A Unified Analytic Framework Based on Minimum Scan Statistics for Wireless Ad Hoc and Sensor Networks

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Abstract—Due to limitations on transmission power of wireless devices, areas with sparse nodes are decisive to some extreme properties of network topology. In this paper, we assume wireless ad hoc and sensor networks are represented by uniform point processes or Poisson point processes. Asymptotic analyses based on *minimum scan statistics* are given for some crucial network properties, including coverage of wireless sensor networks, connectivity of wireless ad hoc networks, the largest edge length of geometric structures, and local-minimum-free geographic routing protocols. We derive explicit formulas of minimum scan statistics. By taking the transmission radius as a major parameter, our results are applied to various network problems. This work offers a unified approach to solve various problems and reveals the evolution of network topology. In addition, boundary effects are thoroughly handled.

Index Terms—Wireless ad hoc networks, wireless sensor networks, minimum scan statistics, random deployment, Poisson point processes, uniform point processes, coverage, connectivity, grid routing, greedy forward routing, Gabriel graphs, relative neighborhood graphs.

1 Introduction

In homogeneous wireless ad hoc networks, all wireless devices have the same transmission radius r, and two nodes have a link between them if they are apart from each other no more than r. A communication session is established either through a single-hop radio transmission if the communication parties are within each other's transmission range, or through relaying by intermediate devices otherwise. The induced network topology is called r-disk graphs, or unit disk graphs (UDGs) if r is scaled to 1. An r-disk graph over a vertex set V is denoted by $G_r(V)$. Because of no need for fixed infrastructures, wireless ad hoc networks can be flexibly deployed at low cost for various missions such as decision making in the battlefield, emergency disaster relief, and environmental monitoring. In many applications, such as wireless sensor networks, a large number of devices need to be deployed in harsh environments. As a result, deterministic deployment usually is not feasible, and random deployment is the only viable solution. Hence, it is natural to model wireless networks by r-disk graphs over random point sets, and asymptotic analyses are interesting to the research community [1], [2], [3], [4], [5].

In wireless communication systems, receivers can decode one signal at a time, so simultaneously arriving signals are interfered and may cause transmission failure at the receiver. Due to short transmission ranges of radio frequency signals, interferences are from nearby nodes. Therefore, for a receiver, nearby nodes are not only potential message senders but also interference sources. Therefore, the number of nodes in one's vicinity is an important topological parameter in a network. In this paper, we introduce an analytical tool called minimum scan statistics that provide an overall lower bound of a network for the number of nodes in one's vicinity.

Let V be a finite point set in a bounded region A, and C be a convex compact set.[1] Let $\#(\cdot)$ be the cardinality function. We say C' is a copy of C, denoted by $C' \cong C$, if C' is obtained from C by reflecting, rotating, and/or shifting. The *minimum scan statistic* of V with respect to scanning set C is the least number of points in V covered by a copy of C, i.e., $\min_{C' \cong C} \#(V \cap C')$. However, without further constraints, since the copy of C can be placed outside of A, the minimum scan statistic is always 0. To prevent meaninglessness and for applying to various applications, we may have several variations depending on supplementary constraints, e.g.

1. C' must be fully contained in A.
2. C' must have at least half of its area contained in A.
3. C must be a disk, and the center of C' must be in A.
4. A has a boundary-free topology such as a sphere or a square with toroidal metrics.

In this paper, we will derive asymptotics for variations 1, 2, and 3. Note that variation 1 was the case studied by Auer and Hornik [6], and variation 4 has similar asymptotics of variation 1.

In literature, most works on scan statistics studied the largest number of points covered by scanning sets. To

1. A set is convex if for any two points u, v in this set, the segment uv is also contained in it. A set is compact if it is bounded and close.

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distinguish our work in this paper from the past results, we called traditional ones maximum scan statistics. In [6], the maximum and minimum scan statistics of a d-dimensional Poisson point process over $[0,T]^d$ were studies, and asymptotic upper bounds and lower bounds with scan sets of volume $o(T^d)$ as $T \to \infty$ were given. With "good" scanning sets, the upper and lower bounds in the Erdős-Rényi regime where the average number of nodes covered by scanning sets is with the order of $\Theta(\ln T)$ are asymptotically tight. Note that in [6], the copy of scanning sets must be fully contained in $[0,T]^d$. In [7], instead of ordinary geometric aspects, coverage was defined by means of Lebesgue integrable 0-1 functions, and it was proved that the probability distribution of chromatic numbers of random geometric graphs focuses on two consecutive integer numbers. For more applications of scan statistics, readers can refer to [8].

Let $\mathbb{D}$ be a unit-area square or disk centered at the origin, and $X_1, X_2, \ldots$ be independent and uniformly distributed random points on $\mathbb{D}$. Let $Po(n)$ be a Poisson random variable with parameter n, independent of $\{X_1, X_2, \ldots\}$. Given a positive integer n, the point process $\{X_1, X_2, \ldots, X_n\}$ is referred to as the *uniform n-point process* and is denoted by $\mathcal{X}_n$. Given a positive number n, the point process $\{X_1, X_2, \ldots, X_{Po(n)}\}$ is referred to as the *Poisson point process* with mean n and is denoted by $\mathcal{P}_n$. In the following, $\mathcal{V}_n$ is shorthand for $\mathcal{P}_n$ and $\mathcal{X}_n$. In this paper, we assume that a wireless ad hoc and sensor network is composed of homogeneous wireless devices represented by $\mathcal{P}_n$ or $\mathcal{X}_n$. Let r_n be the transmission radius or sensing radius given by $n\pi r_n^2 = (\beta + o(1)) \ln n$ for some constant β. The induced network topology is an r_n-disk graph over $\mathcal{V}_n$. If β is fixed, r_n decreases as n goes up. We remark that although r_n is scaled with respect to the parameter n, all results in this paper can be scaled back to a standard unit disk model, like the one used in [5], without too much effort by applying the technique used in [9]. In addition, the simplest method to interpret the model is by reading $n\pi r_n^2$ as the average number of neighbors and β as a tunable parameter for node densities.

In wireless ad hoc and sensor networks, many critical properties are related to the node density of sparse regions. Minimum scan statistics in some sense are correspondents of the minimum node density. In this work, three variations of minimum scan statistics will be studied, and asymptotics in the Erdős-Rényi regime will be derived. Those results will be applied to various network problems, including coverage of wireless sensor networks, connectivity of wireless ad hoc networks, the largest edge length of geometric structures, and local-minimum-free geographic greedy routing.

Connectivity is one of the most extensively studied properties in graph theory and also an essential requirement of wireless networks. For a given constant c, Dette and Henze [10] showed that the probability of $G_{\sqrt{\frac{\ln n + c}{n\pi}}}(\mathcal{X}_n)$ containing no isolated nodes converges to the Gumbel function $\exp(-e^{-c})$, and Penrose [11] further proved that without isolated nodes, $G_{\sqrt{\frac{\ln n + c}{n\pi}}}(\mathcal{V}_n)$ is almost surely connected. In unreliable wireless networks, where nodes may fail independently with the same probability p, Wan et al. [3], [12], [13] reported that the probability of the event that $G_{\sqrt{\frac{\ln n + c}{pn\pi}}}(\mathcal{X}_n)$ is connected converges to $\exp(-pe^{-c})$. The k-connectivity problems were studied in [2] and [4]. Let $r_n = \sqrt{\frac{\ln n + (2k-1)\ln n \ln + \xi_n}{\pi n}}$ for $k \geq 1$ and $\xi_n = o(\ln \ln n)$. Then, $G_{r_n}(\mathcal{V}_n)$ is almost surely $(k+1)$-connected if $\xi_n \to \infty$, and almost surely not $(k+1)$-connected if $\xi_n \to -\infty$.

Coverage is a QoS metrics of wireless sensor networks. In [5], a sufficient condition and a necessary condition for k-coverage of randomly deployed wireless sensor networks were derived to provide upper bounds for sensor network lifetime. In [14], based on the techniques developed in [5], it was proved that if the sensing radius is given by $r_n = \sqrt{\frac{\ln n + (2k+1)\ln \ln n + \xi_n}{\pi n}}$ with $\xi_n = o(\ln \ln n)$ for $k \geq 0$, then the sensor field is almost surely $(k+1)$-covered if $\xi_n \to \infty$, and almost surely not $(k+1)$-covered if $\xi_n \to -\infty$.

In topology control of wireless ad hoc networks, geometric structures, such as Gabriel graphs (GGs), relative neighborhood graphs (RNGs), Yao's graphs, and so forth, are fundamental tools for link management and transmission power (or transmission radius) setting. The largest edge length of geometric structures is a good reference to the maximal transmission radius and, therefore, is an important parameter for setting the maximal transmission power in the hardware design stage. For a given geometric structure, the largest edge length is called the critical transmission radius for this structure.

Geographic greedy routing protocols are interesting due to their advantages in distributed and localized implementation, but a major drawback is the existence of local minima that may trap packets and cause deliverability problems. To solve local minimum problems, besides applying complicated remedy mechanisms to release packets from local minima, we can significantly reduce or even eliminate local minima by increasing the transmission radius. However, to prevent energy waste and signal interferences, a small transmission radius is preferred. The smallest transmission radius for no existence of local minima in the network is called the critical transmission radius for geographic greedy routing.

In this paper, we derive several explicit asymptotic formulas of minimum scan statistics in the Erdős-Rényi regime with respect to various boundary conditions and then apply these results to many network problems. Our major contributions include the following:

- Give an asymptotic formula of sensing radii for $\Theta(\ln n)$-coverage.
- Give an asymptotic formula of transmission radii for $\Theta(\ln n)$-connectivity.
- Give a threshold of the longest Gabriel graph edge.
- Give a threshold of the longest RNG edge.
- Give a threshold of transmission radii for local-minimum-free greedy forward routing.

TABLE 1
Notations

$G_r(V)$	the r-disk graph induced by a point set V
$\mathbb{D}$	a unit-area square or disk (deployment region)
$\mathcal{P}_n$	a Poisson point process with mean n
$\mathcal{X}_n$	a uniform n-point process
$\mathcal{V}_n$	$\mathcal{P}_n$ or $\mathcal{X}_n$
r_n	the transmission radius or sensing radius
$Po(n)$	a Poisson RV with mean n
$\#(A)$	the cardinality of countable set A
$B(x,r)$	a disk with center x and radius r
$\|x\|$	the Euclidean norm of a point x
$\|A\|$	the area of a measurable set A
mc_A	the mass center of a measurable set A
$diam(A)$	the diameter of A
cA	$\{mc_A + c(x - mc_A) : x \in A\}$
a.a.s.	an acronym of asymptotic almost sure
RV	an acronym of random variable
$f = O(g)$	$\exists N > 0, c > 0$ s.t. $f(n) \leq cg(n)$ for all $n \geq N$
$f = \Omega(g)$	$\exists N > 0, c > 0$ s.t. $cg(n) \leq f(n)$ for all $n \geq N$
$f = \Theta(g)$	$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
$f = o(g)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
$f \sim g$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

- Give a threshold of the grid size for local-minimum-free grid routing.

The rest of this paper is organized as follows: In Section 2, we define the notations used in this paper and give a useful probabilistic lemma. In Section 3, we give asymptotics of minimum scan statistics. In Section 4, minimum scan statistics are applied to several problems raised in wireless ad hoc and sensor networks. Our conclusions are provided in Section 5. Some complicated mathematical proofs are left in the Appendix.

2 Preliminaries and Terminologies

In what follows, the disk of radius r centered at x is denoted by $B(x,r)$. For $a, b \in \mathbb{R}^2$ and $A, B \subseteq \mathbb{R}^2$, $a + b$ denotes the addition of two vectors, and $A + b = \{a + b : \forall a \in A\}$. $\|x\|$ is the euclidean norm of a point $x \in \mathbb{R}^2$. $|A|$ is shorthand for the 2D area of a measurable set $A \subset \mathbb{R}^2$. If A is a convex compact set, mc_A denotes the mass center of A. In addition, for a positive real number c, we use cA to denote the set $\{mc_A + c(x - mc_A) : x \in A\}$. The diameter of a set A, denoted as $diam(A)$, is the supreme of the distance between any two points in the set, i.e., $diam(A) = \sup_{x,y\in A} \|x - y\|$. The symbols O, Ω, Θ, o, and $\sim$ are defined in Table 1 and always refer to the limit $n \to \infty$. An event is said to be *asymptotic almost sure* (a.a.s.) if it occurs with a probability converging to one as $n \to \infty$. RV is an acronym of random variable. Table 1 lists notations used in this paper.

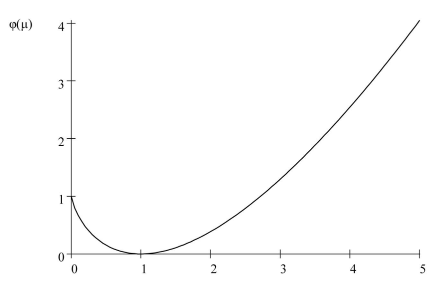

Fig. 1. $\phi(\mu) = 1 + \mu \ln \mu - \mu$.

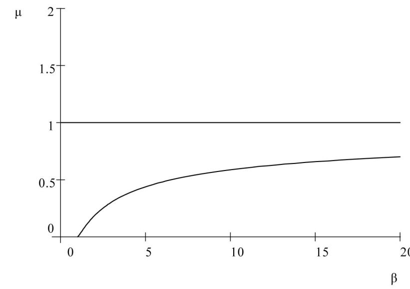

Fig. 2. The curve is of $\mu = \phi_-^{-1}(1/\beta)$, and $\mu = 1$ is the asymptotics as $\beta \to \infty$.

Let ϕ be the function over $(0,\infty)$ defined by $\phi(\mu) = 1 - \mu + \mu \ln \mu$. A straightforward calculation yields $\phi'(\mu) = \ln \mu$ and $\phi''(\mu) = 1/\mu$. Thus, ϕ is strictly convex and has the unique minimum zero at $\mu = 1$ (see Fig. 1). Let $\phi_-^{-1} : [0,1) \to (0,1]$ be the inverse of the restriction of ϕ to (0, 1]. We are interested in the equation $\mu = \phi_-^{-1}(1/\beta)$, and the graph of $\mu = \phi_-^{-1}(1/\beta)$ is depicted in Fig. 2. Define a function $\mathcal{L}$ over $(0,\infty)$ by

$$\mathcal{L}(\beta) = \begin{cases} \beta\phi_-^{-1}(1/\beta), & \text{if } \beta > 1, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{L}$ is a monotonic increasing function of β. The curve of $\mathcal{L}(\beta)$ is illustrated in Fig. 3.

The following lemma is related to the minimum of a collection of Poisson RVs.

Lemma 1. *Assume that* $\lim_{n\to\infty} \frac{\lambda_n}{\ln n} = \beta$ *for some constant* $\beta > 0$. *Let* $Y_1, Y_2, \ldots, Y_{I_n}$ *be* I_n *Poisson RVs with means* λ_n*:*

1. *If* $I_n = o(n\sqrt{\ln n})$, *then for any* $\beta' \in (0,\beta)$, $\min_{i=1,\ldots,I_n} Y_i \geq \mathcal{L}(\beta') \ln n$ *a.a.s.*
2. *If* $I_n = O(\sqrt{\frac{n}{\ln n}})$, *then for any* $\beta' \in (0,\beta)$, $\min_{i=1,\ldots,I_n} Y_i \geq \frac{1}{2}\mathcal{L}(2\beta') \ln n$ *a.a.s.*
3. *If* $Y_1, Y_2, \ldots, Y_{I_n}$ *are independent and* $I_n = \Omega(\frac{n}{\ln n})$, *then for any* $\beta' \in (\beta,\infty)$, $\min_{i=1,\ldots,I_n} Y_i \leq \mathcal{L}(\beta') \ln n$ *a.a.s.*
4. *If* $Y_1, Y_2, \ldots, Y_{I_n}$ *are independent and* $I_n = \Omega(\sqrt{\frac{n}{\ln n}})$, *then for any* $\beta' \in (\beta,\infty)$, $\min_{i=1,\ldots,I_n} Y_i \leq \frac{1}{2}\mathcal{L}(2\beta') \ln n$ *a.a.s.*

Proof. A proof is given in Appendix A. □

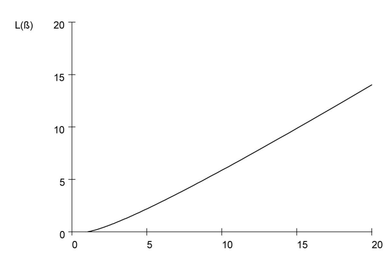

Fig. 3. The curve is of $\mathcal{L}(\beta)$.

3 Minimum Scan Statistics

In what follows, $\mathbf{C}_n$ denotes a collection of convex compact sets, and there exist constants β and c_0 such that for any $C_n \in \mathbf{C}_n$, $n|C_n| = (\beta + o(1)) \ln n$ and $diam(C_n) \leq c_0\sqrt{|C_n|}$. According to the isodiametric inequality [23], [24], [25], we have $|C_n| \leq \frac{1}{4}\pi(diam(C_n))^2$. Hence, $diam(C_n) = \Theta(\sqrt{|C_n|}) = \Theta(\sqrt{\frac{\ln n}{n}})$. In this paper, there are three variations of minimum scan statistics depending on additional constraints on scanning sets to be studied.

First, we consider the case in which each copy of C_n is fully contained in $\mathbb{D}$. Let

$$S_{m1}(V, C) = \min_{C' \cong C, C' \subset \mathbb{D}} \#(V \cap C'), \text{ and}$$
$$\mathcal{S}_{m1}(\mathcal{V}_n, \mathbf{C}_n) = \min_{C_n \in \mathbf{C}_n} S_{m1}(\mathcal{V}_n, C_n).$$

We have the following theorem.

Theorem 2. *For the asymptotics of $\mathcal{S}_{m1}(\mathcal{V}_n, \mathbf{C}_n)$, it is almost sure that*

$$\Pr\left[\frac{\mathcal{S}_{m1}(\mathcal{V}_n, \mathbf{C}_n)}{\ln n} \sim \mathcal{L}(\beta)\right] \to 1.$$

Furthermore, if $\beta < 1$, we almost surely have

$$\Pr[\mathcal{S}_{m1}(\mathcal{V}_n, \mathbf{C}_n) = 0] \to 1.$$

Note that S_{m1} is the case studied by Auer and Hornik [6], and Theorem 2 is consistent with the results in [6]. We also remark that if toroidal metrics is applied, we can have the same asymptotics even if C' is allowed not to be contained in $\mathbb{D}$. So, the asymptotics can be applied to network models using toroidal metrics. In grid routing, the square deployment region is tessellated into equal-size square cells. To eliminate local minima, we set the grid set large enough such that all cells are nonempty. In Section 4.3, Theorem 2 will be applied to find the critical grid size.

Next, we consider the case in which each copy of C_n is with at least one half of its area contained in $\mathbb{D}$. Let

$$S_{m2}(V, C) = \min_{C' \cong C, |C' \cap \mathbb{D}| \geq \frac{1}{2}|C'|} \#(V \cap C'), \text{ and}$$
$$\mathcal{S}_{m2}(\mathcal{V}_n, \mathbf{C}_n) = \min_{C_n \in \mathbf{C}_n} S_{m2}(\mathcal{V}_n, C_n).$$

We have the following theorem.

Theorem 3. *For the asymptotics of $\mathcal{S}_{m2}(\mathcal{V}_n, \mathbf{C}_n)$, it is almost sure that*

$$\Pr\left[\frac{\mathcal{S}_{m2}(\mathcal{V}_n, \mathbf{C}_n)}{\ln n} \sim \frac{1}{2}\mathcal{L}(\beta)\right] \to 1.$$

Furthermore, if $\beta < 1$, we almost surely have

$$\Pr[\mathcal{S}_{m2}(\mathcal{V}_n, \mathbf{C}_n) = 0] \to 1.$$

In GGs and RNGs, each edge involves two nodes and there exists an empty region related to this edge. In Section 4.2, to find the largest edge length, we find the maximal size of a corresponding empty region instead by applying Theorem 3. Similarly, a local minimum in greedy forward routing is associated with an empty lens area. In Section 4.3, Theorem 3 will be applied to find the critical transmission radius for local-minimum-free greedy forward routing.

Last, we consider the case in which $\mathbf{C}_n$ is a collection of disks and each copy of scanning disks has its center in $\mathbb{D}$. Without loss of generality, we may assume that the center of $C_n \in \mathbf{C}_n$ is at the origin. For such a disk C, let

$$S_{m3}(V, C) = \min_{a \in \mathbb{D}} \#(V \cap (C + a)),$$

and

$$\mathcal{S}_{m3}(\mathcal{V}_n, \mathbf{C}_n) = \min_{C_n \in \mathbf{C}_n} S_{m3}(\mathcal{V}_n, C_n).$$

We have the following theorem.

Theorem 4. *For the asymptotics of $\mathcal{S}_{m3}(\mathcal{V}_n, \mathbf{C}_n)$, we have*

1. *If $\mathbb{D}$ is a unit-area square, it is almost sure that*

$$\Pr\left[\frac{\mathcal{S}_{m3}(\mathcal{V}_n, \mathbf{C}_n)}{\ln n} \sim \min\left(\frac{1}{2}\mathcal{L}(\beta), \frac{1}{4}\beta\right)\right] \to 1.$$

2. *If $\mathbb{D}$ is a unit-area disk, it is almost sure that*

$$\Pr\left[\frac{\mathcal{S}_{m3}(\mathcal{V}_n, \mathbf{C}_n)}{\ln n} \sim \frac{1}{2}\mathcal{L}(\beta)\right] \to 1.$$

3. *For any $\beta < 1$, we almost surely have*

$$\Pr[\mathcal{S}_{m3}(\mathcal{V}_n, \mathbf{C}_n) = 0] \to 1.$$

The degree of a node is the number of other nodes in the r-disk centered at this node, and the coverage of a point is the number of nodes in the r-disk centered at this point. Both concepts are related to a disk with its center in the deployment region, so we apply Theorem 4 to both problems. Details will be given in Section 4.

To avoid falling in complicated mathematical reasoning, we leave proofs of Theorems 2, 3, and 4 in Appendix C. In the following section, various applications of minimum scan statistics will be discussed. Once again, we emphasize that if the r_n-disk graph model is scaled back to the UDG model, $n\pi r_n^2$ can be read as the average number of neighbors, and β somehow is proportional to the node density.

4 Topology Evolution of Wireless Ad Hoc and Sensor Networks

Minimum scan statistics have a wide range of applications in the research of wireless ad hoc and sensor networks. For example, coverage of wireless sensor networks can be approximated by minimum scan statistics. A sensor system is said to k-cover the deployment region if every point in the deployment region can be monitored by at least k sensors. Here, k is an application-dependent QoS requirement. For instance, in an object tracking system applying triangulation methods, at least three sensors are needed to decide the location of an object. Thus, 3-coverage is a requirement of such a system. Assume each sensor can monitor the area within distance r, i.e., an object at x can be detected by sensor u if x is in $B(u,r)$. If V denote the set of sensors, we have

$$\begin{aligned} coverage &= \min_{x\in\mathbb{D}} \#(\{u \in V : x \in B(u,r)\}) \\ &= \min_{x\in\mathbb{D}} \#(\{u \in V : u \in B(x,r)\}) \\ &= S_{m3}(V, B(\mathbf{o}, r)). \end{aligned}$$

Hence, under this sensing model, coverage and the minimum scan statistic in some sense are the same, and we can use Theorem 4 to approximate the coverage of sensor networks. Especially, if $\beta > 1$, $\mathbb{D}$ is a.a.s. $\Theta(\ln n)$-covered, and if $\beta < 1$, there a.a.s. exists uncovered area. Hence, $\beta = 1$ is the threshold for sensing coverage. Note that although the boundary effects are not explicitly considered here, they are implicitly handled by the definition of S_{m3}.

A graph property is called *monotone increasing* if all supergraphs of a graph with these properties also have these properties as well. Assume $r_n = (\beta + o(1))\sqrt{\frac{\ln n}{\pi n}}$ for some constant $\beta > 0$. A constant α is the *threshold* of a monotone-increasing property Q if $G_{r_n}(\mathcal{V}_n)$ a.a.s. has Q for any $\beta > \alpha$ and $G_{r_n}(\mathcal{V}_n)$ a.a.s. does not have Q for any $\beta < \alpha$. In the rest of this section, based on minimum scan statistics, we will point out the thresholds of several important topological properties in wireless networks. For convenience, let $\beta_0 = \sqrt{1/(\frac{2}{3} - \frac{\sqrt{3}}{2\pi})} \approx 1.6$ in the following discussion.

4.1 The Minimum Degree and Connectivity

The minimum degree of a graph G, denoted by $\delta(G)$, is the minimum nodal degree over all nodes. In r-disk graphs, the degree of node u is the number of nodes in $B(u,r)$ minus 1. Hence,

$$\begin{aligned} \delta(G_{r_n}(\mathcal{V}_n)) &= \min_{u\in\mathcal{V}_n} \deg(u) \\ &= \min_{u\in\mathcal{V}_n} \#(\mathcal{V}_n \cap B(u, r_n)) - 1 \\ &\geq S_{m3}(\mathcal{V}_n, B(\mathbf{o}, r_n)) - 1. \end{aligned}$$

If we can further prove that $\delta(G_{r_n}(\mathcal{V}_n))$ is asymptotically upper bounded by $S_{m3}(\mathcal{V}_n, B(\mathbf{o}, (1+\varepsilon)r_n))$ for any given $\varepsilon > 0$, then $\delta(G_{r_n}(\mathcal{V}_n))$ can be approximated by $S_{m3}(\mathcal{V}_n, B(\mathbf{o}, r))$.

First, consider $\beta \geq 1$. By Theorem 4, for a given $\varepsilon > 0$, there a.a.s. exists a $(1+\varepsilon)r_n$-disk with center in $\mathbb{D}$ covering $S_{m3}(\mathcal{V}_n, B(\mathbf{o}, (1+\varepsilon)r_n))$ nodes. Draw a concentric εr_n-disk of this $(1+\varepsilon)r_n$-disk. The number of nodes in the εr_n-disk is a binomial RV, and it is a.a.s. that at least one node is in the εr_n-disk. For those nodes in the εr_n-disk, their nodal degrees are less than the number of nodes in the $(1+\varepsilon)r_n$-disk, i.e., $S_{m3}(\mathcal{V}_n, B(\mathbf{o}, (1+\varepsilon)r_n))$. So, for any $\beta \geq 1$ and $\varepsilon > 0$, it is a.a.s. that

$$\delta(G_{r_n}(\mathcal{V}_n)) \leq S_{m3}(\mathcal{V}_n, B(\mathbf{o}, (1+\varepsilon)r_n)).$$

Now, consider $\beta < 1$. According to [10, Theorem 1.2] and [15, Theorem 9], it is a.a.s. that $\delta(G_{r_n}(\mathcal{V}_n)) = 0$. Hence, we have the following theorem.

Theorem 5. *Assume $r_n = (\beta + o(1))\sqrt{\frac{\ln n}{\pi n}}$ for some constant $\beta > 0$. We have*

$$\Pr\left[\frac{\delta(G_{r_n}(\mathcal{V}_n))}{\ln n} \sim \frac{S_{m3}(\mathcal{V}_n, B(\mathbf{o}, r_n))}{\ln n}\right] \to 1.$$

It is known that a random geometric graph is a.a.s. k-connected if its minimum degree is k [2]. Therefore, the connectivity of wireless networks can also be estimated by S_{m3}. In addition, if $\beta > 1$, $G_{r_n}(\mathcal{V}_n)$ is a.a.s. $\Theta(\ln n)$-connected; and if $\beta < 1$, $G_{r_n}(\mathcal{V}_n)$ is a.a.s. disconnected. Therefore, $\beta = 1$ is the threshold for connectivity.

4.2 The Longest Edges of Geometric Structures

Geometric structures such as euclidean minimal spanning trees, RNGs [16], GGs [17], Yao's graphs, and Delauney triangulations are widely used in topology control of wireless ad hoc networks [18], [19], [20]. The largest edge lengths of these structures are good references to the configuration of the maximal transmission radius.

4.2.1 Gabriel Graphs

Two nodes u, v have a Gabriel edge between them whenever the disk with segment uv as a diameter contains no other nodes. In addition, since u, v is in $\mathbb{D}$, the disk has at least half of its area in $\mathbb{D}$. Let $\rho_{GG}(V)$ denote the largest edge length of the GG over V. According to Theorem 3, a disk with diameter larger than $(1+\varepsilon)2\sqrt{\frac{\ln n}{\pi n}}$ for some positive constant ε almost surely contains some nodes. This implies

$$\rho_{GG}(\mathcal{P}_n) \leq (1+\varepsilon)2\sqrt{\frac{\ln n}{\pi n}}.$$

It was further proved in [21] that for any constant $\varepsilon > 0$,

$$\lim_{n\to\infty} \Pr\left[1 - \varepsilon \leq \frac{\rho_{GG}(\mathcal{P}_n)}{2\sqrt{\frac{\ln n}{\pi n}}} \leq 1 + \varepsilon\right] = 1.$$

So, $\beta = 2$ is the threshold of the longest GG edge.

4.2.2 Relative Neighborhood Graphs

If u and v are two nodes, let L_{uv} denote the lens of $B(u, \|u-v\|) \cap B(v, \|u-v\|)$. The segment uv is called the waist of L_{uv}. We have $|L_{uv}| = \pi(\frac{\|u-v\|}{\beta_0})^2$. In RNGs, two nodes u, v have an edge between them if and only if there are no other nodes in L_{uv}. Let $\rho_{RNG}(V)$ denote the largest edge length of the RNG over V. According to Theorem 3, any lens whose waist length is larger than $(1+\varepsilon)\beta_0\sqrt{\frac{\ln n}{\pi n}}$ for some positive constant ε a.a.s. is not empty. This implies

$$\rho_{RNG}(\mathcal{P}_n) \leq (1+\varepsilon)\beta_0\sqrt{\frac{\ln n}{\pi n}}.$$

Furthermore, the following theorem can be proved.

Theorem 6. *For any constant $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \Pr\left[1 - \varepsilon \leq \frac{\rho_{RNG}(\mathcal{P}_n)}{\beta_0\sqrt{\frac{\ln n}{\pi n}}} \leq 1 + \varepsilon\right] = 1.$$

So, $\beta = \beta_0$ is the threshold of the longest RNG edge.

4.3 Local-Minimum-Free Geographic Greedy Routing

Geographic greedy routing protocols such as greedy forward routing and grid routing are widely used in wireless ad hoc and sensor networks. However, being greedy algorithms, those routing protocols are suffered from the local minimum problem.

4.3.1 Greedy Forward Routing

In greedy forward routing, each node discards a packet if none of its neighbors is closer to the destination of the packet than itself, or otherwise forward the packet to the neighbor closest to the destination. Packets are discarded at local minima, and thus deliverability is not guaranteed. To remove local minima, all nodes should have sufficiently large transmission radii. Let $\rho_{GFR}(V)$ denote the smallest transmission radius such that there do not exist local minima in the network. $\rho_{GFR}(V)$ is called the critical transmission radius for (local-minimum-free) greedy forward routing.

Let transmission radius $r_n = (\beta + o(1))\sqrt{\frac{\ln n}{\pi n}}$ for some $\beta > 0$. For any pair of nodes u and v, assume u has packets for v. If $\|u - v\| \leq r_n$, of course, packets can be transmitted from u to v. If $\|u - v\| > r_n$, let w denote the intersection point of the circle of radius r_n centered at u and the segment uv. Since $|L_{uw}| = \pi(\frac{\|u-w\|}{\beta_0})^2 = (\frac{\beta}{\beta_0} + o(1))\frac{\ln n}{n}$, according to Theorem 3, there a.a.s. exist nodes in L_{uw}. Since nodes in L_{uw} are neighbors of u and closer to v than u, u is not a local minimum with respect to v. Therefore, there do not exist local minima in the network. In other words,

$$\rho_{GFR}(\mathcal{P}_n) \leq (1 + \varepsilon)\beta_0\sqrt{\frac{\ln n}{\pi n}}.$$

It was further proved in [22] that

$$\lim_{n\to\infty} \Pr\left[1 - \varepsilon < \frac{\rho_{GFR}(\mathcal{P}_n)}{\beta_0\sqrt{\frac{\ln n}{\pi n}}} \leq 1 + \varepsilon\right] = 1.$$

So, $\beta = \beta_0$ is the threshold of the critical transmission radius for greedy forward routing.

4.3.2 Grid Routing

In grid routing, the plane is tessellated into equal-size square cells. Two cells are called neighboring cells if they share a common edge. For a source-destination pair, the cell containing the source node is called the source cell, and the cell containing the destination node is called the destination cell. In addition, the routing distance is defined as the Manhattan distance between the source and destination cells. A packet is directly transmitted from the source node to the destination node if they are in the same cell; otherwise, the packet is relayed by a node in neighboring cells that are closer to the destination cell. Let l denote the size (edge length) of a cell. Then, $\sqrt{5}l$ is the transmission radius such that two nodes in neighboring cells can communicate with each other.

A node is a local minimum (with respect to a given destination node) if there do not exist neighbors in neighboring cells that are closer to the destination cell. If the transmission radius is at least $\sqrt{5}l$ and every cell contains at least one node, the network is local-minimum-free. Let $N_n = \lceil \frac{1}{\beta}\sqrt{\frac{n}{\ln n}} \rceil$ for some constant β and $l_n = \frac{1}{N_n}$. Tessellate the deployment region $\mathbb{D}$ into N_n^2 square cells with length l_n. According to Theorem 2, if $\beta > 1$, every cell contains at least one node. Hence, if $r_n \geq \sqrt{5}l_n$, the network is a.a.s. local-minimum-free. We have the following theorem.

Theorem 7. *Let $l_n = (\beta + o(1))\sqrt{\frac{\ln n}{n}}$ for some $\beta > 0$ and $r_n = \sqrt{5}l_n$. If $\beta > 1$, the network is a.a.s. local-minimum-free.*

4.4 Evolution of Wireless Ad Hoc Networks

For a set of homogeneous wireless devices, by setting transmission radii large enough, the underlying network topology can have desired properties. However, the maximal transmission radius is specified by the hardware. Hence, threshold information is important in the design of wireless systems in order to fully utilize hardware limitations. In this section, we have obtained the thresholds of several network properties. By lining up those thresholds in increasing order, we can see the evolution of network topology. Let β_Q denote the threshold for property Q. Currently, we have $\beta_{Con} = \beta_{Cov} = 1$, $\beta_{RNG} = \beta_{GFR} = \beta_0 \approx 1.6$, and $\beta_{GG} = 2$.

Although $\beta_{Con} = \beta_{Cov}$, for a connected wireless sensor network, the sensing radius is usually set to two times of the transmission radius in order to guarantee the coverage. Similarly, although $\beta_{RNG} = \beta_{GFR}$, there exist some inherent differences between the longest RNG edge problem and the local-minimum-free GFR problem. Actually, for the same set of nodes, the critical transmission radius for RNG is no more than the critical transmission radius for GFR. Those differences cannot be distinguished in the resolution of order $\ln n$. As a future work, it is worth to study those problems under a finer resolution. Especially, we are interested in the transition at each threshold. For example, for properties Q, we may consider the radius given in the form of $r_n = \beta_Q\sqrt{\frac{\ln n + \xi}{n\pi}}$ or even $r_n = \beta_Q\sqrt{\frac{\ln n + c \ln\ln n + \xi}{n\pi}}$.

5 Conclusions and Future Works

In this paper, an analytic tool called minimum scan statistics is introduced. Corresponding to various boundary conditions, three almost sure asymptotics of minimum scan statistics are derived. These results can be applied to several problems raised in the research of wireless ad hoc and sensor networks, including coverage of wireless sensor systems, connectivity of wireless ad hoc networks, local-minimum-free geographic greedy routing, and the largest edge length of geometric structures. In most previous works, analyses only focused on some particular critical points of transmission radii. On the contrary, our results reveal full line evolution of network topology in the resolution of $\Theta(\ln n)$ average number of neighbors. Moreover, instead of applying techniques like toroidal metrics to avoid boundary effects, we explicitly and carefully handle boundary effects. This helps us to find fundamental differences between deployments over disk regions and square regions.

Although minimum scan statistics provide a big picture of network topology evolution, more works are needed in the future. First, the works given in this paper are in the resolution of $\Theta(\ln n)$, but as pointed out in Section 4.4, it is necessary to study related problems in finer resolutions. Second, disk models are idealized but not realistic. We should relax our assumptions and consider more generalized channel models. Third, the works given in this paper are purely analytical. We ought to consider related issues from application perspectives. The impact of imprecise location information should also be carefully evaluated.

Appendix A
The Minimum of a Collection of Poisson RVs

This section is dedicated to the proof of Lemma 1. For any positive integer n, the factorial of n, $n! = 1 \cdot 2 \cdots n$, is estimated by Sterling's formula

$$n! \sim (2\pi n)^{\frac{1}{2}} n^n e^{-n}. \tag{1}$$

Since

$$\frac{\Pr[Po(\lambda) = k-1]}{\Pr[Po(\lambda) = k]} = \frac{\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}}{\frac{\lambda^k}{k!}e^{-\lambda}} = \frac{k}{\lambda},$$

for any $\mu \in (0,1)$, as $\lambda \to \infty$, the lower tail distribution of a Poisson RV can be given by

$$\begin{aligned}\Pr[Po(\lambda) \le \mu\lambda] &= \sum_{k=\mu\lambda}^{0} \Pr[Po(\lambda) = k] \\ &= \sum_{k=0}^{\mu\lambda} \frac{k!\binom{\mu\lambda}{k}}{\lambda^k} \Pr[Po(\lambda) = \mu\lambda] \\ &\sim \sum_{k=0}^{\mu\lambda} \frac{(\mu\lambda)^k}{\lambda^k} \Pr[Po(\lambda) = \mu\lambda] \\ &\sim \frac{1}{1-\mu} \Pr[Po(\lambda) = \mu\lambda].\end{aligned} \tag{2}$$

We further have the following lemma. Remind that functions ϕ and ϕ_-^{-1} have been defined in Section 2.

Lemma 8. *For any $\mu \in (0,1)$, as $\lambda \to \infty$,*

$$\Pr[Po(\lambda) \le \mu\lambda] \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\mu}(1-\mu)} \frac{1}{\sqrt{\lambda}} e^{-\lambda\phi(\mu)}.$$

Proof. From (2) and then applying (1), we have

$$\begin{aligned}\Pr[Po(\lambda) \le \mu\lambda] &\sim \frac{1}{1-\mu} \frac{\lambda^{\mu\lambda}}{(\mu\lambda)!} e^{-\lambda} \\ &\sim \frac{1}{1-\mu} \frac{\lambda^{\mu\lambda}}{\sqrt{2\pi\mu\lambda}(\mu\lambda)^{\mu\lambda} e^{-\mu\lambda}} e^{-\lambda} \\ &= \frac{1}{1-\mu} \frac{1}{\sqrt{2\pi\mu\lambda}\mu^{\mu\lambda}} e^{-\lambda+\mu\lambda} \\ &= \frac{1}{1-\mu} \frac{1}{\sqrt{2\pi\mu\lambda}} e^{-\lambda+\mu\lambda-\mu\lambda\ln\mu} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\mu}(1-\mu)} \frac{1}{\sqrt{\lambda}} e^{-\lambda(1-\mu+\mu\ln\mu)} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\mu}(1-\mu)} \frac{1}{\sqrt{\lambda}} e^{-\lambda\phi(\mu)}.\end{aligned}$$

Thus, the lemma is proved. □

The next lemma gives a.a.s. bounds for the minimum of a collection of Poisson RVs.

Lemma 9. *Assume that $\lim_{n\to\infty} \frac{\lambda_n}{\ln n} = \beta$ for some $\beta > 0$. Let $Y_1, Y_2, \ldots, Y_{I_n}$ be I_n Poisson RVs with means λ_n:*

1. *If $I_n = o(n\sqrt{\ln n})$ and $\beta > 1$, then for any $0 < \mu < \phi_-^{-1}(1/\beta)$, $\min_{i=1,\ldots,I_n} Y_i > \mu\lambda_n$ a.a.s.*
2. *If $I_n = O(\sqrt{\frac{n}{\ln n}})$ and $\beta > \frac{1}{2}$, then for any $0 < \mu < \phi_-^{-1}(\frac{1}{2\beta})$, $\min_{i=1,\ldots,I_n} Y_i > \mu\lambda_n$ a.a.s.*
3. *If $Y_1, Y_2, \ldots, Y_{I_n}$ are independent and $I_n = \Omega(\frac{n}{\ln n})$, then for any μ such that 1) $\phi_-^{-1}(1/\beta) < \mu < 1$ if $\beta > 1$, 2) $0 < \mu < 1$ if $\beta = 1$, or 3) $\mu = 0$ if $\beta < 1$, it is a.a.s. that $\min_{i=1,\ldots,I_n} Y_i \le \mu\lambda_n$.*
4. *If $Y_1, Y_2, \ldots, Y_{I_n}$ are independent and $I_n = \Omega(\sqrt{\frac{n}{\ln n}})$, then for any μ such that 1) $\phi_-^{-1}(\frac{1}{2\beta}) < \mu < 1$ if $\beta > \frac{1}{2}$, 2) $0 < \mu < 1$ if $\beta = \frac{1}{2}$, or 3) $\mu = 0$ if $\beta < \frac{1}{2}$, it is a.a.s. that $\min_{i=1,\ldots,I_n} Y_i \le \mu\lambda_n$.*

Proof. First of all, we conduct two inequalities. Let Y be a Poisson RV with mean λ_n, X_i be the indicator of the event $Y_i \le \mu\lambda_n$, and $X = X_1 + \cdots + X_{I_n}$. Then, X_i is a Bernoulli RV with probability $\Pr[Y_i \le \mu\lambda_n]$, and $\min_{i=1}^{I_n} Y_i \le \mu\lambda_n$ if and only if $X \ge 1$. Thus,

$$\begin{aligned}\Pr\left[\min_{i=1}^{I_n} Y_i \le \mu\lambda_n\right] &= \Pr[X \ge 1] \le E[X] \\ &= \sum_{i=1}^{I_n} E[X_i] = I_n \Pr[Y \le \mu\lambda_n].\end{aligned} \tag{3}$$

In addition, by Lemma 8,

$$\begin{aligned}I_n \Pr[Y \le \mu\lambda_n] &\sim I_n \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\mu}(1-\mu)} \frac{1}{\sqrt{\lambda_n}} e^{-\lambda_n\phi(\mu)} \\ &\sim \frac{1}{\sqrt{2\pi\beta\mu}(1-\mu)} \frac{I_n}{\sqrt{\ln n}\, n^{-(\lambda_n/\ln n)\phi(\mu)}}.\end{aligned} \tag{4}$$

Assume that $I_n = o(n\sqrt{\ln n})$ and $0 < \mu < \phi_-^{-1}(1/\beta)$. From (3) and (4),

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \mu\lambda_n\right] \lesssim \frac{1}{\sqrt{2\pi\beta\mu}(1-\mu)} \frac{I_n}{n\sqrt{\ln n}} n^{1-(\lambda_n/\ln n)\phi(\mu)}.$$

Since ϕ is decreasing over (0, 1] and $0 < \mu < \phi_-^{-1}(1/\beta)$, we have $\phi(\mu) > 1/\beta$ and

$$1 - (\lambda_n/\ln n)\phi(\mu) \to 1 - \beta\phi(\mu) < 0.$$

Thus,

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \mu\lambda_n\right] = o(1).$$

So, Lemma 9 (condition 1) is proved.

Similarly, if $I_n = O(\sqrt{\frac{n}{\ln n}})$ and $0 < \mu < \phi_-^{-1}(\frac{1}{2\beta})$, we have

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \mu\lambda_n\right] \lesssim \frac{1}{\sqrt{2\pi\beta\mu}(1-\mu)} \frac{I_n}{\sqrt{n\ln n}} n^{1/2-(\lambda_n/\ln n)\phi(\mu)},$$

and

$$1/2 - (\lambda_n/\ln n)\phi(\mu) \to 1/2 - \beta\phi(\mu) < 0.$$

Thus,

$$\Pr\left[\min_{i=1}^{I_n} Y_i \leq \mu\lambda_n\right] = o(1).$$

So, Lemma 9 (condition 2) is proved.

Now, assume Y_i's are independent. Then,

$$\begin{aligned}\Pr\left[\min_{i=1}^{I_n} Y_i \leq \mu\lambda_n\right] &= 1 - \Pr\left[\min_{i=1}^{I_n} Y_i > \mu\lambda_n\right] \\ &= 1 - \prod_{i=1}^{I_n} \Pr[Y_i > \mu\lambda_n] \\ &= 1 - \prod_{i=1}^{I_n}(1 - \Pr[Y_i \leq \mu\lambda_n]) \\ &= 1 - (1 - \Pr[Y \leq \mu\lambda_n])^{I_n} \\ &\geq 1 - \left(e^{-\Pr[Y\leq\mu\lambda_n]}\right)^{I_n} \\ &= 1 - e^{-I_n \Pr[Y\leq\mu\lambda_n]}.\end{aligned}$$

If $I_n = \Omega(\frac{n}{\ln n})$, by (4),

$$I_n \Pr[Y \leq \mu\lambda_n] \sim \frac{1}{\sqrt{2\pi\beta\mu}(1-\mu)} \frac{I_n}{n\sqrt{\ln n}} n^{1-(\lambda_n/\ln n)\phi(\mu)}.$$

For any μ such that either 1) $\phi_-^{-1}(1/\beta) < \mu < 1$ if $\beta > 1$ or 2) $0 < \mu < 1$ if $\beta = 1$, since $1 - (\lambda_n/\ln n)\phi(\mu) \to 1 - \beta\phi(\mu) > 0$, we have

$$I_n \Pr[Y \geq \mu\lambda] \to \infty.$$

In addition, if $\beta < 1$ and $\mu = 0$,

$$\begin{aligned}I_n \Pr[Y \leq \mu\lambda] &= I_n \Pr[Y = 0] \\ &= \Omega\left(\frac{n}{\ln n}\right)e^{-\beta\ln n} \to \infty.\end{aligned}$$

So,

$$\Pr\left[\min_{i=1}^{I_n} Y_i \leq \mu\lambda_n\right] \to 1.$$

Thus, Lemma 9 (condition 3) is proved.

Assume Y_i's are independent and $I_n = \Omega(\sqrt{\frac{n}{\ln n}})$. Similarly, for any μ such that 1) $\phi_-^{-1}(\frac{1}{2\beta}) < \mu < 1$ if $\beta > \frac{1}{2}$, 2) $0 < \mu < 1$ if $\beta = \frac{1}{2}$, or 3) $\mu = 0$ if $\beta < \frac{1}{2}$, we have

$$I_n \Pr[Y \leq \mu\lambda_n] \to \infty,$$

and

$$\Pr\left[\min_{i=1}^{I_n} Y_i \leq \mu\lambda_n\right] \geq 1 - e^{-I_n \Pr[Y\leq\mu\lambda_n]} \to 1.$$

So, Lemma 9 (condition 4) is proved. □

Based on Lemma 9, we give the proof of Lemma 1.

Proof of Lemma 1. Consider the first case in which $I_n = o(n\sqrt{\ln n})$. If $\beta' > 1$, let $\mu = \phi_-^{-1}(1/\beta')$. Then, $\mu < \phi_-^{-1}(1/\beta)$. From Lemma 9 (condition 1), it is a.a.s. that

$$\begin{aligned}\min_{i=1}^{I_n} Y_i > \mu\lambda_n &= \phi_-^{-1}(1/\beta')\beta\ln n \\ &> \phi_-^{-1}(1/\beta')\beta'\ln n = \mathcal{L}(\beta')\ln n.\end{aligned}$$

If $0 < \beta' \leq 1$, $\mathcal{L}(\beta') = 0$. Obviously, $\min_{i=1}^{I_n} Y_i \geq \mathcal{L}(\beta')\ln n$. So, Lemma 1 (condition 1) is proved.

Next, consider the second case in which $I_n = O(\sqrt{\frac{n}{\ln n}})$. If $\beta' > \frac{1}{2}$, let $\mu = \phi_-^{-1}(\frac{1}{2\beta'})$. Then, $\mu < \phi_-^{-1}(\frac{1}{2\beta})$. From Lemma 9 (condition 2), it is a.a.s. that

$$\begin{aligned}\min_{i=1}^{I_n} Y_i > \mu\lambda_n &= \phi_-^{-1}\left(\frac{1}{2\beta'}\right)\beta\ln n \\ &> \frac{1}{2}\phi_-^{-1}\left(\frac{1}{2\beta'}\right)(2\beta')\ln n = \frac{1}{2}\mathcal{L}(2\beta')\ln n.\end{aligned}$$

If $0 < \beta' \leq \frac{1}{2}$, $\mathcal{L}(2\beta') = 0$. Obviously, $\min_{i=1}^{I_n} Y_i \geq \frac{1}{2}\mathcal{L}(2\beta')\ln n$. So, Lemma 1 (condition 2) is proved.

Now, consider the third case in which $Y_1, Y_2, \ldots, Y_{I_n}$ are independent and $I_n = \Omega(\frac{n}{\ln n})$. If $\beta > 1$, let $\mu = \phi_-^{-1}(1/\beta')$. We have $1 > \mu > \phi_-^{-1}(1/\beta)$. From Lemma 9 (condition 3), it is a.a.s. that

$$\begin{aligned}\min_{i=1}^{I_n} Y_i \leq \mu\lambda_n &= \phi_-^{-1}(1/\beta')\beta\ln n \\ &< \phi_-^{-1}(1/\beta')\beta'\ln n = \mathcal{L}(\beta')\ln n.\end{aligned}$$

If $\beta = 1$, let $\mu = \phi_-^{-1}(1/\beta')$. We have $1 > \mu > 0$. From Lemma 9 (condition 3), it is a.a.s. that

$$\begin{aligned}\min_{i=1}^{I_n} Y_i \leq \mu\lambda_n &= \phi_-^{-1}(1/\beta')\beta\ln n \\ &< \phi_-^{-1}(1/\beta')\beta'\ln n = \mathcal{L}(\beta')\ln n.\end{aligned}$$

If $\beta < 1$, let $\mu = 0$. From Lemma 9 (condition 3), it is a.a.s. that

$$\min_{i=1}^{I_n} Y_i = 0 \leq \mathcal{L}(\beta')\ln n.$$

So, Lemma 1 (condition 3) is proved.

Finally, consider the last case in which $Y_1, Y_2, \ldots, Y_{I_n}$ are independent and $I_n = \Omega(\sqrt{\frac{n}{\ln n}})$. Similarly, if $\beta \geq \frac{1}{2}$, let $\mu = \phi_-^{-1}(\frac{1}{2\beta'})$; and if $\beta < \frac{1}{2}$, let $\mu = 0$. No matter which one, from Lemma 9 (condition 4), it is a.a.s. that

$$\min_{i=1}^{I_n} Y_i \leq \frac{1}{2}\mathcal{L}(2\beta')\ln n.$$

So, Lemma 1 (condition 4) is proved. □

Appendix B

Geometric Preliminaries

To prove Theorems 2, 3, and 4, we need more geometric techniques.

B.1 Partition of Deployment Regions

If $\mathbb{D}$ is a unit-area square, it is partitioned into $\mathbb{D}_r(0)$, $\mathbb{D}_r(1)$, and $\mathbb{D}_r(2)$ according to r. $\mathbb{D}_r(0)$ consists of all points in $\mathbb{D}$ apart from $\partial\mathbb{D}$ by at least r, $\mathbb{D}_r(1)$ consists of all points in $\mathbb{D}$ apart from some side of $\mathbb{D}$ by less than r and from all other sides by at least r, and $\mathbb{D}_r(2)$ consists of the rest points in $\mathbb{D}$ (see Fig. 4).

If $\mathbb{D}$ is a unit-area disk, it is partitioned into $\mathbb{D}_r(0)$ and $\mathbb{D}_r(1)$ according to r. $\mathbb{D}_r(0) = B(\mathbf{o}, \frac{1}{\sqrt{\pi}} - r)$ is a disk consisting of all points in $\mathbb{D}$ apart from $\partial\mathbb{D}$ by at least r,

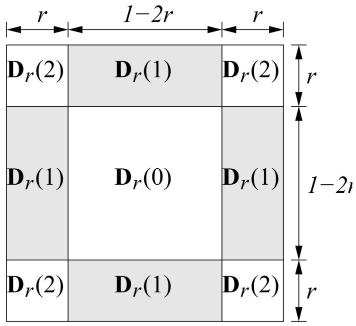

Fig. 4. Partition of the unit-area square $\mathbb{D}$.

and $\mathbb{D}_r(1) = \mathbb{D} \setminus \mathbb{D}_r(0)$ is an annulus centered at $\mathbf{o}$ with radii $\frac{1}{\sqrt{\pi}} - r$ and $\frac{1}{\sqrt{\pi}}$ (see Fig. 5).

B.2 Isodiametric Inequalities

Assume $C \subset \mathbb{R}^2$ is a convex compact set. The r-neighborhood of C, denoted as C_r, is the union of all r-disks with centers in C, i.e., $C_r = \bigcup_{x \in C} B(x, r)$. We use C_{-r} to denote the set of points of C that are apart from ∂C by at least r and use $peri(C)$ to denote the perimeter of C. According to the isodiametric inequality [23], [24], [25], the disk of diameter d has the largest area $\frac{1}{4}\pi d^2$ over all measurable sets with diameter d and also has the longest perimeter πd over all convex compact sets with diameter d.

Lemma 10. *Suppose that $C \subset \mathbb{R}^2$ is a convex compact set with diameter at most d. We have*

$$|C_t - C| < \pi dt + \pi t^2 \quad \text{and} \quad |C_{-t}| \geq |C| - \pi dt.$$

Proof. We will explicitly prove the inequalities for C being a polygon. If C is a convex compact set, the lemma can be proved by applying the fact that C can be approximated by a sequence of polygons contained in C. Thus, we assume C is a polygon.

First, we prove $|C_t - C| < \pi dt + \pi t^2$. At each vertex of C, draw two perpendicular lines to the edges of C. The area $C_t - C$ is divided into disjoint rectangles and sectors (see Fig. 6). All rectangles are with the same width t, and the sum of their length is equal to the perimeter of C. All sectors (marked by x in Fig. 6) are with radius t, and since the angle of each sector is supplementary to its interior angle, the sum of their angles is equal to 2π. Let l denote the perimeter of C, then $|C_t - C| = lt + \pi t^2$. For $diam(C) \leq d$, we have $l < \pi d$ [25]. So, the inequality follows.

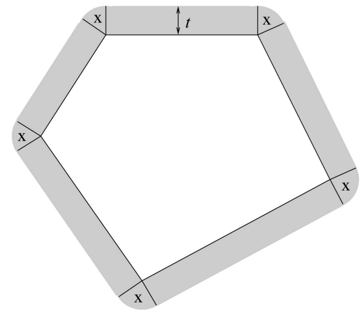

Fig. 6. $C_t - C$ is the shaded area, composed of rectangles and sectors.

Now, we prove $|C_{-t}| \geq |C| - \pi dt$. For each edge of C, draw a rectangle by it with width t toward the inner of C. Since $C - C_{-t}$ is fully covered by these rectangles, we have $|C_{-t}| \geq |C| - peri(C)t$. For all compact sets with diameter d, we have $peri(C) < \pi d$. Thus, $|C_{-r}| \geq |C| - \pi dt$ is proved. □

B.3 Tessellations

A ε-tessellation divides the plane by vertical and horizontal lines into a grid in which each grid cell has width ε. Without loss of generality, we assume the origin is a corner of some cells. In a tessellation, a polyquadrate is a collection of cells intersecting with a convex compact set. For example, in Fig. 7, the shaded cells form a polyquadrate induced by a polygon. The horizontal span of a polyquadrate is the horizontal distance measured in the number of cells from the left to the right. The vertical span of a polyquadrate is defined similarly but in the vertical direction. If the diameter of a polygon is d, the span of the corresponding polyquadrate in a ε-tessellation is at most $\lceil \frac{d}{\varepsilon} \rceil + 1$.

Lemma 11. *Let S be a region composed of m cells. For a positive constant integer τ, the number of polyquadrates with span at most τ and intersecting with S is $\Theta(m)$.*

Proof. For a specified cell, since τ is a constant, the number of polyquadrates that contain the cell and have span at most τ is also a constant (depending on τ). For each cell in S, the number of polyquadrates that contain the cell and have span at most τ is $\Theta(1)$. Since there are m cells in S, the total number of polyquadrates with span at most τ and intersecting with S is $\Theta(m)$. □

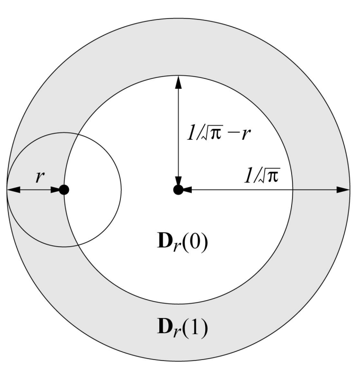

Fig. 5. Partition of the unit-area disk $\mathbb{D}$.

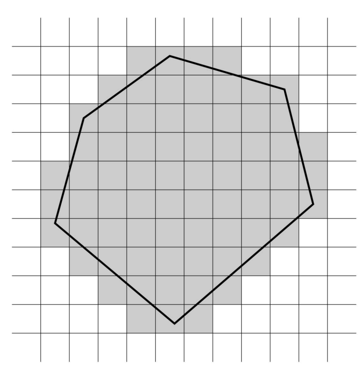

Fig. 7. The cells intersecting with the polygon form a polyquadrate.

APPENDIX C
PROOFS OF THEOREMS 2, 3, AND 4

First, we focus only on Poisson point processes. So, in Appendices C.1, C.2, and C.3, all lemmas are on $\mathcal{P}_n$. Then, applying a de-Poissonization technique given in Appendix C.4, we can extend results to uniform point processes.

C.1 Proof of Theorem 2

Lemma 12. *For any constant $\beta' \in (\beta, \infty)$, it is almost sure that*

$$\Pr[\mathcal{S}_{m1}(\mathcal{P}_n, \mathbf{C}_n) \le \mathcal{L}(\beta') \ln n] \to 1.$$

Especially, if $\beta < 1$, we almost surely have

$$\Pr[\mathcal{S}_{m1}(\mathcal{V}_n, \mathbf{C}_n) = 0] \to 1.$$

Proof. For any $C_n \in \mathbf{C}_n$, let I_n be the number of copies of C_n that can be packed into $\mathbb{D}$, and Y_i denote the number of nodes in the ith copy of C_n. Then, $Y_1, Y_2, \ldots, Y_{I_n}$ are i.i.d. Poisson RVs with rate $\beta \ln n$, and $\mathcal{S}_{m1}(\mathcal{V}_n, \mathbf{C}_n) \le \min_{1 \le i \le I_n} Y_i$. We can tile $\Theta(\frac{n}{\ln n})$ squares with edge length $c_0\sqrt{|C_n|}$ in $\mathbb{D}$ and then place one copy of C_n in each square. These copies of C_n are pairwise disjoint. Thus, $I_n = \Omega(\frac{n}{\ln n})$. By Lemma 1 (condition 3), we have

$$\mathcal{S}_{m1}(\mathcal{P}_n, \mathbf{C}_n) \le \min_{1 \le i \le I_n} Y_i \le \mathcal{L}(\beta') \ln n.$$

Note that if $\beta < 1$, we may choose a $\beta' \in (\beta, 1)$. Since $\mathcal{L}(\beta') = 0$, $\mathcal{S}_{m1}(\mathcal{P}_n, \mathbf{C}_n) = 0$ is a.a.s. implied. Thus, the lemma is proved. □

Lemma 13. *For any constant $0 < \beta' < \beta$, it is almost sure that*

$$\Pr[\mathcal{S}_{m1}(\mathcal{P}_n, \mathbf{C}_n) \ge \mathcal{L}(\beta') \ln n] \to 1.$$

Proof. Let r_n be the inferior over any $C_n \in \mathbf{C}_n$ of the (smallest) distance from the mass center of C_n to ∂C_n. First of all, we prove that $r_n = \Theta(\sqrt{\frac{\ln n}{n}})$. Let a denote the mass center of C_n and b be a point in ∂C_n such that $\|a - b\| = r_n$. Draw two lines l_1 and l_2 that are perpendicular to ab and through a and b, respectively. Since a is the mass center and C_n is convex, a half of C_n is between l_1 and l_2. Draw two more lines l_3 and l_4 parallel with ab such that l_3 and l_4 intersect with ∂C_n and C_n is between l_3 and l_4. Let c_n denote the distance between l_3 and l_4. The rectangle bounded by l_1, l_2, l_3, and l_4 contains one half of C_n, so $r_n \cdot c_n = \Omega(\frac{\ln n}{n})$. Since $diam(C_n) = \Theta(\sqrt{\frac{\ln n}{n}})$, we have $c_n = O(\sqrt{\frac{\ln n}{n}})$. This implies $r_n = \Omega(\sqrt{\frac{\ln n}{n}})$. In addition, $r_n \le diam(C_n)$. Therefore, $r_n = \Theta(\sqrt{\frac{\ln n}{n}})$ is true. Let $\varepsilon_n = \frac{1}{\sqrt{2}}(1 - \sqrt{\frac{\beta'}{\beta}})r_n$, and $M_n = (1/\varepsilon_n)^2$. Since β' and β are fixed, we have $\varepsilon_n = \Theta(\sqrt{\frac{\ln n}{n}})$, and $M_n = \Theta(\frac{n}{\ln n})$.

Divide $\mathbb{D}$ by a ε_n-tessellation. The distance between any two points in a cell is at most $\sqrt{2}\varepsilon_n$. We claim that any copy of C_n that is fully contained in $\mathbb{D}$ must contain a polyquadrate that is with span at most $\Theta(1)$ and with area at least $(\beta' + o(1))\frac{\ln n}{n}$. Let A be a copy of C_n fully contained in $\mathbb{D}$, and P be the maximal polyquadrate contained in A. P contains $\sqrt{\frac{\beta'}{\beta}}A$ since the (smallest) distance between ∂A and $\partial\sqrt{\frac{\beta'}{\beta}}A$ is at least $(1 - \sqrt{\frac{\beta'}{\beta}})r_n = \sqrt{2}\varepsilon_n$. Thus,

$$|P| \ge \left|\sqrt{\frac{\beta'}{\beta}}A\right| = \frac{\beta'}{\beta}|C_n| = (\beta' + o(1))\frac{\ln n}{n},$$

and the span of P is at most $m = \lceil \frac{diam(C_n)}{\varepsilon_n} \rceil + 1 = \Theta(1)$, asymptotically depending only on β, β', and c_0. In addition, if Y is the number of nodes in P, Y is a Poisson RV with rate at least $(\beta' + o(1))\ln n$. So, our claim is true. Now, consider all polyquadrates that are contained in $\mathbb{D}$ with span at most m and area at least $(\beta' + o(1))\frac{\ln n}{n}$. Let I_n denote the number of those polyquadrates, and Y_i denote the number of nodes in the ith polyquadrate. Then, we have $\mathcal{S}_{m1}(\mathcal{P}_n, \mathbf{C}_n) \ge \min_{i=1}^{I_n} Y_i$. In addition, from Lemma 11, $I_n = \Theta(M_n) = \Theta(\frac{n}{\ln n})$. Then, applying Lemma 1 (condition 1), we a.a.s. have

$$\mathcal{S}_{m1}(\mathcal{P}_n, \mathbf{C}_n) \ge \mathcal{L}(\beta') \ln n.$$

Thus, the lemma is proved. □

Theorem 2 is proved by Lemmas 12 and 13 and the de-Poissonization technique given in Appendix C.4.

C.2 Proof of Theorem 3

Lemma 14. *For any constant $\beta' \in (\beta, \infty)$, we have*

$$\Pr\left[\mathcal{S}_{m2}(\mathcal{P}_n, \mathbf{C}_n) \le \frac{1}{2}\mathcal{L}(\beta') \ln n\right] \to 1.$$

Especially, if $\beta < 1$, it is almost sure that

$$\Pr[\mathcal{S}_{m2}(\mathcal{P}_n, \mathbf{C}_n) = 0] \to 1.$$

Proof. For any $C_n \in \mathbf{C}_n$, place pairwise disjoint copies of C_n along the boundary of $\mathbb{D}$ such that each of them has exactly one half area in $\mathbb{D}$. Let I_n be the number of copies of C_n, and Y_i denote the number of nodes in the ith copy. Then, $Y_1, Y_2, \ldots, Y_{I_n}$ are i.i.d. Poisson RVs with rate $(\frac{1}{2}\beta + o(1))\ln n$. Since $diam(C_n) = \Theta(\sqrt{\frac{\ln n}{n}})$ for any $C_n \in \mathbf{C}_n$, we may have $I_n = \Theta(\sqrt{\frac{n}{\ln n}})$. From Lemma 1 (condition 4), we have

$$\mathcal{S}_{m2}(\mathcal{P}_n, \mathbf{C}_n) \le \min_{1 \le i \le I_n} Y_i \le \frac{1}{2}\mathcal{L}(\beta') \ln n.$$

Furthermore, if $\beta < 1$, we may choose a $\beta' \in (\beta, 1)$, and then $\mathcal{S}_{m2}(\mathcal{P}_n, \mathbf{C}_n) = 0$ is a.a.s. implied. So, the lemma is proved. □

Lemma 15. *For any constant $0 < \beta' < \beta$, we have*

$$\Pr\left[\mathcal{S}_{m2}(\mathcal{P}_n, \mathbf{C}_n) > \frac{1}{2}\mathcal{L}(\beta') \ln n\right] \to 1.$$

Proof. We will apply a similar argument used in the proof of Lemma 13. Let r_n be the inferior of the (smallest) distance from the mass center of C_n to ∂C_n over all $C_n \in \mathbf{C}_n$, and $\varepsilon_n = \frac{1}{2\sqrt{2}}(1-\sqrt{\frac{\beta'}{\beta}})r_n$. We have $r_n = \Theta(\sqrt{\frac{\ln n}{n}})$ and $\varepsilon_n = \Theta(\sqrt{\frac{\ln n}{n}})$. Divide $\mathbb{D}$ by a ε_n-tessellation. Obviously, the distance between any two points in a cell is at most $\sqrt{2}\varepsilon_n$. Consider the collection of polyquadrates each of which is the maximal one contained in the intersection of $\mathbb{D}$ and a copy of C_n with at least half area in $\mathbb{D}$. Let I_n denote the number of those polyquadrates, and Y_i denote the number of nodes in the ith polyquadrate. Then,

$$\mathcal{S}_{m2}(\mathcal{P}_n, \mathbf{C}_n) \geq \min_{1\leq i\leq I_n} Y_i.$$

$Y_1, Y_2, \ldots, Y_{I_n}$ are categorized into two groups. First, we consider polyquadrates that are contained in copies of C_n fully contained in $\mathbb{D}$. All these polyquadrates are with span at most $m = \lceil\frac{diam(C_n)}{\varepsilon_n}\rceil + 1 = \Theta(1)$ and with area at least $(\beta' + o(1))\frac{\ln n}{n}$. Let $I_{0,n}$ denote the number of polyquadrates, and $Y_{0,i}$ denote the number of nodes in the ith polyquadrate. All $Y_{0,i}$'s are Poisson RVs with rate at least $(\beta' + o(1))\ln n$. From Lemma 11, $I_{0,n} = \Theta(\frac{1}{\varepsilon_n^2}) = \Theta(\frac{n}{\ln n})$. Applying Lemma 1 (condition 1), it is a.a.s. that

$$\min_{i=1}^{I_{0,n}} Y_{0,i} \geq \mathcal{L}(\beta')\ln n.$$

Next, we consider polyquadrates that are contained in copies of C_n not fully contained but with at least half area in $\mathbb{D}$. All these polyquadrates are with span at most $m = \lceil\frac{diam(C_n)}{\varepsilon_n}\rceil + 1 = \Theta(1)$ and with area at least $(\frac{1}{2}\beta' + o(1))\frac{\ln n}{n}$. Let $I_{1,n}$ denote the number of polyquadrates, and $Y_{1,i}$ denote the number of nodes in the ith polyquadrate. All $Y_{1,i}$'s are Poisson RVs with rate at least $(\frac{1}{2}\beta' + o(1))\ln n$. From Lemma 11, $I_{1,n} = \Theta(\frac{1}{\varepsilon_n}) = \Theta(\sqrt{\frac{n}{\ln n}})$. Applying Lemma 1 (condition 2), it is a.a.s. that

$$\min_{i=1}^{I_{1,n}} Y_{1,i} \geq \frac{1}{2}\mathcal{L}(\beta')\ln n.$$

Thus, we have

$$\mathcal{S}_{m2}(\mathcal{P}_n, C_n) \geq \min\left(\min_{i=1}^{I_{0,n}} Y_{0,i}, \min_{i=1}^{I_{1,n}} Y_{1,i}\right) \geq \frac{1}{2}\mathcal{L}(\beta')\ln n,$$

and the lemma is proved. □

Theorem 3 is proved by Lemmas 14 and 15 and the de-Poissonization argument.

C.3 Proof of Theorem 4

Remind that here $\mathbf{C}_n$ is a collection of disks whose centers are at the origin.

Lemma 16. *For any constant $\beta' \in (\beta, \infty)$, if $\mathbb{D}$ is a square, we a.a.s. have*

$$\Pr\left[\mathcal{S}_{m3}(\mathcal{P}_n, \mathbf{C}_n) \leq \min\left(\frac{1}{2}\mathcal{L}(\beta'), \frac{1}{4}\beta'\right)\ln n\right] \to 1,$$

if $\mathbb{D}$ is a disk, we a.a.s. have

$$\Pr\left[\mathcal{S}_{m3}(\mathcal{P}_n, \mathbf{C}_n) \leq \frac{1}{2}\mathcal{L}(\beta')\ln n\right] \to 1.$$

Especially, if $\beta < 1$, it is almost sure that

$$\Pr[\mathcal{S}_{m3}(\mathcal{P}_n, \mathbf{C}_n) = 0] \to 1.$$

Proof. For any $C_n \in \mathbf{C}_n$, place pairwise disjoint copies of C_n with centers in $\partial\mathbb{D}$. Let I_n be the number of copies of C_n and Y_i denote the number of nodes in the ith copy. Then, $Y_1, Y_2, \ldots, Y_{I_n}$ are i.i.d. Poisson RVs with rate at most $(\frac{1}{2}\beta + o(1))\ln n$, and we may have $I_n = \Theta(\sqrt{\frac{n}{\ln n}})$. From Lemma 1 (condition 4), we have

$$\mathcal{S}_{m3}(\mathcal{P}_n, \mathbf{C}_n) \leq \min_{1\leq i\leq I_n} Y_i \leq \frac{1}{2}\mathcal{L}(\beta')\ln n.$$

In addition, if $\mathbb{D}$ is a square, we consider the copy of C_n with its center at $(\frac{1}{2}, \frac{1}{2})$, a vertex of $\mathbb{D}$. Let Y denote the number of nodes in the intersection of the disk and $\mathbb{D}$. We have $Y = Po(\frac{1}{4}\beta\ln n)$, and

$$\mathcal{S}_{m3}(\mathcal{P}_n, \mathbf{C}_n) \leq Y \leq \frac{1}{4}\beta'\ln n.$$

Therefore, the lemma is proved. □

Lemma 17. *For any constant $0 < \beta' < \beta$, if $\mathbb{D}$ is a square, we almost surely have*

$$\Pr\left[\mathcal{S}_{m3}(\mathcal{P}_n, \mathbf{C}_n) \geq \min\left(\frac{1}{2}\mathcal{L}(\beta'), \frac{1}{4}\beta'\right)\ln n\right] \to 1,$$

if $\mathbb{D}$ is a disk, we almost surely have

$$\Pr\left[\mathcal{S}_{m3}(\mathcal{P}_n, \mathbf{C}_n) \geq \frac{1}{2}\mathcal{L}(\beta')\ln n\right] \to 1.$$

Proof. Choose a $\beta_1 \in (\beta', \beta)$. Let r and r' be given by $n\pi r^2 = \beta\ln n$ and $n\pi r'^2 = \beta_1\ln n$, respectively. Let $M_n = \frac{\sqrt{2}}{r-r'}$ and $\varepsilon_n = 1/M_n$. Divide $\mathbb{D}$ by a ε_n-tessellation, and then for each cell, draw a r'-disk with its center in the intersection of this cell and $\mathbb{D}$. Since the distance between any two points in a cell is at most $\sqrt{2}\varepsilon_n = r - r'$, any r-disk with center in $\mathbb{D}$ must contain at least one of these r'-disks. Let I_n denote the number of these r'-disks, and Y_i denote the number of nodes in the ith r'-disk. Then,

$$\mathcal{S}_{m3}(\mathcal{P}_n, \mathbf{C}_n) \geq \min_{1\leq i\leq I_n} Y_i.$$

If $\mathbb{D}$ is a square, we partition $Y_1, Y_2, \ldots, Y_{I_n}$ into three groups. First, we consider cells contained in $\mathbb{D}(0)$, and let N_0 denote the number of cells. For these cells, we have $Y_i = Po(\beta_1\ln n)$ and

$$N_0 \sim \left(\frac{1-2r}{\frac{r-r'}{\sqrt{2}}}\right)^2 = \left(\frac{\sqrt{2}(1-2r)}{r\left(\frac{1-r'}{r}\right)}\right)^2 = \Theta\left(\frac{n}{\ln n}\right).$$

From Lemma 1 (condition 1), it is a.a.s. that

$$\min_{1\leq i\leq N_0} Y_i \geq \mathcal{L}(\beta')\ln n.$$

Next, we consider cells intersecting with $\mathbb{D}(1)$ but not with $\mathbb{D}(2)$, and let N_1 denote the number of cells. For these cells, we have $Y_i \geq Po(\frac{1}{2}\beta_1 \ln n)$ and

$$N_1 \sim \frac{r(1-2r)}{\left(\frac{r-r'}{\sqrt{2}}\right)^2} = \frac{2(1-2r)}{r\left(1-\frac{r'}{r}\right)^2} = \Theta\left(\sqrt{\frac{n}{\ln n}}\right).$$

From Lemma 1 (condition 2), it is a.a.s. that

$$\min_{1\leq i\leq N_1} Y_i \geq \frac{1}{2}\mathcal{L}(\beta')\ln n.$$

Last, we consider cells intersecting with $\mathbb{D}(2)$, and let N_2 denote the number of cells. For these cells, we have $Y_i \geq Po(\frac{1}{4}\beta_1 \ln n)$ and

$$N_2 < \left(\frac{r}{\varepsilon}+1\right)^2 < \left(\frac{\sqrt{2}+(r-r')}{1-\frac{r'}{r}}+1\right)^2 = \Theta(1).$$

Then, it is a.a.s. that

$$\min_{1\leq i\leq N_2} Y_i \geq \frac{1}{4}\beta' \ln n.$$

So, if $\mathbb{D}$ is a square, the lemma is proved.

If $\mathbb{D}$ is a disk, we separate $Y_1, Y_2, \ldots, Y_{I_n}$ into two groups. First, we consider cells contained in $\mathbb{D}(0)$, and let N_0 denote the number of cells. For these cells, we have $Y_i = Po(\beta_1 \ln n)$ and $N_0 = \Theta(\frac{n}{\ln n})$. From Lemma 1 (condition 1), it is a.a.s. that

$$\min_{1\leq i\leq N_0} Y_i \geq \mathcal{L}(\beta')\ln n.$$

Next, we consider cells not fully contained in $\mathbb{D}(0)$, and let N_1 denote the number of cells. For these cells, we have $Y_i \geq Po((\frac{1}{2}\beta_1 + o(1))\ln n)$ and $N_1 = \Theta(\sqrt{\frac{n}{\ln n}})$. From Lemma 1 (condition 2),

$$\min_{1\leq i\leq N_1} Y_i \geq \frac{1}{2}\mathcal{L}(\beta')\ln n.$$

So, if $\mathbb{D}$ is a disk, the lemma is proved. □

Theorem 4 is proved by Lemmas 16 and 17 and the de-Poissonization argument.

C.4 De-Poissonization

By Chebyshev inequality, it is almost sure that $Po(n - n^{\frac{3}{4}}) \leq n \leq Po(n + n^{\frac{3}{4}})$. Thus, an instance of $\mathcal{P}_{n-n^{\frac{3}{4}}}$ may be generated by $\mathcal{X}_n$ followed by removing $n - Po(n - n^{\frac{3}{4}})$ points, and an instance of $\mathcal{P}_{n+n^{\frac{3}{4}}}$ may be generated by $\mathcal{X}_n$ followed by adding $Po(n + n^{\frac{3}{4}}) - n$ random points. So, it is almost sure that

$$\mathcal{S}_m\left(\mathcal{P}_{n-n^{\frac{3}{4}}}, C_n\right) \leq \mathcal{S}_m(\mathcal{X}_n, C_n) \leq \mathcal{S}_m\left(\mathcal{P}_{n+n^{\frac{3}{4}}}, C_n\right).$$

For $|C_n| = (\beta + o(1))\frac{\ln n}{n}$, we have

$$\frac{\left(n-n^{\frac{3}{4}}\right)|C_n|}{\ln\left(n-n^{\frac{3}{4}}\right)} \sim \frac{\left(n+n^{\frac{3}{4}}\right)|C_n|}{\ln\left(n+n^{\frac{3}{4}}\right)} \sim \frac{n|C_n|}{\ln n} = \beta + o(1),$$

and

$$\mathcal{S}_m\left(\mathcal{P}_{n-n^{\frac{3}{4}}}, C_n\right) \sim \mathcal{S}_m\left(\mathcal{P}_{n+n^{\frac{3}{4}}}, C_n\right) \sim \mathcal{S}_m(\mathcal{P}_n, C_n).$$

Therefore,

$$\mathcal{S}_m(\mathcal{P}_n, C_n) \sim \mathcal{S}_m(\mathcal{X}_n, C_n).$$

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