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Mutually independent hamiltonian cycles for the pancake graphs and the star graphs

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ABSTRACT

A *hamiltonian cycle* C of a graph G is an ordered set $\langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ of vertices such that $u_i \neq u_j$ for $i \neq j$ and u_i is adjacent to u_{i+1} for every $i \in \{1, 2, \ldots, n(G) - 1\}$ and $u_{n(G)}$ is adjacent to u_1, where $n(G)$ is the order of G. The vertex u_1 is the starting vertex and u_i is the ith vertex of C. Two hamiltonian cycles $C_1 = \langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_{n(G)}, v_1\rangle$ of G are *independent* if $u_1 = v_1$ and $u_i \neq v_i$ for every $i \in \{2, 3, \ldots, n(G)\}$. A set of hamiltonian cycles $\{C_1, C_2, \ldots, C_k\}$ of G is *mutually independent* if its elements are pairwise independent. The *mutually independent hamiltonicity IHC*(G) of a graph G is the maximum integer k such that for any vertex u of G there exist k mutually independent hamiltonian cycles of G starting at u.

In this paper, the mutually independent hamiltonicity is considered for two families of Cayley graphs, the n-dimensional pancake graphs P_n and the n-dimensional star graphs S_n. It is proven that $IHC(P_3) = 1$, $IHC(P_n) = n - 1$ if $n \geq 4$, $IHC(S_n) = n - 2$ if $n \in \{3, 4\}$ and $IHC(S_n) = n - 1$ if $n \geq 5$.

1. Introduction

In 1969, Lovász [32] asked whether every finite connected vertex transitive graph has a hamiltonian path, that is, a simple path that traverses every vertex exactly once. All known vertex transitive graphs have a hamiltonian path and moreover, only four vertex transitive graphs without a hamiltonian cycle are known. Since none of these four graph is a Cayley graph there is a folklore conjecture [9] that every Cayley graph with more than two vertices has a hamiltonian cycle. In the last decades this problem was extensively studied (see [2–5,7,12,19,33–36]) and for those Cayley graphs for which the existence of hamiltonian cycles is already proven, further properties related to this problem, such as edge-hamiltonicity, Hamilton-connectivity and Hamilton-laceability, are investigated (see [4,8]). In this paper we introduce one of such properties, the concept of mutually independent hamiltonian cycles which is related to the number of hamiltonian cycles in a given graph. In particular, mutually independent hamiltonian cycles of pancake graphs P_n and star graphs S_n (for definitions see Sections 4 and 5) are studied.

The paper is organized as follows. In Section 2 definitions and notations needed in the subsequent sections are introduced. In Section 3 applications of the mutually independent hamiltonicity concept are given. In Sections 4 and 5 the mutually independent hamiltonicity of pancake graphs P_n and star graphs S_n, respectively, is computed. And in the last section, Section 6, directions for further research on this topic are discussed.

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2. Definitions

For definitions and notations not defined here see [6]. Let V be a finite set and E a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. Then $G = (V, E)$ is a *graph* with *vertex set* V and *edge set* E. The *order* of G, that is, the cardinality of the set V, is denoted by $n(G)$. For a subset S of V the graph $G[S]$ induced by S is a graph with vertex set $V(G[S]) = S$ and edge set $E(G[S]) = \{(x, y) \mid (x, y) \in E(G)$ and $x, y \in S\}$. Two vertices u and v are *adjacent* if (u, v) is an edge of G. For a vertex u the set $N_G(u) = \{v \mid (u, v) \in E\}$ is called the *set of neighbors* of u. The *degree* $\deg_G(u)$ of a vertex u in G, is the cardinality of the set $N_G(u)$. The *minimum degree* of G, $\delta(G)$, is $\min\{\deg_G(x) \mid x \in V\}$. A graph G is *k-regular* if $\deg_G(u) = k$ for every vertex u in G. The *connectivity* of G is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. A *path* between vertices v_0 and v_k is a sequence of vertices represented by $\langle v_0, v_1, \ldots, v_k\rangle$ such that there is no repeated vertex and (v_i, v_{i+1}) is an edge of G for every $i \in \{0 \ldots k-1\}$. We use $Q(i)$ to denote the ith vertex v_i of $Q = \langle v_1, v_2, \ldots, v_k\rangle$. We also write the path $\langle v_0, v_1, \ldots, v_k\rangle$ as $\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k\rangle$, where Q is a path form v_i to v_j. A path is a *hamiltonian path* if it contains all vertices of G. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two distinct vertices of G. A *cycle* is a sequence of vertices represented by $\langle v_0, v_1, \ldots, v_k, v_0\rangle$ such that $v_i \neq v_j$ for all $i \neq j$, (v_0, v_k) is an edge of G, and (v_i, v_{i+1}) is an edge of G for every $i \in \{0, \ldots, k-1\}$. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G. A graph is *hamiltonian* if it has a hamiltonian cycle.

A hamiltonian cycle C of graph G is described as $\langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ to emphasize the order of vertices in C. Thus, u_1 is the starting vertex and u_i is the ith vertex in C. Two hamiltonian cycles $C_1 = \langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_{n(G)}, v_1\rangle$ of G are *independent* if $u_1 = v_1$ and $u_i \neq v_i$ for every $i \in \{2, \ldots, n(G)\}$. A set of hamiltonian cycles $\{C_1, C_2, \ldots, C_k\}$ of G are *mutually independent* if its elements are pairwise independent. The *mutually independent hamiltonicity* $IHC(G)$ of graph G the maximum integer k such that for any vertex u of G there exist k mutually independent hamiltonian cycles of G starting at u. Obviously, $IHC(G) \leq \delta(G)$ if G is a hamiltonian graph.

The mutually independent hamiltonicity of a graph can be interpreted as a Latin rectangle. A *Latin square* of order n is an $n \times n$ array made from the integers 1 to n with the property that any integer occurs once in each row and column. If we delete some rows from a Latin square, we will get a Latin rectangle. Let K_5 be the complete graph with vertex set $\{0, 1, 2, 3, 4\}$ and let $C_1 = \langle 0, 1, 2, 3, 4, 0\rangle$, $C_2 = \langle 0, 2, 3, 4, 1, 0\rangle$, $C_3 = \langle 0, 3, 4, 1, 2, 0\rangle$, and $C_4 = \langle 0, 4, 1, 2, 3, 0\rangle$. Obviously, C_1, C_2, C_3, and C_4 are mutually independent. Thus, $IHC(K_5) = 4$. We rewrite C_1, C_2, C_3, and C_4 into the following Latin square:

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

In general, a Latin square of order n can be viewed as n mutually independent hamiltonian cycles with respect to the complete graph K_{n+1}.

Let H be a group and let S be the generating set of H such that $S^{-1} = S$. Then the *Cayley graph Cayley*$(S; H)$ of the group H with respect to the generating set S is the graph with vertex set H and two vertex u and v are adjacent in *Cayley*$(S; H)$ if and only if $u^{-1}v \in S$. Hamiltonian cycles in Cayley graphs naturally arise in computer science [25], in the study of word-hyperbolic groups and automatic groups [14], in changing-ringing [40], in creating Escher-like repeating patterns in hyperbolic plane [13], and in combinatorial designs [11].

3. Applications of the concept of mutually independent hamiltonian cycles

Mutually independent hamiltonicity of graphs can be applied to many areas. Consider the following scenario. In Christmas, we have a holiday of 10-days. A tour agency will organize a 10-day tour to Italy. Suppose that there will be a lot of people joining this tour. However, the maximum number of people stay in each local area is limited, say 100 people, for the sake of hotel contract. One trivial solution is on the First-Come-First-Serve basis. So only 100 people can attend this tour. (Note that we cannot schedule the tour in a pipelined manner because the holiday period is fixed.) Nonetheless, we observe that a tour is like a hamiltonian cycle based on a graph, in which a vertex is denoted as a hotel and any two vertices are joined with an edge if the associated two hotels can be traveled in a reasonable time. Therefore, we can organize several subgroups, that is, each subgroup has its own tour. In this way, we do not allow two subgroups stay in the same area during the same time period. In other words, any two different tours are indeed independent hamiltonian cycles. Suppose that there are 10 mutually independent hamiltonian cycles. Then we may allow 1000 people to visit Italy on Christmas vacation. For this reason, we would like to find the maximum number of mutually independent hamiltonian cycles. Such applications are useful for task scheduling and resource placement, which are also important for compiler optimization to exploit parallelism.

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph in which the vertices correspond to processors and the edges correspond to connections. Hence, we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer networks. The n-cube is one of the most popular topologies [27]. The n-dimensional star network S_n was proposed in [1] as n attractive alternative to the n-cube topology for interconnecting processors in parallel computers. Since its introduction, the network

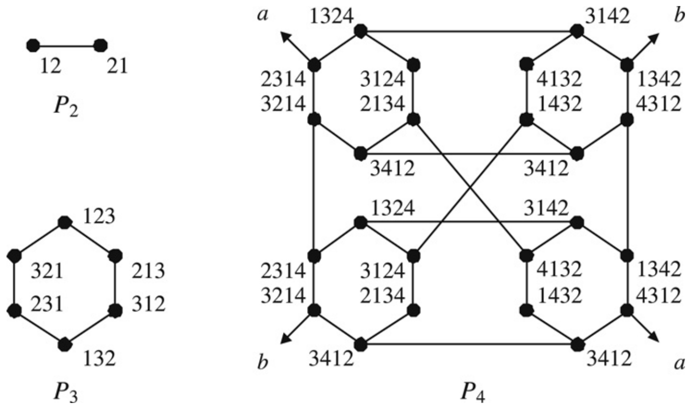

Fig. 1. The pancake graphs P_2, P_3, and P_4.

has received considerable attention. Akers and Krishnameurthy [1] showed that the star graphs are vertex transitive and edge transitive. The diameter and fault diameters were computed in [1,26,37]. The hamiltonian and hamiltonian laceability of star graphs are studied in [16,17,21,23,31]. The spanning container of star graph is studied in [28].

Akers and Krishnameurthy [1] proposed another family of interesting interconnection networks, the n-dimensional pancake graph P_n. Hung et al. [22] studied the hamiltonian connectivity on the faulty pancake graphs. The embedding of cycles and trees into the pancake graphs were discussed in [10,15,22,24]. The spanning container of pancake graph is studied in [28]. Gates and Papadimitriou [18] studied the diameter of the pancake graphs. Up to now, we do not know the exact value of the diameter of the pancake graphs [20].

4. The pancake graphs

Let n be a positive integer. We use $\langle n\rangle$ to denote the set $\{1, 2, \dots, n\}$. The *n-dimensional pancake graph*, P_n, is a graph with the vertex set $V(P_n) = \{u_1u_2\dots u_n \mid u_i \in \langle n\rangle \text{ and } u_j \neq u_k \text{ for } j \neq k\}$. The adjacency is defined as follows: $u_1u_2\dots u_i\dots u_n$ is adjacent to $v_1v_2\dots v_i\dots v_n$ through an edge of dimension i with $2 \le i \le n$ if $v_j = u_{i-j+1}$ for all $1 \le j \le i$ and $v_j = u_j$ for all $i < j \le n$. We will use boldface to denote a vertex of P_n. Hence, $\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n}$ denote a sequence of vertices in P_n. In particular, $\mathbf{e}$ denotes the vertex $12\dots n$. The pancake graphs P_2, P_3, and P_4 are illustrated in Fig. 1.

By definition, P_n is an $(n-1)$-regular graph with $n!$ vertices. Akers and Krishnameurthy [1] showed that the connectivity of P_n is $(n-1)$. Let $\mathbf{u} = u_1u_2\dots u_n$ be an arbitrary vertex of P_n. We use $(\mathbf{u})_i$ to denote the ith component u_i of $\mathbf{u}$, and use $P_n^{\{i\}}$ to denote the ith subgraph of P_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Then P_n can be decomposed into n vertex disjoint subgraphs $P_n^{\{i\}}$, $1 \le i \le n$, and each $P_n^{\{i\}}$ is isomorphic to P_{n-1} for all i, $i \le n$. Thus, the pancake graph can be constructed recursively. Let H be any subset of $\langle n\rangle$. We use P_n^H to denote the subgraph of P_n induced by $\cup_{i\in H} V(P_n^{\{i\}})$. By definition, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-dimensional edge with $2 \le i \le n$. We use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$. We have $((\mathbf{u})^i)^i = \mathbf{u}$ and $(\mathbf{u})^n \in P_n^{\{(\mathbf{u})_1\}}$. For any two distinct elements i and j in $\langle n\rangle$, we use $E_n^{i,j}$ to denote the set of edges between $P_n^{\{i\}}$ and $P_n^{\{j\}}$.

Lemma 1. *Let i and j be any two distinct elements in $\langle n\rangle$ with $n \ge 3$. Then $|E_n^{i,j}| = (n-2)!$.*

Lemma 2. *Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct vertices of P_n with $d(\mathbf{u}, \mathbf{v}) \le 2$. Then $(\mathbf{u})_1 \neq (\mathbf{v})_1$.*

Theorem 1 ([22]). *Suppose that F is a subset of $V(P_n)$ with $|F| \le n-4$. Then $P_n - F$ is hamiltonian connected.*

Theorem 2. *Let $\{a_1, a_2, \dots, a_r\}$ be a subset of $\langle n\rangle$ for some positive integer $r \in \langle n\rangle$ with $n \ge 5$. Assume that $\mathbf{u}$ and $\mathbf{v}$ are two distinct vertices of P_n with $\mathbf{u} \in P_n^{\{a_1\}}$ and $\mathbf{v} \in P_n^{\{a_r\}}$. Then there is a hamiltonian path $\langle \mathbf{u} = \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \dots, \mathbf{x_r}, H_r, \mathbf{y_r} = \mathbf{v}\rangle$ of $\cup_{i=1}^r P_n^{\{a_i\}}$ joining $\mathbf{u}$ to $\mathbf{v}$ such that $\mathbf{x_1} = \mathbf{u}$, $\mathbf{y_r} = \mathbf{v}$, and H_i is a hamiltonian path of $P_n^{\{a_i\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for every i, $1 \le i \le r$.*

Proof. We set $\mathbf{x_1}$ as $\mathbf{u}$ and $\mathbf{y_r}$ as $\mathbf{v}$. We know that $P_n^{\{a_i\}}$ is isomorphic to P_{n-1} for every $i \in \langle r\rangle$. By Theorem 1, this statement holds for $r = 1$. Thus, we assume that $r \ge 2$. By Lemma 1, $|E_n^{a_i,a_{i+1}}| = (n-2)! \ge 6$ for every $i \in \langle r-1\rangle$. We choose $(\mathbf{y_i}, \mathbf{x_{i+1}}) \in E_n^{a_i,a_{i+1}}$ for every $i \in \langle r-1\rangle$ with $\mathbf{y_1} \neq \mathbf{x_1}$ and $\mathbf{x_r} \neq \mathbf{y_r}$. By Theorem 1, there is a hamiltonian path H_i of $P_n^{\{a_i\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for every $i \in \langle r\rangle$. Then $\langle \mathbf{u} = \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \dots, \mathbf{x_r}, H_r, \mathbf{y_r} = \mathbf{v}\rangle$ is the desired path. See Fig. 2 for illustration on P_n. □

Lemma 3. *Let $k \in \langle n\rangle$ with $n \ge 4$, and let $\mathbf{x}$ be a vertex of P_n. There is a hamiltonian path P of $P_n - \{\mathbf{x}\}$ joining the vertex $(\mathbf{x})^n$ to some vertex $\mathbf{v}$ with $(\mathbf{v})_1 = k$.*

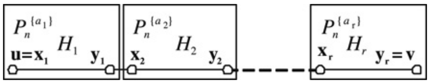

Fig. 2. Illustration for Theorem 2 on P_n.

Proof. Suppose that $n = 4$. Since P_4 is vertex transitive, we may assume that $\mathbf{x} = 1234$. The required paths of $P_4 - \{1234\}$ are listed below:

$k = 1$	⟨4321, 3421, 2431, 4231, 1324, 3124, 2134, 4312, 1342, 2143, 4132, 2314, 3214, 4123, 2143, 3412, 1432, 2341, 3241, 1423, 2413, 4213, 1243⟩
$k = 2$	⟨4321, 3421, 2431, 4231, 1324, 3124, 2134, 4312, 1342, 3142, 2413, 4213, 1243, 2143, 3412, 1432, 4132, 2314, 3214, 4123, 1423, 3241, 2341⟩
$k = 3$	⟨4321, 3421, 2431, 4231, 1324, 3124, 2134, 4312, 1342, 3142, 4132, 2314, 3214, 4123, 1423, 2413, 4213, 1342, 2143, 3412, 1432, 2341, 3241⟩
$k = 4$	⟨4321, 3421, 2431, 1342, 3142, 4132, 2314, 3214, 4123, 2143, 1243, 4213, 2413, 1423, 3241, 2341, 1432, 3412, 4312, 2134, 3124, 1324, 4231⟩

With Theorem 1, we can find the required hamiltonian path in P_n for every n, $n \geq 5$. □

Lemma 4. *Let a and b be any two distinct elements in $\langle n \rangle$ with $n \geq 4$, and let $\mathbf{x}$ be a vertex of P_n. There is a hamiltonian path P of $P_n - \{\mathbf{x}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$.*

Proof. Suppose that $n = 4$. Since P_4 is vertex transitive, we may assume that $\mathbf{x} = 1234$. Without loss of generality, we may assume that $a < b$. The required paths of $P_4 - \{1234\}$ are listed below:

$a = 1$ and $b = 2$
⟨1423, 4123, 3214, 2314, 1324, 3124, 4213, 2413, 3142, 4132, 1432, 3412, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 4312, 2134⟩
$a = 1$ and $b = 3$
⟨1423, 4123, 2143, 1243, 4213, 2413, 3142, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 3124, 2134, 4312, 3412, 1432, 4132, 2314, 3214⟩
$a = 1$ and $b = 4$
⟨1423, 2413, 3142, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 2314, 3214, 4123, 2143, 1243, 4213, 3124, 2134, 4312, 3412, 1432, 4132⟩
$a = 2$ and $b = 3$
⟨2134, 4312, 1342, 3142, 2413, 4213, 1243, 2143, 3412, 1432, 4132, 2314, 3214, 4123, 1423, 3241, 2341, 4321, 3421, 2431, 4231, 1324, 3124⟩
$a = 2$ and $b = 4$
⟨2134, 3124, 1324, 2314, 3214, 4123, 2143, 1243, 4213, 2413, 1423, 3241, 4231, 2431, 3421, 4321, 2341, 1432, 3412, 4312, 1342, 3142, 4132⟩
$a = 3$ and $b = 4$
⟨3214, 4123, 2143, 1243, 4213, 3124, 2134, 4312, 3412, 1432, 2341, 4321, 3421, 2431, 1342, 3142, 2413, 1423, 3241, 4231, 1324, 2314, 4132⟩

With Theorem 1, we can find the required hamiltonian path on P_n for every n, $n \geq 5$. □

Lemma 5. *Let a and b be any two distinct elements in $\langle n \rangle$ with $n \geq 4$. Assume that $\mathbf{x}$ and $\mathbf{y}$ are two adjacent vertices of P_n. There is a hamiltonian path P of $P_n - \{\mathbf{x}, \mathbf{y}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$.*

Proof. Since P_n is vertex transitive, we may assume that $\mathbf{x} = \mathbf{e}$ and $\mathbf{y} = (\mathbf{e})^i$ for some $i \in \{2, 3, \ldots, n\}$. Without loss of generality, we assume that $a < b$. Thus, $a \neq n$ and $b \neq 1$. We prove this statement by induction on n. For $n = 4$, the required paths of $P_4 - \{1234, (1234)^i\}$ are listed below:

$\mathbf{y} = 2134$
$a = 1$ and $b = 2$
⟨1432, 2413, 3142, 4132, 1432, 3412, 4312, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 3124, 4213, 1243, 2143, 4123, 3214, 2314⟩
$a = 1$ and $b = 3$
⟨1432, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 4312, 3412, 1432, 4132, 3142, 2413, 4213, 3124, 1324, 2314, 3214⟩
$a = 1$ and $b = 4$
⟨1432, 4123, 3214, 2314, 1324, 3124, 4213, 2413, 3142, 4132, 1432, 2341, 3241, 4231, 2431, 1342, 4312, 3412, 2143, 1243, 3421, 4321⟩
$a = 2$ and $b = 3$
⟨2314, 3214, 4123, 2143, 1243, 4213, 3124, 1324, 4231, 2431, 1342, 4312, 3412, 1432, 4132, 3142, 2413, 1423, 3241, 2341, 4321, 3421⟩
$a = 2$ and $b = 4$
⟨2314, 3214, 4123, 2143, 3412, 4312, 1342, 2431, 3421, 1243, 4213, 3124, 1324, 4231, 3241, 1423, 2413, 3142, 4132, 1432, 2341, 4321⟩
$a = 3$ and $b = 4$
⟨3214, 4123, 2143, 1243, 3421, 2431, 4231, 3241, 1423, 2413, 4213, 3124, 1324, 2314, 4132, 3142, 1342, 4312, 3412, 1432, 2341, 4321⟩

$\mathbf{y} = 3214$
$a = 1$ and $b = 2$
⟨1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 3142, 2413, 4213, 3124, 1324, 2314, 4132, 1432, 3412, 4312, 2134⟩
$a = 1$ and $b = 3$
⟨1423, 4123, 2143, 1243, 4213, 2413, 3142, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 2314, 4132, 1432, 3412, 4312, 2134, 3124⟩
$a = 1$ and $b = 4$
⟨1423, 4123, 2143, 1243, 3421, 2431, 1342, 3142, 2413, 4213, 3124, 2134, 4312, 3412, 1432, 4132, 2314, 1324, 4231, 3241, 2341, 4321⟩
$a = 2$ and $b = 3$
⟨2134, 4312, 1342, 2431, 4231, 3241, 1423, 4123, 2143, 3412, 1432, 2341, 4321, 3421, 1243, 4213, 2413, 3142, 4132, 2314, 1324, 3124⟩
$a = 2$ and $b = 4$
⟨2134, 3124, 4213, 2413, 3142, 1342, 4312, 3412, 1432, 4132, 2314, 1324, 4231, 2431, 3421, 1243, 2143, 4123, 1423, 3241, 2341, 4321⟩
$a = 3$ and $b = 4$
⟨3124, 2134, 4312, 1342, 3142, 2413, 4213, 1243, 3421, 2431, 4231, 1324, 2314, 4132, 1432, 3412, 2143, 4123, 1423, 3241, 2341, 4321⟩

$\mathbf{y} = 4321$
$a = 1$ and $b = 2$
⟨1423, 4123, 3214, 2314, 4132, 3142, 2413, 4213, 3124, 1324, 4231, 3241, 2341, 1432, 3412, 2143, 1243, 3421, 2431, 1342, 4312, 2134⟩
$a = 1$ and $b = 3$
⟨1423, 4123, 2143, 3412, 1432, 2341, 3241, 4231, 1324, 3124, 2134, 4312, 1342, 2431, 2431, 1243, 4213, 2413, 3142, 4132, 2314, 3214⟩
$a = 1$ and $b = 4$
⟨1423, 2413, 4213, 3124, 2134, 4312, 3412, 2143, 1243, 3421, 2431, 1342, 3142, 4132, 1432, 2341, 3241, 4231, 1324, 2314, 3214, 4123⟩
$a = 2$ and $b = 3$
⟨2134, 4312, 1342, 3142, 4132, 2314, 3214, 4123, 2143, 3412, 1432, 2341, 3241, 1423, 2413, 4213, 1243, 3421, 2431, 4231, 1324, 3124⟩
$a = 2$ and $b = 4$
⟨2134, 3124, 4213, 2413, 1423, 3241, 2341, 1432, 4132, 3142, 1342, 4312, 3412, 2143, 1243, 3421, 2431, 4231, 1324, 2314, 3214, 4123⟩
$a = 3$ and $b = 4$
⟨3214, 2314, 1324, 4231, 3241, 2341, 1432, 4132, 3142, 1342, 2431, 3421, 1243, 2143, 3412, 4312, 2134, 3124, 4213, 2413, 1423, 4123⟩

Suppose that this statement holds for P_k for every k, $4 \leq k < n$. We have the following cases:

Case 1. $\mathbf{y} = (\mathbf{e})^i$ for some $i \neq 1$ and $i \neq n$, that is, $\mathbf{y} \in P_n^{\{n\}}$. Let c be an element in $\langle n-1\rangle - \{a\}$. By induction, there is a hamiltonian path R of $P_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^i\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = c$. We choose a vertex $\mathbf{v}$ in $P_n^{\langle n-1\rangle - \{c\}}$ with $(\mathbf{v})_1 = b$. By Theorem 2, there is a hamiltonian path H of $P_n^{\langle n-1\rangle}$ joining the vertex $(\mathbf{z})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, R, \mathbf{z}, (\mathbf{z})^n, H, \mathbf{v}\rangle$ is the desired path.

Case 2. $\mathbf{y} = (\mathbf{e})^n$, that is, $\mathbf{y} \in P_n^{\{1\}}$. Let c be an element in $\langle n-1\rangle - \{1, a\}$, and let d be an element in $\langle n-1\rangle - \{1, b, c\}$. By Lemma 4, there is a hamiltonian path R of $P_n^{\{n\}} - \{\mathbf{e}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{w}$ with $(\mathbf{w})_1 = c$. Again, there is a hamiltonian path H of $P_n^{\{1\}} - \{(\mathbf{e})^n\}$ joining a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = d$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$. By Theorem 2, there is a hamiltonian path Q of $P_n^{\langle n-1\rangle - \{1\}}$ joining the vertex $(\mathbf{w})^n$ to the vertex $(\mathbf{z})^n$. Then $\langle \mathbf{u}, R, \mathbf{w}, (\mathbf{w})^n, Q, (\mathbf{z})^n, \mathbf{z}, H, \mathbf{v}\rangle$ is the desired path. □

Lemma 6. *Let a and b be any two distinct elements in $\langle n\rangle$ with $n \geq 4$. Let $\mathbf{x}$ be any vertex of P_n. Assume that $\mathbf{x_1}$ and $\mathbf{x_2}$ are two distinct neighbors of $\mathbf{x}$. There is a hamiltonian path P of $P_n - \{\mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$.*

Proof. Since P_n is vertex transitive, we may assume that $\mathbf{x} = \mathbf{e}$. Moreover, we assume that $\mathbf{x_1} = (\mathbf{e})^i$ and $\mathbf{x_2} = (\mathbf{e})^j$ for some $\{i, j\} \subset \langle n\rangle - \{1\}$ with $i < j$. Without loss of generality, we assume that $a < b$. Thus, $a \neq n$ and $b \neq 1$. We prove this lemma by induction on n. For $n = 4$, the required paths of $P_4 - \{1234, (1234)^i, (1234)^j\}$ are listed below:

$\mathbf{x_1} = 2134$ and $\mathbf{x_2} = 3214$
$a = 1$ and $b = 2$
⟨1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 4312, 3412, 1432, 4132, 3142, 2413, 4213, 3124, 1324, 2314⟩
$a = 1$ and $b = 3$
⟨1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 4312, 3412, 1432, 4132, 2314, 1324, 3124, 4213, 2413, 3142⟩
$a = 1$ and $b = 4$
⟨1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 3142, 2413, 4213, 3124, 1324, 2314, 4132, 1432, 3412, 4312⟩
$a = 2$ and $b = 3$
⟨2143, 4123, 1423, 3241, 4231, 2431, 1342, 4312, 3412, 1432, 2341, 4321, 3421, 1243, 4213, 2413, 3142, 4132, 2314, 1324, 3124⟩
$a = 2$ and $b = 4$
⟨2143, 4123, 1423, 2413, 3142, 1342, 4312, 3412, 1432, 4132, 2314, 1324, 3124, 4213, 1243, 3421, 2431, 4231, 3241, 2341, 4321⟩
$a = 3$ and $b = 4$
⟨3124, 4213, 2413, 3142, 1342, 4312, 3412, 1432, 4132, 2314, 1324, 4231, 2431, 3421, 1243, 2143, 4123, 1423, 3241, 2341, 4321⟩

$\mathbf{x_1} = 2134$ and $\mathbf{x_2} = 4321$
$a = 1$ and $b = 2$
⟨1423, 2413, 3142, 4132, 1432, 2341, 3241, 4231, 1324, 3124, 4213, 1243, 3421, 2431, 1342, 4312, 3412, 2143, 4123, 3214, 2314⟩
$a = 1$ and $b = 3$
⟨1423, 4123, 2143, 1243, 3421, 2431, 1342, 4312, 3412, 1432, 2341, 3241, 4231, 1342, 3124, 4213, 2413, 3142, 4132, 2314, 3214⟩
$a = 1$ and $b = 4$
⟨1423, 4123, 3214, 2314, 1324, 3124, 4213, 2413, 3142, 1342, 4312, 3412, 2143, 1243, 3421, 2431, 4231, 3241, 2341, 1432, 4132⟩
$a = 2$ and $b = 3$
⟨2314, 3214, 4123, 2143, 3412, 4312, 1342, 3142, 4132, 1432, 2341, 3241, 1423, 2413, 4213, 1243, 3421, 2431, 4231, 1324, 3124⟩
$a = 2$ and $b = 4$
⟨2314, 3214, 4123, 2143, 1243, 3421, 2431, 4231, 1324, 3124, 4213, 2413, 1423, 3241, 2341, 1432, 3412, 4312, 1342, 3142, 4132⟩
$a = 3$ and $b = 4$
⟨3214, 2314, 4132, 3142, 1342, 4312, 3412, 1432, 2341, 3241, 1423, 2413, 4213, 3124, 1324, 4231, 2431, 3421, 1243, 2143, 4123⟩

$\mathbf{x_1} = 3214$ and $\mathbf{x_2} = 4321$
$a = 1$ and $b = 2$ ⟨1423, 4123, 2143, 1243, 3421, 2431, 1342, 4312, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 2413, 4213, 3124, 2134⟩
$a = 1$ and $b = 3$ ⟨1423, 4123, 2143, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 2413, 4213, 1243, 3421, 2431, 1342, 4312, 2134, 3124⟩
$a = 1$ and $b = 4$ ⟨1423, 2413, 4213, 3124, 2134, 4312, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 1342, 2431, 3421, 1243, 2143, 4123⟩
$a = 2$ and $b = 3$ ⟨2134, 4312, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 1341, 2431, 3421, 1243, 2143, 4123, 1423, 2413, 4213, 3124⟩
$a = 2$ and $b = 4$ ⟨2134, 3124, 4213, 2413, 3142, 1342, 4312, 3412, 1432, 2341, 3241, 1423, 4123, 2143, 1243, 3421, 2431, 4231, 1324, 2314, 4132⟩
$a = 3$ and $b = 4$ ⟨3124, 2134, 4312, 3412, 1432, 2341, 3241, 4231, 1324, 2314, 4132, 3142, 1342, 2431, 3421, 1243, 2143, 4123, 1423, 2413, 4213⟩

Suppose that this statement holds for P_k for every k, $4 \le k < n$. We have the following cases:

Case 1. $j \neq n$, that is, $\mathbf{x_1} \in P_n^{\{n\}}$ and $\mathbf{x_2} \in P_n^{\{n\}}$. Let $c \in \langle n-1\rangle - \{1, a\}$. By induction, there is a hamiltonian path R of $P_n^{\{n\}} - \{\mathbf{e}, \mathbf{x_1}, \mathbf{x_2}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = c$. We choose a vertex $\mathbf{v}$ in $P_n^{\{1\}}$ with $(\mathbf{v})_1 = b$. By Theorem 2, there is a hamiltonian path H of $P_n^{\langle n-1\rangle}$ joining the vertex $(\mathbf{z})^n$ to $\mathbf{v}$. We set $P = \langle \mathbf{u}, R, \mathbf{z}, (\mathbf{z})^n, H, \mathbf{v}\rangle$. Then P is the desired path.

Case 2. $j = n$, that is, $\mathbf{x_1} \in P_n^{\{n\}}$ and $\mathbf{x_2} \in P_n^{\{1\}}$. Let $c \in \langle n-1\rangle - \{1, a\}$ and $d \in \langle n-1\rangle - \{1, b, c\}$. By Lemma 5, there is a hamiltonian path R of $P_n^{\{n\}} - \{\mathbf{e}, \mathbf{x_1}\}$ joining a vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = c$. By Lemma 4, there is a hamiltonian path H of $P_n^{\{1\}} - \{\mathbf{x_2}\}$ joining a vertex $\mathbf{w}$ with $(\mathbf{w})_1 = d$ to a vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$. By Theorem 2, there is a hamiltonian Q of $P_n^{\langle n-1\rangle - \{1\}}$ joining the vertex $(\mathbf{z})^n$ to the vertex $(\mathbf{w})^n$. We set $P = \langle \mathbf{u}, R, \mathbf{z}, (\mathbf{z})^n, Q, (\mathbf{w})^n, \mathbf{w}, H, \mathbf{v}\rangle$. Then P is the desired path. □

Our main result for the pancake graph P_n is stated in the following theorem.

Theorem 3. *$IHC(P_3) = 1$ and $IHC(P_n) = n - 1$ if $n \ge 4$.*

Proof. It is easy to see that P_3 is isomorphic to a cycle with six vertices. Thus, $IHC(P_3) = 1$. Since P_n is $(n-1)$-regular graph, it is clear that $IHC(P_n) \le n-1$. Since P_n is vertex transitive, we only need to show that there exist $(n-1)$ mutually independent hamiltonian cycles of P_n starting form the vertex $\mathbf{e}$. For $n = 4$, we prove that $IHC(P_4) \ge 3$ by listing the required hamiltonian cycles as follows:

$C_1 =$ ⟨1234, 2134, 4312, 3412, 2143, 1243, 4213, 3124, 1324, 4231, 3241, 2341, 1432, 4132, 2314, 3214, 4123, 1423, 2413, 3142, 1342, 2431, 3421, 4321, 1234⟩
$C_2 =$ ⟨1234, 3214, 2314, 1324, 3124, 4213, 2413, 1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 3142, 4132, 1432, 3412, 4312, 2134, 1234⟩
$C_3 =$ ⟨1234, 4321, 2341, 1432, 4132, 2314, 1324, 4231, 3241, 1423, 2413, 3142, 1342, 2431, 3421, 1243, 4213, 3124, 2134, 4312, 3412, 2143, 4123, 3214, 1234⟩

Suppose that $n \ge 5$. Let B be the $(n-1) \times n$ matrix with

$$b_{i,j} = \begin{cases} i+j-1 & \text{if } i+j-1 \le n, \\ i+j-n+1 & \text{if } n \ge i+j. \end{cases}$$

More precisely,

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 2 & 3 & 4 & 5 & \cdots & n & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n & 1 & 2 & \cdots & n-3 & n-2 \end{bmatrix}.$$

It is not hard to see that $b_{i,1}b_{i,2}\dots b_{i,n}$ forms a permutation of $\{1, 2, \dots, n\}$ for every i with $1 \le i \le n-1$. Moreover, $b_{i,j} \neq b_{i',j}$ for any $1 \le i < i' \le n-1$ and $1 \le j \le n$. In other words, B forms a Latin rectangle with entries in $\{1, 2, \dots, n\}$.

For every $k \in \langle n-1\rangle$, we construct C_k as follows:

(1) $k = 1$. By Lemma 3, there is a hamiltonian path H_1 of $P_n^{\{b_{1,n}\}} - \{\mathbf{e}\}$ joining a vertex $\mathbf{x}$ with $\mathbf{x} \neq (\mathbf{e})^{n-1}$ and $(\mathbf{x})_1 = n-1$ to the vertex $(\mathbf{e})^{n-1}$. By Theorem 2, there is a hamiltonian path H_2 of $\cup_{t=1}^{n-1} P_n^{\{b_{1,t}\}}$ joining the vertex $(\mathbf{e})^n$ to the vertex $(\mathbf{x})^n$ with $H_2(i+(j-1)(n-1)!) \in P_n^{\{b_{1,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-1\rangle$. We set $C_1 = \langle \mathbf{e}, (\mathbf{e})^n, H_2, (\mathbf{x})^n, \mathbf{x}, H_1, (\mathbf{e})^{n-1}, \mathbf{e}\rangle$.

(2) $k = 2$. By Lemma 5, there is a hamiltonian path Q_1 of $P_n^{\{b_{2,n-1}\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining a vertex $\mathbf{y}$ with $(\mathbf{y})_1 = n-1$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = 1$. By Theorem 2, there is a hamiltonian Q_2 of $\cup_{t=1}^{n-2} P_n^{\{b_{2,t}\}}$ joining the vertex $((\mathbf{e})^2)^n$ to the vertex $(\mathbf{y})^n$ such that $Q_2(i+(j-1)(n-1)!) \in P_n^{\{b_{2,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-2\rangle$. By Theorem 1, there is a hamiltonian path Q_3 of $P_n^{\{b_{2,n}\}}$ joining the vertex $(\mathbf{z})^n$ to the vertex $(\mathbf{e})^n$. We set $C_2 = \langle \mathbf{e}, (\mathbf{e})^2, ((\mathbf{e})^2)^n, Q_2, (\mathbf{y})^n, \mathbf{y}, Q_1, \mathbf{z}, (\mathbf{z})^n, Q_3, (\mathbf{e})^n, \mathbf{e}\rangle$.

(3) $k \in \langle n-1\rangle - \{1, 2\}$. By Lemma 6, there is a hamiltonian path R_1^k of $P_n^{\{b_{k,n-k+1}\}} - \{\mathbf{e}, (\mathbf{e})^{k-1}, (\mathbf{e})^k\}$ joining a vertex $\mathbf{w_k}$ with $(\mathbf{w_k})_1 = n-1$ to a vertex $\mathbf{v_k}$ with $(\mathbf{v_k})_1 = 1$. By Theorem 2, there is a hamiltonian path R_2^k of $\cup_{t=1}^{n-k} P_n^{\{b_{k,t}\}}$

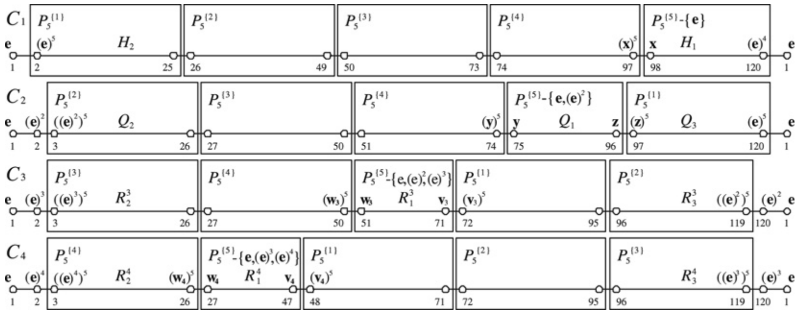

Fig. 3. Illustration for Theorem 3 on P_5.

joining the vertex $((\mathbf{e})^k)^n$ to the vertex $(\mathbf{w_k})^n$ such that $R_2^k(i + (j-1)(n-1)!) \in P_n^{\{b_{k,j}\}}$ for every $i \in \langle (n-1)! \rangle$ and for every $j \in \langle n-k \rangle$. Again, there is a hamiltonian path R_3^k of $\cup_{t=n-k+2}^{n} P_n^{\{b_{k,t}\}}$ joining the vertex $(\mathbf{v_k})^n$ to the vertex $((\mathbf{e})^{k-1})^n$ such that $R_3^k(i + (j-1)(n-1)!) \in P_n^{\{b_{k,n-k+j+1}\}}$ for every $i \in \langle (n-1)! \rangle$ and for every $j \in \langle k-1 \rangle$. We set $C_k = \langle \mathbf{e}, (\mathbf{e})^k, ((\mathbf{e})^k)^n, R_2^k, (\mathbf{w_k})^n, \mathbf{w_k}, R_1^k, \mathbf{v_k}, (\mathbf{v_k})^n, R_3^k, ((\mathbf{e})^{k-1})^n, (\mathbf{e})^{k-1}, \mathbf{e} \rangle$.

Then $\{C_1, C_2, \ldots, C_{n-1}\}$ forms a set of $(n-1)$ mutually independent hamiltonian cycles of P_n starting from the vertex $\mathbf{e}$. □

Example. We illustrate the proof of Theorem 3 with $n = 5$ as follows:

We set

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix}.$$

Then we construct $\{C_1, C_2, C_3, C_4\}$ as follows:

(1)$k = 1$. By Lemma 3, there is a hamiltonian path H_1 of $P_5^{\{b_{1,5}\}} - \{\mathbf{e}\}$ joining a vertex $\mathbf{x}$ with $\mathbf{x} \neq (\mathbf{e})^4$ and $(\mathbf{x})_1 = 4$ to the vertex $(\mathbf{e})^4$. By Theorem 2, there is a hamiltonian path H_2 of $\cup_{t=1}^{4} P_5^{\{b_{1,t}\}}$ joining the vertex $(\mathbf{e})^5$ to the vertex $(\mathbf{x})^5$ with $H_2(i + 24(j-1)) \in P_5^{\{b_{1,j}\}}$ for every $i \in \langle 24 \rangle$ and for every $j \in \langle 4 \rangle$. We set $C_1 = \langle \mathbf{e}, (\mathbf{e})^5, H_2, (\mathbf{x})^5, \mathbf{x}, H_1, (\mathbf{e})^4, \mathbf{e} \rangle$.

(2)$k = 2$. By Lemma 5, there is a hamiltonian path Q_1 of $P_5^{\{b_{2,4}\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining a vertex $\mathbf{y}$ with $(\mathbf{y})_1 = 4$ to a vertex $\mathbf{z}$ with $(\mathbf{z})_1 = 1$. By Theorem 2, there is a hamiltonian Q_2 of $\cup_{t=1}^{3} P_5^{\{b_{2,t}\}}$ joining the vertex $((\mathbf{e})^2)^5$ to the vertex $(\mathbf{y})^5$ such that $Q_2(i + 24(j-1)) \in P_5^{\{b_{2,j}\}}$ for every $i \in \langle 24 \rangle$ and for every $j \in \langle 3 \rangle$. By Theorem 1, there is a hamiltonian path Q_3 of $P_5^{\{b_{2,5}\}}$ joining the vertex $(\mathbf{z})^5$ to the vertex $(\mathbf{e})^5$. We set $C_2 = \langle \mathbf{e}, (\mathbf{e})^2, ((\mathbf{e})^2)^5, Q_2, (\mathbf{y})^5, \mathbf{y}, Q_1, \mathbf{z}, (\mathbf{z})^5, Q_3, (\mathbf{e})^5, \mathbf{e} \rangle$.

(3)$k \in \{3, 4\}$. By Lemma 6, there is a hamiltonian path R_1^k of $P_5^{\{b_{k,6-k}\}} - \{\mathbf{e}, (\mathbf{e})^{k-1}, (\mathbf{e})^k\}$ joining a vertex $\mathbf{w_k}$ with $(\mathbf{w_k})_1 = 4$ to a vertex $\mathbf{v_k}$ with $(\mathbf{v_k})_1 = 1$. By Theorem 2, there is a hamiltonian path R_2^k of $\cup_{t=1}^{5-k} P_5^{\{b_{k,t}\}}$ joining the vertex $((\mathbf{e})^k)^5$ to the vertex $(\mathbf{w_k})^5$ such that $R_2^k(i + 24(j-1)) \in P_5^{\{b_{k,j}\}}$ for every $i \in \langle 24 \rangle$ and for every $j \in \langle 5-k \rangle$. Again, there is a hamiltonian path R_3^k of $\cup_{t=7-k}^{5} P_5^{\{b_{k,t}\}}$ joining the vertex $(\mathbf{v_k})^5$ to the vertex $((\mathbf{e})^{k-1})^5$ such that $R_3^k(i + 24(j-1)) \in P_5^{\{b_{k,6-k+j}\}}$ for every $i \in \langle 24 \rangle$ and for every $j \in \langle k-1 \rangle$. We set $C_k = \langle \mathbf{e}, (\mathbf{e})^k, ((\mathbf{e})^k)^5, R_2^k, (\mathbf{w_k})^5, \mathbf{w_k}, R_1^k, \mathbf{v_k}, (\mathbf{v_k})^5, R_3^k, ((\mathbf{e})^{k-1})^5, (\mathbf{e})^{k-1}, \mathbf{e} \rangle$.

Then $\{C_1, C_2, C_3, C_4\}$ forms a set of 4 mutually independent hamiltonian cycles of P_5 starting from the vertex $\mathbf{e}$. See Fig. 3 for illustration.

5. The star graphs

Let n be a positive integer. The *n-dimensional star graph*, S_n, is a graph with the vertex set $V(S_n) = \{u_1 \ldots u_n \mid u_i \in \langle n \rangle$ and $u_j \neq u_k$ for $j \neq k\}$. The adjacency is defined as follows: $u_1 \ldots u_i \ldots u_n$ is adjacent to $v_1 \ldots v_i \ldots v_n$ through an edge of dimension i with $2 \leq i \leq n$ if $v_j = u_j$ for every $j \in \langle n \rangle - \{1, i\}$, $v_1 = u_i$, and $v_i = u_1$. The star graphs S_2, S_3, and S_4 are illustrated in Fig. 4. In [1], it showed that the connectivity of S_n is $(n-1)$. We use boldface to denote vertices in S_n. Hence, $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ denotes a sequence of vertices in S_n.

By definition, S_n is an $(n-1)$-regular graph with $n!$ vertices. We use $\mathbf{e}$ to denote the vertex $12\ldots n$. It is known that S_n is a bipartite graph with one partite set containing the vertices corresponding to odd permutations and the other partite set containing those vertices correspond to even permutations. We use white vertices to represent those even permutation vertices and we use black vertices to represent those odd permutation vertices. Let $\mathbf{u} = u_1u_2\ldots u_n$ be an arbitrary vertex of the star graph S_n. We say that u_i is the ith coordinate of $\mathbf{u}$, $(\mathbf{u})_i$, for $1 \leq i \leq n$. For $1 \leq i \leq n$, let $S_n^{\{i\}}$ be the subgraph of S_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Then S_n can be decomposed into n subgraph $S_n^{\{i\}}$, $1 \leq i \leq n$, and each $S_n^{\{i\}}$ is isomorphic to S_{n-1}. Thus, the star graph can also be constructed recursively. Let I be any subset of $\langle n \rangle$. We use S_n^I to denote

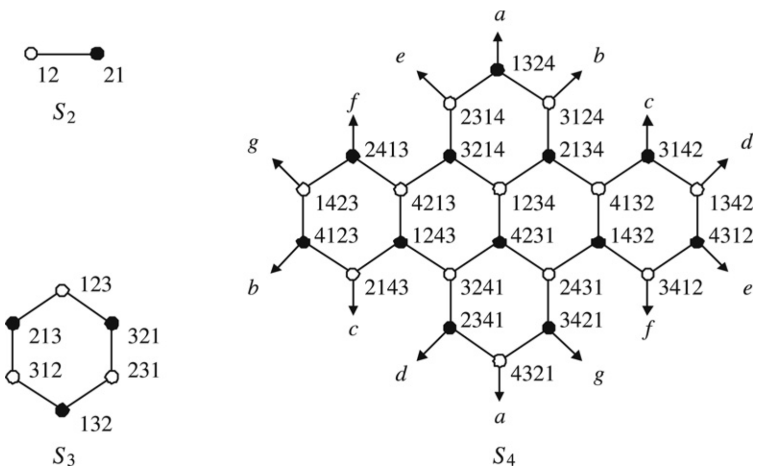

Fig. 4. The star graphs S_2, S_3, and S_4.

the subgraph of S_n induced by $\cup_{i\in I} V(S_n^{\{i\}})$. For any two distinct elements i and j in $\langle n\rangle$, we use $E_n^{i,j}$ to denote the set of edges between $S_n^{\{i\}}$ and $S_n^{\{j\}}$. By the definition of S_n, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-dimensional edge with $2 \le i \le n$. For this reason, we use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$. We have $((\mathbf{u})^i)^i = \mathbf{u}$ and $(\mathbf{u})^n \in S_n^{\{(\mathbf{u})_1\}}$.

Lemma 7. *Let i and j be any two distinct elements in $\langle n\rangle$ with $n \ge 3$. Then $|E_n^{i,j}| = (n-2)!$. Moreover, there are $(n-2)!/2$ edges joining black vertices of $S_n^{\{i\}}$ to white vertices of $S_n^{\{j\}}$.*

Lemma 8. *Let $\mathbf{u}$ and $\mathbf{v}$ be two distinct vertices of S_n with $d(\mathbf{u}, \mathbf{v}) \le 2$. Then $(\mathbf{u})_1 \ne (\mathbf{v})_1$.*

Theorem 4 ([21]). *Let $n \ge 4$. Suppose that $\mathbf{u}$ is a white vertex of S_n and $\mathbf{v}$ is a black vertex of S_n. Then there is a hamiltonian path of S_n joining $\mathbf{u}$ to $\mathbf{v}$.*

Theorem 5. *Let $\{a_1, a_2, \dots, a_r\}$ be a subset of $\langle n\rangle$ for some $r \in \langle n\rangle$ with $n \ge 5$. Assume that $\mathbf{u}$ is a white vertex in $S_n^{\{a_1\}}$ and $\mathbf{v}$ is a black vertex in $S_n^{\{a_r\}}$. Then there is a hamiltonian path $\langle \mathbf{u} = \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \dots, \mathbf{x_r}, H_r, \mathbf{y_r} = \mathbf{v}\rangle$ of $\cup_{i=1}^{r} S_n^{\{a_i\}}$ joining $\mathbf{u}$ to $\mathbf{v}$ such that $\mathbf{x_1} = \mathbf{u}$, $\mathbf{y_r} = \mathbf{v}$, and H_i is a hamiltonian path of $S_n^{\{a_i\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for every i, $1 \le i \le r$.*

Proof. We set $\mathbf{x_1}$ as $\mathbf{u}$ and $\mathbf{y_r}$ as $\mathbf{v}$. By Theorem 4, this theorem holds on $r = 1$. Suppose that $r \ge 2$. By Lemma 7, there are $(n-2)!/2 \ge 3$ edges joining black vertices of $S_n^{\{a_i\}}$ to white vertices of $S_n^{\{a_{i+1}\}}$ for every $i \in \langle r-1\rangle$. We can choose an edge $(\mathbf{y_i}, \mathbf{x_{i+1}}) \in E_n^{a_i, a_{i+1}}$ with $\mathbf{y_i}$ being a black vertex and $\mathbf{x_{i+1}}$ being a white vertex for every $i \in \langle r-1\rangle$. By Theorem 4, there is a hamiltonian path H_i of $S_n^{\{a_i\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for every $i \in \langle r\rangle$. Then the path $\langle \mathbf{u} = \mathbf{x_1}, H_1, \mathbf{y_1}, \mathbf{x_2}, H_2, \mathbf{y_2}, \dots, \mathbf{x_r}, H_r, \mathbf{y_r} = \mathbf{v}\rangle$ is the desired path. □

Theorem 6 ([21]). *Let $\mathbf{w}$ be a black vertex of S_n with $n \ge 4$. Assume that $\mathbf{u}$ and $\mathbf{v}$ are two distinct white vertices of $S_n - \{\mathbf{w}\}$. Then there is a hamiltonian path H of $S_n - \{\mathbf{w}\}$ joining $\mathbf{u}$ to $\mathbf{v}$.*

Lemma 9 ([30]). *Let i be any element in $\langle n\rangle$ with $n \ge 4$. Assume that $\mathbf{r}$ and $\mathbf{s}$ are two adjacent vertices of S_n and $\mathbf{u}$ is a white vertex of $S_n - \{\mathbf{r}, \mathbf{s}\}$. Then there is a hamiltonian path of $S_n - \{\mathbf{r}, \mathbf{s}\}$ joining $\mathbf{u}$ to some black vertex $\mathbf{v}$ with $(\mathbf{v})_1 = i$.*

Lemma 10. *Let a and b be any two distinct elements in $\langle n\rangle$ with $n \ge 4$. Assume that $\mathbf{x}$ is a white vertex of S_n, and assume that $\mathbf{x_1}$ and $\mathbf{x_2}$ are two distinct neighbors of $\mathbf{x}$. Then there is a hamiltonian path P of $S_n - \{\mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\}$ joining a white vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a white vertex $\mathbf{v}$ with $(\mathbf{v})_1 = b$.*

Proof. Since S_n is vertex transitive and edge transitive, we may assume that $\mathbf{x} = \mathbf{e}$, $\mathbf{x_1} = (\mathbf{e})^2$, and $\mathbf{x_2} = (\mathbf{e})^3$. Without loss of generality, we may also assume that $a < b$. We have $a \ne n$ and $b \ne 1$. We prove this statement by induction on n. For $n = 4$, the required paths of $S_4 - \{1234, 2134, 3214\}$ are listed below:

$a = 1$ and $b = 2$	⟨1324, 3142, 4132, 1432, 3412, 4312, 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431⟩
$a = 1$ and $b = 3$	⟨1423, 2413, 4213, 1243, 2143, 4123, 3124, 1324, 2314, 4312, 3412, 1432, 4132, 3142, 1342, 2341, 4321, 3421, 2431, 4231, 3241⟩
$a = 1$ and $b = 4$	⟨1324, 3142, 4132, 1432, 3412, 4312, 2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 2431, 4231, 3241, 2341, 4321⟩
$a = 2$ and $b = 3$	⟨2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431, 1432, 4132, 3142, 1342, 4312, 3412⟩
$a = 2$ and $b = 4$	⟨2314, 1324, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 4321, 2341, 3241, 4231, 2431, 1432, 3412, 4312, 1342, 3142, 4132⟩
$a = 3$ and $b = 4$	⟨3124, 1324, 2314, 4312, 3412, 1432, 4132, 3142, 1342, 2341, 4321, 3421, 2431, 4231, 3241, 1243, 2143, 4123, 1423, 2413, 4213⟩

Suppose that this statement holds for S_k for every k, $4 \le k \le n-1$. Let c be any element in $\langle n-1\rangle - \{1, a\}$. By induction, there is a hamiltonian path H of $S_n^{\{n\}} - \{\mathbf{e}, (\mathbf{e})^2, (\mathbf{e})^3\}$ joining a white vertex $\mathbf{u}$ with $(\mathbf{u})_1 = a$ to a white vertex $\mathbf{z}$ with $(\mathbf{z})_1 = c$. We choose a white vertex $\mathbf{v}$ in $S_n^{\{1\}}$ with $(\mathbf{v})_1 = b$. By Theorem 5, there is a hamiltonian path R of $S_n^{\langle n-1\rangle}$ joining the black vertex $(\mathbf{z})^n$ to $\mathbf{v}$. Then $\langle \mathbf{u}, H, \mathbf{z}, (\mathbf{z})^n, R, \mathbf{v}\rangle$ is the desired path of $S_n - \{\mathbf{e}, (\mathbf{e})^2, (\mathbf{e})^3\}$. □

The following theorem is our main result for the star graph S_n.

Theorem 7. *$IHC(S_3) = 1$, $IHC(S_4) = 2$, and $IHC(S_n) = n-1$ if $n \ge 5$.*

Proof. It is easy to see that S_3 is isomorphic to a cycle with six vertices. Thus, $IHC(S_3) = 1$. Using a computer, we have $IHC(S_4) = 2$ by brute force checking. Thus, we assume that $n \ge 5$. We know that S_n is $(n-1)$-regular graph. Hence, $IHC(S_n) \le n-1$. Since S_n is vertex transitive, we only need to show that there are $(n-1)$ mutually independent hamiltonian cycles of S_n starting from $\mathbf{e}$. Let B be the $(n-1) \times n$ matrix with

$$b_{i,j} = \begin{cases} i+j-1 & \text{if } i+j-1 \le n, \\ i+j-n+1 & \text{if } n < i+j-1. \end{cases}$$

We construct $\{C_1, C_2, \ldots, C_{n-1}\}$ as follows:

(1)$k = 1$. We choose a black vertex $\mathbf{x}$ in $S_n^{\{b_{1,n}\}} - \{(\mathbf{e})^{n-1}\}$ with $(\mathbf{x})_1 = n-1$. By Theorem 6, there is a hamiltonian path H_1 of $S_n^{\{b_{1,n}\}} - \{\mathbf{e}\}$ joining $\mathbf{x}$ to the black vertex $(\mathbf{e})^{n-1}$. By Theorem 5, there is a hamiltonian path H_2 of $\cup_{t=1}^{n-1} S_n^{\{b_{1,t}\}}$ joining the black vertex $(\mathbf{e})^n$ to the white vertex $(\mathbf{x})^n$ with $H_2(i + (j-1)(n-1)!) \in S_n^{\{b_{1,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-1\rangle$. We set $C_1 = \langle \mathbf{e}, (\mathbf{e})^n, H_2, (\mathbf{x})^n, \mathbf{x}, H_1, (\mathbf{e})^{n-1}, \mathbf{e}\rangle$.

(2)$k = 2$. We choose a white vertex $\mathbf{y}$ in $S_n^{\{b_{2,n-1}\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ with $(\mathbf{y})_1 = n-1$. By Lemma 9, there is a hamiltonian path Q_1 of $S_n^{\{b_{2,j}\}} - \{\mathbf{e}, (\mathbf{e})^2\}$ joining $\mathbf{y}$ to a black vertex $\mathbf{z}$ with $(\mathbf{z})_1 = 1$. By Theorem 5, there is a hamiltonian Q_2 of $\cup_{t=1}^{n-2} S_n^{\{b_{2,t}\}}$ joining the white vertex $((\mathbf{e})^2)^n$ to the black vertex $(\mathbf{y})^n$ such that $Q_2(i + (j-1)(n-1)!) \in S_n^{\{b_{2,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-2\rangle$. Again, there is a hamiltonian path Q_3 of $S_n^{\{b_{2,n}\}}$ joining the white vertex $(\mathbf{z})^n$ to the black vertex $(\mathbf{e})^n$. We set $C_2 = \langle \mathbf{e}, (\mathbf{e})^2, ((\mathbf{e})^2)^n, Q_2, (\mathbf{y})^n, \mathbf{y}, Q_1, \mathbf{z}, (\mathbf{z})^n, Q_3, (\mathbf{e})^n, \mathbf{e}\rangle$.

(3)$3 \le k \le n-1$. By Lemma 10, there is a hamiltonian path R_1^k of $S_n^{\{b_{k,n-k+1}\}} - \{\mathbf{e}, (\mathbf{e})^{k-1}, (\mathbf{e})^k\}$ joining a white vertex $\mathbf{w_k}$ with $(\mathbf{w_k})_1 = n-1$ to a white vertex $\mathbf{v_k}$ with $(\mathbf{v_k})_1 = 1$. By Theorem 5, there is a hamiltonian path R_2^k of $\cup_{t=1}^{n-k} S_n^{\{b_{k,t}\}}$ joining the white vertex $((\mathbf{e})^k)^n$ to the black vertex $(\mathbf{w_k})^n$ such that $R_2^k(i + (j-1)(n-1)!) \in S_n^{\{b_{k,j}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle n-k-1\rangle$. Again, there is a hamiltonian path R_3^k of $\cup_{t=n-k+2}^{n} S_n^{\{b_{k,t}\}}$ joining the black vertex $(\mathbf{v_k})^n$ to the black vertex $((\mathbf{e})^{k-1})^n$ such that $R_3^k(i + (j-1)(n-1)!) \in S_n^{\{b_{k,n-k+j+1}\}}$ for every $i \in \langle (n-1)!\rangle$ and for every $j \in \langle k-1\rangle$. We set $C_k = \langle \mathbf{e}, (\mathbf{e})^k, ((\mathbf{e})^k)^n, R_2^k, (\mathbf{w_k})^n, \mathbf{w_k}, R_1^k, \mathbf{v_k}, (\mathbf{v_k})^n, R_3^k, ((\mathbf{e})^{k-1})^n, (\mathbf{e})^{k-1}, \mathbf{e}\rangle$.

Then $\{C_1, C_2, \ldots, C_{n-1}\}$ forms a set of $(n-1)$ mutually independent hamiltonian cycles of S_n starting form the vertex $\mathbf{e}$. □

6. Discussion

In this paper, we discuss the mutually independent hamiltonian cycles for the pancake graphs and the star graphs. The concept of mutually independent hamiltonian cycle can be viewed as a generalization of Latin rectangles. Perhaps one of the most interesting topics in Latin square is orthogonal Latin square. Two Latin squares of order n are *orthogonal* if the n-squared pairs formed by juxtaposing the two arrays are all distinct. Similarly, two Latin rectangles of order $n \times m$ are *orthogonal* if the $n \times m$ pairs formed by juxtaposing the two arrays are all distinct. With this in mind, let G be a hamiltonian graph and C_1 and C_2 be two sets of mutually independent hamiltonian cycles of G from a given vertex x. We say C_1 and C_2 are *orthogonal* if their corresponding Latin rectangles are orthogonal. For example, we know that $IHC(P_4) = 3$. The following Latin rectangle represents three mutually independent hamiltonian cycles beginning at 1234.

2134, 4312, 1342, 2431, 3421, 1243, 4213, 3124, 1324, 4231, 3241, 1423, 2413, 3142, 4132, 2314, 3214, 4123, 2143, 3412, 1432, 2341, 4321
3214, 2314, 4132, 1432, 3412, 4312, 1342, 3142, 2413, 4213, 1243, 2143, 4123, 1423, 3241, 2341, 4321, 3421, 2431, 4231, 1324, 3124, 2134
4321, 2341, 1432, 3412, 2143, 4123, 1423, 3241, 4231, 1324, 3124, 2134, 4312, 1342, 2431, 3421, 1243, 4213, 2413, 3142, 4132, 2314, 3214

Yet, the following Latin rectangle also represents three mutually independent hamiltonian cycles beginning at 1234.

2134, 3124, 4213, 1243, 2143, 4123, 1423, 2413, 3142, 4132, 1432, 3412, 4312, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 2314, 3214
3214, 2314, 4132, 3142, 2413, 4213, 1243, 3421, 2431, 1342, 4312, 2134, 3124, 1324, 4231, 3241, 1423, 4123, 2143, 3412, 1432, 2341, 4321
4321, 3421, 1243, 2143, 3412, 4312, 1342, 2431, 4231, 1324, 2314, 3214, 4123, 1423, 3241, 2341, 1432, 4132, 3142, 2413, 4213, 3124, 2134

We can check that these two Latin rectangles are orthogonal. Thus, we have two sets of three mutually independent hamiltonian cycles that are orthogonal. With this example in mind, we can consider the following problem. Let G be any hamiltonian graph. We can define $MOMH(G)$ as the largest integer k such that there exist k sets of mutually independent hamiltonian cycle of G beginning from any vertex x such that each set contains exactly $IHC(G)$ hamiltonian cycles and any two different sets are orthogonal. It would be interesting to study the value of $MOMH(G)$ for some hamiltonian graphs G.

We can also discuss mutually independent hamiltonian paths for some graphs. Let $P_1 = \langle v_1, v_2, \ldots, v_n \rangle$ and $P_2 = \langle u_1, u_2, \ldots, u_n \rangle$ be two hamiltonian paths of a graph G. We say that P_1 and P_2 are independent if $u_1 = v_1$, $u_n = v_n$, and $u_i \neq v_i$ for $1 < i < n$. We say a set of hamiltonian paths $\{P_1, P_2, \ldots, P_s\}$ of G between two distinct vertices are mutually independent if any two distinct paths in the set are independent. There are some study on mutually independent hamiltonian paths [29,39].

Recently, people are interested in a mathematical puzzle, called Sudoku [38]. Sudoku can be viewed as a 9×9 Latin square with some constraints. There are several variations of Sudoku have been introduced. Mutually independent hamiltonian cycles can also be considered as a variation of Sudoku.

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