \le \alpha_1 + a_1p + b_{12}q \le \widehat{a}_1 - 1$ and $0 \le \alpha_2 + a_2q \le \widehat{a}_2 - 1$,

$$u_{(\alpha_1+a_1p+b_{12}q, \alpha_2+a_2q, \alpha_3)_\gamma} = u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma}; \tag{67}$$

(ii) for $0 \le \alpha_i \le \widehat{a}_i - 1$, $1 \le i \le 3$, $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}\setminus\{0\}$, if $0 \le \alpha_1 + a_1p + b_{12}q + b_{13}r \le \widehat{a}_1 - 1$, $0 \le \alpha_2 + a_2q + b_{23}r \le \widehat{a}_2 - 1$ and $0 \le \alpha_3 + a_3r \le \widehat{a}_3 - 1$,

$$\begin{aligned}&u_{(\alpha_1+a_1p+b_{12}q+b_{13}r, \alpha_2+a_2q+b_{23}r, \alpha_3+a_3r)_\gamma}\\&\qquad = u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma}.\end{aligned} \tag{68}$$

Then, for $h \ge 1$, the set of all local patterns on $\mathbb{Z}_{\gamma;\widehat{a}_1\times\widehat{a}_2\times h}$ that satisfy Eq. (67) with $0 \le \alpha_3 \le h-1$ is denoted by $\mathbb{P}_{\gamma;a_1,a_2;b_{12};h}$.

Similar to Eq. (18), the counting function $\overline{\psi}_\gamma$ for patterns $\overline{U}_\gamma$ in $\mathbb{P}_{\gamma;a_1,a_2;b_{12};h}$ is defined by

$$\begin{aligned}\overline{\psi}_\gamma(\overline{U}_\gamma) = 1 &+ \sum_{\alpha_1=0}^{a_1-1}\sum_{\alpha_2=0}^{a_2-1}\sum_{\alpha_3=0}^{h-1} u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma}\\&\times 2^{a_1a_2(h-1-\alpha_3)+a_1(a_2-1-\alpha_2)+a_1-1-\alpha_1}.\end{aligned}$$

A local pattern $\overline{U}_\gamma$ in $\mathbb{P}_{\gamma;a_1,a_2;b_{12};h}$ can be represented as

$$\overline{U}_\gamma = \overline{U}_{\gamma;0} \oplus_{\gamma_3} \overline{U}_{\gamma;1} \oplus_{\gamma_3} \cdots \oplus_{\gamma_3} \overline{U}_{\gamma;h-1},$$

where $\overline{U}_{\gamma;i} \in \mathbb{P}_{\gamma;a_1,a_2;b_{12};1}$, $0 \le i \le h-1$, and $\overline{U}'_\gamma \oplus_z \overline{U}''_\gamma$ means that $\overline{U}''_\gamma$ is put on the top (in the γ_3-direction) of $\overline{U}'_\gamma$. For $0 \le i \le j \le h-1$, let $\overline{U}_{\gamma;i:j} = \overline{U}_{\gamma;i} \oplus_{\gamma_3} \cdots \oplus_{\gamma_3} \overline{U}_{\gamma;j}$. Therefore, for $h \ge \widehat{a}_3$, the cylindrical ordering matrix $\mathbb{C}_{\gamma;a_1,a_2;b_{12};h} = [C_{\gamma;a_1,a_2;b_{12};h;i,j}]_{2^{a_1a_2(h-1)}\times 2^{a_1a_2(h-1)}}$ of patterns in $\mathbb{P}_{\gamma;a_1,a_2;b_{12};h}$ is defined by

$$\begin{aligned}&C_{\gamma;a_1,a_2;b_{12};h;i,j}\\&\quad = \{\overline{U}_\gamma \in \mathbb{P}_{\gamma;a_1,a_2;b_{12};h} : \overline{\psi}_\gamma(\overline{U}_{\gamma;0:\widehat{a}_3-2}) = i \text{ and}\\&\qquad \overline{\psi}_\gamma(\overline{U}_{\gamma;h-\widehat{a}_3+1:h-1}) = j\}.\end{aligned}$$

In particular, for $h = \widehat{a}_3$, $\mathbb{C}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3}$ can be used to construct the associated trace operator. Notably the set $\mathbb{C}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3;i,j}$ either contains exactly one pattern or is an empty set.

Now, given $\mathcal{B} \subset \Sigma_{2\times2\times2}$, the associated trace operator $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B}) = [t_{\gamma;a_1,a_2;b_{12};i,j}]$, with $t_{\gamma;a_1,a_2;b_{12};i,j} \in \{0,1\}$, can be defined by $t_{\gamma;a_1,a_2;b_{12};i,j} = 1$ if and only if

$$\begin{aligned}&\mathbb{C}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3;i,j} \ne \emptyset \quad \text{and the pattern in}\\&\qquad \mathbb{C}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3;i,j} \text{ is } \mathcal{B}\text{-admissible.}\end{aligned} \tag{69}$$

Now, the shift (to the left) in the γ_1-direction of any pattern $\overline{U}_\gamma = (u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})$ in $\mathbb{P}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3-1}$, $u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma} \in \{0,1\}$, is defined by

$$\sigma_{\gamma_1;a_1,a_2;b_{12}}((u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})) = (u^{(1)}_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})_{0\le\alpha_1\le\widehat{a}_1-1,0\le\alpha_2\le\widehat{a}_2-1,0\le\alpha_3\le\widehat{a}_3-2}$$

where

$$u^{(1)}_{(\alpha_1,\alpha_2,\alpha_3)_\gamma} = \begin{cases} u_{(\alpha_1+1,\alpha_2,\alpha_3)_\gamma} & \text{if } 0 \le \alpha_1 \le \widehat{a}_1 - 2 \\ u_{([\alpha_1+1]_{a_1},\alpha_2,\alpha_3)_\gamma} & \text{if } \alpha_1 = \widehat{a}_1 - 1. \end{cases} \tag{70}$$

Similarly, the shift (to the below) in the γ_2-direction is defined by

$$\sigma_{\gamma_2;a_1,a_2;b_{12}}((u_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})) = (u^{(2)}_{(\alpha_1,\alpha_2,\alpha_3)_\gamma})_{0\le\alpha_1\le\widehat{a}_1-1,0\le\alpha_2\le\widehat{a}_2-1,0\le\alpha_3\le\widehat{a}_3-2}$$

where

$$u^{(2)}_{(\alpha_1,\alpha_2,\alpha_3)_\gamma} = \begin{cases} u_{(\alpha_1,\alpha_2+1,\alpha_3)_\gamma} & \text{if } 0 \le \alpha_2 \le \widehat{a}_2 - 2 \\ u_{([\alpha_1-b_{12}]_{a_1},\alpha_2+1-a_2,\alpha_3)_\gamma} & \text{if } \alpha_2 = \widehat{a}_2 - 1. \end{cases} \tag{71}$$

Notably, $\sigma_{\gamma_1;a_1,a_2;b_{12}}$ and $\sigma_{\gamma_2;a_1,a_2;b_{12}}$ are automorphism on $\mathbb{P}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3-1}$. Furthermore,

$$\sigma_{\gamma_2;a_1,a_2;b_{12}} \circ \sigma_{\gamma_1;a_1,a_2;b_{12}} = \sigma_{\gamma_1;a_1,a_2;b_{12}} \circ \sigma_{\gamma_2;a_1,a_2;b_{12}}$$

and

$$\begin{aligned} \sigma^{a_1}_{\gamma_1;a_1,a_2;b_{12}} &= \sigma^{b_{12}}_{\gamma_1;a_1,a_2;b_{12}}(\sigma^{a_2}_{\gamma_2;a_1,a_2;b_{12}}) \\ &= \text{identity map.} \end{aligned}$$

Now, the rotational matrices with respect to γ are defined as follows.

Definition 4.2. The $2^{a_1a_2(\widehat{a}_3-1)} \times 2^{n_1n_2(\widehat{a}_3-1)}$ γ_1-rotational matrix $R_{\gamma_1;a_1,a_2;b_{12}} = [R_{\gamma_1;a_1,a_2;b_{12};i,j}]$, $R_{\gamma_1;a_1,a_2;b_{12};i,j} \in \{0,1\}$, is defined by

$$R_{\gamma_1;a_1,a_2;b_{12};i,j} = 1 \quad \text{if and only if } i = \overline{\psi}_\gamma(\overline{U}_\gamma) \text{ and } j = \overline{\psi}_\gamma(\sigma_{\gamma_1;a_1,a_2;b_{12}}(\overline{U}_\gamma)), \tag{72}$$

where $\overline{U}_\gamma \in \mathbb{P}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3-1}$. From Eq. (72), for convenience, denote by

$$j = \sigma_{\gamma_1}(i). \tag{73}$$

Similarly, the $2^{a_1a_2(\widehat{a}_3-1)} \times 2^{n_1n_2(\widehat{a}_3-1)}$ γ_2-rotational matrix $R_{\gamma_2;a_1,a_2;b_{12}} = [R_{\gamma_2;a_1,a_2;b_{12};i,j}]$, $R_{\gamma_2;a_1,a_2;b_{12};i,j} \in \{0,1\}$, is defined by

$$R_{\gamma_2;a_1,a_2;b_{12};i,j} = 1 \quad \text{if and only if } i = \overline{\psi}_\gamma(\overline{U}_\gamma) \text{ and } j = \overline{\psi}_\gamma(\sigma_{\gamma_2;a_1,a_2;b_{12}}(\overline{U}_\gamma)), \tag{74}$$

where $\overline{U}_\gamma \in \mathbb{P}_{\gamma;a_1,a_2;b_{12};\widehat{a}_3-1}$. From Eq. (74), for convenience, denote by

$$j = \sigma_{\gamma_2}(i). \tag{75}$$

Moreover,

$$\mathbf{R}_{\gamma;a_1,a_2;b_{12}} = \sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1} R^{b_{13}}_{\gamma_1;a_1,a_2;b_{12}} R^{b_{23}}_{\gamma_2;a_1,a_2;b_{12}}. \tag{76}$$

The main results for $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma\right)$ as in Theorems 2.9 and 2.10 are obtained as follows and the proofs are omitted.

Theorem 4.3. *Given a basic set* $\mathcal{B} \subset \Sigma_{2\times2\times2}$, *for* $a_i \ge 1, 1 \le i \le 3$, $0 \le b_{ij} \le a_i - 1$, $i+1 \le j \le 3$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma\right) = \text{tr}(\mathbf{T}^{a_3}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B}) R^{b_{13}}_{\gamma_1;a_1,a_2;b_{12}} R^{b_{23}}_{\gamma_2;a_1,a_2;b_{12}}) \tag{77}$$

and

$$\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1} \Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{bmatrix}_\gamma\right) = \text{tr}(\mathbf{T}^{a_3}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B}) \mathbf{R}_{\gamma;a_1,a_2;b_{12}}). \tag{78}$$

Furthermore,

$$\zeta_{\gamma;a_1,a_2;b_{12}}(s) = \exp\left(\frac{1}{a_1a_2}\sum_{a_3=1}^{\infty} \frac{1}{a_3} tr(\mathbf{T}^{a_3}_{\gamma;a_1,a_2;b_{12}}(\mathcal{B}) \times \mathbf{R}_{\gamma;a_1,a_2;b_{12}}) s^{a_1a_2a_3}\right). \tag{79}$$

The equivalent class $\mathcal{C}_{\gamma;a_1,a_2;b_{12}}(i)$, the cardinal number $\omega_{\gamma;a_1,a_2;b_{12};i}$ of $\mathcal{C}_{\gamma;a_1,a_2;b_{12}}(i)$, the index set $\mathcal{I}_{\gamma;a_1,a_2;b_{12}}$ and the cardinal number of $\chi_{\gamma;a_1,a_2;b_{12}}$ can be defined as in Sec. 3 and are omitted here.

Definition 4.4. For $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$, the reduced trace operator $\tau_{\gamma;a_1,a_2;b_{12}} = [\tau_{\gamma;a_1,a_2;b_{12};i,j}]$ of $\mathbf{T}_{\gamma;a_1,a_2;b_{12}} = [t_{\gamma;a_1,a_2;b_{12};i,j}]$ is a $\chi_{\gamma;a_1,a_2;b_{12}} \times \chi_{\gamma;a_1,a_2;b_{12}}$ matrix defined by

$$\tau_{\gamma;a_1,a_2;b_{12};i,j} = \sum_{k \in \mathcal{C}_{\gamma;a_1,a_2;b_{12}}(j)} t_{\gamma;a_1,a_2;b_{12};i,k} \tag{80}$$

for each $i, j \in \mathcal{I}_{\gamma;a_1,a_2;b_{12}}$.

By the argument as in Sec. 3, the rotational symmetry of $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}$ can be obtained, yielding then the rationality of the $(a_1, a_2; b_{12})$-th zeta function $\zeta_{\mathcal{B};\gamma;a_1,a_2;b_{12}}$. The results are stated as follows.

Theorem 4.5. *Given $\mathcal{B} \subset \Sigma_{2\times2\times2}$ and $\gamma \in GL_3(\mathbb{Z})$. For $a_i \geq 1$, $1 \leq i \leq 3$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq 3$,*

$$\begin{aligned}\frac{1}{a_1a_2}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3\end{bmatrix}_\gamma\right) &= tr(\tau^{a_3}_{\gamma;a_1,a_2;b_{12}}) \\ &= \sum_{\lambda\in\Sigma(\tau_{\gamma;a_1,a_2;b_{12}})} \chi_{\gamma;a_1,a_2;b_{12}}(\lambda)\lambda^{a_3},\end{aligned} \tag{81}$$

where $\Sigma(\tau_{\gamma;a_1,a_2;b_{12}})$ is the spectrum of $\tau_{\gamma;a_1,a_2;b_{12}}$ and $\chi_{\gamma;a_1,a_2;b_{12}}(\lambda)$ is the algebraic multiplicity of $\tau_{\gamma;a_1,a_2;b_{12}}$ with eigenvalue λ. Moreover,

$$\begin{aligned}\zeta_{\gamma;a_1,a_2;b_{12}}(s) &= (\det(I - s^{a_1a_2}\tau_{\gamma;a_1,a_2;b_{12}}))^{-1} \\ &= \prod_{\lambda\in\Sigma(\tau_{\gamma;a_1,a_2;b_{12}})} (1-\lambda s^{a_1a_2})^{-\chi_{\gamma;a_1,a_2;b_{12}}(\lambda)},\end{aligned} \tag{82}$$

and

$$\zeta_\gamma(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1}(\det(I - s^{a_1a_2}\tau_{\gamma;a_1,a_2;b_{12}}))^{-1}. \tag{83}$$

Corollary 4.6. *For any $\mathcal{B} \subset \Sigma_{2\times2\times2}$ and $\gamma \in GL_3(\mathbb{Z})$, the Taylor series expansions for $\zeta_{\mathcal{B};\gamma}$ at $s = 0$ has integer coefficients.*

Proof. Since $\tau_{\gamma;a_1,a_2;b_{12}}$ has integer entries for any $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$, the result follows. ■

Now, that $\zeta_{\mathcal{B};\gamma}$ are meromorphic extensions of $\zeta^0_{\mathcal{B}}$ is obtained as follows.

Theorem 4.7. *Given $\mathcal{B} \subset \Sigma_{2\times2\times2}$. For any $\gamma \in GL_3(\mathbb{Z})$,*

$$\zeta_{\mathcal{B};\gamma}(s) = \zeta^0_{\mathcal{B}}(s) \tag{84}$$

for $|s| < \exp(-g(\mathcal{B}))$, where

$$g(\mathcal{B}) = \limsup_{[L]\to\infty}\frac{1}{[L]}\log\Gamma_{\mathcal{B}}(L). \tag{85}$$

Moreover, $\zeta_{\mathcal{B};\gamma}$ has the same (integer) coefficients in its Taylor series expansions at $s = 0$, for all $\gamma \in GL_3(\mathbb{Z})$.

Proof. By [Lind, 1996], $\zeta^0_{\mathcal{B}}$ has radius of convergence $\exp(-g(\mathcal{B}))$ and is analytic in $|s| < \exp(-g(\mathcal{B}))$. Since $\zeta_{\mathcal{B};\gamma}$ is a rearrangement of $\zeta^0_{\mathcal{B}}$, Eq. (84) holds. From [Lind, 1996] or Corollary 4.6, $\zeta_{\mathcal{B};\gamma}$ has the same integer coefficients in its Taylor series expansions at $s = 0$. The proof is complete. ■

Remark 4.8. From Theorem 4.5, for any $\mathcal{B} \subset \Sigma_{2\times2\times2}$, there exists a family of zeta functions $\{\zeta_{\mathcal{B};\gamma} : \gamma \in GL_3(\mathbb{Z})\}$. For certain $\mathcal{B}$, the other $\gamma \in GL_3(\mathbb{Z})$ may give a different description to $\zeta_{\mathcal{B}}$; see Example 3.7 and the following Example 4.9. Those different descriptions of $\zeta^0_{\mathcal{B}}$ may be useful in studying zeta functions.

Example 4.9. Consider the basic set $\mathcal{B}$ in Example 3.7 and $\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. It is easy to verify that

$$\mathbf{T}_{\gamma;a_1,a_2;b_{12}} = \mathbf{T}_{\gamma;a_1,a_2;0}$$

for $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$. Moreover, after the zero columns and rows of $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}$ (or $\tau_{\gamma;a_1,a_2;b_{12}}$) were deleted, $\mathbf{T}_{\gamma;a_1,a_2;b_{12}}$ ($\tau_{\gamma;a_1,a_2;b_{12}}$) is reduced to $\mathbf{T}_{\gamma;1,a_2;0}$ ($\tau_{\gamma;1,a_2;0}$). Clearly

$$\mathbf{T}_{\gamma;1,a_2;0} = I_{2^{a_2}}$$

and

$$\tau_{\gamma;1,a_2;0} = I_{\chi_{a_2}},$$

where

$$\chi_n = \frac{1}{n}\sum_{d|n}\phi(d)2^{n/d},$$

and $\phi(d)$ is the Euler totient function. Note that χ_n is the number of necklaces that can be made from n beads of two colors when the necklaces can be rotated but not turned over [Plouffe & Sloane, 1995].

Hence,

$$\zeta_{\gamma;a_1,a_2;b_{12}} = (1-s^{a_1 a_2})^{-\chi_{a_2}} \tag{86}$$

and

$$\zeta_\gamma = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}(1-s^{a_1 a_2})^{-a_1\chi_{a_2}}. \tag{87}$$

It can be proved that $g(\mathcal{B}) = \log 2$. Therefore, from Example 3.7 and Theorem 4.7,

$$\prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}(1-s^{a_1 a_2})^{-a_1\chi_{a_2}} = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}(1-2s^{a_1 a_2})^{-a_1} \tag{88}$$

for $|s| < \frac{1}{2}$, and they have the same integer coefficients in their Taylor series expansions at $s = 0$.

5. Further Results

This section briefly describes the results for $\mathbb{Z}^d$, $d \geq 4$, and more symbols on larger lattice. The thermodynamic zeta function for the three-dimensional Ising model with finite range interactions is also studied.

5.1. *Higher-dimensional shifts of finite type*

This subsection considers the zeta functions for shifts of finite type on $\mathbb{Z}^d$, $d \geq 4$. Only brief statements are made here.

As in [Lind, 1996], $\mathcal{L}_d$ can be parameterized by using Hermite normal form [MacDuffie, 1956]:

$$\mathcal{L}_d = \left\{ \begin{bmatrix} a_1 & b_{12} & b_{13} & \cdots & b_{1d} \\ 0 & a_2 & b_{23} & \cdots & b_{2d} \\ 0 & 0 & a_3 & \cdots & b_{3d} \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & a_d \end{bmatrix} \mathbb{Z}^d : a_i \geq 1, 1 \leq i \leq d, 0 \leq b_{ij} \leq a_i - 1, i+1 \leq j \leq d \right\}. \tag{89}$$

Let the lattice $\mathbb{L}_d = \{(n_1, n_2, \ldots, n_d) : 0 \leq n_i \leq 1, 1 \leq i \leq d\}$. Fix a basic set $\mathcal{B} \subset \{0,1\}^{\mathbb{L}_d}$. For $a_i \geq 1$, $1 \leq i \leq d-1$, $0 \leq b_{ij} \leq a_i - 1$, $i+1 \leq j \leq d-1$, the (a_i, b_{ij})-th zeta function is defined by

$$\zeta_{\mathcal{B};(a_i,b_{ij})}(s) \equiv \exp\left(\frac{1}{a_1\cdots a_{d-1}} \sum_{a_d=1}^{\infty}\sum_{i=1}^{d-1}\sum_{b_{id}=0}^{a_i-1} \frac{1}{a_d}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13} & \cdots & b_{1d} \\ 0 & a_2 & b_{23} & \cdots & b_{2d} \\ 0 & 0 & a_3 & \cdots & b_{3d} \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & a_d \end{bmatrix} \right) s^{a_1\cdots a_d} \right) \tag{90}$$

and

$$\zeta_{\mathcal{B}}(s) \equiv \prod_{i=1}^{d-1}\prod_{a_i=1}^{\infty}\prod_{j=i+1}^{d-1}\prod_{b_{ij}=0}^{a_i-1}\zeta_{\mathcal{B};(a_i,b_{ij})}(s). \tag{91}$$

As in Secs. 2 and 3, the cylindrical ordering matrix, the trace operator, the rotational matrices and the reduced trace operator can be defined. The method in Secs. 2 and 3 can also be applied to verify that $\zeta_{\mathcal{B};(a_i,b_{ij})}$ is a rational function. Therefore, $\zeta_{\mathcal{B}}$ is an infinite product of rational functions. Furthermore, given any $\gamma \in GL_d(\mathbb{Z})$, the result also holds in γ-coordinates. Hence, a family of zeta functions exists with the same integer coefficients in their Taylor series expansions at $s = 0$, and yields a family of identities in number theory.

5.2. *More symbols on larger lattice*

This subsection extends the results of the previous sections and subsections to any finite number of symbols and any finite lattice. For simplicity, only the zeta functions for three-dimensional shifts of finite type are discussed. Given a set of symbols $\mathcal{S}_p = \{0, 1, \ldots, p-1\}$, $p \geq 2$, a set of finite lattice points $\mathbb{L} \subset \mathbb{Z}^3$ and a basic set $\mathcal{B}(\mathbb{L}) \subset \mathcal{S}_p^{\mathbb{L}}$. Let $\mathbb{Z}_{m\times m\times m}$ be the smallest cubic lattice that contains $\mathbb{L}$ and $\mathcal{B}(\mathbb{Z}_{m\times m\times m})$ be the set of all admissible patterns that are generated by $\mathcal{B}(\mathbb{L})$. Then, it is easy to verify that

$$\mathcal{P}(\mathcal{B}(\mathbb{Z}_{m\times m\times m})) = \mathcal{P}(\mathcal{B}(\mathbb{L})).$$

Therefore, only $\mathcal{B} \subset \mathcal{S}_p^{\mathbb{Z}_{m\times m\times m}}$, for $m \geq 2$, need to be considered. The definitions of cylindrical ordering matrix and the rotational matrices must be adjusted and the details are omitted here. Then, the associated trace operator and reduced trace operator can also be defined. Hence, by the arguments similar to those made in Secs. 2–4, the results for $\mathcal{B} \subset \mathcal{S}_p^{\mathbb{Z}_{m\times m\times m}}$ also hold.

5.3. *Ising model with finite range interactions*

This subsection will extend the results to the $\mathbb{Z}^3$ lattice Ising model with finite range interactions. For simplicity, only the case of the nearest neighbor interactions is considered. Let the $\mathbb{Z}^3$ lattice Ising model be with the external field $\mathcal{H}$, the coupling constant $\mathcal{J}_1$ in the x-direction, the coupling constant $\mathcal{J}_2$ in the y-direction and the coupling constant $\mathcal{J}_3$ in the z-direction. Each site $(\alpha_1, \alpha_2, \alpha_3)$ of $\mathbb{Z}^3$ lattice has a spin $u_{\alpha_1,\alpha_2,\alpha_3}$ with two possible values, $+1$ or -1. Assume that the state space is given by $\mathcal{B} \subset \{0,1\}^{\mathbb{Z}_{2\times 2\times 2}}$. Given a state $U = (u_{\alpha_1,\alpha_2,\alpha_3}) \in \{0,1\}^{\mathbb{Z}^3}$, denote by $U_{n_1\times n_2\times n_3} = U|_{\mathbb{Z}_{n_1\times n_2\times n_3}} = (u_{\alpha_1,\alpha_2,\alpha_3})_{0\leq\alpha_i\leq n_i-1, 1\leq i\leq 3}$.

Now, the Hamiltonian (energy) $\mathcal{E}(U_{n_1\times n_2\times n_3})$ is defined by

$$\mathcal{E}(U_{n_1\times n_2\times n_3}) = -\mathcal{J}_1 \sum_{\substack{0\leq\alpha_1\leq n_1-2\\ 0\leq\alpha_3\leq n_2-1\\ 0\leq\alpha_3\leq n_3-1}} u_{\alpha_1,\alpha_2,\alpha_3}u_{\alpha_1+1,\alpha_2,\alpha_3} - \mathcal{J}_2 \sum_{\substack{0\leq\alpha_1\leq n_1-1\\ 0\leq\alpha_2\leq n_2-2\\ 0\leq\alpha_3\leq n_3-1}} u_{\alpha_1,\alpha_2,\alpha_3}u_{\alpha_1,\alpha_2+1,\alpha_3}$$
$$- \mathcal{J}_3 \sum_{\substack{0\leq\alpha_1\leq n_1-1\\ 0\leq\alpha_2\leq n_2-1\\ 0\leq\alpha_3\leq n_3-2}} u_{\alpha_1,\alpha_2,\alpha_3}u_{\alpha_1,\alpha_2,\alpha_3+1} - \mathcal{H} \sum_{\substack{0\leq\alpha_1\leq n_1-1\\ 0\leq\alpha_2\leq n_2-1\\ 0\leq\alpha_3\leq n_3-1}} u_{\alpha_1,\alpha_2,\alpha_3}. \tag{92}$$

Given $L = \begin{bmatrix} a_1 & b_{12} & b_{13}\\ 0 & a_2 & b_{23}\\ 0 & 0 & a_3\end{bmatrix}\mathbb{Z}^3 \in \mathcal{L}_3$, the set of all $\mathcal{B}$-admissible and L-periodic patterns is denoted by $\mathcal{P}_{\mathcal{B}}(L)$. Then, the partition function for $\mathcal{B}$ with L-periodic patterns is defined as

$$\mathcal{Z}_{\mathcal{B}}(L) = \mathcal{Z}_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13}\\ 0 & a_2 & b_{23}\\ 0 & 0 & a_3\end{bmatrix}\right)$$
$$= \sum_{U\in\mathcal{P}_{\mathcal{B}}(L)} \exp\left[\sum_{\substack{0\leq\alpha_1\leq n_1-1\\ 0\leq\alpha_2\leq n_2-1\\ 0\leq\alpha_3\leq n_3-1}} u_{\alpha_1,\alpha_2,\alpha_3}(\mathbf{K}_1 u_{\alpha_1+1,\alpha_2,\alpha_3} + \mathbf{K}_2 u_{\alpha_1,\alpha_2+1,\alpha_3} + \mathbf{K}_3 u_{\alpha_1,\alpha_2,\alpha_3+1} + \mathbf{h})\right] \tag{93}$$

where $\mathbf{K}_i = \mathcal{J}_i/k_B T$, $1 \leq i \leq 3$, k_B is Boltzmann's constant and T is the temperature. Therefore, the thermodynamic zeta function is defined by

$$\zeta^0_{\text{Ising};\mathcal{B}}(s) \equiv \exp\left(\sum_{L\in\mathcal{L}_3} \mathcal{Z}_{\mathcal{B}}(L)\frac{s^{[L]}}{[L]}\right). \tag{94}$$

As Eqs. (8) and (9), for any $a_1, a_2 \geq 1$, $0 \leq b_{12} \leq a_1 - 1$, the $(a_1, a_2; b_{12})$-th thermodynamic zeta function $\zeta_{Ising;\mathcal{B};a_1,a_2;b_{12}}(s)$ is defined as

$$\zeta_{\text{Ising};\mathcal{B};a_1,a_2;b_{12}}(s) \equiv \exp\left(\frac{1}{a_1a_2}\sum_{a_3=1}^{\infty}\sum_{b_{13}=0}^{a_1-1}\sum_{b_{23}=0}^{a_2-1}\frac{1}{a_3}\mathcal{Z}_{\mathcal{B}}\left(\begin{bmatrix} a_1 & b_{12} & b_{13}\\ 0 & a_2 & b_{23}\\ 0 & 0 & a_3\end{bmatrix}\right)s^{a_1a_2a_3}\right) \tag{95}$$

and the thermodynamic zeta function $\zeta_{\text{Ising};\mathcal{B}}(s)$ is given by

$$\zeta_{\text{Ising};\mathcal{B}}(s) \equiv \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1} \zeta_{\text{Ising};\mathcal{B};a_1,a_2;b_{12}}(s). \tag{96}$$

Since the spin $u_{\alpha_1,\alpha_2,\alpha_3} \in \{+1,-1\}$, the cylindrical ordering matrix $\mathbb{C}_{\text{Ising};a_1,a_2;b_{12};h} = [C_{\text{Ising};a_1,a_2;b_{12};h;i,j}]$ is obtained by replacing all symbols "0" in $\mathbb{C}_{a_1,a_2;b_{12};h}$ with the symbols "-1". Notably, exactly one pattern exists in $C_{\text{Ising};a_1,a_2;b_{12};2;i,j}$ and the pattern is given by $U_{\text{Ising};a_1,a_2;b_{12};2;i,j} = (u_{\alpha_1,\alpha_2,\alpha_3})$. Define

$$\mathcal{Z}(U_{\text{Ising};a_1,a_2;b_{12};2;i,j}) \equiv \exp\left[\sum_{\substack{0\le\alpha_1\le a_1-1\\ 0\le\alpha_2\le a_2-1}} u_{\alpha_1,\alpha_2,0}(\mathbf{K}_1 u_{\alpha_1+1,\alpha_2,0} + \mathbf{K}_2 u_{\alpha_1,\alpha_2+1,0} + \mathbf{K}_3 u_{\alpha_1,\alpha_2,1} + \mathbf{h})\right]. \tag{97}$$

Then, the trace operator $\mathbf{T}_{\text{Ising};a_1,a_2;b_{12}} = [t_{\text{Ising};a_1,a_2;b_{12};i,j}]$ is defined by

$$\begin{cases} t_{\text{Ising};a_1,a_2;b_{12};i,j} = 0 & \text{if } U_{\text{Ising};a_1,a_2;b_{12};2;i,j} \text{ is not } \mathcal{B}\text{-admissible,} \\ t_{\text{Ising};a_1,a_2;b_{12};i,j} = \mathcal{Z}(U_{\text{Ising};a_1,a_2;b_{12};2;i,j}) & \text{if } U_{\text{Ising};a_1,a_2;b_{12};2;i,j} \text{ is } \mathcal{B}\text{-admissible.} \end{cases} \tag{98}$$

Therefore, the associated reduced operator $\tau_{\text{Ising};a_1,a_2;b_{12}}$ can be defined as in Definition 3.4. Since all arguments for the rationality of $\zeta_{\text{Ising};\mathcal{B};a_1,a_2;b_{12}}$ are similar to those in Secs. 2 and 3, only the final result is stated, as follows.

Theorem 5.1. *For* $a_1, a_2 \ge 1$, $0 \le b_{12} \le a_1 - 1$,

$$\zeta_{\text{Ising};\mathcal{B};a_1,a_2;b_{12}}(s) = (\det(I - s^{a_1a_2}\tau_{\text{Ising};a_1,a_2;b_{12}}))^{-1} \tag{99}$$

and

$$\zeta_{\text{Ising};\mathcal{B}}(s) = \prod_{a_1=1}^{\infty}\prod_{a_2=1}^{\infty}\prod_{b_{12}=0}^{a_1-1} \times(\det(I - s^{a_1a_2}\tau_{\text{Ising};a_1,a_2;b_{12}}))^{-1}. \tag{100}$$

Notably, this result also holds in γ-coordinates for $\gamma \in GL_3(\mathbb{Z})$.

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