



Multiple lot-sizing decisions with an interrupted geometric yield and variable production time

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ABSTRACT

This study examines a multiple lot-sizing problem for a single-stage production system with an interrupted geometric distribution, which is distinguished in involving variable production lead-time. In a finite number of setups, this study determined the optimal lot-size for each period that minimizes total expected cost. The following cost items are considered in optimum lot-sizing decisions: setup cost, variable production cost, inventory holding cost, and shortage cost. A dynamic programming model is formulated in which the duration between current time and due date is a stage variable, and remaining demand and work-in-process status are state variables. This study then presents an algorithm for solving the dynamic programming problem. Additionally, this study examines how total expected costs of optimal lot-sizing decisions vary when parameters are changed. Numerical results show that the optimum lot-size as a function of demand is not always monotonic.

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1. Introduction

Multiple lot-sizing production-to-order (MLPO) problems have been studied for several decades (Bowman, 1955). Such problems typically arise from variations in production yield. Consider a production system with an uncertain process yield. To fulfill a particular customer demand, lots may need to be released several times to minimize total expected costs. The MLPO problem is to determine the optimal lot-size for each possible lot release.

This study describes and formulates a single-stage MLPO problem with one salient feature—uncertain production lead-time. According to Yano (1987), this feature may arise due to many factors such as unreliable vendors, unreliable transportation time, job queuing, machine breakdowns, and rework. Uncertain lead-time characteristic has seldom been considered in MLPO studies; although it has been examined in production control studies (Hsu, Wee, & Teng, 2007). In this study, we assume production lead-time is a random variable; the probability for one period is p and that for two periods is $1 - p$.

In the MLPO problem, process yield follows an interrupted geometric (IG) distribution. The delivery agreement includes due dates; that is, customers will not accept products after delivery due dates, and salvage values of products are negligible. In contrast, finished goods produced ahead of the due date become

inventory and incur holding costs. The following cost items are included: setup cost, variable production cost, inventory holding cost, and shortage cost.

An example of the MLPO problem in this study is a process of drawing special steel coils. The manufacturing process has two operations: pickling and wire drawing. The pickling operation removes rust from steel coils. The processing time required for pickling a steel coil varies. In practice, a steel coil undergoes one or two pickling operations depending on the duration the coil has been in air. The drawing operation reduces the size of the input coil. Drawing speed is very fast. All coils in a lot are inspected when the whole lot is complete. The drawing operation involves a die that is worn gradually over time. When this die is excessively worn, the output does not meet specifications. This implies that the integrated drawing process follows an IG distribution, and production lead-time for a lot from release to output takes one or two periods. Special steels are customized products that in most cases cannot be sold to other customers. Thus, we assume product salvage value is negligible.

This study develops a dynamic programming (DP) approach to solve the MLPO problem. Several lemmas are proposed to reduce the DP problem solution space. Numerical experiments show that the optimum lot-size, as a function of demand, is not necessarily monotonic. This study experimentally investigated how total expected costs of optimal lot-sizing decisions vary when various parameters change.

The remainder of this paper is organized as follows. A literature review is given in Section 2. Section 3 presents the MLPO problem as a DP model by including a simple example to facilitate understanding the formulation. Lemmas for reducing the DP solution

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Notation

| | | | |
|----------|---|---|--|
| D | quantity required by a customer | $1 - p$ | probability of producing a lot in two periods |
| T | number of periods in the decision time horizon | θ | probability that the production system is in-control |
| t | index of time, $t = 0$ is the due date, $t = 0, 1, 2, \dots, T$ | Y_{k_t} | a random variable for the number of output units for lot k_t |
| α | setup cost incurred at each lot input, $\alpha > 0$ | $R_t(k_{t+1})$ | number of work-in-process (WIP) at t , |
| β | variable production cost per unit, $\beta > 0$ | $R_t(k_{t+1}) = \begin{cases} 0 & \text{if the realized production time for } k_{t+1} \text{ is one period,} \\ k_{t+1} & \text{otherwise} \end{cases}$ | |
| k_t | lot-size released at t | $s_t = (D_t, R_t(k_{t+1}))$ | the production system status at t , also called state t |
| W_t | a binary variable indicating the demand of a setup, $W_t = \begin{cases} 0 & \text{if } k_t = 0 \\ 1 & \text{if } k_t > 0 \end{cases}$ | $C_t(s_t, k_t)$ | total expected cost incurred after t |
| D_t | remaining demand at t (number of demand units still not fulfilled at t) | $C_t^*(s_t) = \underset{0 \leq k_t \leq \infty}{\text{Min}} \{C_t(s_t, k_t)\}$ | minimum total expected cost incurred after t |
| h | inventory holding cost per unit per period (\$/unit-period), $h > 0$ | $N_t(s_t)$ | optimal lot-size at state s_t ; that is, $\underset{0 \leq k_t \leq \infty}{\text{Min}} \{C_t(s_t, k_t)\} = C_t(s_t, N_t(s_t))$ |
| m | shortage cost per unit, $m > 0$ | | |
| p | probability of producing a lot in one period | | |

space are presented in Section 4. An algorithm for solving the DP is presented in Section 5. Numerical examples are given in Section 6. Conclusions are provided in Section 7.

2. Related literature

Grosfeld-Nir and Gerchak (2004) and Yano and Lee (1995) comprehensively surveyed studies of MLPO problem. Such studies can be categorized as: *single-stage* and *multiple-stage*. This study is in the category of single-stage MLPO problems; thus, recent studies in this category are reviewed.

Recent single-stage MLPO studies can be analyzed from multiple perspectives. The first perspective is associated with customer demand and delivery requirements. Customer demand may be *stochastic* (Gerchak & Grosfeld-Nir, 1998) or *deterministic*. Delivery requirements can be based on due dates or quantities. In a quantity-based agreement (also called *rigid-demand delivery*), the quantity ordered must be delivered in full; that is, partial delivery is unacceptable. In due-date-based agreements (also called *non-rigid demand delivery*), customers will not accept products after the due date. Prior studies are either based on rigid-demand (e.g., Anily, 1995; Anily, Beja, & Mendel, 2002; Beja, 1977; Zhang & Guu, 1998), or non-rigid demand (e.g., Guu & Zhang, 2003; Pentico, 1988; Sepehri, Silver, & New, 1986; Wang & Gerchak, 2000).

The second perspective is associated with production characteristics such as process yield, lead-time, and quality classifications. Previous studies assumed process yield is governed by a probability distribution, which includes the discrete uniform (Anily, 1995), the binomial distribution (Beja, 1977; Pentico, 1988; Sepehri et al., 1986), the interrupted geometric (Anily et al., 2002; Guu & Zhang, 2003; Zhang & Guu, 1998), the general distribution (Zhang & Guu, 1997), and the stochastically proportional (Grosfeld-Nir & Gerchak, 1990; Wang & Gerchak, 2000). In terms of lead-time, few studies (Wang & Gerchak, 2000) addressed an MLPO problem in which production lead-time is longer than the time epochs between any two lot releases. Most researchers assumed production outcomes have only two possible states, either acceptable or unacceptable quality, while a few other studies (Gerchak & Grosfeld-Nir, 1999) examined scenarios that may have three or more outcomes—for example, high quality, medium quality and unacceptable quality.

The third perspective is associated with cost items and objective functions for the MLPO decision making. The most widely addressed cost items include setup cost, variable production cost, inventory cost, and shortage cost. A few researchers also considered inspection cost (Grosfeld-Nir, Gerchak, & He, 2000) and dis-

posal cost (Wang & Gerchak, 2000). For the objective function, most researchers attempted to minimize total expected cost, while a few considered the impact of risk caused by cost variance (Grosfeld-Nir & Gerchak, 1996).

The fourth perspective is associated with the solution approach. Most formulations of MLPO problems include recursive formulas and have been widely interpreted as DP problems. Therefore, DP has been widely used to solve MLPO problems; however, such a solution approach may be very demanding computationally. Some researchers proposed lemmas to reduce the solution space (Anily, 1995; Beja, 1977; Zhang & Guu, 1998); some others attempted to develop near-optimal heuristic rules (Pentico, 1988; Sepehri et al., 1986); and a few others approximately model the DP problem using a relatively simpler non-DP problem for cases with extremely large/small demand quantities (Anily et al., 2002).

The four perspectives highlight the various complex scenarios that can occur in single-stage MLPO problems. Some researchers investigated multiple-stage MLPO problems, in which additional complexity may arise due to inclusion of lot-sizing decisions made at the start of each stage. For example, a production system with two stages needs a lot-sizing decision for the first stage. Releasing all output items of the first stage immediately to the next stage may not be an optimal decision. A lot-sizing decision at the start of the second stage is needed. Example studies that addressed the multiple-stage MLPO problem include Grosfeld-Nir (2005) and Grosfeld-Nir and Robinson (1995).

Compared to those in literature, the single-stage MLPO problem in this study is unique in that it includes one salient feature—*uncertain production lead-time*. This feature has rarely been considered in either single-stage or multiple-stage MLPO studies.

3. Modeling

To model the MLPO problem, the notation is first presented, followed by a description of the IG distribution. A simple example is then given to explain the idea of the formulation. Finally, the cost function of the MLPO problem is modeled using a recursive formula, and its boundary conditions (BCs) are defined.

3.1. IG distribution

As process yield is governed by an IG distribution, the production system manufactures each unit in a one-by-one manner and operates in two possible states *in-control* or *out-of-control*. The output unit is non-defective when the system is in-control, and is defective when the system is out-of-control. The process can

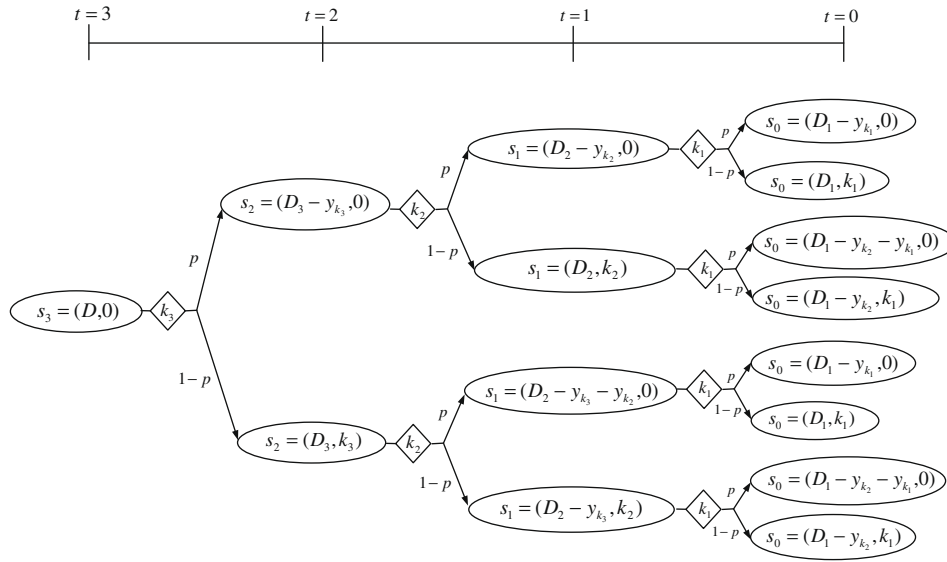


Fig. 1. A simple example illustrating the decision structure.

switch from an in-control state to an out-of-control state, but not vice versa. This *irreversible* characteristic leads to the naming of “interrupted” geometric distribution. The IG distribution can be interpreted as follows. To produce exactly y_{k_t} non-defective units, the system must be in-control for the first y_{k_t} units and out-of-control at the $(y_{k_t} + 1)^{th}$ unit; the probability is $(1 - \theta)^{y_{k_t}}$. In contrast, the probability of producing k_t non-defective units is θ^{k_t} because the system must be in-control for each unit produced. The IG distribution is as follows:

$$P(Y_{k_t} = y_{k_t}) = \begin{cases} (1 - \theta)^{y_{k_t}} & y_{k_t} = 0, 1, 2, \dots, k_t - 1, \\ \theta^{k_t} & y_{k_t} = k_t. \end{cases}$$

3.2. Simple example

Consider an order demand D with $T=3$ as the current time (Fig. 1). Three lot-sizing decisions must be made at $t=3, 2,$ and $1,$ respectively. At $t=3,$ consider the state $s_3 = (D, 0),$ where lot k_3 is released. If lot k_3 is completed at $t=2,$ then $D_2 = D_3 - y_{k_3}, R_2(k_3) = 0,$ and $s_2 = (D_3 - y_{k_3}, 0).$ If lot k_3 is not completed at $t=2,$ then $D_2 = D_3, R_2(k_3) = k_3,$ and $s_2 = (D_3, k_3).$

At $t=2,$ consider the state $s_2 = (D_3 - y_{k_3}, 0),$ where lot k_2 is released. If lot k_2 is completed at $t=1,$ then $D_1 = D_2 - y_{k_2}, R_1(k_2) = 0,$ and $s_1 = (D_2 - y_{k_2}, 0).$ If lot k_2 is not completed at $t=1,$ then $D_1 = D_2, R_1(k_2) = k_2,$ and $s_1 = (D_2, k_2).$

At $t=2,$ consider the state $s_2 = (D_3, k_3),$ where lot k_2 is released. If lot k_2 is completed at $t=1,$ then $D_1 = D_2 - y_{k_3} - y_{k_2}.$ That is, two lots, k_3 and $k_2,$ are now completed, where the realized production time of lot k_3 is two periods and that of lot k_2 is one period. Nota-

bly, $R_1(k_2) = 0$ because all released lots (k_3 and k_2) are now completed. Therefore, $s_1 = (D_2 - y_{k_3} - y_{k_2}, 0).$ If lot k_2 is not completed at $t=1,$ then $D_1 = D_2 - y_{k_3}, R_1(k_2) = k_2,$ and $s_1 = (D_2 - y_{k_3}, k_2).$ The other portions in Fig. 1 about the production system status can be likewise derived by following the above procedure.

3.3. Cost function formulation

Fig. 2 shows the general representation of a lot-sizing decision made at $s_t = (D_t, R_t(k_{t+1})).$ Cost function $C_t(s_t, k_t)$ in the intermediate stage, when $t \geq 1,$ can be formulated as follows:

$$C_t(s_t, k_t) = H_1 + p \cdot \sum_{y_{R_t(k_{t+1})}=0}^{R_t(k_{t+1})} \sum_{y_{k_t}=0}^{k_t} p(y_{R_t(k_{t+1})}) \cdot p(y_{k_t}) \cdot (H_2 + H_3) + (1 - p) \cdot \sum_{y_{R_t(k_{t+1})}=0}^{R_t(k_{t+1})} p(y_{R_t(k_{t+1})}) \cdot (H_4 + H_5) \tag{1}$$

where

$$\begin{aligned} H_1 &= \alpha W_t + \beta k_t, \\ H_2 &= h(t - 1)(y_{R_t(k_{t+1})} + y_{k_t}), \\ H_3 &= C_{t-1}^*(s_{t-1} = (D_t - y_{R_t(k_{t+1})} - y_{k_t}, 0)), \\ H_4 &= h(t - 1)y_{R_t(k_{t+1})}, \\ H_5 &= C_{t-1}^*(s_{t-1} = (D_t - y_{R_t(k_{t+1})}, k_t)). \end{aligned}$$

where H_1 is the aggregated production cost for lot $k_t,$ including both setup and variable production costs. At $s_t = (D_t, R_t(k_{t+1}))$ with a lot k_t released, the possible outcomes of s_{t-1} can be represented in two

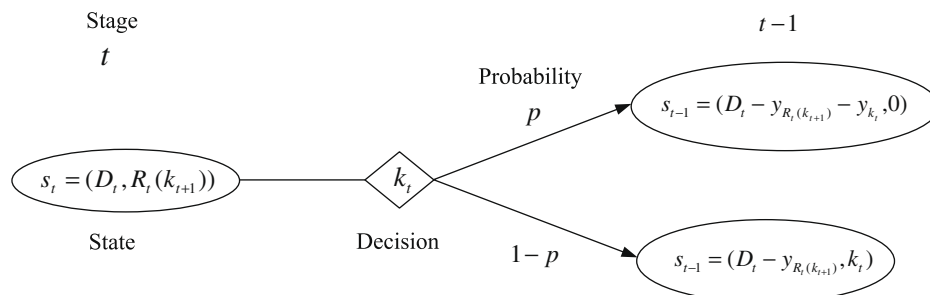


Fig. 2. A decision structure in the DP model.

cases, which lead to the derivation of the second and third terms in (1) (Fig. 2).

Case 1. $s_{t-1} = (D_t - y_{R_t(k_{t+1})} - y_{k_t}, 0)$, with probability p . The number of total output units is $y_{R_t(k_{t+1})} + y_{k_t}$, with a joint probability $p(y_{R_t(k_{t+1})}) \cdot p(y_{k_t})$. These output units are produced at $t-1$ and incur an inventory holding cost of H_2 . The expected inventory holding cost can then be expressed as $\sum_{y_{R_t(k_{t+1})}=0}^{R_t(k_{t+1})} \sum_{y_{k_t}=0}^{k_t} p(y_{R_t(k_{t+1})}) \cdot p(y_{k_t}) \cdot H_2$. The term $\sum_{y_{R_t(k_{t+1})}=0}^{R_t(k_{t+1})} \sum_{y_{k_t}=0}^{k_t} p(y_{R_t(k_{t+1})}) \cdot p(y_{k_t}) \cdot H_3$ is the minimum total expected cost incurred after $t-1$.

Case 2. $s_{t-1} = (D_t - y_{R_t(k_{t+1})}, k_t)$ with probability $1-p$. The number of total output units is $y_{R_t(k_{t+1})}$, with an occurrence probability $p(y_{R_t(k_{t+1})})$. These output units are produced at $t-1$ and incur an inventory holding cost H_4 . The expected inventory holding cost can then be represented as $\sum_{y_{R_t(k_{t+1})}=0}^{R_t(k_{t+1})} p(y_{R_t(k_{t+1})}) \cdot H_4$. The term $\sum_{y_{R_t(k_{t+1})}=0}^{R_t(k_{t+1})} p(y_{R_t(k_{t+1})}) \cdot H_5$ denotes the minimum total expected cost incurred after $t-1$.

3.4. Boundary conditions

As a recursive formula, (1) has two BCs. The first BC is intended to address costs incurred at t while $D_t = 0$; that is, $s_t = (0, R_t(k_{t+1}))$. Since demand now has been fulfilled, no lot needs to be released. Therefore, we can conclude:

$$C_t^*(s_t = (0, R_t(k_{t+1}))) = 0. \quad (2)$$

The second BC addresses the costs incurred at $t=0$ with a status $s_0 = (D_0, R_0(k_1))$. The WIP $R_0(k_1)$ is produced after $t=0$ and cannot be used to fulfill the customer demand. Shortage cost for unfulfilled demand $D_0 > 0$ is mD_0 . At $t=0$, no lot is released and the cost incurred is

$$C_0^*(s_0 = (D_0, R_0(k_1))) = \begin{cases} mD_0 & \text{if } D_0 > 0, \\ 0 & \text{if } D_0 = 0. \end{cases} \quad (3)$$

In summary, (1) is a recursive formula for determining costs prior to the due date, (2) is a BC for cost in the situation in which demand has already been satisfied, and (3) is a BC for cost in the case in which demand has not been satisfied at the due date.

4. Solution space reduction

The recursive formula in (1), as well as its two BCs, define a DP problem, where $N_t(s_t)$ is to be found. To reduce the solution space, Lemma 1 is proposed to define an upper bound for $N_t(s_t)$, with Proposition 1 as a prerequisite to its proof.

Proposition 1. Given $t \geq 1$, $D_t \geq 1$, $R_t(k_{t+1}) \geq D_t$, $s_t = (D_t, R_t(k_{t+1}))$, and $s'_t = (D_t, R_t(k_{t+1}) + 1)$, then $C_t^*(s'_t) > C_t^*(s_t)$.

Proof.

$$\forall k_t, \quad C_t(s'_t, k_t) - C_t(s_t, k_t) = h(t-1)(E[Y_{R_t(k_{t+1})+1}] - E[Y_{R_t(k_{t+1})}]) \\ = h(t-1)(\theta^{R_t(k_{t+1})+1}) > 0.$$

That is, $C_t(s'_t, k_t) > C_t(s_t, k_t)$, $\forall k_t$.

Let $k_t = N_t(s'_t)$. Then $C_t(s'_t, N_t(s'_t)) > C_t(s_t, N_t(s'_t))$.

By definition,

$$C_t^*(s'_t) = \min_{k_t} \{C_t(s'_t, k_t)\} = C_t(s'_t, N_t(s'_t)),$$

$$C_t^*(s_t) = \min_{k_t} \{C_t(s_t, k_t)\} = C_t(s_t, N_t(s_t)).$$

Therefore,

$$C_t^*(s_t) = C_t(s_t, N_t(s_t)) \leq C_t(s_t, N_t(s'_t)) < C_t(s'_t, N_t(s'_t)) = C_t^*(s'_t)$$

. \square

This proposition implies that while the WIP is greater than the remaining demand, changing a state by including one more unit in WIP typically increases cost. It is intuitively rational when considering a case in which the remaining demand is 12 units. With an IG distribution, we infer that all the states with 12 units or more in WIP lead to the same probability for meeting remaining demand. However, for a state with additional quantity in WIP, holding cost of finished goods increases.

Lemma 1. Given $t \geq 1$, $D_t \geq 1$, and $s_t = (D_t, R_t(k_{t+1}))$, then $N_t(s_t) \leq D_t$.

Proof. If $N_t(s_t) = 0$, trivially, one can obtain $N_t(s_t) = 0 < 1 \leq D_t$. If $N_t(s_t) \geq 1$, $\forall k_t > D_t$,

$$C_t(s_t, k_t) - C_t(s_t, D_t) = \beta(k_t - D_t) + ph(t-1)(E[Y_{k_t}] - E[Y_{D_t}]) \\ + (1-p) \cdot \sum_{y_{R_t(k_{t+1})}=0}^{R_t(k_{t+1})} p(y_{R_t(k_{t+1})}) \cdot [C_{t-1}^*(s_{t-1} = (D_t - y_{R_t(k_{t+1})}, k_t)) \\ - C_{t-1}^*(s_{t-1} = (D_t - y_{R_t(k_{t+1})}, D_t))] > \beta(k_t - D_t) + ph(t-1)(E[Y_{k_t}] \\ - E[Y_{D_t}]) > 0 \text{ (by Proposition 1)}$$

That is, $C_t(s_t, k_t) > C_t(s_t, D_t)$ for any $k_t > D_t$. This implies that $N_t(s_t) \leq D_t$. \square

This lemma implies that the optimal lot size should always be less than or equal to remaining demand. It is helpful to reducing the solution space of the dynamic program. This lemma is intuitively rational. Likewise, consider a case in which the remaining demand is 12 units. With an IG distribution, releasing a lot with at least 12 units would lead to the same probability of meeting remaining demand. Thus, at most 12 units should be released in this case.

Lemmas 2 and 3 are intended to quickly compute $N_t(s_t)$ and $C_t^*(s_t)$ for cases of $s_t = (1, R_t(k_{t+1}))$. In Lemmas 2 and 3, thresholds a and b are derived to determine whether to release a lot at $s_t = (1, 0)$ and $s_t = (1, R_t(k_{t+1}) > 0)$, respectively.

Lemma 2. For $s_t = (1, R_t(k_{t+1}))$, where $R_t(k_{t+1}) = 0$,

if $C_{t-1}^*(s_{t-1} = (1, 0)) \leq a$, then $N_t(s_t) = 0$ and

$$C_t^*(s_t) = C_{t-1}^*(s_{t-1} = (1, 0));$$

if $C_{t-1}^*(s_{t-1} = (1, 0)) > a$, then $N_t(s_t) = 1$ and

$$C_t^*(s_t) = \alpha + \beta + ph(t-1)\theta + p(1-\theta)C_{t-1}^*(s_{t-1} = (1, 0)) \\ + (1-p)C_{t-1}^*(s_{t-1} = (1, 1)),$$

where $a = [\alpha + \beta + ph(t-1)\theta + (1-p)C_{t-1}^*(s_{t-1} = (1, 1))]/[1-p(1-\theta)]$.

Proof. If $k_t = 0$, then $C_t(s_t, 0) = C_{t-1}^*(s_{t-1} = (1, 0))$. If $k_t = 1$, then $C_t(s_t, 1) = \alpha + \beta + ph(t-1)\theta + p(1-\theta)C_{t-1}^*(s_{t-1} = (1, 0)) + (1-p)C_{t-1}^*(s_{t-1} = (1, 1))$. Then,

$$C_t(s_t, 0) - C_t(s_t, 1) = C_{t-1}^*(s_{t-1} = (1, 0)) \\ - [\alpha + \beta + ph(t-1)\theta + p(1-\theta)C_{t-1}^*(s_{t-1} = (1, 0)) \\ + (1-p)C_{t-1}^*(s_{t-1} = (1, 1))] \\ = C_{t-1}^*(s_{t-1} = (1, 0)) \cdot [1-p(1-\theta)] \\ - [\alpha + \beta + ph(t-1)\theta + (1-p)C_{t-1}^*(s_{t-1} = (1, 1))].$$

Let $a = [\alpha + \beta + ph(t-1)\theta + (1-p)C_{t-1}^*(s_{t-1} = (1, 1))]/[1-p(1-\theta)]$. Then, $C_t(s_t, 0) - C_t(s_t, 1) = [C_{t-1}^*(s_{t-1} = (1, 0)) - a] \cdot [1-p(1-\theta)]$. Therefore, we conclude that

if $C_{t-1}^*(s_{t-1} = (1, 0)) \leq a$, then $N_t(s_t) = 0$;

if $C_{t-1}^*(s_{t-1} = (1, 0)) > a$, then $N_t(s_t) = 1$.

And $C_t^*(s_t)$ for each case can be accordingly computed. \square

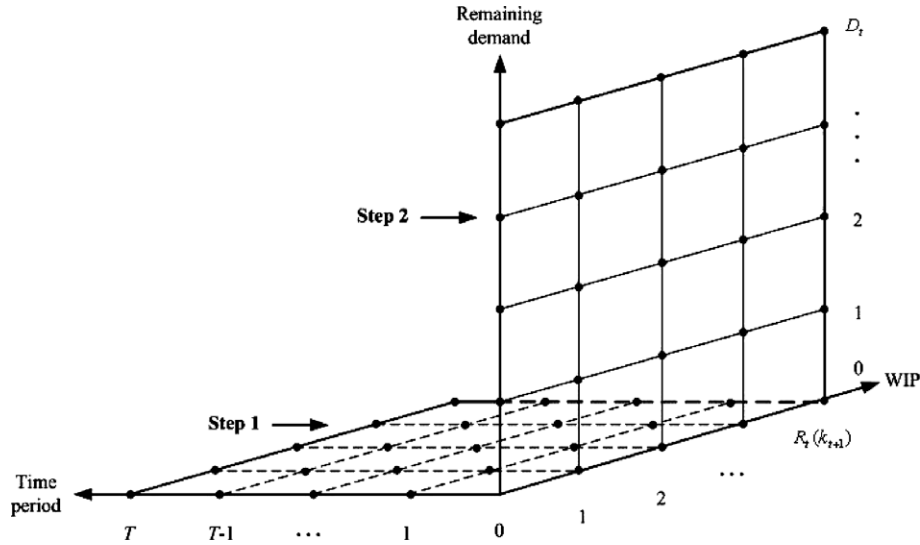


Fig. 3. The set of states considered in Steps 1 and 2.

Lemma 3. For $s_t = (1, R_t(k_{t+1}))$, where $R_t(k_{t+1}) > 0$,
 if $C_{t-1}^*(s_{t-1} = (1, 0)) \leq b$, then $N_t(s_t) = 0$ and
 $C_t^*(s_t) = h(t-1)E[Y_{R_t(k_{t+1})}] + (1-\theta)C_{t-1}^*(s_{t-1} = (1, 0))$,
 if $C_{t-1}^*(s_{t-1} = (1, 0)) > b$, then $N_t(s_t) = 1$ and
 $C_t^*(s_t) = \alpha + \beta + h(t-1)E[Y_{R_t(k_{t+1})}] + ph(t-1)\theta + p(1-\theta)^2 C_{t-1}^*$
 $\times (s_{t-1} = (1, 0)) + (1-p)(1-\theta)C_{t-1}^*(s_{t-1} = (1, 1))$,
 where $b = [\alpha + \beta + ph(t-1)\theta + (1-p)(1-\theta)C_{t-1}^*(s_{t-1} = (1, 1))] / \{(1-\theta)[1-p(1-\theta)]\}$.

Proof. If $k_t = 0$, then $C_t(s_t, 0) = h(t-1)E[Y_{R_t(k_{t+1})}] + (1-\theta)C_{t-1}^*(s_{t-1} = (1, 0))$. If $k_t = 1$, we have

$$C_t(s_t, 1) = \alpha + \beta + h(t-1)E[Y_{R_t(k_{t+1})}] + ph(t-1)\theta + p(1-\theta)^2 C_{t-1}^*(s_{t-1} = (1, 0)) + (1-p)(1-\theta)C_{t-1}^*(s_{t-1} = (1, 1)).$$

Let $b = [\alpha + \beta + ph(t-1)\theta + (1-p)(1-\theta)C_{t-1}^*(s_{t-1} = (1, 1))] / \{(1-\theta)[1-p(1-\theta)]\}$. Then, $C_t(s_t, 0) - C_t(s_t, 1) = [C_{t-1}^*(s_{t-1} = (1, 0)) - b] \cdot \{(1-\theta)[1-p(1-\theta)]\}$. This implies that

if $C_{t-1}^*(s_{t-1} = (1, 0)) \leq b$, then $N_t(s_t) = 0$;
 if $C_{t-1}^*(s_{t-1} = (1, 0)) > b$, then $N_t(s_t) = 1$.
 Thus, $C_t^*(s_t)$ for each case can be accordingly computed. \square

In summary, Proposition 1 is a prerequisite of Lemma 1. Lemma 1 is used to define the upper bound for releasing a lot, which helps reduce the solution space of the DP. Both Lemmas 2 and 3 are used to accelerate decision making when $D_t = 1$.

5. Dynamic programming algorithm

Based on the above lemmas, we propose an algorithm for computing $N_T(s_T)$, beginning with $s_T = (D, 0)$.

Algorithm Computing Optimal Lot Size ($s_T = (D, 0)$)

- Step 1: Based on the first BC, compute $N_t(s_t)$ and $C_t^*(s_t)$ at $s_t = (0, R_t(k_{t+1}))$.
- Step 2: Based on the second BC, compute $N_0(s_0)$ and $C_0^*(s_0)$ at $s_0 = (D_0, R_0(k_1))$.
- Step 3: Based on Lemma 2, compute $N_t(s_t)$ and $C_t^*(s_t)$ at $s_t = (1, R_t(k_{t+1}) = 0)$ for $1 \leq t \leq T$.

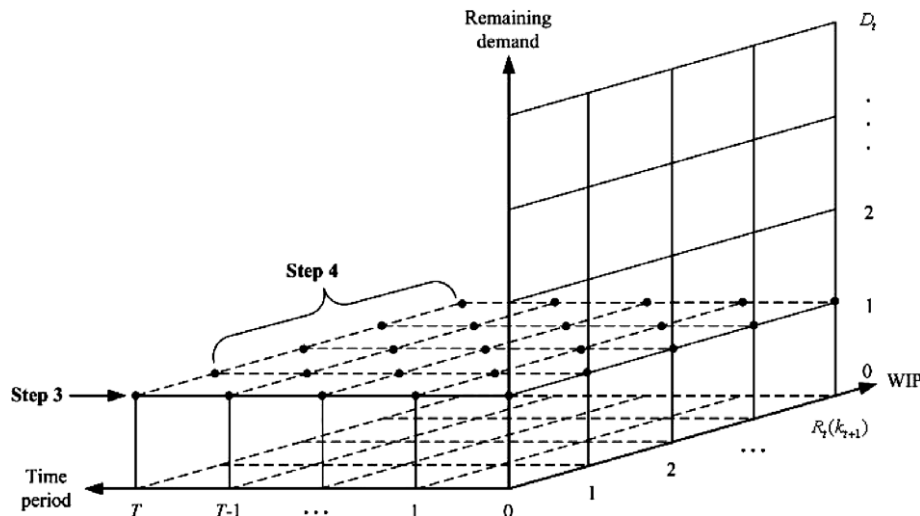


Fig. 4. The set of states considered in Steps 3 and 4.

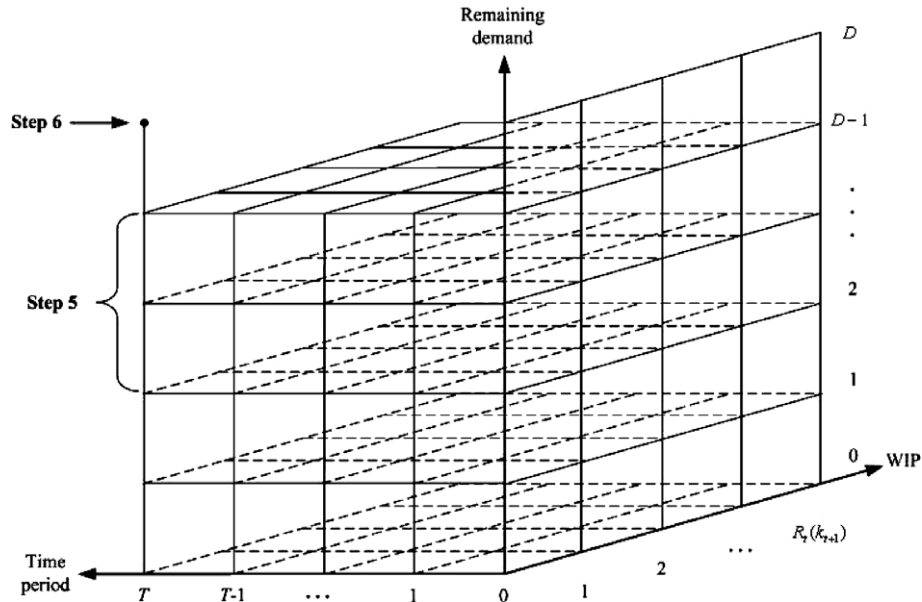


Fig. 5. The set of states considered in Steps 5 and 6.

- Step 4: Based on Lemma 3, compute $N_t(s_t)$ and $C_t^*(s_t)$ at $s_t = (1, R_t(k_{t+1}) > 0)$ for $1 \leq t \leq T$.
- Step 5: Based on (1) and Lemma 1, compute $N_t(s_t)$ and $C_t^*(s_t)$ at $s_t = (2 \leq D_t \leq D - 1, R_t(k_{t+1}) \geq 0)$ for $1 \leq t \leq T$.
- Step 6: Based on (1) and Lemma 1, compute $N_T(s_T)$ and $C_T^*(s_T)$ at $s_T = (D, 0)$.

The DP algorithm is utilized to calculate the cost function of each state in the DP decision tree, whose number of states could be quite huge. The complexity of the DP decision tree can be understood by examining the three component variables of a state s_t , which involves t , D_t , and $R_t(k_{t+1})$. Step 1 determines $N_t(s_t)$ and $C_t^*(s_t)$ for plane $D_t = 0$ based on the first BC, and Step 2 is for the plane $t = 0$ based on the second BC (Fig. 3). With the results obtained from Steps 1 and 2, Step 3 together with Step 4 determine $N_t(s_t)$ and $C_t^*(s_t)$ for plane $D_t = 1$ (Fig. 4) based on Lemmas 2 and 3. The results for the remaining planes are computed by Steps 5 and 6 (Fig. 5).

6. Numerical examples

The properties of the multiple lot-sizing problems are examined using numerical examples. First, this work examines how the value

of decision parameters, $T, D, p, \theta, \alpha, \beta, h$, and m , affect total expected cost. Second, this work examines whether $N_T(s_T)$ against the order size D is monotonically increasing.

6.1. Properties of decision parameters

We could readily justify that the total expected cost decreases as the values of cost parameters, α, β, h , and m decrease, while total expected cost decreases as the value of θ increases. However, the relationships between total expected cost and parameters p, T , and D are not explicit and must be examined by numerical tests.

To examine the relationship between p and total expected cost, 700 cases are used, which are designed by setting $D = 30, m = 200, \alpha = 100$ and $\beta = 1$, and varying the other parameter values as follows: $T = 3, 4, 6$, and 10 ; $p = 0, 0.1, 0.3, 0.5, 0.7, 0.9$, and 1 ; $\theta = 0.5, 0.6, 0.7, 0.8$, and 0.9 ; and $h = 0, 1, 3, 5$, and 10 .

Experimental results imply that total expected cost decreases with p when the value of p is sufficiently large (Fig. 6). This implies that a company that with shorter production lead-time tends to incur less cost. Suppose a motivational mechanism is established by sharing with workers a certain percentage (say, 20%) of the cost-saving amount. The proposed model can then be used to determine the percentage that should be shared with workers for a particular p .

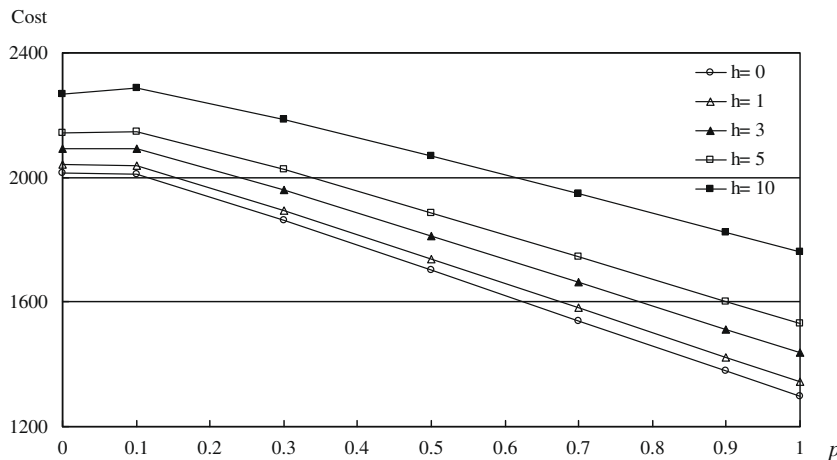


Fig. 6. Total expected cost as a function of p ($T = 4, D = 30, \theta = 0.9, \alpha = 100, \beta = 1$, and $m = 200$).

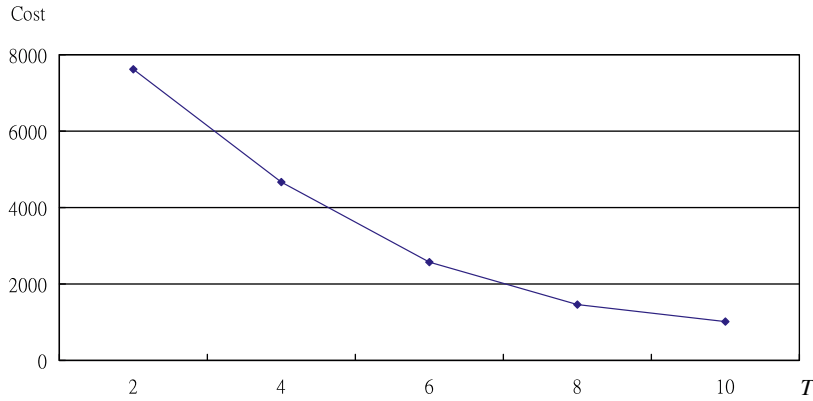


Fig. 7. Total expected cost as a function of T ($T = 2, 4, 6, 8, 10$, $D = 50$, $p = 0.5$, $\theta = 0.9$, $\alpha = 100$, $\beta = 1$, $h = 0.025$, and $m = 200$).

To examine the impacts of T and D on total expected cost, we used 700 cases which are designed by setting $\alpha = 100$, $\beta = 1$ and $m = 200$, and varying the other parameter values as follows: $T = 2, 4, 6, 8$, and 10 ; $D = 10, 20, 30, 40$, and 50 ; $p = 0, 0.1, 0.3, 0.5, 0.7, 0.9$, and 1 ; $h = 0.025, 2.5$; and $\theta = 0.6, 0.9$.

Total expected cost decreases with T (Fig. 7), implying that the production of rush orders (with a small T value) increases cost. To ensure a constant contribution margin, say, 30%, for any lead-time commitment, one must adopt a price discrimination policy. That is, as lead-time commitment decreases the unit price charged increases. Thus, the proposed model is helpful in determining the pricing policy in terms of lead-time commitment.

Total expected cost increases as D increases in a convex manner (Fig. 8). In Fig. 8, a scenario is considered in which unit price for any order size is constant. Based on revenue and cost curves, profit, as a function of order size, does not increase monotonically (Fig. 8). This highlights the need for computing an optimal order size, which is one application of the proposed MLPO model.

6.2. Counter-examples of monotonic property

To justify whether $N_T(s_T)$ is increasing monotonically with respect to D , this work uses two scenarios, $p=0.7$ and $p=1.0$. The other parameters are $T = 6$, $\alpha = 50$, $\beta = 1$, $h = 1$, $m = 200$, and $\theta = 0.95$. The proposed model is used to compute $N_T(s_T)$ for $1 \leq D \leq 100$. Fig. 9 presents computational results, revealing that $N_T(s_T)$ as a function of D is not necessarily monotonic, further supporting the impor-

tance of applying the proposed model to the lot-sizing decision. The size of $N_T(s_T)$ for $p = 0.7$ is in general larger than that for $p = 1.0$ (Fig. 9). Thus, the optimal lot-size for this scenario with variable lead-times was larger than that with a constant lead-time.

7. Conclusions

This study addresses a new single-stage MLPO problem, which is distinguished by its inclusion of one salient feature—production lead time is uncertain with two possible outcomes. That is, production lead time is either one or two periods. Such a problem has appeared in various production processes, such as when the drawing of steel coils; however, it has scarcely been studied in literature. This study formulates the MLPO problem as a dynamic problem and examines its properties via numerical experiments.

Some properties of decision variables (optimal lot-sizes) are summarized as follow. First, the optimal lot-size at any period is less than or equal to remaining demand, as proved in Lemma 1. Second, optimal lot-size as a function of demand is not necessarily monotonic. Third, optimal lot-size with variable lead-time tends to be larger than that with a fixed lead-time.

Properties of decision parameters T and p are also summarized. Total cost appears to decreases with T , implying that the unit selling price can be lowered when customers accept an extended lead-time. While the value of fp is large enough, the higher is p , the lower the total expected cost tends to be. This implies that a production system would be more cost-competitive if its production lead-time

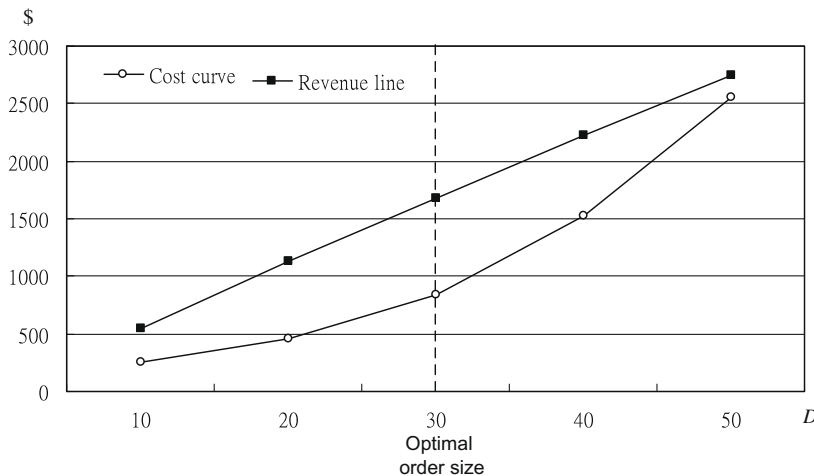


Fig. 8. Total expected cost and revenue as a function of D ($T = 6$, $D = 10, 20, 30, 40, 50$, $p = 0.5$, $\theta = 0.9$, $\alpha = 100$, $\beta = 1$, $h = 0.025$, and $m = 200$).

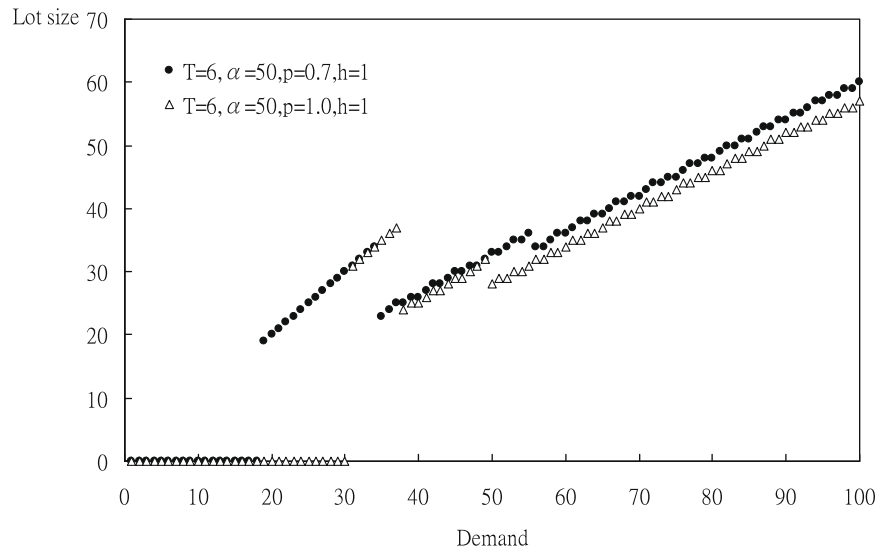


Fig. 9. The optimum lot-sizing decisions as a function of D for two particular scenarios.

could become shorter, in terms of probability. With the proposed DP model, this study determined the lowest total expected cost for any production scenario, and in turn determined the appropriate quoted price.

If salvage costs for after-due products are not negligible, some lemmas in this work may not be valid. Therefore, one possible extension is to develop an MLPO model that includes substantial salvage costs. Another extension is to investigate the MLPO problem with more than two possible outcomes in production lead-time. The proposed approach appears to be applicable to such an extension; however, formulating and solving a relatively much more complex DP problem is challenging. Additionally, some other extensions include investigating different probability distributions for modeling process yield and a scenario of a multiple-stage production system.

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