Applied Mathematics Letters 22 (2009) 1429–1431

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Mutually orthogonal hamiltonian connected graphs

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ARTICLE INFO

Article history:
Received 11 June 2007
Received in revised form 6 January 2009
Accepted 6 January 2009

Keywords:
Hamiltonian
Hamiltonian connected
Interconnection networks

ABSTRACT

In this work, we concentrate on those n-vertex graphs G with $n \geq 4$ and $\bar{e} \leq n-4$. Let $P_1 = \langle u_1, u_2, \dots, u_n \rangle$ and $P_2 = \langle v_1, v_2, \dots, v_n \rangle$ be any two hamiltonian paths of G. We say that P_1 and P_2 are *orthogonal* if $u_1 = v_1$, $u_n = v_n$, and $u_q \neq v_q$ for $q \in \{2, n-1\}$. We say that a set of hamiltonian paths $\{P_1, P_2, \dots, P_s\}$ of G are *mutually orthogonal* if any two distinct paths in the set are orthogonal. We will prove that there are at least two orthogonal hamiltonian paths of G between any two different vertices. Furthermore, we classify the cases such that there are exactly two orthogonal hamiltonian paths of G between any two different vertices. Aside from these special cases, there are at least three mutually orthogonal hamiltonian paths of G between any two different vertices.

1. Introduction

In this work, a network is represented as a loopless undirected graph. For graph definitions and notation we follow [1]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. Let S be a subset of V. The subgraph of G *induced by* S is the graph $G[S]$ with $V(G[S]) = S$ and $E(G[S]) = \{(u, v) \mid (u, v) \in E,$ and $u, v \in S\}$. The *complement* $\bar{G}$ of a graph G is with the same vertex set $V(G)$ defined by $(u, v) \in E(\bar{G})$ if and only if $(u, v) \notin E(G)$. We use $\bar{e}$ to denote $|E(\bar{G})|$. The *degree* of a vertex u of G, $\deg_G(u)$, is the number of edges incident with u. A *path*, $\langle v_0, v_1, v_2, \dots, v_k \rangle$, is an ordered list of distinct vertices such that v_i and v_{i+1} are adjacent for $0 \leq i \leq k-1$. A path is a *hamiltonian path* if its vertices are distinct and span V. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. A *cycle*, $\langle v_0, v_1, \dots, v_k, v_0 \rangle$, is a path with at least three vertices such that the first vertex is the same as the last vertex. A cycle is a *hamiltonian cycle* if it traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle.

Let $P_1 = \langle u_1, u_2, \dots, u_n \rangle$ and $P_2 = \langle v_1, v_2, \dots, v_n \rangle$ be any two hamiltonian paths of an n-vertex hamiltonian connected graph G. We say that P_1 and P_2 are *orthogonal* if $u_1 = v_1$, $u_n = v_n$, and $u_q \neq v_q$ for $q \in \{2, n-1\}$. We say that a set of hamiltonian paths $\{P_1, P_2, \dots, P_s\}$ of G are *mutually orthogonal* if any two distinct paths in the set are orthogonal.

In this work, we concentrate on those n-vertex graphs G with $n \geq 4$ and $\bar{e} \leq n-4$. By the famous Ore's Theorem [2], G is hamiltonian connected. Yet, we will prove that there are at least two orthogonal hamiltonian paths of G between any two different vertices. Furthermore, we classify the cases such that there are exactly two orthogonal hamiltonian paths of G between any two different vertices. Thus, there are at least three mutually orthogonal hamiltonian paths of G between any two different vertices except for the cases mentioned above. This result can be used to compute the fault-tolerant hamiltonian connectivity of the WK-recursive networks [3].

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0893-9659/$ – see front matter
doi:10.1016/j.aml.2009.01.058

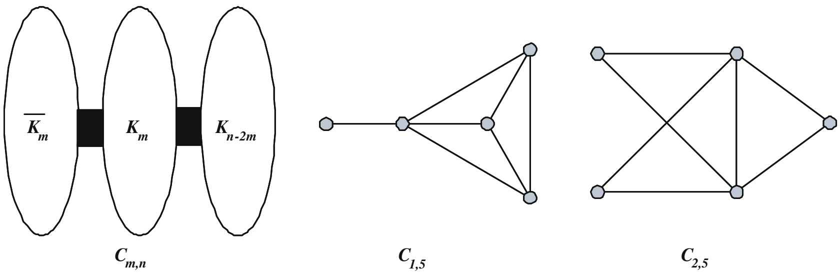

Fig. 1. Illustration of $C_{m,n}$.

2. Mutually orthogonal hamiltonian paths

The following theorem is proved by Ore [2].

Theorem 1 ([2]). *Assume that G is an n-vertex graph with $n \geq 4$. Then G is hamiltonian if $\bar{e} \leq n-3$, and is hamiltonian connected if $\bar{e} \leq n-4$.*

Let G and H be two graphs. We use $G+H$ to denote the disjoint union of G and H. We use $G \vee H$ to denote the graph obtained from $G+H$ by joining each vertex of G to each vertex of H. For $1 \leq m < n/2$, let $C_{m,n}$ be the graph $(\bar{K}_m + K_{n-2m}) \vee K_m$. See Fig. 1 for an illustration.

The following theorem is proved by Chvátal [4].

Theorem 2 ([4]). *If G is an n-vertex graph where $n \geq 3$ and $|E(G)| > C_2^{n-1} + 1$, then G is hamiltonian. Moreover, the only non-hamiltonian graphs with n vertices and $C_2^{n-1} + 1$ edges are $C_{1,n}$ and, for $n = 5$, $C_{2,5}$.*

Suppose that G is an n-vertex graph with $\bar{e} \leq n-4$. Assume that $n = 4$. Obviously, G is isomorphic to K_4. It is easy to check that there are exactly two orthogonal hamiltonian paths between any two distinct vertices of G.

Assume that $n = 5$. Obviously, G is isomorphic either to K_5 or to $K_5 - e$ where e is any edge of K_5. We label the vertices of K_5 with $\{1, 2, 3, 4, 5\}$ and we set $e = (1, 2)$. Suppose that G is isomorphic to K_5. It is easy to check that there are exactly three mutually orthogonal hamiltonian paths of G between any two vertices. Suppose that G is isomorphic to $K_5 - (1, 2)$. By brute force, we can check that there are exactly three mutually orthogonal hamiltonian paths between vertices 1 and 2. However, there are exactly two orthogonal hamiltonian paths between the remaining pairs.

Now, we assume that $n \geq 6$. Let s and t be any two distinct vertices of G. Let H be the subgraph of G induced by the remaining $(n-2)$ vertices of G. We have the following two cases:

Case 1: H is hamiltonian. We can label the vertices of H with $\{0, 1, 2, \ldots, n-3\}$ such that $\langle 0, 1, 2, \ldots, n-3, 0\rangle$ forms a hamiltonian cycle of H. We use the notation $[i]$ to denote i **mod** $(n-2)$. Let Q denote the set $\{i \mid (s, [i+1]) \in E(G) \text{ and } (i, t) \in E(G)\}$. Since $\bar{e} \leq n-4$, $|Q| \geq n-2-(n-4) = 2$. There are at least two elements q_1, q_2 in Q. We set P_j as $\langle s, [q_j+1], [q_j+2], \ldots, [q_j], t\rangle$ for $j = 1, 2$. Then P_1 and P_2 are two orthogonal hamiltonian paths between s and t.

Suppose that $\bar{e} \leq n-5$, $(s, t) \notin E$, or H is not isomorphic to the complete graph K_{n-2}. Then $|Q| \geq 3$. Let q_1, q_2, and q_3 be the three elements in Q. For $j = 1, 2$, and 3, we set P_j as $\langle s, [q_j+1], [q_j+2], \ldots, [q_j], t\rangle$. Then P_1, P_2, and P_3 are three mutually orthogonal hamiltonian paths between s and t.

Thus, we consider $\bar{e} = n-4$, $(s, t) \in E$, and H is isomorphic to the complete graph K_{n-2}. Let ST be the set of vertices in H that are adjacent to s and t, let $S\bar{T}$ be the set of vertices in H that are adjacent to s but not adjacent to t, let $\bar{S}T$ be the set of vertices in H that are not adjacent to s but adjacent to t, and let $\bar{S}\bar{T}$ be the set of vertices in H that are neither adjacent to s nor adjacent to t.

Let $a = |ST|$, $b = |S\bar{T}|$, $c = |\bar{S}T|$, and $d = |\bar{S}\bar{T}|$. Without loss of generality, we assume that $\deg_G(s) \geq \deg_G(t)$. Then $b \geq c$, $b + c + 2d = n-4$, and $a + b + c + d = n-2$. Thus, $a - d = 2$. Hence, $a \geq 2$.

Suppose $a \geq 3$. Let q_1, q_2, and q_3 be three vertices in ST and $q_4, q_5, \ldots, q_{n-2}$ be the remaining vertices of H. We set P_1 as $\langle s, q_1, q_2, X, q_3, t\rangle$, P_2 as $\langle s, q_2, q_3, Y, q_1, t\rangle$, and P_3 as $\langle s, q_3, Z, q_1, q_2, t\rangle$ where X, Y, and Z are any permutations of $q_4, q_5, \ldots, q_{n-2}$. Obviously, P_1, P_2, and P_3 are three mutually orthogonal hamiltonian paths between s and t.

Suppose $a = 2$. Then $d = 0$. Suppose $c \geq 1$. Then $b \geq 1$. We rearrange the vertices of H so that 0 is a vertex in $S\bar{T}$, 1 and 2 are the vertices in ST, 3 is a vertex in $\bar{S}T$, and $4, 5, \ldots, n-3$ are the remaining vertices. Obviously, $\langle 0, 1, 2, \ldots, n-3, 0\rangle$ forms a hamiltonian cycle of H. Let Q denote the set $\{i \mid (s, i) \in E(G) \text{ and } ([i+1], t) \in E(G)\}$. Obviously, $|Q| \geq 3$. Thus, there are three mutually orthogonal hamiltonian paths between s and t.

Finally, we consider $a = 2$, $d = 0$, and $c = 0$. Thus, $b = n-4$. In this case, s is adjacent to t and all the vertices in H; t is adjacent to s and exactly two vertices in H, say q_1 and q_2. Let $\langle s = v_1, v_2, \ldots, v_n = t\rangle$ be a hamiltonian path of G between s and t. Obviously, v_{n-1} is either q_1 or q_2. Therefore, there are exactly two orthogonal hamiltonian paths between s and t.

Case 2: H is non-hamiltonian. There are exactly $(n-2)$ vertices in H. By Theorem 2, there are exactly $(n-4)$ edges in the complement of H and H is isomorphic to $C_{1,n-2}$ or $C_{2,5}$. Hence, s is adjacent to $V(G) - \{s\}$ and t is adjacent to $V(G) - \{t\}$.

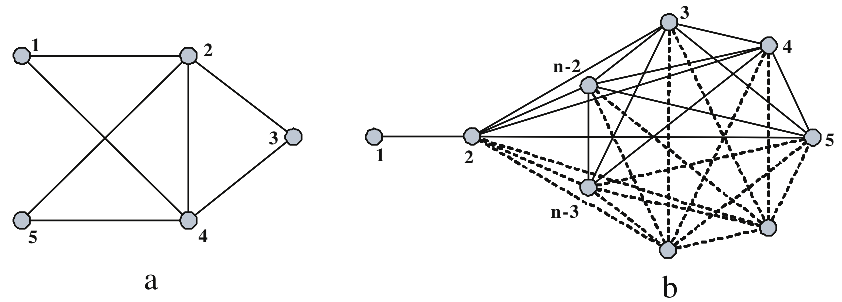

Fig. 2. (a) $C_{2,5}$ and (b) $C_{1,n-2}$.

We can construct two orthogonal hamiltonian paths of G between s and t as the following cases:

Subcase 2.1: H is isomorphic to $C_{2,5}$. We label the vertices of $C_{2,5}$ with $\{1, 2, 3, 4, 5\}$ as shown in Fig. 2(a). Let $P_1 = \langle s, 1, 2, 3, 4, 5, t\rangle$ and $P_2 = \langle s, 3, 4, 5, 2, 1, t\rangle$. Then P_1 and P_2 form the required orthogonal paths. By brute force, we can check that there are exactly two orthogonal hamiltonian paths between s and t.

Subcase 2.2: H is isomorphic to $C_{1,n-2}$. We label the vertices of $C_{1,n-2}$ with $\{1, 2, \ldots, n-2\}$ as shown in Fig. 2(b). Let $P_1 = \langle s, 1, 2, 3, \ldots, n-2, t\rangle$ and $P_2 = \langle s, 3, 4, \ldots, n-2, 2, 1, t\rangle$. Then P_1 and P_2 form the orthogonal hamiltonian paths. Let $\langle s = v_1, v_2, \ldots, v_n = t\rangle$ be any hamiltonian path of G between s and t. Obviously, 1 is either v_2 or v_{n-1}. Therefore, there are exactly two orthogonal hamiltonian paths between s and t.

From the above discussions, we have the following theorem.

Theorem 3. *Assume that G is an n-vertex graph with $n \geq 4$ and $\bar{e} \leq n-4$. Let s and t be any two vertices of G. Then there are at least two orthogonal hamiltonian paths of G between s and t. Moreover, there are at least three mutually orthogonal hamiltonian paths of G between s and t except for the following cases:*

(1) *G is isomorphic to K_4 where s and t are any two vertices of G.*
(2) *G is isomorphic to $K_5 - (1, 2)$ where s and t are any two vertices except for $\{s, t\} = \{1, 2\}$.*
(3) *The subgraph H induced by $V(G) - \{s, t\}$ is a complete graph with $n \geq 6$ where s is adjacent to t and all the vertices in H and t is adjacent to s and exactly two vertices in H.*
(4) *The subgraph induced by $V(G) - \{s, t\}$ is isomorphic to $C_{2,5}$ where s is adjacent to $V(G) - \{s\}$ and t is adjacent to $V(G) - \{t\}$.*
(5) *The subgraph induced by $V(G) - \{s, t\}$ is isomorphic to $C_{1,n-2}$ with $n \geq 6$ where s is adjacent to $V(G) - \{s\}$ and t is adjacent to $V(G) - \{t\}$.*

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