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Habit formation and chaotic dynamics in an *n*-dimensional cash-in-advance economy

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Abstract This paper investigates the global dynamics of an n-dimensional cash-in-advance model where external habits persist for n periods. We find that habit formation is an important determinant to the dynamic property of the economy and identifies conditions generating entropic chaos. Our results indicate that the possibility of chaotic motion increases with the depreciation rate of habits.

Keywords Cash-in-advance economy · Chaos · Persistent habit

1 Introduction

Recently, there is considerable interest in studying the formation of consumption habits and their impact on economic performance. Empirical estimations of habit formation can be found in [6, 14] and [26]. These papers all highlight the significance of persistent habits. These empirical results lead theoretical studies to consider persistent habits in macroeconomic models; see, among others, [1–4, 12, 13, 27] and [7]. Although these authors may use different terms to describe the habit persistence, they all mean that current consumption is affected by past consumptions.[1]

Most of these studies considering persistent habits restrict their analysis to a complete depreciation of habits after one period due to empirical and theoretical tractability. However, it is quite unreasonable to think that consumption habits only last for one period and will be gone thereafter. An attempt to estimate persistent habits lasting for two periods can be found in [14]. In theoretical studies, an example of incomplete depreciation of habits is given by Ryder and Heal [27], who assume that habit formation is composed by "all" past consumptions with decreasing weights.

The purpose of this paper is to define a more satisfactory and general habit formation in which habits depreciate gradually and persist for n periods, where n is an arbitrary finite positive integer. It has been shown by previous studies that complex dynamics may arise when including habit formation into the model.[2] When complex dynamics is present, the economic transition becomes very sensitive to the initial condition. A slight difference between two initial conditions may lead to

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[1] Based on the model setting, Becker [4] calls persistent habits as "consumption addiction" while de la Croix [12] describes them as "bequeathed tastes". Besides, some of these papers consider "internal habits," while the others—"external habits".

[2] See [2, 12, 13] and [7].

two different dynamical paths. Hence, when a dynamical system exhibits complicated motion, a traditional method to study the linear property around steady states does not suffice and an analysis of the global property is needed.

When dynamics is rather complex, cycles or even chaos may emerge under certain conditions. If so, fluctuations can also be generated endogenously by a nonlinear deterministic model without any shocks to the fundamentals. This is different from traditional macroeconomic models which tend to explain oscillations by using unexpected shocks. Day [10, 11] is one of the pioneers who investigated the possibility of chaotic motion in the Solow growth model.[3]

It is well known now that complex dynamics can be easily generated in monetary models.[4] Complicated motion in money-in-the-utility models with nonseparable utility function and separable utility function are considered by Matsuyama [21] and Fukuda [15], respectively. Michener and Ravikumar [24] and Chen and Li [7] show that chaos can occur in a Lucas–Stokey type monetary model where consumers consume cash goods and credit goods. Auray et al. [2] introduce habit persistence as a way to generate complex dynamics in a cash-in-advance (CIA) economy and find that chaotic dynamics can emerge for reasonable degrees of habit.

In this paper, we follow this research line to consider the habit persistence of consumptions in a monetary model. However, there are two features to distinguish this paper from the previous studies. First, we use a more general formation of consumption habits to investigate how their formation affects the dynamic transition of the economy. We assume that habits persist for n periods which can be easily adjusted to any situations.[5] For example, with a complete depreciation of habits after one period, our model returns to the one of Auray et al. [2]. On the other hand, with a choice of duration of habits equal to the current time, the habit formation is the same as the one used in Ryder and Heal [27], who assume that all past consumptions affect current consumption.

Second, from the methodological point of view, this paper also contributes to the literature on economic dynamics of chaos by providing a new technique to study a high-dimensional dynamical system (a high-order difference equation).[6] Since an investigation of a high-dimensional dynamical system is quite difficult, there are only very few papers studying economic models with high-dimensional dynamical systems. Even for a 1-D dynamical system (a first-order difference equation), to trace the dynamic transition of the economy is not easy. This is why most studies of 1-D dynamical systems restrict their analysis to the Li–Yorke chaos because of the appeal of easy verification. However, with more advanced mathematical methods and computer techniques, some economists explore the possibility of complicated dynamics in a two-dimensional (2-D) dynamical system by showing either numerical simulation or solid proofs; this approach can be found in [7, 13, 22, 29] and [8]. Since the Li–Yorke chaos is only defined for 1-D dynamical systems, studies of 2-D dynamical systems need to utilize other types of chaos. The most often used type of chaos in 2-D nonlinear systems is the horseshoe structure.[7] The utilization of the horseshoe structure is quite popular in 2-D dynamical systems because it can be analyzed by geometric methods.

Depending on the length of persistent habit, the economic transition can be represented by a high-dimensional dynamical system. A monetary model with the persistent habits lasting for n periods will generate an n-dimensional (n-D) dynamical system. So far as we know, no economic research focuses on the occurrence of chaotic motion in an n-D system. Although it might be difficult, a geometric construction of a horseshoe structure is possible for a 2-D system. However, the construction becomes quite impossible for high-dimensional systems. This leads us to select another type of chaos which is defined for an n-D dynamical system. The entropic chaos is then chosen and used in this paper. It is very difficult to study the dynamic motion and to prove the occurrence of chaos in an n-D dynamical system because of the lack of geometric intuition and the analysis is much more abstract. Our goal in the present paper is to give rigorous

[3] See [5] and [25] for chaotic dynamics in one-dimensional (1-D) growth models.

[4] Beside growth models and monetary economies, overlapping-generations (OLG) models can also display complex dynamics; see [16, 23] and [29].

[5] The parameter n can be any finite positive integer.

[6] Here we refer to a difference equation with the order greater than or equal to 2 as a high-dimensional system.

[7] See [13] and [29].

proofs (instead of numerical experiments) for conditions generating entropic chaos. Hence, we adopt a newly-developed method by Juang et al. [17] and Li and Malkin [19] to approximate an n-D dynamical system by using a 1-D dynamical system and study the possibility of entropic chaos; also refer to Li et al. [20].

We begin our analysis by assuming that habits last for one or two periods and then extend the duration of habits to n periods. We find that the dimension of the dynamical system increases with the duration of habit persistence. For all cases, we provide sufficient conditions for the occurrence of entropic chaos. Our results indicate that habit formation (the duration and the depreciation rate of persistent habits), the degree of persistent habits and the elasticity of labor supply are important determinants to the dynamic property of the economy. Including incomplete depreciation of habit will lower the critical value of the degree of persistent habits for the chaotic motion to occur. Therefore, if the depreciation rate is quite low, the critical value of the degree of persistent habits will not remain within the range supported by the empirical data even when the elasticity of labor supply is high.

The remainder of the paper is organized as follows. In the next section, we develop a CIA economy with persistent habits of consumptions. The dynamic property is analyzed in Sect. 3. The conclusion is given in Sect. 4.

2 The model

The model is based on [2]. We consider an economy with infinitely living identical agents who were born in period 0. We use p_t and M_t to denote the common price and the nominal money demand, respectively. In period t, individuals use the real money balance $\frac{M_t}{p_t}$ brought from the previous period to buy goods. Hence, agents face the following cash-in-advance constraint:

$$c_t \leq \frac{M_t}{p_t}, \tag{1}$$

where c_t represents households' consumption.

Households supply h_t units of time for work to earn the real wage w_t. The production function is $Y_t = h_t$. In period t, government injects money into the economy by giving a nominal lump-sum transfer T_t to households. We assume that nominal money supply $\bar{M}_t$ grows at the rate of μ. That is, $\bar{M}_{t+1} = (1+\mu)\bar{M}_t$. Thus, $T_t = \bar{M}_{t+1} - \bar{M}_t$. Households allocate the transfer, the real money balance carried from the previous period, and their wage income on the consumption and the money balance they plan to carry to the next period. Therefore, the budget constraint for households is

$$c_t + \frac{M_{t+1}}{p_t} = w_t h_t + \frac{M_t}{p_t} + \frac{T_t}{p_t}. \tag{2}$$

The preference is separable in consumption and labor, and is represented as

$$\sum_{t=0}^{\infty} \beta^t \left[\log(d_t) - \frac{h_t^{1+\varphi}}{1+\varphi}\right], \tag{3}$$

where $\beta \in (0, 1)$ is the discount factor and $\varphi \geq 0$ is the inverse of the labor supply elasticity.

We assume that agents have persistent habits of consumption and will compare their current consumption with the past consumptions. We go with the literature of the *catching up with the Joneses* by assuming that agents have external habits of consumption. Unlike previous studies which usually assume that persistent habits depreciate completely after one period, we assume that habits depreciate gradually at the rate $\eta \in [0, 1]$ within n periods. Hence, habits are composed by

$$v_t(\bar{c}_{t-1}, \bar{c}_{t-2}, \ldots, \bar{c}_{t-n}) = \sum_{i=0}^{n-1} (1-\eta)^i \bar{c}_{t-i-1}, \tag{4}$$

where $\bar{c}_{t-i}$ is the average consumption in period $t-i$. Note that when $\eta = 1$ (or $n = 1$), the depreciation of habits is complete after one period since (4) only depends on the average consumption in the previous period ($v_t(\bar{c}_{t-1}) = \bar{c}_{t-1}$), and we will return to the simple case studied by Auray et al. [2]. With $n = t$, (4) relies on all past consumptions and this kind of habit formation is considered by several theoretical and empirical studies (see [6, 9, 26, 27]).[8]

Agents compare their current consumption with the past average consumptions (habits) and only the

[8]The habit formation (x_t) in period t used in these papers is

$$x_t = x_0 e^{-at} + b \int_0^t e^{a(s-t)} c_s \, ds,$$

where a and b are constant. The variables x_0 and c_s denote the initial condition and consumption at time s, respectively; see [26].

component of current consumption which is above the habits will be beneficial to utility. Hence, the variable d_t is composed by the current consumption, c_t, and habits. We set d_t to $d_t = c_t - \theta v_t(\bar{c}_{t-1}, \bar{c}_{t-2}, \dots, \bar{c}_{t-n})$, where $\theta \in (0, 1)$ measures the degree of persistent habits.

The equilibrium is defined as follows. Given the money supply rule and the initial habit, the perfect foresight equilibrium comprises the sequences of $\{w_t, p_t, c_t, h_t, \quad M_t, \bar{M}_t\}_{t=0}^{\infty}$ such that: (i) given $\{w_t, p_t, \bar{M}_t\}$, $\{c_t, h_t, M_{t+1}\}_{t=0}^{\infty}$ are optimal choices of the representative household; (ii) the equilibrium wage rate is $w_t = 1$; (iii) goods market clears, $Y_t = c_t$; and (iv) money market clears, $\bar{M}_t = M_t$.

Households then maximize (3) subject to (1) and (2). The optimization decisions are

$$\frac{1}{d_{t+1}} = \lambda_{t+1} + \gamma_{t+1}, \tag{5}$$

$$h_t^{\varphi} = \gamma_t w_t, \tag{6}$$

$$\frac{\gamma_t}{p_t} = \frac{\beta}{p_{t+1}}(\lambda_{t+1} + \gamma_{t+1}), \tag{7}$$

where λ and γ are the Lagrangian multipliers of (1) and (2), respectively. As is common in the literature, we assume that (1) is binding in the equilibrium.

Defining the inverse of real money balance as $\rho_t = \frac{p_t}{M_t}$ and combining (5), (6) and (7) as well as applying the equilibrium condition, we obtain

$$\left(\frac{M_t}{p_t}\right)^{1+\varphi} = \frac{\beta}{\mu}\frac{M_{t+1}}{p_{t+1}} d_{t+1}^{-1}. \tag{8}$$

Substituting the definitions of d_t and ρ_t into (8) implies that at the equilibrium, the economy can be represented by the following difference equation:

$$\begin{aligned}(\rho_t^{-1})^{1+\varphi} = \frac{\beta}{\mu}(\rho_{t+1}^{-1})\big[\rho_{t+1}^{-1} \\ - \theta v_{t+1}(\rho_t^{-1}, \rho_{t-1}^{-1}, \dots, \rho_{t-n+1}^{-1})\big]^{-1}.\end{aligned} \tag{9}$$

Equation (9) shows that the dynamic behavior of the economy can be represented by a difference equation in the inverse of real money balance.

3 Chaotic dynamics

In this section, we use (9) to study the dynamic behavior of the economy. Substituting (4) into (9), we obtain:

$$\begin{aligned}\rho_{t+1} &= \frac{1}{\theta}\frac{\rho_t\rho_{t-1}\cdots\rho_{t-n+1}(1-\frac{\beta}{\mu}\rho_t^{1+\varphi})}{\rho_{t-1}\rho_{t-2}\cdots\rho_{t-n+1} + (1-\eta)\rho_t\rho_{t-2}\cdots\rho_{t-n+1} + \cdots + (1-\eta)^{n-1}\rho_t\rho_{t-1}\cdots\rho_{t-n+2}} \\ &= \frac{1}{\theta}\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\sum_{j=0}^{n-1}[(1-\eta)^j(\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\rho_{t-j}})]}\left(1-\frac{\beta}{\mu}\rho_t^{1+\varphi}\right),\end{aligned} \tag{10}$$

where, by convention, we define $0^0 = 1$. Equation (10) shows that the order of the difference equation which represents the dynamic behavior of the economy equals the length of persistent habits. This implies that the dynamic property depends on the duration of habits. To study the dynamic behavior of (10), we begin from two simpler cases: persistent habits that last for 1 and 2 periods, and then extend our analysis to a general case where habits last for n periods. We investigate sufficient conditions generating entropic chaos in all three cases. When persistent habits last only for 1 period, the appearance of the Li–Yorke chaos provides a sufficient condition for the occurrence of entropic chaos. Due to convenience and easiness of verification, we follow the literature to study the possibility of the Li–Yorke chaos in this case. When the persistent habits last for 2 periods, we analyze the possibility of chaos in the horseshoe structure which provides a sufficient condition for the occurrence of entropic chaos. When persistent habits last for n periods, no graphic methodology can be used and we utilize the theorems developed by Juang et al. [17] and Li and Malkin [19] to provide a sufficient condition for the existence of entropic chaos. This result in the last case can be applied to any positive integer n.

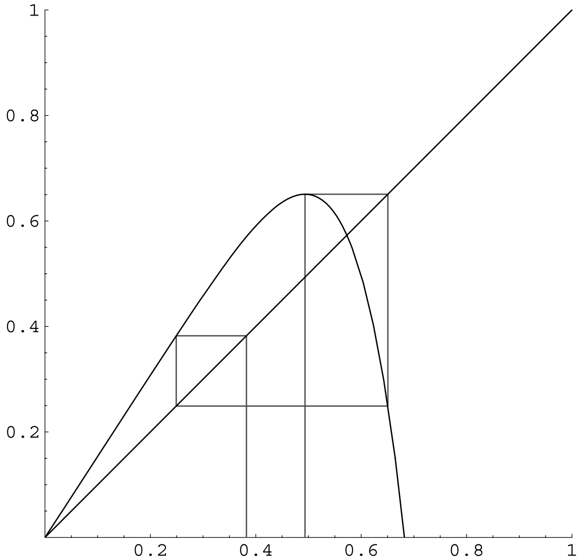


Fig. 1 The graphs of f_θ with $\theta = 0.65$, $\beta = 0.99$, $\mu = 0.1$, and $\varphi = 5$, the diagonal line, and the first three iterations of the critical point of f_θ

3.1 Persistent habits of one period

When habits persist for one period ($\eta = 1$ or $n = 1$), (10) can be written as

$$\rho_{t+1} = f_\theta(\rho_t), \tag{11}$$

where $f_\theta(\rho_t) = \frac{1}{\theta}\rho_t(1 - \frac{\beta}{\mu}\rho_t^{1+\varphi})$. For iterations, we write $f_\theta = f_\theta(\rho_t)$ to denote the identity function by f_θ^0 and inductively define $f_\theta^\tau = f_\theta \circ f_\theta^{\tau-1}$ for positive integer τ. The appearance of a 3-period cycle under certain parameter values has been verified by Auray et al. [2]. To complement their study, we show a numerical simulation of the dynamic behavior of f_θ with $\theta = 0.65$ in Fig. 1.[9] It exhibits that the first three iterations of the critical point $\bar{\rho}$ where $f_\theta'(\bar{\rho}) = 0$ satisfy the condition $f_\theta(\bar{\rho}) > \bar{\rho} > f_\theta^3(\bar{\rho}) > f_\theta^2(\bar{\rho})$ and hence, chaotic dynamics in the sense of Li and Yorke occur; refer to Proposition 1 of [2].

3.2 Persistent habits of two periods

When habits persist for two periods ($n = 2$), (10) becomes

$$\rho_{t+1} = \frac{1}{\theta}\frac{\rho_{t-1}\rho_t}{\rho_{t-1} + (1-\eta)\rho_t}\left(1 - \frac{\beta}{\mu}\rho_t^{1+\varphi}\right). \tag{12}$$

Although (12) looks quite similar to (11), the study of its global property is much more difficult. In the following theorems we show that the dynamic property of (12) could be quite complex and is sensitive to the parameter values. First, we show that cycles can occur.

Theorem 1 *Equation (12) has a steady state at $\rho_* = \sqrt[1+\varphi]{\frac{\mu(1-2\theta+\eta\theta)}{\beta}}$, which loses its stability at the critical value $\theta_* = \frac{1+\varphi}{(2+\varphi)(2-\eta)+\eta}$ and then generates a two-period cycle.*

Proof Let $\rho_{t-1} = \rho_t = \rho_* = \sqrt[1+\varphi]{\frac{\mu(1-2\theta+\eta\theta)}{\beta}}$. Then

$$\begin{aligned}\rho_{t+1} &= \frac{1}{\theta}\frac{\rho_{t-1}\rho_t}{\rho_{t-1} + (1-\eta)\rho_t}\left(1 - \frac{\beta}{\mu}\rho_t^{1+\varphi}\right)\\ &= \frac{1}{\theta}\frac{\rho_*\rho_*}{\rho_* + (1-\eta)\rho_*}\left(1 - \frac{\beta}{\mu}\rho_*^{1+\varphi}\right)\\ &= \frac{1}{\theta}\frac{\rho_*}{1 + (1-\eta)}(1 - 1 + 2\theta - \eta\theta) = \rho_*.\end{aligned}$$

Thus, ρ_* is a steady state of (12).

The Jacobian matrix of (12) is

$$\begin{bmatrix} 0 & 1 \\ \frac{(1-\eta)\rho_t^2}{\theta(\rho_{t-1}+(1-\eta)\rho_t)^2}(1 - \frac{\beta}{\mu}\rho_t^{1+\varphi}) & \frac{\rho_{t-1}^2}{\theta(\rho_{t-1}+(1-\eta)\rho_t)^2}(1 - \frac{\beta}{\mu}\rho_t^{1+\varphi}) - \frac{\beta\rho_{t-1}\rho_t}{\mu\theta(\rho_{t-1}+(1-\eta)\rho_t)}(1+\varphi)\rho_t^\varphi \end{bmatrix}$$

and, at the steady state ρ_*, it becomes $\begin{bmatrix} 0 & 1 \\ \frac{1-\eta}{2-\eta} & (1+\varphi) + \frac{1}{2-\eta} - \frac{1+\varphi}{\theta(2-\eta)} \end{bmatrix}$ and has the characteristic polynomial $p(x) = x^2 - [(1+\varphi) + \frac{1}{2-\eta} - \frac{1+\varphi}{\theta(2-\eta)}]x - \frac{1-\eta}{2-\eta}$. Let $\theta_* = \frac{1+\varphi}{(2+\varphi)(2-\eta)+\eta}$. Then $p(-1) = 0$, hence the value -1 is an eigenvalue of the Jacobian matrix of (12) with $\theta = \theta_*$. Therefore, the steady state ρ_* undergoes a period-doubling bifurcation and a two-period cycle appears. □

[9] Empirical studies indicate that the elasticity of labor supply is less than 1, so we assign $\varphi = 5$ in Fig. 1.

Theorem 1 illustrates that the degree of habit persistence (θ), the depreciation rate of habit persistence (η) and the inverse of labor supply elasticity (φ) are crucial factors to determine the emergence of a two-period cycle of (12). Taking the derivatives of θ_* with respect to η and φ, we obtain

$$\frac{\partial\theta_*}{\partial\eta} = \frac{(1+\varphi)^2}{[(2+\varphi)(2-\eta)+\eta]^2} > 0, \tag{13}$$

$$\frac{\partial\theta_*}{\partial\varphi} = \frac{2}{[(2+\varphi)(2-\eta)+\eta]^2} > 0. \tag{14}$$

Equations (13) and (14) imply that the critical value θ_* increases with η and φ. A numerical example of $\theta_*(\eta)$ is given in Fig. 2(a). It shows that $\theta_* \approx 0.7075$ if $\eta = 0.92$ and $\theta_* = 0.75$ if $\eta = 1$. Empirical estimations by Ferson and Constantinides [14] and Braun et al. [6] suggest that a reasonable value of θ is between 0.5 and 0.9 when habit lasts for one period. Hence, Fig. 2(a) indicates that for η equal to or close to 1, cycles may occur under reasonable range of θ. Setting $\eta = 0.92$, a numerical example of $\theta_*(\varphi)$ is given in Fig. 2(b).

We now turn to show that under certain parameter values, chaotic dynamics will occur in the sense of topological entropy and entropic chaos, the definitions of which are given as follows:

Definition 1 Let $g : X \to X$ be a continuous map on the space X with metric d. For $n \in \mathbb{N}$ and $\varepsilon > 0$, a set $S \subset X$ is called an (n,ε)-*separated set for* g if for every pair of points $x, y \in S$ with $x \neq y$, there exists an integer k with $0 \le k < n$ such that $d(g^k(x), g^k(y)) > \varepsilon$. The *topological entropy of* g is defined to be

$$h_{\text{top}}(g|X) = \lim_{\varepsilon\to 0,\varepsilon>0}\ \limsup_{n\to\infty} \frac{\log(\max\{\#(S) : S \subset X \text{ is an } (n,\varepsilon)\text{-separated set for } g\})}{n},$$

where $\#(S)$ is the cardinality of elements of S.

We say that g has *entropic chaos* on X if $h_{\text{top}}(g|X) > 0$.

Topological entropy describes the total exponential complexity of the orbit structure with a single number in a rough but expressive way. The topological entropy is positive for chaotic systems and is zero for non-chaotic systems. To say that a system has entropic chaos means that at least some part of its phase space has complicated behaviors so that every two nearby orbits diverge tremendously and their long-time behaviors cannot be predicted precisely. It is well known that for 1-D systems with entropic chaos, there are infinitely many numbers of periodic cycles with different periods.

In the following theorem, we give a sufficient condition for the presence of entropic chaos in a 2-D system, by following the pioneer article of Smale [28] in the theory of chaotic dynamical systems, to show that (12) has a so-called *Smale's horseshoe* and hence has entropic chaos.

Theorem 2 *If* $1-\eta+\theta < \frac{1+\varphi}{(2+\varphi)^{\frac{2+\varphi}{1+\varphi}}}$, *then* (12) *has entropic chaos.*

Proof The dynamics of (12) with $(\rho_t, \rho_{t+1}) \mapsto (\rho_{t+1}, \rho_{t+2})$ is equivalent to the dynamics of the family of maps $(x, y) \mapsto F(x, y)$, where

$$F(x, y) = \left(y, \frac{1}{\theta}\frac{xy}{x+(1-\eta)y}\left(1-\frac{\beta}{\mu}y^{1+\varphi}\right)\right).$$

Since $1-\eta+\theta < \frac{1+\varphi}{(2+\varphi)^{\frac{2+\varphi}{1+\varphi}}}$, $\frac{1}{\theta} < \frac{1}{\theta(1-\eta)}\left(\frac{1+\varphi}{(2+\varphi)^{\frac{2+\varphi}{1+\varphi}}}-\theta\right)$; hence we can take a positive real number ω such that

$$\frac{1}{\theta} < \omega < \frac{1}{\theta(1-\eta)}\left(\frac{1+\varphi}{(2+\varphi)^{\frac{2+\varphi}{1+\varphi}}}-\theta\right). \tag{15}$$

Let S be the following trapezoid in the plane:

$$S = \left\{(x, y) \in \mathbb{R}^2 : 0 < y < \sqrt[1+\varphi]{\frac{\mu}{\beta}} \text{ and } \frac{y}{\omega} < x < \sqrt[1+\varphi]{\frac{\mu}{\beta}}\right\}.$$

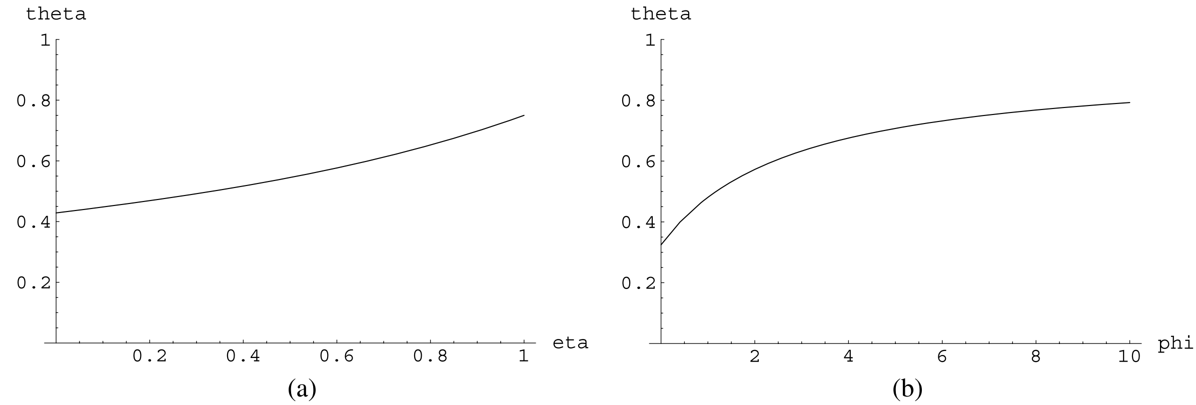


Fig. 2 The graph of the critical value $\theta_* = \frac{1+\varphi}{(2+\varphi)(2-\eta)+\eta}$ with **a** $\varphi = 5$ and **b** $\eta = 0.92$

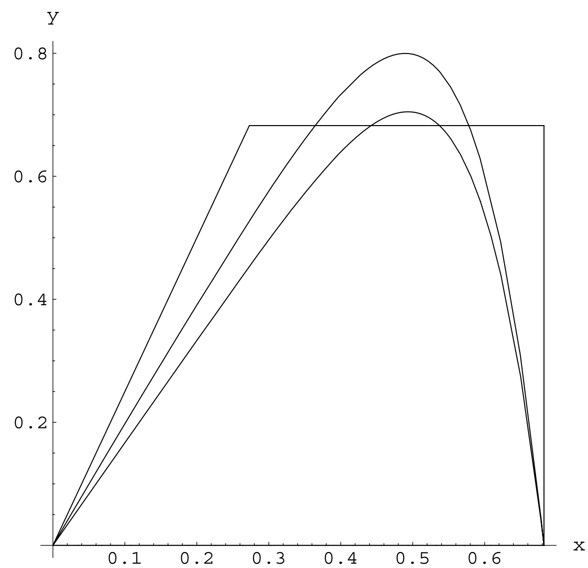


Fig. 3 The set S and its image $F(S)$ form a "horseshoe", where $\theta = 0.5$, $\beta = 0.99$, $\mu = 0.1$, $\varphi = 5$, $\eta = 0.92$, $\omega = 2.5$

Let L be the left boundary of S, that is, in a parameterization form,

$$L = \left\{ \left(\frac{z}{\omega}, z \right) \in \mathbb{R}^2 : 0 < z < \sqrt[1+\varphi]{\frac{\mu}{\beta}} \right\}.$$

Let $g(z)$ be the projection of the map F on L to the y-axis, that is, $g(z) = \pi_y F(\frac{z}{\omega}, z)$ where π_y is the y-coordinate projection. Then

$$\begin{aligned} g(z) &= \frac{1}{\theta} \frac{\frac{z}{\omega} z}{\frac{z}{\omega} + (1-\eta) z} \left(1 - \frac{\beta}{\mu} z^{1+\varphi} \right) \\ &= \frac{1}{\theta} \frac{z}{1 + \omega(1-\eta)} \left(1 - \frac{\beta}{\mu} z^{1+\varphi} \right) \end{aligned}$$

and the graph of $y = g(z)$ for $z \in (0, \sqrt[1+\varphi]{\frac{\mu}{\beta}})$ is the bottom boundary of the image $F(S)$; see Fig. 3. The maximum of g on the interval $(0, \sqrt[1+\varphi]{\frac{\mu}{\beta}})$ is

$$\begin{aligned} & g\left(\sqrt[1+\varphi]{\frac{\mu}{\beta(2+\varphi)}} \right) \\ &\quad = \sqrt[1+\varphi]{\frac{\mu}{\beta}} \frac{1+\varphi}{(2+\varphi)^{\frac{2+\varphi}{1+\varphi}}} \frac{1}{\theta(1+\omega(1-\eta))} \end{aligned}$$

which is greater than $\sqrt[1+\varphi]{\frac{\mu}{\beta}}$ due to inequality (15). Moreover, the graph of $y = wz$ with $0 < y < \sqrt[1+\varphi]{\frac{\mu}{\beta}}$

represents the left boundary of S and is always above the graph of $y = g(z)$; indeed, inequality (15) gives us that

$$\omega z > \frac{z}{\theta} > \frac{1}{\theta}\frac{z}{1+\omega(1-\eta)} > \frac{1}{\theta}\frac{z}{1+\omega(1-\eta)}\left(1-\frac{\beta}{\mu}z^{1+\varphi}\right) = g(z).$$

Thus, $F(S) \cap S$ has two vertical strips, namely, V_1 on the left and V_2 on the right. Similarly, $F^{-1}(S) \cap S$ has two horizontal strips, namely, H_1 on the bottom and H_2 on the top. We have $F(H_k) = V_k$ for $k = 1, 2$.

For integers $m \le 0$ and $n \ge 0$, let $S_m^n = \bigcap_{i=m}^n F^i(S)$. Then $S_0^1 = V_1 \cup V_2$ is a union of two vertical strips in S. As in the one-dimensional case, for $n \ge 1$,

$$\begin{aligned} S_0^n &= F\big(S_0^{n-1}\big) \cap S \\ &= \big[F\big(S_0^{n-1}\big) \cap V_1\big] \cup \big[F\big(S_0^{n-1} \cap V_2\big)\big] \\ &= F\big(S_0^{n-1} \cap H_1\big) \cup F\big(S_0^{n-1} \cap H_2\big). \end{aligned}$$

In particular, for $n = 2$, $S_0^2 = F(S_0^1 \cap H_1) \cup F(S_0^1 \cap H_2) = F([V_1 \cup V_2] \cap H_1) \cup F([V_1 \cup V_2] \cap H_2)$ is the union of 2^2 vertical strips in S_0^1. By induction, S_0^n is the union of 2^n vertical strips. Taking $n \to \infty$, we have that $S_0^\infty = \bigcap_{n=1}^\infty S_0^n$ is the union of infinitely many vertical strips or segments (occurring while the widths of strips converge to zero as $n \to \infty$). If $z \in S_0^\infty$, then $z \in F^i(S)$ and $F^{-i}(z) \in S$ for all $i \ge 0$. Thus S_0^∞ is the set of points whose backward iterates stay in S.

Considering the sets S_m^0, we have that $S_{-1}^0 = H_1 \cup H_2$ is the union of two horizontal strips in S. Then S_{-2}^0 is the union of four horizontal strips in S_{-1}^0. Continuing by induction, we have that for $m \le 0$, S_m^0 is the union of 2^{-m} horizontal strips and $S_{-\infty}^0 = \bigcap_{m=-\infty}^0 S_m^0$ is the union of infinitely many horizontal strips or segments (occurring while the heights of strips converge to zero as $m \to -\infty$). If $z \in S_{-\infty}^0$, then $z \in F^{-i}(S)$ and $F^i(z) \in S$ for all $i \ge 0$. Thus $S_{-\infty}^0$ is the set of points whose forward iterates stay in S.

By the definition of Λ, we have that $\Lambda = S_0^\infty \cap S_{-\infty}^0$ is the intersection of infinitely many vertical strips (or segments) and infinitely many horizontal strips (or segments), and Λ is the set of points such that both the forward and backward iterates stay in S.

Let $\Sigma_2^2 = \{\mathbf{i} = (\ldots, i_{-1}, i_0, i_1, \ldots) : i_k \in \{1, 2\}$ for all $k \in \mathbb{Z}\}$ be the two-sided sequence space with the metric $d(\mathbf{i}, \mathbf{j}) = \sum_{k=-\infty}^\infty \frac{\delta(i_k, j_k)}{4^{|k|}}$, where $\delta(s,t)$ is 0 if $s = t$ and is 1 if $s \ne t$. The shift map σ on Σ_2^2 is defined by $\sigma(\mathbf{i}) = \mathbf{j}$ where $j_k = i_{k+1}$ for all $k \in \mathbb{Z}$. Let $\bar{\Sigma}_2^2$ be the space obtained form Σ_2^2 by identifying $(\ldots, i_{-1}, i_0, i_1, \ldots)$ and $(\ldots, j_{-1}, j_0, j_1, \ldots)$ if either $i_k = j_m = 1$ or $i_k = j_m = 2$ for all $k \in \mathbb{Z}$ and all $m \le 1$; that is, by identifying two sequences if they are itineraries of the same point in S. Define $h : \Lambda \to \bar{\Sigma}_2^2$ by $h(z) = (\ldots, i_{-1}, i_0, i_1, \ldots)$ where $F^k(z) \in H_{i_k}$ for all $k \in \mathbb{Z}$. We prove that h is a semi-conjugacy from $F|\Lambda$ to $\sigma|\bar{\Sigma}_2^2$.

First we prove that $\sigma \circ h = h \circ F$ on Λ. Let $h(z) = (\ldots, i_{-1}, i_0, i_1, \ldots)$ and $h(F(z)) = (\ldots, j_{-1}, j_0, j_1, \ldots)$. Then $F^{k+1}(z) \in H_{i_{k+1}}$ but also $F^{k+1}(z) = F^k(F(z)) \in H_{j_k}$. Thus $i_{k+1} = j_k$ and $\sigma(h(z)) = h(F(z))$.

Next we prove the continuity of h. Let $h(z) = (\ldots, i_{-1}, i_0, i_1, \ldots)$. A neighborhood of $(\ldots, i_{-1}, i_0, i_1, \ldots)$ is given by $U = \{(\ldots, j_{-1}, j_0, j_1, \ldots) : j_k = i_k$ for $-k_0 \le k \le k_0\}$. With k_0 fixed, the continuity of F insures that there is a $\delta > 0$ such that if $w \in \Lambda$ with $|w - z| \le \delta$, then $F^k(w) \in H_{i_k}$ for $-k_0 \le k \le k_0$. Thus if $w \in \Lambda$ with $|w - z| \le \delta$ then $h(w) \in U$.

Last we check whether h is surjective. We apply induction on n to show that $\bigcap_{k=1}^n F^k(H_{i_{-k}})$ is a vertical strip for all strings of symbols $(\ldots, i_{-1}, i_0, i_1, \ldots) \in \bar{\Sigma}_2^2$. Let $(\ldots, i_{-1}, i_0, i_1, \ldots) \in \bar{\Sigma}_2^2$. For $n = 1$, this set is just $F(H_{i_{-1}}) = V_{i_{-1}}$, which is a vertical strip. Then

$$\bigcap_{k=1}^n F^k(H_{i_{-k}}) = F\left(\bigcap_{k=2}^n F^{k-1}(H_{i_{-k}})\right) \cap F(H_{i_{-1}})$$

is a vertical strip. Letting n go to infinity, $\bigcap_{k=1}^\infty F^k(H_{i_{-k}})$ is a vertical strip or segment. Similarly, $\bigcap_{k=-\infty}^0 F^k(H_{i_{-k}})$ is a horizontal strip or segment. Thus $\bigcap_{k=-\infty}^\infty F^k(H_{i_{-k}})$ is nonempty; say z is in this intersection. Therefore, $h(z) = (\ldots, i_{-1}, i_0, i_1, \ldots)$ and h is surjective. This completes the proof that h is a semi-conjugacy from $F|\Lambda$ to $\sigma|\bar{\Sigma}_2^2$.

The space $\bar{\Sigma}_2^2$ is obtained from Σ_2^2 by identifying two sequences if they are itineraries of the same point in S. Thus the shift map σ on Σ_2^2 and $F|\Lambda$ both naturally project to the shift map σ on $\bar{\Sigma}_2^2$. Notice that the semi-conjugacy $\bar{g} : \Sigma_2^2 \to \bar{\Sigma}_2^2$ is injective outside a countable set, namely, the itineraries of points in the backward orbits of turning points. The *Variational Principle* says that topological entropy is the supremum of metric theoretic entropies; more precisely, if $g : X \to X$ is a homeomorphism of a compact metric space (X, d) then $h_{\text{top}}(g) = \sup\{h_\mu(g) :$

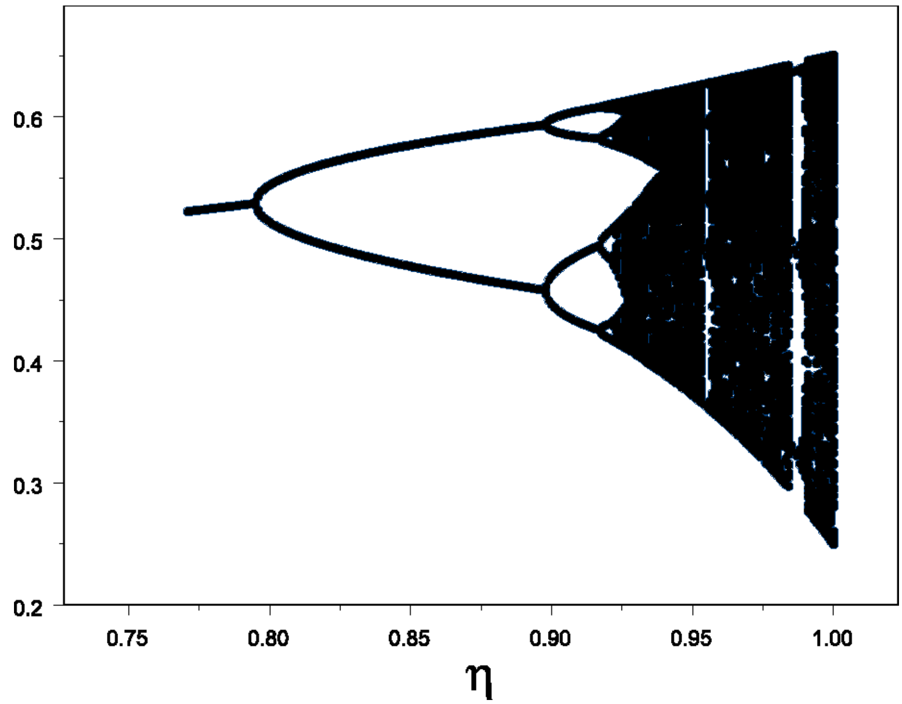


Fig. 4 Orbit diagram in η with $\theta = 0.65$, $n = 2$, $\beta = 0.99$, $\mu = 0.1$, and $\varphi = 5$

μ is an f-invariant Borel probability measures on $X\}$; refer to Theorem 4.5.3 of Katok and Hasselblatt [18]. Therefore, by the Variational Principle it suffices to consider non-atomic measures, since purely atomic measures have zero entropy. Consider a non-atomic σ-invariant measure ς on Σ_2^2 and pull it back via the semi-conjugacy $\bar{g}$ to a measure $\bar{g}_*\varsigma$ on $\bar{\Sigma}_2^2$. Thus $\bar{g}$ establishes a bijective correspondence between ς and $\bar{g}_*\varsigma$ so the measure-theoretic entropies coincide. By the Variational Principle, we have $h_{\text{top}}(\sigma|\bar{\Sigma}_2^2) = \sup_\zeta h_\zeta(\sigma|\bar{\Sigma}_2^2) \geq \sup_\varsigma h_{\bar{g}_*\varsigma}(\sigma|\bar{\Sigma}_2^2) = \sup_\varsigma h_\varsigma(\sigma|\Sigma_2^2) = h_{\text{top}}(\sigma|\Sigma_2^2)$. Since g is a semi-conjugacy from $F|\Lambda$ to $\sigma|\bar{\Sigma}_2^2$, $h_{\text{top}}(f|[0,1]) \geq h_{\text{top}}(\sigma|\bar{\Sigma}_2^2) = h_{\text{top}}(\sigma|\Sigma_2^2) = \log(2)$. The proof of the theorem is complete. □

Theorem 2 demonstrates that chaotic motion will emerge if the depreciation of habits is large enough. Figure 4 presents the bifurcation diagram with varying η and we can see that the economy undergoes from simple to complex dynamics as η increases.

To compare results under complete and incomplete habit depreciation, the plots (a) and (b) in Fig. 5 show the bifurcation diagrams with varying θ for $\eta = 1$ and $\eta = 0.81$, respectively. From these two figures, we find that a decrease in η reduces the possibility of chaotic motion. Figure 5(a) illustrates that with complete depreciation ($\eta = 1$), chaotic motion will occur for the value of $\theta \in [0.5, 0.9]$. However, Fig. 5(b) shows that with a minor decrease in η ($\eta = 0.81$) and a slightly longer persistence of habits ($n = 2$), only simple dynamics and periodic cycles will be present for $\theta \in [0.5, 0.9]$. This is because when the depreciation of habits decreases, consumptions will have larger impacts on the current consumption. Hence, it is more difficult for agents to change their behavior unless the degree of habit persistence is sufficiently low. Similar results can be obtained if the persistence of habits becomes longer (i.e., if n increases).

3.3 A general case: persistent habits of n periods

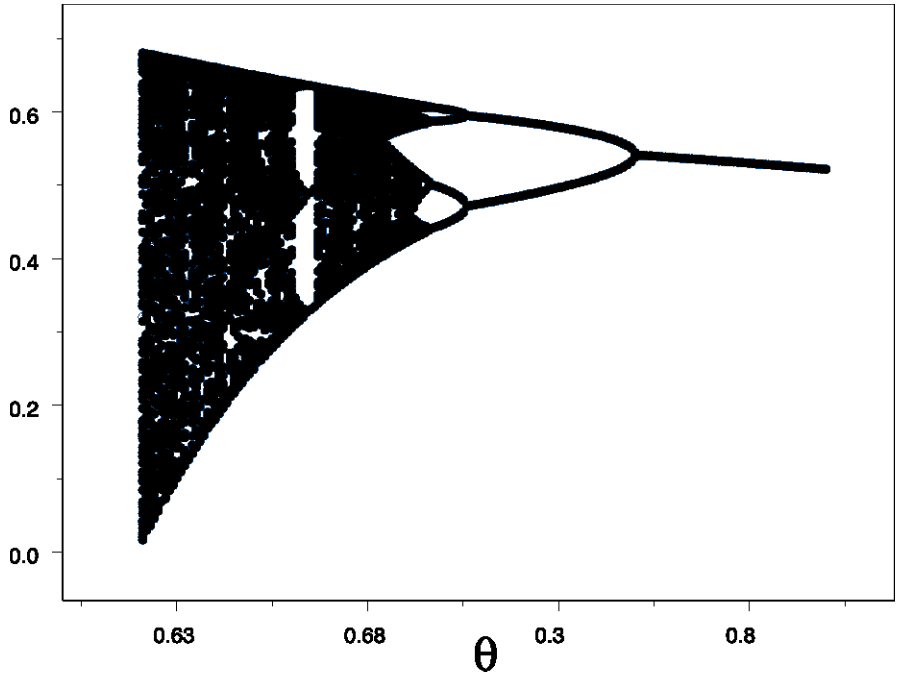
In this section, we analyze the general case where habits last for n periods and (10) represents an n-D system. We first identify conditions generating period-doubling bifurcation of a steady state for (10).

Theorem 3 *Equation* (10) *has a steady state at* $\rho_{**} = \sqrt[1+\varphi]{\frac{\mu}{\beta}\left(1 - \frac{\theta}{\eta}[1-(1-\eta)^n]\right)}$, *which loses its stability at the critical value*

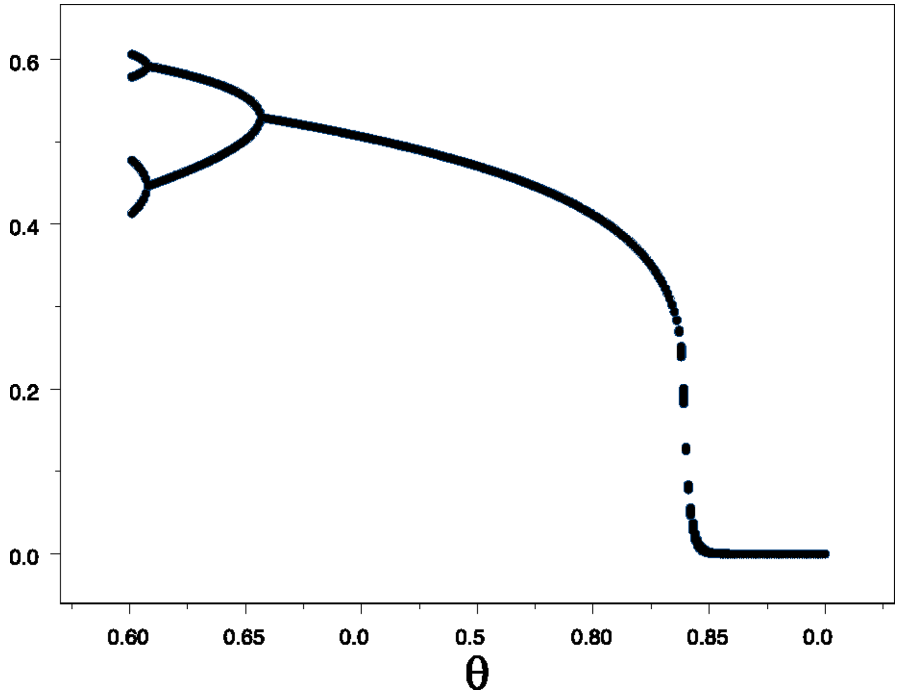
$$\theta_{**} = \frac{\eta(2-\eta)(1+\varphi)}{(2-\eta)(2+\varphi) + \eta - (1-\eta)^n[(2-\eta)(2+\varphi) + (-1)^n \eta]}$$

and then generates a two-period cycle.

Fig. 5 Orbit diagram in θ with $\beta = 0.99$, $\mu = 0.1$, $\varphi = 5$, $n = 2$, and **a** $\eta = 1$ and **b** $\eta = 0.81$



(a)



(b)

Proof Let $\rho_{t-i} = \rho_{**} = \sqrt[1+\varphi]{\frac{\mu}{\beta}(1 - \frac{\theta}{\eta}[1-(1-\eta)^n])}$ for $0 \le i \le n-1$. Then

$$\rho_{t+1} = \frac{1}{\theta}\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\sum_{j=0}^{n-1}[(1-\eta)^j(\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\rho_{t-j}})]}\left(1-\frac{\beta}{\mu}\rho_t^{1+\varphi}\right)$$
$$= \frac{1}{\theta}\frac{\rho_{**}}{\sum_{j=0}^{n-1}(1-\eta)^j}\left(1-\frac{\beta}{\mu}\rho_{**}^{1+\varphi}\right)$$
$$= \frac{1}{\theta}\frac{\rho_{**}}{\sum_{j=0}^{n-1}(1-\eta)^j}\left(\frac{\theta}{\eta}\big[1-(1-\eta)^n\big]\right) = \rho_{**}.$$

Thus, ρ_{**} is a steady state of (10).

The Jacobian matrix of (10) is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1\\ \frac{\partial\rho_{t+1}}{\partial\rho_{t-n+1}} & \frac{\partial\rho_{t+1}}{\partial\rho_{t-n+2}} & \frac{\partial\rho_{t+1}}{\partial\rho_{t-n+3}} & \frac{\partial\rho_{t+1}}{\partial\rho_{t-n+4}} & \cdots & \frac{\partial\rho_{t+1}}{\partial\rho_{t-1}} & \frac{\partial\rho_{t+1}}{\partial\rho_t} \end{bmatrix},$$

where, for $1 \le k \le n-1$,

$$\frac{\partial\rho_{t+1}}{\partial\rho_{t-k}} = \frac{(1-\eta)^k}{\theta}\left(\frac{\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\rho_{t-k}}}{\sum_{j=0}^{n-1}[(1-\eta)^j(\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\rho_{t-j}})]}\right)^2 \times\left(1-\frac{\beta}{\mu}\rho_t^{1+\varphi}\right)$$

and

$$\frac{\partial\rho_{t+1}}{\partial\rho_t} = \frac{1}{\theta}\left(\frac{\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\rho_t}}{\sum_{j=0}^{n-1}[(1-\eta)^j(\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\rho_{t-j}})]}\right)^2\left(1-\frac{\beta}{\mu}\rho_t^{1+\varphi}\right) - \frac{\beta(1+\varphi)}{\theta\mu}\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\sum_{j=0}^{n-1}[(1-\eta)^j(\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\rho_{t-j}})]}\rho_t^{\varphi}.$$

At the steady state ρ_{**}, we have that for $1 \le k \le n-1$,

$$\left.\frac{\partial\rho_{t+1}}{\partial\rho_{t-k}}\right|_{\rho_{t-i}=\rho_{**},0\le i\le n-1} = \frac{(1-\eta)^k}{\theta}\left(\frac{1}{\sum_{j=0}^{n-1}(1-\eta)^j}\right)^2\left(1-\frac{\beta}{\mu}\rho_{**}^{1+\varphi}\right) = \frac{(1-\eta)^k}{\sum_{j=0}^{n-1}(1-\eta)^j} = \frac{\eta(1-\eta)^k}{1-(1-\eta)^n}$$

and

$$\left.\frac{\partial\rho_{t+1}}{\partial\rho_t}\right|_{\rho_{t-i}=\rho_{**},0\le i\le n-1} = \frac{1}{\theta}\left(\frac{1}{\sum_{j=0}^{n-1}(1-\eta)^j}\right)^2\left(1-\frac{\beta}{\mu}\rho_{**}^{1+\varphi}\right) - \frac{\beta(1+\varphi)}{\theta\mu}\frac{\rho_{**}}{\sum_{j=0}^{n-1}(1-\eta)^j}\rho_{**}^{\varphi}$$
$$= \frac{\eta}{1-(1-\eta)^n} + 1 + \varphi - \frac{1+\varphi}{\theta}\frac{\eta}{1-(1-\eta)^n}.$$

Thus the Jacobian matrix of (10) at the steady state ρ_{**} has the characteristic polynomial

$$p(x) = x^n - \sum_{k=0}^{n-1}\left.\frac{\partial\rho_{t+1}}{\partial\rho_{t-k}}\right|_{\rho_{t-i}=\rho_{**},0\le i\le n-1} x^{n-1-k}.$$

Since

$$p(-1) = (-1)^n - \sum_{k=0}^{n-1}\left.\frac{\partial\rho_{t+1}}{\partial\rho_{t-k}}\right|_{\rho_{t-i}=\rho_{**},0\le i\le n-1}(-1)^{n-1-k}$$
$$= (-1)^n - \frac{(-1)^{n-1}\eta}{1-(1-\eta)^n} - (-1)^{n-1}(1+\varphi) + \frac{1+\varphi}{\theta}\frac{(-1)^{n-1}\eta}{1-(1-\eta)^n} - \sum_{k=1}^{n-1}(-1)^{n-1-k}\frac{\eta(1-\eta)^k}{1-(1-\eta)^n}$$
$$= (-1)^n - \frac{(-1)^{n-1}\eta}{1-(1-\eta)^n} - (-1)^{n-1}(1+\varphi) + \frac{1+\varphi}{\theta}\frac{(-1)^{n-1}\eta}{1-(1-\eta)^n} + \frac{(-1)^{n-1}\eta}{1-(1-\eta)^n}\frac{1-\eta+(-1)^n(1-\eta)^n}{2-\eta}$$
$$= \frac{(-1)^{n-1}\eta}{1-(1-\eta)^n}\left[\frac{1+\varphi}{\theta} - \frac{(2-\eta)(2+\varphi)+\eta-(1-\eta)^n[(2-\eta)(2+\varphi)+(-1)^n\eta]}{\eta(2-\eta)}\right],$$

we have $p(-1)=0$ if θ is equal to the critical value θ_{**}. This says that one of the eigenvalues of the Jacobian matrix of (10) at the steady state ρ_{**} passes through -1 if θ varies through the critical value θ_{**} and hence a period-doubling bifurcation occurs and a two-period cycle appears. □

Theorem 3 provides a sufficient condition for the emergence of a two-period cycle. It shows that the critical value θ_{**} depends on the length of the persistent habits (n). To study the possibility of chaos for an n-D dynamical system, we modify the theorems of [17] and [19] to prove the occurrence of entropic chaos.[10]

Theorem 4 *Let* $\underline{\theta}=\frac{1+\varphi}{(2+\varphi)^{\frac{2+\varphi}{1+\varphi}}}$ *and* $\bar{\theta}=\frac{1+\varphi}{2+\varphi}$, *then there exists a unique* $\theta^*\in(\underline{\theta},\bar{\theta})$ *such that if* $\theta\in(\underline{\theta},\theta^*]$ *then for any* η *close to* 1, (10) *has entropic chaos.*

Proof Consider f_θ in (11). By Proposition 1 of [2], there exists a unique $\theta^*\in(\underline{\theta},\bar{\theta})$ such that if $\theta\in(\underline{\theta},\theta^*]$, then f_θ has Li–Yorke chaos and hence $h_{\text{top}}(f_\theta)>0$. Let $\theta\in(\underline{\theta},\theta^*]$. Then there exists a unique $\bar{\rho}\in(0,(\frac{\mu}{\beta})^{\frac{1}{1+\varphi}})$ such that $f'_\theta(\bar{\rho})=0$. Let $B=f_\theta(f_\theta(\bar{\rho}))$ and $C=f_\theta(\bar{\rho})$. Then $0<B<C$. For $\eta<1$, define $\Phi_\eta:[B,C]^{n+1}\to\mathbb{R}$ by

$$\Phi_\eta(\rho_{t-n+1},\dots,\rho_t,\rho_{t+1})$$
$$=\rho_{t+1}-\frac{1}{\theta}\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\sum_{j=0}^{n-1}[(1-\eta)^j(\frac{\prod_{i=0}^{n-1}\rho_{t-i}}{\rho_{t-j}})]}\times\left(1-\frac{\beta}{\mu}\rho_t^{1+\varphi}\right).$$

Let Y_η be the set of solutions of the difference equation

$$\Phi_\eta(\rho_{t-n+1},\dots,\rho_t,\rho_{t+1})=0, \tag{16}$$

i.e., the set of sequences $\underline{\rho}=(\rho_t)=(\dots,\rho_{-1},\rho_0,\rho_1,\dots)$ such that for any $t\ge n-1$,

1. $\rho_t\in[B,C]$; and
2. $n+1$ consecutive components $\rho_{t-n+1},\dots,\rho_t,\rho_{t+1}$ of $\underline{\rho}$ satisfy (16).

Let σ be the shift map on Y_η, i.e., $(\sigma(\underline{\rho}))_t=\rho_{t+1}$ for all $t\ge n-1$. Then on $[B,C]^{n+1}$, the function Φ_η is C^1 for each η and is continuous in η and so are the partial derivatives $\partial_i\Phi_\eta$ for $1\le i\le n+1$. Now letting $\eta=1$, we have the limit function $\Phi_\eta(\rho_{t-n+1},\dots,\rho_t,\rho_{t+1})=\rho_{t+1}-f_\theta(\rho_t)$, where f_θ is given in (11). By Theorem 5 in the Appendix, for all η near 1, there is a closed σ-invariant subset Γ_η of Y_η in the product topology, such that $h_{\text{top}}(\sigma|\Gamma_\eta)>0$. Therefore, the dynamics of the economy system has entropic chaos. □

The result of Theorem 4 fits well with the numerical results in Fig. 4. If $\varphi=5$, then $\underline{\theta}\approx 0.6197$ and $\bar{\theta}\approx 0.8571$. Figure 4 with $\theta=0.65$ indicates that the system has complex dynamics for all η close to one. Moreover, it provides a sufficient condition for the occurrence of entropic chaos when persistent habit lasts for n periods. The approach we use is to approximate an n-dimensional dynamical system by a 1-dimensional dynamical system. This method is useful when studying the global property of a high-dimensional dynamical system. Furthermore, Theorem 4 illustrates occurrence of chaos crucially depending on the formation of habits (the length and the depreciation rate of persistent habits).

4 Conclusion

We investigate the global dynamics of a CIA model where consumption habits persist for n periods. We find that the length and the depreciation rate of persistent habits, the degree of habits in consumption, and the labor supply elasticity are important determinants to the dynamic property of a monetary economy. In economic research, only very few papers concern the global property of a high-dimensional dynamical system due to the difficulty of analysis and tractability. With the technique to approximate an n-D dynamical system by a 1-D dynamical system, economists can explore many other interesting questions without the limitation to reduce economic models to 1-D or 2-D systems.

In this paper, we show that the qualitative properties are very different when considering the occurrence of chaotic dynamics of a monetary economy with complete or incomplete habit depreciation. Hence, a more precise estimation of habit formation is needed for future research. However, Ferson and Constantinides [14] find that estimation of a two-lag consumption model is problematic and is hard to be precise. Therefore, advanced econometric techniques are

[10] A simple version of the theorems of [17] and [19] is given in the Appendix.

needed to overcome the estimation problems concerning habit formation.

Appendix

The following theorem is a simple version of Theorem 3 of [17] and Theorem 3.3 of [19].

Theorem 5 *Consider a difference equation of order n in the form*

$$\Phi_\eta(\rho_{t-n+1},\ldots,\rho_t,\rho_{t+1})=0,\quad t\ge n-1, \tag{17}$$

where $\eta\in[0,1]$ *is a parameter and the real-valued function* Φ_λ *is defined on a* $(n+1)$*-dimensional cube* $[B,C]^{n+1}\subset\mathbb{R}^{n+1}$ *with constants* $0<B<C$. *Assume that* (i) Φ_η *is* C^1 *on* $[B,C]^{n+1}$ *for each* $\eta\in[0,1]$; (ii) *the function* $\eta\mapsto\Phi_\eta$ *is continuous on* $[0,1]$; *and* (iii) *for* $i=1,2,\ldots,n+1$, *the function* $\eta\mapsto\partial_i\Phi_\eta$ *is continuous on* $[0,1]$, *where* $\partial_i\Phi_\eta$ *is the partial derivative of* Φ_η *with respect to the ith variable. Suppose that for* $\eta=1$, *the difference equation* (17) *reduces to a difference equation of order one in the form* $\rho_{t+1}-\varphi(\rho_t)=0$, $t\ge n-1$, *where* $\varphi:[B,C]\to\mathbb{R}$ *is a* C^2 *function with positive topological entropy. Let* Y_η *be the set of solutions for* (17), *i.e. the set of sequences* $\underline{\rho}=(\rho_0,\rho_1,\rho_2,\ldots)$ *such that for any* $t\ge n-1$,

1. $\rho_t\in[B,C]$; *and*
2. $n+1$ *consecutive components* $\rho_{t-n+1},\ldots,\rho_t,\rho_{t+1}$ *of* $\underline{\rho}$ *satisfy* (17)

Let σ *be the shift map on* Y_η, *i.e.,* $\sigma(\underline{\rho})=\underline{\rho}'$, *where* $\rho'_t=\rho_{t+1}$ *for all* $t\ge n-1$. *Then there exists* $\varepsilon>0$ *such that for any* $1-\varepsilon<\eta<1$, *there is a closed* σ*-invariant subset* Γ_η *of* Y_η *in the product topology such that* $h_{\text{top}}(\sigma|\Gamma_\eta)>0$.

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