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Citation: Chaos: An Interdisciplinary Journal of Nonlinear Science **19**, 033131 (2009); doi: 10.1063/1.3212941
View online: http://dx.doi.org/10.1063/1.3212941
View Table of Contents: http://scitation.aip.org/content/aip/journal/chaos/19/3?ver=pdfcov
Published by the AIP Publishing

Coordinate transformation and matrix measure approach for synchronization of complex networks

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(Received 29 April 2009; accepted 6 August 2009; published online 8 September 2009)

Global synchronization in complex networks has attracted considerable interest in various fields. There are mainly two analytical approaches for studying such time-varying networks. The first approach is Lyapunov function-based methods. For such an approach, the connected-graph-stability (CGS) method arguably gives the best results. Nevertheless, CGS is limited to the networks with cooperative couplings. The matrix measure approach (MMA) proposed by Chen, although having a wider range of applications in the network topologies than that of CGS, works for smaller numbers of nodes in most network topologies. The approach also has a limitation with networks having partial-state coupling. Other than giving yet another MMA, we introduce a new and, in some cases, optimal coordinate transformation to study such networks. Our approach fixes all the drawbacks of CGS and MMA. In addition, by merely checking the structure of the vector field of the individual oscillator, we shall be able to determine if the system is globally synchronized. In summary, our results can be applied to rather general time-varying networks with a large number of nodes.
 [DOI: 10.1063/1.3212941]

Synchronization of networks of dynamical systems is frequently observed in nature and technology.[1,2] Recently, the study of synchronization phenomena in complex networks with different topologies has received much attention.[3–15] There are mainly two analytical approaches for studying such time-varying networks. The first approach is Lyapunov function-based methods. For such an approach, the connected-graph-stability (CGS) method arguably gives the best results. Nevertheless, CGS is limited to the networks with cooperative couplings. The matrix measure approach (MMA) proposed by Chen, despite a wider range of applications in the network topologies than that of CGS, works for smaller numbers of nodes in most network topologies. The approach also has a limitation with networks having partial-state coupling. In the current work, generalizing our previous work,[26] which considered time-independent networks, we are able to fix all the drawbacks of CGS and MMA. In addition, by merely checking the structure of the vector field of the individual oscillator, we shall be able to determine if the system is globally synchronized. In summary, our results can be applied to rather general time-varying networks with a large number of nodes.

I. INTRODUCTION

During the past few decades the study of networks of dynamical systems has generated a rapidly growing interest in theoretical physics and other fields of science. Particularly, an increasing interest has been focused on complex networks with different topologies.[3–30] Complex networks, including the Internet, the World Wide Web, and electrical power grids, are prominent candidates to describe sophisticated collaborative dynamics in many sciences.[3–17]

As one of the basic characteristics of a dynamical network, synchronizing a crowd of dynamical nodes within the complex networks has become an important and interesting research topic in many fields.[3–30] General approaches to local synchronization of coupled chaotic systems have been proposed, including the master stability function-based criteria[6,7,31,32] and the MMA.[17,33] Typically, in networks of coupled chaotic systems, the synchronous solution becomes stable when the coupling strength between the oscillators exceeds a critical value. However, a few examples[34,35] were reported to be inconsistent with this pattern. Among them is a lattice of the x-component coupled Rössler systems in which the stability of synchronization regime is lost with an increase in coupling strengths. Furthermore, even if the coupled system always stays in a compact set and local synchronization occurs, global synchronization can be absent due to the possible presence of different invariant sets lying outside the synchronous manifold (in certain cases, this is a multistability effect). As a result, global synchronization of coupled chaotic systems was also intensively studied.

The methods to deal with global synchronization include but not limited to Lyapunov function-based criteria[12,14,23,35–38] as well as the MMA.[26,33,39] Among the Lyapunov function-based criteria, the CGS[22–25] has the widest range of the applicability. Indeed, the method can be applied to the asymmetrically coupled networks that are time varying. However, the couplings in the network are assumed to be non-negative. In fact, there exist some networks with both negative/competitive and positive/cooperative couplings. Recently, the MMA proposed by Chen[17,33] has been

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TABLE I. The table gives the matrix measures of $\overline{\boldsymbol{G}}_{C_i}$, $i=1,2$ with various size of $\boldsymbol{G}$, which is given in Eq. (9). Since $\boldsymbol{G}$ is a circular matrix, the matrix measures of $\overline{\boldsymbol{G}}$ with respect to $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are equal. Note that the matrix measure of $\overline{\boldsymbol{G}}_{\boldsymbol{C}}$ is $\lambda_2(\boldsymbol{G})$, $\forall \boldsymbol{C} \in \mathfrak{D}$, which is negative regardless of the size of $\boldsymbol{G}$.

m	4	5	6	7	8	9
$\boldsymbol{C}_1$	−1.78	−1	−0.51	−0.19	0.05	0.23
$\boldsymbol{C}_2$	−1.78	−1	−0.51	−0.19	0.05	0.23

very successful in treating local synchronization with complex network topologies. In Refs. 33 and 39 some global synchronization theorems were also obtained via a similar MMA. Even though the theorems can be applied to a wider range of complex networks than those obtained by CGS, there are two drawbacks. First, the number of nodes considered may be limited. The matrix measure of the diffusive synchronization stability matrix (see, e.g., Refs. 33 and 39), which is equivalent to our $\overline{\boldsymbol{G}}_{C_1}(t)$ [see Eq. (5b)], is size dependent. Its corresponding matrix measure can go from negative to positive as the size of the nodes increases. Indeed, given a near-neighbor coupling with periodic boundary conditions if the number of nodes is greater than 7, then $\overline{\boldsymbol{G}}_{C_1}$ has positive matrix measure (see Table I). Second, their approach works better for systems of the full-state coupling between connected nodes as opposed to those of the partial-state coupling. It should be noted that the partial-state coupling also finds applications in various fields. For instance, in self-pulsating laser diode equations (see, e.g., Ref. 40), only the photon density can be coupled with the electron density of the active region. Moreover, in the case of coupled chaotic systems, the systems that are partial state coupled may exhibit different dynamic behavior. For instance, it is well known (see, e.g., Ref. 26) that for the coupled Lorentz systems, if the x-component or y-component is coupled, the resulting system then achieves synchronization. In contrast, the network would fail to be synchronized provided that only the z-component is coupled.

The purpose of this paper is to give a different MMA, which was originated in Ref. 26 to study global synchronization in time-varying complex networks. In particular, a new and, in some cases, optimal coordinate transformation is introduced to remedy the first drawback of MMA. Moreover, by taking account of the structure of the uncoupled parts of the vector field of the individual oscillators, we are able to avoid the second drawback of MMA. In short, our approach fixes both drawbacks of Chen's approach and preserves their salient feature of wider applicability of complex networks. In addition, by merely checking the structure of the vector field of the individual oscillator, we shall be able to determine if the system is globally synchronized. Moreover, a rigorous lower bound on the coupling strength for global synchronization of all oscillators is also obtained. The paper is organized as follows. Section II is to lay down the foundation of our paper. The properties of the new coordinate transformation and its resulting coupling matrix $\overline{\boldsymbol{G}}(t)$ are studied in Sec. III. The main results are contained in Sec. IV. Some examples to illustrate the effectiveness of our approach and to compare with the existing methods are recorded in Sec. V. The examples include some complex networks such as the star type, the wavelet transformed type, the pristine world joining with some randomness, the generalized wheel type, and the prism type. In Sec. VI, we summarize our main results and give some concluding remarks. The needed definitions and properties of matrix measures of matrices and some technical proofs leading to the main results of our paper are recorded in Appendices A and B, respectively.

II. BASIC FRAMEWORK

In this paper, we will denote scalar variables in lower case, matrices in bold type upper case, and vectors (or vector-valued functions) in bold type lower case. We consider an array of m nodes/oscillators, coupled linearly together, with each node/oscillator being an n-dimensional system. The entire array is a system of nm ordinary differential equations. In particular, the state equations are

$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{f}(\boldsymbol{x}_i,t) + d \cdot \sum_{j=1}^{m} g_{ij}(t)\boldsymbol{D}\boldsymbol{x}_j, \quad i=1,2,\ldots,m, \tag{1a}$$

where $\boldsymbol{D}=(d_{ij})_{n\times n}$ is the inner coupling matrix, $\boldsymbol{x}_i =(x_{i1},x_{i2},\ldots,x_{in})^T \in \mathbb{R}^n$, and $\boldsymbol{f}$ is a vector-valued function form $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ denoted by

$$\boldsymbol{f}(\boldsymbol{x}_i,t) = \begin{pmatrix} f_1(\boldsymbol{x}_i,t) \\ \vdots \\ f_n(\boldsymbol{x}_i,t) \end{pmatrix}. \tag{1b}$$

Let $\boldsymbol{x}=(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_m)^T$ and $\boldsymbol{G}(t)=(g_{ij}(t))_{m\times m}$. Then $\boldsymbol{G}(t)$ represents the (outer) coupling configuration of the network at time t. Equivalently, Eq. (1a) becomes

$$\dot{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{f}(\boldsymbol{x}_1,t) \\ \vdots \\ \boldsymbol{f}(\boldsymbol{x}_m,t) \end{pmatrix} + d(\boldsymbol{G}(t)\otimes \boldsymbol{D})\boldsymbol{x} =: \boldsymbol{F}(\boldsymbol{x},t) + d(\boldsymbol{G}(t)\otimes \boldsymbol{D})\boldsymbol{x}, \tag{2}$$

where $\otimes$ denotes the Kronecker product. To study the synchronization of Eq. (2), we assume, throughout the paper, that

$$\boldsymbol{G}(t)\boldsymbol{e} = \boldsymbol{0} \ \forall\, t, \tag{3a}$$

where $\boldsymbol{e}=1/\sqrt{m}(1,1,\ldots,1)^T$. Such assumption above is to ensure the invariant property of the synchronization manifold $\mathfrak{M}=\{\boldsymbol{x}:\boldsymbol{x}_i=\boldsymbol{x}_j, 1\le i,j\le m\}$.

We further assume that the inner coupling matrix $\boldsymbol{D}$ is, without loss of generality, of the form

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}_{n\times n}. \tag{3b}$$

The index k, $1\le k\le n$, means that the first k components of the individual system are coupled. If $k\ne n$, then the system is said to be partial state coupled. Otherwise, it is said to be full state coupled.

Definition 1: System (1a) is said to have global synchronization if for each initial condition $\boldsymbol{x}(0)\in\mathbb{R}^{nm}$, the trajectory $\boldsymbol{x}(t)$ satisfies

$$\lim_{t\to\infty}\sum_{1\le i<j\le m}\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|=0.$$

Permute the state variables in the following way:

$$\tilde{\boldsymbol{x}}_i=\begin{pmatrix}x_{1i}\\ \vdots\\ x_{mi}\end{pmatrix}, \quad \text{and} \quad \tilde{\boldsymbol{x}}=\begin{pmatrix}\tilde{\boldsymbol{x}}_1\\ \vdots\\ \tilde{\boldsymbol{x}}_n\end{pmatrix}. \tag{4a}$$

Then Eq. (2) can be written equivalently as

$$\dot{\tilde{\boldsymbol{x}}}=\begin{pmatrix}\tilde{\boldsymbol{f}}_1(\tilde{\boldsymbol{x}},t)\\ \vdots\\ \tilde{\boldsymbol{f}}_n(\tilde{\boldsymbol{x}},t)\end{pmatrix}+d(\boldsymbol{D}\otimes\boldsymbol{G}(t))\tilde{\boldsymbol{x}}=:\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}},t)+d(\boldsymbol{D}\otimes\boldsymbol{G}(t))\tilde{\boldsymbol{x}}, \tag{4b}$$

where

$$\tilde{\boldsymbol{f}}_i(\tilde{\boldsymbol{x}},t)=\begin{pmatrix}f_i(\boldsymbol{x}_1,t)\\ \vdots\\ f_i(\boldsymbol{x}_m,t)\end{pmatrix}. \tag{4c}$$

The purpose of such a reformulation is twofold. First, a transformation of coordinates of $\tilde{\boldsymbol{x}}$ is to be applied to Eq. (4b) so as to isolate the synchronous manifold. Second, once the synchronous manifold is isolated, proving synchronization of Eq. (2) is then equivalent to showing that the origin is asymptotically stable with respect to reduced system (7a). To this end, we first make a coordinate change to isolate the synchronous subspace. Let $\boldsymbol{C}$ be an $(m-1)\times m$ full-rank matrix with all its row sums being zero. Such a matrix is to be termed as *coordinate transformation*. Define

$$\boldsymbol{A}=\begin{pmatrix}\boldsymbol{C}\\ \boldsymbol{e}^T\end{pmatrix}. \tag{5a}$$

Then $\boldsymbol{A}^{-1}=(\boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{C}^T)^{-1},\boldsymbol{e})$ and

$$\boldsymbol{A}\boldsymbol{G}(t)\boldsymbol{A}^{-1}=\begin{pmatrix}\boldsymbol{C}\boldsymbol{G}(t)\boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{C}^T)^{-1} & \boldsymbol{0}\\ \boldsymbol{e}^T\boldsymbol{G}(t)\boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{C}^T)^{-1} & 0\end{pmatrix}=:\begin{pmatrix}\overline{\boldsymbol{G}}_C(t) & \boldsymbol{0}\\ \boldsymbol{h}(t)^T & 0\end{pmatrix}. \tag{5b}$$

Let $\boldsymbol{E}=\boldsymbol{I}_n\otimes\boldsymbol{A}$ and $\tilde{\boldsymbol{y}}=\boldsymbol{E}\tilde{\boldsymbol{x}}$. Multiplying $\boldsymbol{E}$ to both sides of Eq. (4b), we get

$$\dot{\tilde{\boldsymbol{y}}}=\boldsymbol{E}\tilde{\boldsymbol{F}}(\boldsymbol{E}^{-1}\tilde{\boldsymbol{y}},t)+d\left(\boldsymbol{D}\otimes\begin{pmatrix}\overline{\boldsymbol{G}}_C(t) & \boldsymbol{0}\\ \boldsymbol{h}(t)^T & 0\end{pmatrix}\right)\tilde{\boldsymbol{y}}.$$

Let $\tilde{\boldsymbol{y}}=(\tilde{\boldsymbol{y}}_1,\ldots,\tilde{\boldsymbol{y}}_n)^T$. Then

$$\tilde{\boldsymbol{y}}_i=\begin{pmatrix}\boldsymbol{C}\tilde{\boldsymbol{x}}_i\\ \sum_{j=1}^{m}x_{ji}/\sqrt{m}\end{pmatrix}=:\begin{pmatrix}\overline{\boldsymbol{y}}_i\\ \boldsymbol{e}_i\end{pmatrix}. \tag{6}$$

Setting $\overline{\boldsymbol{y}}=(\overline{\boldsymbol{y}}_1,\ldots,\overline{\boldsymbol{y}}_n)^T$, we have that the dynamics of $\overline{\boldsymbol{y}}$ is now satisfied by the following equation:

$$\dot{\overline{\boldsymbol{y}}}=d(\boldsymbol{D}\otimes\overline{\boldsymbol{G}}_C(t))\overline{\boldsymbol{y}}+\overline{\boldsymbol{F}}(\tilde{\boldsymbol{y}},t), \tag{7a}$$

where

$$\overline{\boldsymbol{F}}(\tilde{\boldsymbol{y}},t)=(\boldsymbol{I}_n\otimes\boldsymbol{C})\cdot\tilde{\boldsymbol{F}}(\boldsymbol{E}^{-1}\tilde{\boldsymbol{y}},t). \tag{7b}$$

Since the rank and the row sums of $\boldsymbol{C}$ are $m-1$ and 0, respectively, we conclude that the task of obtaining global synchronization of system (1a) is now reduced to showing that the origin is globally and asymptotically stable with respect to system (7a). The choice of a coordination transformation will greatly influence how negative the matrix measure of $\overline{\boldsymbol{G}}_C(t)$ could be, which plays the important role, among others, to determine the global stability of Eq. (7a) with respect to the origin.

III. MATRICES OF THE COORDINATE TRANSFORMATION

In what follows we shall address the question of how to choose a matrix $\boldsymbol{C}$ of the coordinate transformation and its corresponding properties. To make the origin an asymptotically stable equilibrium of system (7a), one would like to have the matrix measure of $\overline{\boldsymbol{G}}_C(t)$ as smaller a negative number as possible. In fact, such an optimal choice $\boldsymbol{C}$ can be achieved provided that the outer coupling matrix $\boldsymbol{G}(t)$ is symmetric, nonpositive definite.

Definition 2: Denote by $\mathfrak{C}$ the set of $(m-1)\times m$ coordinate transformations, i.e.,

$$\mathfrak{C}=\{\boldsymbol{C}\in\mathbb{R}^{(m-1)\times m}:\boldsymbol{C} \text{ is full rank, and all its row sums are zero}\}.$$

Let $\mathfrak{O}\subseteq\mathfrak{C}$ be such that

$$\mathfrak{O}=\{\boldsymbol{C}\in\mathfrak{C}:\boldsymbol{C} \text{ such that matrix } \boldsymbol{A}=(\boldsymbol{C}^T,\boldsymbol{e})^T \text{ is orthogonal}\}.$$

Theorem 1: *Assume that all eigenvalues of outer coupling matrix $\boldsymbol{G}(t)$ have nonpositive real parts. Then* $\inf_{\boldsymbol{C}\in\mathfrak{C}}\mu_2(\overline{\boldsymbol{G}}_C(t))\ge\operatorname{Re}\lambda_2(\boldsymbol{G}(t))$. *Here* $\operatorname{Re}\lambda_2(\boldsymbol{G}(t))$ *is the second largest real part of eigenvalues of $\boldsymbol{G}(t)$. If, in addition, $\boldsymbol{G}(t)$ is symmetric for all t, then the above equality can be achieved by choosing any $\boldsymbol{C}$ in $\mathfrak{O}$.*

Proof: It follows from Eq. (5b) that the spectrum $\sigma(\overline{\boldsymbol{G}}_C(t))$ of $\overline{\boldsymbol{G}}_C(t)$ is equal to $\sigma(\boldsymbol{G}(t))-\{0\}$. Using the fact that $\operatorname{Re}\lambda(\boldsymbol{K})\le\lambda_{\max}(\boldsymbol{K}+\boldsymbol{K}^T/2)$ for any real matrix $\boldsymbol{K}$, we have, via Eq. (A1), that $\mu_2(\overline{\boldsymbol{G}}_C(t))\ge\operatorname{Re}\lambda_2(\boldsymbol{G}(t))$. In particular, if $\boldsymbol{C}\in\mathfrak{O}$ and $\boldsymbol{G}(t)$ is symmetric, then $\overline{\boldsymbol{G}}_C(t)(=\boldsymbol{C}\boldsymbol{G}(t)\boldsymbol{C}^T)$ is symmetric and $\boldsymbol{h}(t)=\boldsymbol{0}$. Here $\boldsymbol{h}(t)$ is given as in Eq. (5b).

Therefore, $\mu_2(\bar{G}_C(t))=\lambda_2(G(t))$. We have just completed the proof of the theorem.

The theorem above amounts to saying that if $G(t)$ is symmetric, nonpositive definite, then any choice of C in $\mathfrak{O}$ yields the smallest possible matrix measure of $\bar{G}_C(t)$. This, in turn, gives one the best possible position to study the stability of Eq. (7a) with respect to the origin.

Remark 1: In those earlier papers (see, e.g., Refs. 12, 26, and 33), the choice of the coordinate transformations is either

$$C_1=\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix} \quad \text{or}$$

$$C_2=\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}. \tag{8}$$

The drawback for such a choice of C is that even if $G(t)(\equiv G)$ is the diffusive matrix with periodic boundary conditions, i.e.,

$$G(t)\equiv\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}_{m\times m}, \tag{9}$$

the corresponding matrix measure of $\bar{G}_{C_i}$, $i=1,2$ is positive whenever $m>7$ (see Table I), while $\mu_2(\bar{G}_C)=\lambda_2(G)<0$ for all $C\in\mathfrak{O}$ regardless of the size of G.

Theorem 2: *For any outer coupling matrix $G(t)$ and any coordinate transformations C_p, C_q in $\mathfrak{O}$, $\mu_2(\bar{G}_{C_p}(t))=\mu_2(\bar{G}_{C_q}(t))$.*

Proof: Since for any $x\in\mathbb{R}^{m-1}$, there is $z=C_qC_p^Tx$ such that

$$x^TC_p(G(t)+G(t)^T)C_p^Tx=z^TC_q(G(t)+G(t)^T)C_q^Tz.$$

By the definition of matrix measure, we have that $\mu_2(\bar{G}_{C_p}(t))=\mu_2(\bar{G}_{C_q}(t))$.

From here on, the matrix C in Eq. (7a) is assumed to lie in $\mathfrak{O}$ unless otherwise stated. For ease of the notations, we shall drop the subscript C of $\bar{G}_C(t)$ if $C\in\mathfrak{O}$. The remainder of the section is devoted to finding the matrix measure of $\bar{G}(t)$ where its corresponding coupling matrix $G(t)$ appears often in many applications.

Proposition 1: Assume that for each t, $G(t)$ is a node-balancing matrix, i.e., its row sums and column sums are equal. Then

$$\mu_2(\bar{G}(t))=\lambda_2\left(\frac{G(t)+G(t)^T}{2}\right), \tag{10}$$

whenever all eigenvalues of $G(t)+G(t)^T$ are nonpositive.

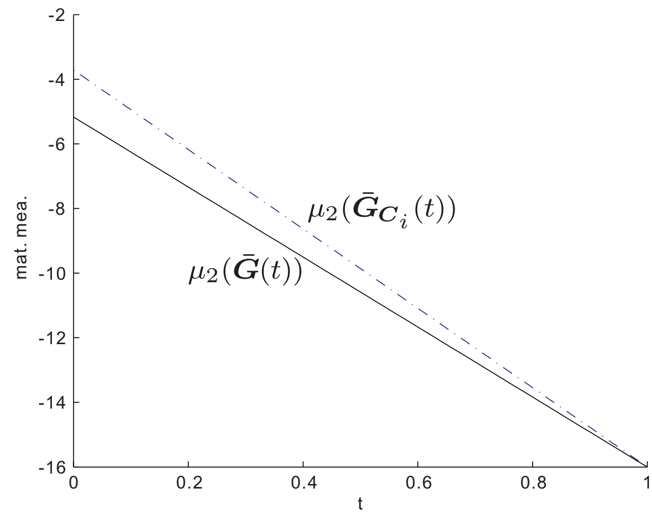

FIG. 1. (Color online) The matrix measures of $\bar{G}(t)$ and $\bar{G}_{C_i}(t)$, $i=1,2$ with G being given in Eq. (11) and $p(t)=t$ are, respectively, represented by the solid line and the dotted lines above. Lines for $\bar{G}_{C_i}(t)$, $i=1,2$ are coincided since $G(t)$ is circular for all t.

Proof: If $G(t)$ is as assumed, then it follows from Eq. (5b) that

$$AG(t)A^{-1}=\begin{pmatrix} \bar{G}(t) & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}.$$

Consequently, Eq. (10) holds as asserted.

In what follows, some outer coupling matrices are to be provided. Their corresponding matrix measures of $\bar{G}(t)$ and $\bar{G}_{C_i}(t)$, $i=1$ or 2, are to be compared.

Example 1: [Belykh *et al.* (Ref. 22)] Consider the regular coupled network by adding to the pristine world G (the ring of $2K$-nearest coupled oscillators) an additional global coupling such that the coupling $p(t)$, $0\le p(t)\le 1$ is placed on all free spots of the matrix G (see, e.g., Ref. 22). Specifically, the resulting coupling matrix $G(t)$ can be represented by a circular matrix of the form

$$G(t)=\mathrm{circ}(-g(t),\overbrace{1,\ldots,1}^{K},\overbrace{p(t),\ldots,p(t)}^{m-2K-1},\overbrace{1,\ldots,1}^{K}), \tag{11}$$

where $g(t)=2K+(m-2K-1)p(t)$. Since $G(t)$ is symmetric, we have that

$$\mu_2(\bar{G}(t))=\lambda_2(G(t)) = \max_{1\le j\le m-1}\left(-g(t)+\sum_{l=1}^{K}(\omega^{lj}+\omega^{(m-l)j}) + p(t)\sum_{l=K+1}^{m-K-1}\omega^{lj}\right).$$

Here $\omega=\exp(2\pi i/m)$. The matrix measures $\mu_2(\bar{G}(t))$ and $\mu_2(\bar{G}_{C_i}(t))$, $i=1,2$, with $p(t)=t$, $t\in[0,1]$ are recorded in Fig. 1.

Example 2: [Wei *et al.* (Ref. 41); Juang *et al.* (Ref. 42)] Let $G=G_\beta^{(m)}$, $0\le\beta\le 1$ be the diffusive matrix of size $m\times m$ with mixed boundary conditions. That is, if $m>2$,

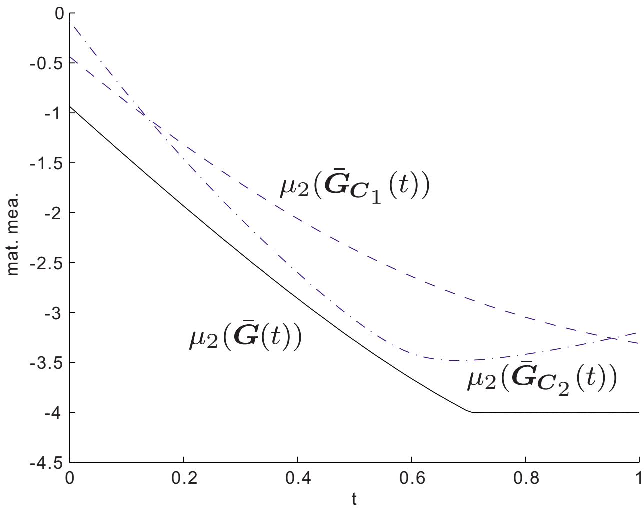

FIG. 2. (Color online) The matrix measures of $\overline{G}(t)$ and $\overline{G}_{C_i}(t)$, $i=1,2$ with G given in Eq. (13) and $p(t)=t$ are, respectively, represented by the solid line and the dotted lines above.

$$G_\beta^{(m)} = \begin{pmatrix} -1-\beta & 1 & 0 & \cdots & 0 & \beta \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ \beta & 0 & \cdots & 0 & 1 & -1-\beta \end{pmatrix}_{m\times m}, \tag{12}$$

and if $m=2$,

$$G_\beta^{(2)} = \begin{pmatrix} -1-\beta & 1+\beta \\ 1+\beta & -1-\beta \end{pmatrix}.$$

For such G, $\mu_2(\overline{G})=\lambda_2(G)<0$. However, $\lambda_2(G)$ would move closer to the origin as the number of nodes increases. As a result, synchronization of the network is more difficult to be realized as the number m of nodes increases. In Refs. 41 and 42 a wavelet transformation method is proposed to alter the connectivity topology of the network. In doing so, $\lambda_2(G(t)) = \lambda_2(p(t))$ becomes a quantity depending on wavelet parameter $p(t)$. By choosing suitable $p(t)$, which is a wavelet transformation method[41,42] applied to the coupling matrix $G_\beta^{(m)}$, one would expect that $\lambda_2(p(t))$ will move away from the origin regardless of the number of the nodes. Under such a reconstruction, the resulting coupling matrix $G(t)$ is of the following form:

$$G(t) = G_\beta^{(m)} + p(t)(G_\beta^{(m/k)} \otimes ee^T), \tag{13}$$

where $e=(1,\ldots,1)^T$. Here we assume $p(t)\geq 0$ and $k=2^l$ for some $l\in\mathbb{N}$, and $m=Nk$ for some $N\in\mathbb{N}-\{1\}$. Since the reconstructed matrix $G(t)$ is symmetric, $\mu_2(\overline{G}(t))=\lambda_2(G(t))<0$. The matrix measures $\mu_2(\overline{G}(t))$ and $\mu_2(\overline{G}_{C_i}(t))$, $i=1,2$, with $p(t)=t$, $t\in[0,1]$ are recorded in Fig. 2.

Example 3: Let

$$G(t) = \mathrm{circ}(\overbrace{-2,2,0,\ldots,0}^{m}).$$

Since $G(t)$ is a node-balancing matrix, $\mu_2(\overline{G}(t))=\lambda_2(G(t))<0$. Note that the values of $\mu_2(\overline{G}_{C_i})$, $i=1,2$, are positive provided that $m>5$ (see Table II).

TABLE II. The table gives the matrix measures of $\overline{G}_{C_i}$, $i=1,2$ with various size of G, which is given in Example 3.

m	4	5	6	7	8	9
C_1	−0.83	−0.17	0.24	0.54	0.78	0.98
C_2	−0.83	−0.17	0.24	0.54	0.78	0.98

Proposition 2: Let $C=(c_1,\ldots,c_{m-1})^T\in\mathfrak{O}$. If, in addition, $\{c_i\}_{i=1}^{m-1}$ are pairwise $G(t)$-conjugate, i.e., $c_i^T G(t) c_j=0$, $\forall 1\leq i\neq j\leq m-1$, then $\overline{G}(t)$ is a diagonal matrix. Moreover,

$$\mu_2(\overline{G}(t)) = \lambda_2(G(t)), \tag{14}$$

whenever all eigenvalues of $G(t)$ are nonpositive.

Proof: Note that $\overline{G}(t)=CG(t)C^T=(c_i^T G(t) c_j)$. Hence, $\overline{G}(t)$ is a diagonal matrix. Therefore, the assertion in Eq. (14) holds as asserted.

Example 4: [Chen (Ref. 33)] Let $G(t)$ describe a star-typed coupled network of the form

$$G(t) = \begin{pmatrix} -d_1(t) & & & d_1(t) \\ & \ddots & & \vdots \\ & & -d_1(t) & d_1(t) \\ 1 & \cdots & 1 & -(m-1) \end{pmatrix}_{m\times m}. \tag{15}$$

Here $d_1(t)$ is a real number. We next show that a set $\{c_i\}_{i=1}^{m-1}$ of column vectors can be chosen so that $C=(c_i,\ldots,c_{m-1})\in\mathfrak{O}$ and that $\{c_i\}_{i=1}^{m-1}$ are pairwise $G(t)$-conjugate. Define $\alpha_i=(i(i+1))^{-1/2}$, $i=1,\ldots,m-1$. Let

$$c_i^T = (\overbrace{\alpha_i,\ldots,\alpha_i}^{i}, -i\alpha_i, \overbrace{0,\ldots,0}^{m-i-1})$$

for all $i=1,\ldots,m-1$. Then c_i, $i=1,\ldots,m-1$ are orthonormal vectors. Moreover, they are also $G(t)$-conjugate. To see this, we first note that $d_1(t)$ is an eigenvalue of $G(t)$ and its associated eigenvectors are c_i, $i=1,\ldots,m-2$. Therefore, $c_i^T G(t) c_j=0$ for all $1\leq i\neq j\leq m-2$. Some direct computation would yield that $c_i^T G(t) c_{m-1}=0$ for $i=1,\ldots,m-2$ and that $c_{m-1}^T G(t) c_{m-1}=-d_1(t)-(m-1)$. By Proposition 2, we have that

$$\mu_2(\overline{G}_C(t)) = \max\{-d_1(t), -d_1(t)-(m-1)\} = -d_1(t). \tag{16}$$

The matrix measures $\mu_2(\overline{G}(t))$ and $\mu_2(\overline{G}_{C_i}(t))$, $i=1,2$, with $p(t)=t$, $t\in[0,1]$ are demonstrated in Fig. 3.

The remainder of the section is to address the system with even more complex topology.

Proposition 3: Let $G(t)=O(t)+P(t)$ with $O(t)$ and $P(t)$ having all its row sums zero. Suppose further that $P(t)$ is node balancing. Then

$$\mu_2(\overline{G}(t)) \leq \mu_2(\overline{O}(t)) + \lambda_2\left(\frac{P(t)+P(t)^T}{2}\right),$$

whenever all eigenvalues of $P(t)+P(t)^T$ are nonpositive.

Proof: Noting that $\overline{G}(t)=CG(t)C^T=\overline{O}(t)+CP(t)C^T$, we easily conclude that the above inequality holds as asserted.

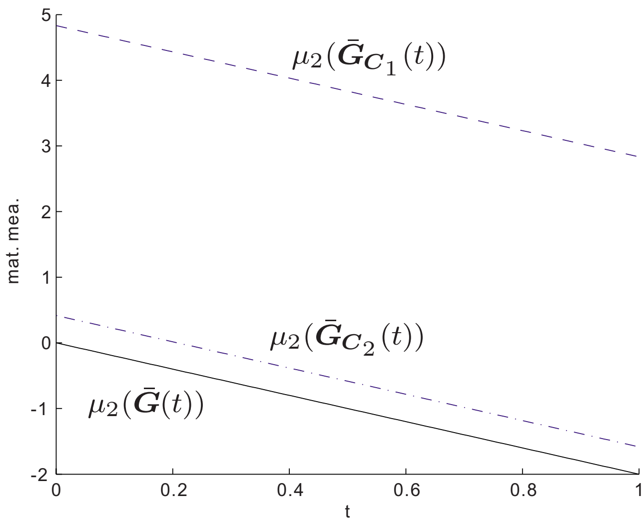

FIG. 3. (Color online) The matrix measures of $\overline{G}(t)$ and $\overline{G}_{C_i}(t)$, $i=1,2$ with G being given in Eq. (15) and $p(t)=t$ are, respectively, represented by the solid line and the dotted lines above.

Example 5: [Belykh *et al.* (Ref. 22)] Consider the outer coupling matrix $G(t)$ to be of random type. Specifically, $G(t)$ is of the form

$$G(t) = \mathrm{circ}(-2K,\overbrace{1,\ldots,1}^{K},\overbrace{0,\ldots,0}^{m-2K-1},\overbrace{1,\ldots,1}^{K}) + P(t) =: O + P(t), \quad (17)$$

where $P(t) =: (p_{ij}(t))$ is a symmetric matrix with all its row sums being zero and satisfies $p_{ij}(t) \equiv 0$ for (i,j) with $i - j \bmod m \le K$ or $j - i \bmod m \le K$, and $p_{ij}(t) = S_{ij}(q)$ for $(q-1)\tau \le t < q\tau$ for all remaining pairs (i,j) with $i \ne j$. Here each of $S_{ij}(q)$ is a random variable that takes the value 1 with probability p and 0 with probability $1-p$.

The random variables $S_{ij}(q)$ are assumed to be all independent. To each realization ω of this stochastic process $S(1), S(2), \ldots,$ where $S(q) = \{S_{ij}(q), i=1,\ldots,n, j=i+l \bmod n, l=K+1,\ldots,[n/2]\}$, i.e., to each switching sequence ω, there corresponds a time-varying system described by Eq. (2).

Since $P(t)$ is symmetric, by Proposition 3,

$$\mu_2(\overline{G}(t)) \le \mu_2(\overline{O}) + \lambda_2(P(t)) \le \mu_2(\overline{O}) = \lambda_2(O) < 0.$$

Let $G(t) \equiv G$. Generally speaking, $\inf_{\overline{C}\in\mathfrak{C}} \mu_2(\overline{G}_{\overline{C}}) \ne \mu_2(\overline{G}_C)$ for any $C \in \mathfrak{D}$. Nevertheless, $\mu_2(\overline{G}_C)$ produces a good upper bound of $\inf_{\overline{C}\in\mathfrak{C}} \mu_2(\overline{G}_{\overline{C}})$.

To support the observation, we conclude this section by providing some additional network topologies where the matrix measure of its corresponding $\overline{G}_C(t)$, $C \in \mathfrak{D}$ is smaller than that of $\overline{G}_{C_i}$, $i=1,2$. As a matter of fact, $\mu_2(\overline{G}_{C_i})$, $i=1,2$, switch signs as the number of nodes increases. In contrast, $\mu_2(\overline{G}_C)$ mostly remains negative as the size of the system grows.

Example 6: Consider a *generalized wheel-typed coupled network* of the form as illustrated in Fig. 4(a). The inner nodes have the strong all-to-all connections. The outer nodes are only directly connected with their nearest neighbors. The communications between the inner and outer nodes are through one way going from each inside node to its nearest outside node. Specifically, such a network can be written as the following:

$$G(t) \equiv \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}_{m\times m}, \quad (18)$$

where

$$G_1 = \begin{pmatrix} -\left(\frac{m}{2}-1\right) & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & -\left(\frac{m}{2}-1\right) \end{pmatrix}_{m/2\times m/2},$$

corresponding to the all-to-all coupling, $G_2 = 0$, $G_3 = 0.1I$, and $G_4 = G_1^{(m/2)} - 0.1I$. Here $G_1^{(m/2)}$ is the diffusive matrix with periodic boundary conditions and of size $m/2 \times m/2$. The numerical computation suggests that the matrix measures of $\overline{G}_{C_i}$, $i=1,2$, are positive provided that $m \ge 4$ while that of $\overline{G}_C$, $C \in \mathfrak{D}$, remains negative (see Table III).

Example 7: Consider the *prism-typed coupled network* of the form as illustrated in Fig. 4(b). The difference between the generalized wheel-typed network and the one considered here lies only on how the inner nodes communicate with each other (see Fig. 4). Specifically, such a network can be written as the following:

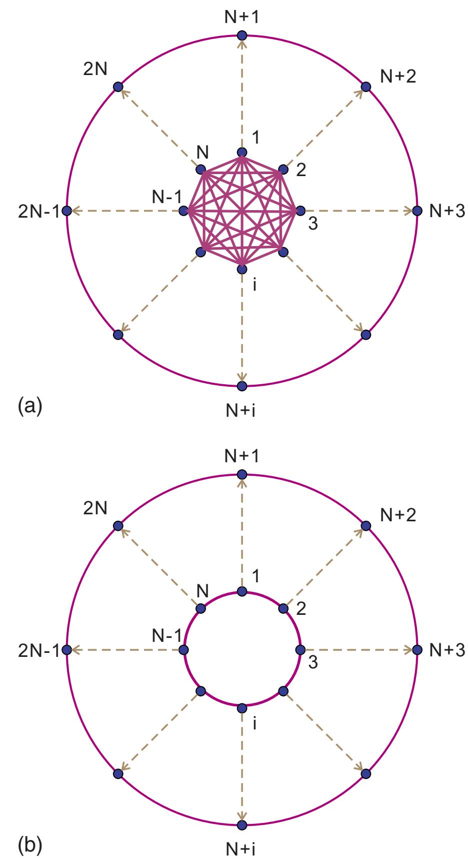

FIG. 4. (Color online) Coupling topologies: (a) generalized wheel-typed coupled network with $m=2N$ and (b) prism-typed coupled network with $m=2N$. Networks (a) and (b) appear in Examples 6 and 7, respectively.

TABLE III. The table gives the matrix measures of $\overline{G}_{C_i}$, $i=1,2$ and $\overline{G}_C$, $C \in \mathfrak{D}$ with various size of G, which is given in Eq. (18).

m	4	6	8	10	5000
C_1	0.11	0.32	0.53	0.74	517.47
C_2	0.23	0.56	0.96	1.44	34 843.01
C	−0.1	−0.1	−0.1	−0.1	−0.1

$$G(t) \equiv \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}_{m\times m},$$

where $G_1 = G_1^{(m/2)}$, $G_2=0$, $G_3=0.1I$, and $G_4=G_1^{(m/2)}-0.1I$. The numerical computation suggests (see Table IV) that the matrix measures of $\overline{G}_{C_i}$, $i=1,2$, are positive provided that $m\geq 4$, while that of $\overline{G}_C$, $C\in\mathfrak{D}$, stays negative until $m=86$. The example demonstrates that a coordinate transformation C, $C\in\mathfrak{D}$, is indeed a good candidate among all coordinate transformations.

IV. MAIN RESULTS

In the section, we turn our attention back to the dynamics of Eq. (7a) and analyze the stability of the origin of the system. As in Ref. 26, we break the space $\bar{y}$ into two parts: $\bar{y}_c$, the coupled space, and $\bar{y}_u$, the uncoupled space. Specifically, let

$$\bar{y} = \begin{pmatrix} \bar{y}_c \\ \bar{y}_u \end{pmatrix}, \quad \text{and} \quad \overline{F}(\bar{y},t) = \begin{pmatrix} \overline{F}_c(\bar{y},t) \\ \overline{F}_u(\bar{y},t) \end{pmatrix}. \tag{19}$$

Here

$$\bar{y}_c = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_k \end{pmatrix}, \quad \text{and} \quad \bar{y}_u = \begin{pmatrix} \bar{y}_{k+1} \\ \vdots \\ \bar{y}_n \end{pmatrix}.$$

Then Eq. (7a) can be rewritten in the form

$$\begin{pmatrix} \dot{\bar{y}}_c \\ \dot{\bar{y}}_u \end{pmatrix} = \begin{pmatrix} d(I_k \otimes \overline{G}(t)) & 0 \\ 0 & U(t) \end{pmatrix} \begin{pmatrix} \bar{y}_c \\ \bar{y}_u \end{pmatrix} + \begin{pmatrix} \overline{F}_c(\bar{y},t) \\ \overline{R}_u(\bar{y},t) \end{pmatrix}, \tag{20}$$

where $\overline{R}_u(\bar{y},t) =: \overline{F}_u(\bar{y},t) - U(t)\bar{y}_u$ for some matrix $U(t)$. Note that form (20) can always be achieved since the remainder term $\overline{R}_u$ still depends on the whole space $\bar{y}$. In what follows, we shall give some intuitive explanations as to why assumptions on system (20) would make the origin into a globally attracting equilibrium.

The dynamics on the coupled space with respect to the linear part is under the influence of $\overline{G}(t)$, which is assumed to have the negative matrix measure. The dynamics of the nonlinear part on coupled space can then be controlled by choosing a large coupling strength. On the other hand, the uncoupled space has no stable matrix $\overline{G}(t)$ to play with. Thus, its corresponding vector field $\overline{F}_u(\bar{y},t)$ must have a certain structure to make the trajectory stay closer to the origin as time progresses, which we shall explain more later. Specifically, the following list of assumptions is needed for our first main theorem:

(H1†) System (20) or Eq. (7a) is *bounded dissipative* with respect to α. By that, we mean that there is a bounded region $\mathbf{B} =: \{y : \|y\| \leq \alpha\}$ such that for each parameter $d>0$, and each initial value $y(0)$, there is a time t_0, such that $y(t)$ lies in $\mathbf{B}$ whenever $t\geq t_0$.

(H2†) There is some $\lambda>0$ such that $\mu_2(\overline{G}(t)) \leq -\lambda$, $\forall t \geq 0$.

(H3†) For any $0<\beta\leq\alpha$, $\|\overline{F}_c(\bar{y},t)\| \leq b_1\beta$ whenever $\|\bar{y}\| \leq \beta$. Here b_1 is independent of β and t.

(H4†) Matrix $U(t)$ is of block diagonal form, i.e., $U(t) = \mathrm{diag}(U_1(t),\ldots,U_l(t))$. Here the sizes of $U_j(t)$, $j=1,\ldots,l$, are $(m-1)k_j \times (m-1)k_j$. Moreover, there is some $\gamma>0$ such that the matrix measures $\mu_2(U_j(t)) \leq -\gamma$ for all t sufficiently large and all j.

(H5†) Let

$$\overline{R}_u(\bar{y},t) = \begin{pmatrix} R_{u1}(\bar{y},t) \\ \vdots \\ R_{ul}(\bar{y},t) \end{pmatrix}$$

with each $R_{uj}(\bar{y},t) \in \mathbb{R}^{(m-1)\times k_j}$, $\forall j=1,\ldots,l$, where l, k_j are given as in (H4†). There is some $b_2>0$ such that for each $j=1,\ldots,l$, $\|R_{uj}(\bar{y},t)\| \leq b_2\beta$ whenever $\|(\bar{y}_c,\bar{y}_{u1},\ldots,\bar{y}_{uj-1})\| \leq \beta$ and $\|\bar{y}\|\leq\alpha$. Here

$$\bar{y}_{ui} = \begin{pmatrix} \bar{y}_{(m-1)\cdot\left(k+\sum_{j=1}^{i-1}k_j\right)+1} \\ \vdots \\ \bar{y}_{(m-1)\cdot\left(k+\sum_{j=1}^{i}k_j\right)} \end{pmatrix}$$

for all $i=1,\ldots,l$.

Remark 2: (i) Although the nonlinear terms $R_{uj}(\bar{y},t)$ could possibly depend on the whole space, their norm estimates are required to depend only on the coupled space and the uncoupled subspaces with their indices proceeding j. (ii) The size of the partition matrices $U_j(t)$, $j=1,\ldots,l$ from $U(t)$ depends on how the uncoupled part of the vector field of the single oscillator is structured. To determine how to partition $U(t)$, we begin with checking the case for $l=1$. That is, if for $l=1$, hypotheses (H4†) and (H5†) are satisfied, then no further partition is necessary. Otherwise, we further partition $U(t)$ into a set of smaller pieces to see if the resulting inequalities in (H4†) and (H5†) are fulfilled.

We are now in a position to state our first main theorem.

Theorem 3: *Let the outer coupling matrix* $G(t)$ *satisfying Eq. (3a) and the inner coupling matrix* D *be given as in Eq. (3b). Suppose hypotheses* (H1†), (H2†), (H3†), (H4†), *and* (H5†) *hold true, then* $\lim_{t\to\infty} \bar{y}(t) = 0$ *for any initial value provided that the coupling strength* d *satisfies the following inequality:*

TABLE IV. The table gives the matrix measures of $\overline{G}_{C_i}$, $i=1,2$ and $\overline{G}_C$, $C \in \mathfrak{D}$ with various size of G, which is given in Example 7.

m	4	6	8	86	88
C_1	0.34	0.32	0.35	4.65	4.72
C_2	0.34	0.56	0.73	4.79	4.86
C	−0.1	−0.1	−0.1	−0.0006	0.0004

$$d > \frac{b_1}{\lambda}\left(1+\frac{b_2^2}{\gamma^2}\right)^{l/2}. \tag{21}$$

Proof: For any initial condition $\bar{y}(0)$, there is $t_0>0$ such that $\|\bar{y}(t)\|\le\alpha$ for all $t\ge t_0$. Without loss of generality, t_0 is chosen sufficiently large so that the inequalities in (H4†) hold. Applying the matrix measure inequality (A2) and hypotheses (H2†), (H3†) on $\bar{y}_c$, for any $t\ge t_0$, we have that

$$\|\bar{y}_c(t)\| \le \|\bar{y}_c(t_0)\|e^{-\lambda d(t-t_0)} + \frac{b_1\alpha}{\lambda d}$$
$$\le \left(e^{-\lambda d(t-t_0)}+\frac{b_1}{\lambda d}\right)\alpha =: \left(e^{-\lambda d(t-t_0)}+c_0\frac{1}{d}\right)\alpha.$$

Let $\delta>1$. We see that

$$\|\bar{y}_c(t)\| \le \frac{\alpha}{d}c_0\delta \tag{22a}$$

whenever $t\ge t_{0,1}$ for some $t_{0,1}>t_0$. Similarly, applying inequality (A2) and hypotheses (H4†), (H5†) on $\bar{y}_{u1}$,

$$\|\bar{y}_{u1}(t)\| \le \frac{\alpha}{d}\left(\frac{b_2}{\gamma}c_0\right)\delta^2 =: \frac{\alpha}{d}c_1\delta^2, \tag{22b}$$

whenever $t\ge t_{1,1}$ for some $t_{1,1}>t_{0,1}$. Inductively, we have

$$\|\bar{y}_{uj}(t)\| \le \frac{\alpha}{d}c_j\delta^{j+1} \tag{22c}$$

whenever $t\ge t_{j,1}$ for all $j=2,\ldots,l$. Here $c_j=b_2/\gamma\sqrt{\Sigma_{i=0}^{j-1}c_i^2}$. Letting $t_1=t_{l,1}$ and summing up Eqs. (22a)–(22c), we get

$$\|\bar{y}(t)\| \le \frac{\alpha}{d}\left(1+\frac{b_2^2}{\gamma^2}\right)^{l/2}\frac{b_1}{\lambda}\delta^{l+1} =: h\alpha$$

whenever $t>t_1$. Choosing $d>(1+b_2^2/\gamma^2)^{l/2}(b_1/\lambda)\delta^{l+1}$, we see that the contraction factor h is strictly less than 1, and $\|\bar{y}(t)\|$ contracts to zero as time progresses. Since $\delta>1$ can be made arbitrarily close to 1, consequently, if d is chosen as assumed, then h can still be made to be less than 1. The assertion of the theorem now follows.

Note that the verification of hypotheses (H3†), (H4†), and (H5†) is a nontrivial matter since those assumptions depend on the coordinate transformation $\boldsymbol{C}$. Furthermore, these hypotheses are made for system (7a) or Eq. (20). Hence, it is desirable to derive some easily verifiable hypotheses for system (1a). Indeed, we are able to derive a set of hypotheses for system (1a) that can be easily checked. In fact, by merely checking the structure of the vector field $\boldsymbol{f}$ of the individual oscillator, one would be able to verify if those hypotheses hold true. Since the derivation of such a new set of hypotheses is rather long and technical, we shall refer the interested readers to Appendix B, which contains Propositions 4 and 5. We summarize these derived hypotheses in the following:

- (H1) System (1a) is bounded dissipative with respect to α.
- (H2) There is some $\lambda>0$ such that $\mu_2(\overline{\boldsymbol{G}}(t))\le-\lambda$, $\forall t\ge 0$.
- (H3) Functions $f_i(\cdot,t)$, $i=1,\ldots,k$ in Eq. (1a) are uniformly Lipschitz in region $\boldsymbol{B}$ given in (H1). That is, there is a constant $r>0$ such that $|f_i(\boldsymbol{u},t)-f_i(\boldsymbol{v},t)|\le r\|\boldsymbol{u}-\boldsymbol{v}\|$, whenever t is sufficiently large, and $\boldsymbol{u}$, $\boldsymbol{v}$ in $\boldsymbol{B}$.
- (H4) The matrix $\boldsymbol{Q}(\boldsymbol{v},t)$, which is given as in Eq. (B1), is of block diagonal form, i.e., $\boldsymbol{Q}(\boldsymbol{v},t)=\mathrm{diag}(\boldsymbol{Q}_1(\boldsymbol{v},t),\ldots,\boldsymbol{Q}_l(\boldsymbol{v},t))$. Here the sizes of $\boldsymbol{Q}_j(\boldsymbol{v},t)$, $j=1,\ldots,l$, are $k_j\times k_j$. Moreover, there is some $\gamma>0$ such that matrix measures $\mu_2(\boldsymbol{Q}_j(\boldsymbol{v},t))\le-\gamma$ for all j, whenever t is sufficiently large, and $\boldsymbol{v}$ in $\boldsymbol{B}$.
- (H5) Denoted by $s_1=k$ and $s_j=k+\Sigma_{i=1}^{j-1}k_i$, $j=2,\ldots,l$, where k_i and l are defined in (H4). Suppose, for any $1\le j\le l$, there is a $\delta>0$ such that

$$\|[\boldsymbol{r}(\boldsymbol{u},\boldsymbol{v},t)]_{s_j+1}^{s_j+k_j}\| \le \delta\|[\boldsymbol{u}-\boldsymbol{v}]_1^{s_j}\|$$

for t sufficiently large, and $\boldsymbol{u}$, $\boldsymbol{v}$ in $\boldsymbol{B}$. Here $[\boldsymbol{u}]_i^j$ is defined to be $(u_i,\ldots,u_j)^T$.

Remark 3: (i) Using the similar techniques as developed in the proof of Propositions 4 and 5, we may also conclude that the global theorems obtained in Ref. 33 may still be valid by using the coordinate transformation developed here in this paper. Consequently, the first drawback of their approach can be removed. (ii) Examples are given in Sec. V to illustrate how hypotheses (H4) and (H5) can be easily checked.

The main result of the paper is now stated in the following. The proof of the main theorem follows directly from Theorem 3 and Propositions 4 and 5.

Theorem 4: *Let the outer coupling matrix* $\mathbf{G}(t)$ *satisfying Eq. (3a) and the inner coupling matrix* $\boldsymbol{D}$ *be given as in Eq. (3b). Suppose hypotheses (H1), (H2), (H3), (H4), and (H5) hold true, then coupled system (1a) achieves global synchronization whenever*

$$d > \frac{r\sqrt{k}\,\mathrm{cond}(\boldsymbol{C}_1\boldsymbol{C}^T)}{\lambda}\left(1+\frac{\delta^2\|\tilde{\boldsymbol{C}}\|^2\|\boldsymbol{C}_1\boldsymbol{C}^T\|^2}{\gamma^2}\right)^{l/2}, \tag{23}$$

where $\boldsymbol{C}$, $\boldsymbol{C}_1$, *and* $\tilde{\boldsymbol{C}}$ *are given as in Theorem 1, Eq. (8), and Eq. (B4c), respectively.*

Remark 4: The small price to pay by introducing the coordinate transformation $\boldsymbol{C}$ is that the lower bound given as in the right hand side of Eq. (23) on the coupling strength d is size dependent.

V. APPLICATIONS AND COMPARISONS

To see the effectiveness of our main results and to compare our results with existing methods, we consider coupled Lorentz equations with various coupling configurations. The vector field of the individual Lorentz oscillator under consideration is recognized as $\boldsymbol{f}(\boldsymbol{x})=(\sigma(x_2-x_1),rx_1-x_2-x_1x_3,-bx_3+x_1x_2)^T=:(f_1(\boldsymbol{x}),f_2(\boldsymbol{x}),f_3(\boldsymbol{x}))^T$. Here $\sigma=10$, $r=28$, and $b=8/3$. We shall illustrate, via the first three cases, how one should examine the structure of $\boldsymbol{f}(\boldsymbol{x})$ to see if hypotheses (H3)–(H5) are fulfilled or not.

Case 1: Let the inner coupling matrix $\boldsymbol{D}$ correspond to y-component partial-state coupling, i.e.,

$$\boldsymbol{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

Let the outer coupling matrix $\boldsymbol{G}(t)$ be either of the forms given as in Eqs. (9), (11)–(13), and (17). Then hypothesis (H1) of bounded dissipation of system (1a) is well known (see, e.g., Ref. 23), which is the ball $\boldsymbol{B}$ containing the topological product of an absorbing domain

$$B = \left\{ x_1^2 + x_2^2 + (x_3 - r - \sigma)^2 < \frac{b^2(r+\sigma)^2}{4(b-1)} \right\}.$$

Hypothesis (H2) of matrix measure $\mu_2(\overline{\boldsymbol{G}})$ is clearly held as shown in Sec. III. Since the "coupled" nonlinearity $f_2(\cdot)$ satisfies

$$\|f_2(\boldsymbol{u}) - f_2(\boldsymbol{v})\| = \|(r - v_3)(u_1 - v_1) - (u_2 - v_2) - u_1(u_3 - v_3)\| \le r\|\boldsymbol{u} - \boldsymbol{v}\|$$

for some constant $r > 0$ in region $\mathbf{B}$, hypothesis (H3) holds true. Moreover, the difference of "uncoupled" nonlinearities f_1 and f_3 is given as follows:

$$f_1(\mathbf{u}) - f_1(\mathbf{v}) = [-\sigma(u_1 - v_1)] + \sigma(u_2 - v_2),$$

$$f_3(\mathbf{u}) - f_3(\mathbf{v}) = [-b(u_3 - v_3)] + u_1(u_2 - v_2) + v_2(u_1 - v_1).$$

It is readily seen that one should break the uncoupled space into two parts. That is, if we choose $l=2$ and pick the space of the first (respectively, second) diagonal block being the one associated with the nonlinearity f_1 (respectively, f_3), then $\mathbf{Q}_1(\mathbf{v},t) = (-\sigma)$ and $\mathbf{Q}_2(\mathbf{v},t) = (-b)$, it then follows that hypothesis (H4) is held. Furthermore, since $[\boldsymbol{r}(\boldsymbol{u},\boldsymbol{v},t)]_2^2 = r_2(\boldsymbol{u},\boldsymbol{v},t) = \sigma(u_2 - v_2)$, which depends only on coupled space, and $[\boldsymbol{r}(\boldsymbol{u},\boldsymbol{v},t)]_3^3 = r_3(\boldsymbol{u},\boldsymbol{v},t) = u_1(u_2 - v_2) + v_2(u_1 - v_1)$, which depends on the coupled system and the uncoupled subspace with the proceeding index, hypothesis (H5) is fulfilled as well. Hence, by Theorem 4, coupled system (1a) has global synchronization provided coupling strength d is large enough. A numerical result is also presented to support our analytic result, see Fig. 5(a).

Case 2: Let the inner coupling matrix $\boldsymbol{D}$ correspond to either x-component partial-state coupling or full-state coupling, i.e.,

$$\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

Let the outer coupling matrix $\boldsymbol{G}(t)$ be either of the forms given as in Eqs. (9), (11)–(13), and (17). In this case, the coupled nonlinearity f_1 satisfies the following uniformly Lipschitz condition:

$$\|f_1(\mathbf{u}) - f_1(\mathbf{v})\| = \|-\sigma(u_1 - v_1) + \sigma(u_2 - v_2)\| \le \sqrt{2}\sigma\|\boldsymbol{u} - \boldsymbol{v}\|.$$

For uncoupled nonlinearities, we see that

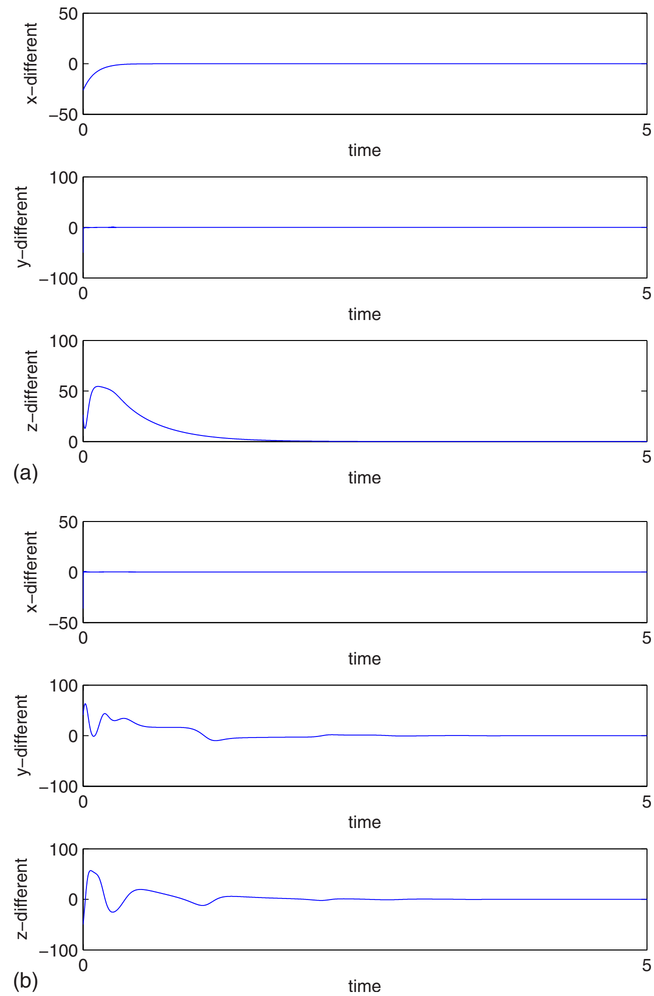

FIG. 5. (Color online) The difference of components of the first two coupled oscillators: (a) the y-component partial-state coupling addressed in Case 1 and (b) the x-component partial-state coupling addressed in Case 2. In both cases $m=8$ and the outer coupling matrix is given as in Eq. (9).

$$\begin{pmatrix} f_2(\boldsymbol{u}) - f_2(\boldsymbol{v}) \\ f_3(\boldsymbol{u}) - f_3(\boldsymbol{v}) \end{pmatrix} = \begin{pmatrix} -1 & -v_1 \\ v_1 & -b \end{pmatrix} \begin{pmatrix} u_2 - v_2 \\ u_3 - v_3 \end{pmatrix} + \begin{pmatrix} (r - u_3)(u_1 - v_1) \\ u_2(u_1 - v_1) \end{pmatrix} =: \boldsymbol{Q}(\boldsymbol{v},t) \begin{pmatrix} u_2 - v_2 \\ u_3 - v_3 \end{pmatrix} + \boldsymbol{r}(\boldsymbol{u},\boldsymbol{v},t).$$

Clearly, $\mu_2(\boldsymbol{Q}(\boldsymbol{v},t)) = \max\{-1, -b\} = -1 < 0$ and $\|\boldsymbol{r}(\boldsymbol{u},\boldsymbol{v},t)\| \le r|u_1 - v_1|$ for some constant $r > 0$ in region $\boldsymbol{B}$. Hence, hypotheses (H4) and (H5) are satisfied, and we can conclude that coupled system (1a) has global synchronization provided coupling strength d is large enough. A numerical result is also presented to support our analytic result, see Fig. 5(b).

Case 3: Consider the z-component partial-state coupling. Since the remainder term in the difference of uncoupled nonlinearities f_1 and f_2 contains each other, the only feasible breaking is to be given in the following:

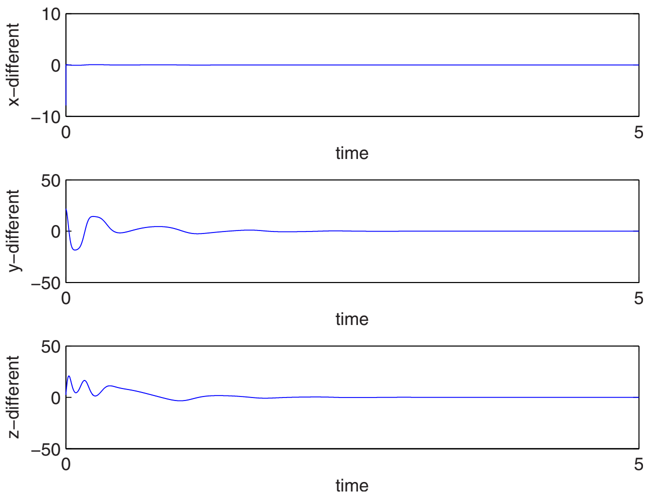

FIG. 6. (Color online) The difference of components of the first two coupled oscillators considered in Case 4. Here the x-component partial-state coupling is considered with $m=8$ and the outer coupling matrix given as in Eq. (15).

$$\begin{pmatrix} f_1(\boldsymbol{u})-f_1(\boldsymbol{v}) \\ f_2(\boldsymbol{u})-f_2(\boldsymbol{v}) \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r-v_3 & -1 \end{pmatrix}\begin{pmatrix} u_1-v_1 \\ u_2-v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -u_1(u_3-v_3) \end{pmatrix}.$$

With $r=28$, the matrix measure of the associated $\boldsymbol{Q}(\boldsymbol{v},t)$ cannot stay negative. Consequently, the conclusion of our main theorem cannot be assumed, which is in consistence with the numerical results.

In the next two cases various (outer) coupling matrices addressed in Sec. III are considered.

Case 4: Let the coupling matrix $\boldsymbol{G}(t)$ be given in Eq. (15) with $d_1(t)=\frac{3}{2}-\sin(t)$. The numerical results are demonstrated in Fig. 6. The synchronization of the coupled Lorentz systems with the coupling matrices $\boldsymbol{G}(t)$ studied in Examples 5–7 of Sec. III is also verified numerically. To save the space, we will not provide such figures. It should be noted that for $\boldsymbol{G}(t)$ considered in Example 7, our numerical results demonstrate that the synchronization of the corresponding system still occurs with $m=88$.

Comparison 1: [Chen (Ref. 33)] Consider the case that the inner coupling matrix $\boldsymbol{D}$ is given in Eq. (25) and the outer coupling matrix $\boldsymbol{G}(t)$ is of the form

$$\boldsymbol{G} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

To apply the global theorem proposed by Chen,[33] one needs to verify the following:

(a) There exists a matrix $\boldsymbol{K}\in\mathbb{R}^{3\times 3}$ such that $\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{K}\boldsymbol{x}$ is $\boldsymbol{V}$-uniformly decreasing for some symmetric positive definite matrix $\boldsymbol{V}\in\mathbb{R}^{3\times 3}$. That is, there is a positive constant μ such that $(\boldsymbol{x}-\boldsymbol{y})^T\boldsymbol{V}(\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{K}\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{K}\boldsymbol{y}) \le -\mu\|\boldsymbol{x}-\boldsymbol{y}\|^2$ for all $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3$.

(b) There exists a diagonal matrix $\boldsymbol{U}\in\mathbb{R}^{2\times 2}$ and $\boldsymbol{M}\in\mathbb{R}^{6\times 6}$ with $\boldsymbol{M}^T\boldsymbol{M}=\boldsymbol{U}\otimes\boldsymbol{V}$ such that

$$\mu_2(\boldsymbol{M}(d\bar{\boldsymbol{G}}_{C_1}\otimes\boldsymbol{D}-\boldsymbol{I}_2\otimes\boldsymbol{K})\boldsymbol{M}^{-1})<0.$$

Here $\bar{\boldsymbol{G}}_{C_1}=-3\boldsymbol{I}_2$.

For the choices of $\boldsymbol{U}=\boldsymbol{I}_2$ and $\boldsymbol{V}=\boldsymbol{I}_3$, we shall show that conditions (a) and (b) cannot be satisfied simultaneously. Indeed, suppose condition (b) holds true for some $\boldsymbol{K}$, then $\mu_2(\boldsymbol{M}(d\bar{\boldsymbol{G}}_{C_1}\otimes\boldsymbol{D}-\boldsymbol{I}_2\otimes\boldsymbol{K})\boldsymbol{M}^{-1})=\mu_2(-3d\boldsymbol{D}-\boldsymbol{K})<0$ and, hence, $\boldsymbol{z}^T\boldsymbol{K}\boldsymbol{z}\ge -3d\boldsymbol{z}^T\boldsymbol{D}\boldsymbol{z}$ for any $\boldsymbol{z}\in\mathbb{R}^3$. Let $\boldsymbol{x}=(x_1,x_2,x_3)^T$ with $x_2\ge 10$, x_1, $x_3\in\mathbb{R}$, and $\boldsymbol{y}=\boldsymbol{x}-k(1,0,1)^T$ with $k\in\mathbb{R}-\{0\}$. Then there is an α, $0<\alpha<1$ such that

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{y})^T&(\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{K}\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{K}\boldsymbol{y}) \\ &= (\boldsymbol{x}-\boldsymbol{y})^T(\boldsymbol{K}+D\boldsymbol{f}(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y}))(\boldsymbol{x}-\boldsymbol{y}) \\ &\ge (\boldsymbol{x}-\boldsymbol{y})^T(D\boldsymbol{f}(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y})-3d\boldsymbol{D})(\boldsymbol{x}-\boldsymbol{y}) \ge \tfrac{2}{3}k^2.\end{aligned}$$

Here

$$D\boldsymbol{f}(\bar{\boldsymbol{x}}) = \begin{pmatrix} -10 & 10 & 0 \\ 28-\bar{x}_3 & -1 & -\bar{x}_1 \\ \bar{x}_2 & \bar{x}_1 & -\frac{8}{3} \end{pmatrix}.$$

And so, condition (a) fails.

Comparison 2: [Chen (Ref. 39)] Let the outer coupling matrix $\boldsymbol{G}(t)$ be given in Eq. (9), and the inner coupling matrix $\boldsymbol{D}$ be $\boldsymbol{I}_3$. To verify the criterion for global synchronization in Ref. 39 it suffices to show

$$\mu_2(\boldsymbol{I}_{m-1}\otimes\boldsymbol{A}+\bar{\boldsymbol{G}}_{C_1}\otimes\boldsymbol{I}_n)<0, \tag{26}$$

where $\boldsymbol{A}=\mathrm{diag}(a_1,\ldots,a_n)$ and $a_i\ge 0$, $i=1,2,\ldots,n$. However, if the number of oscillators is greater than 7, i.e., $m>7$, then $\mu_2(\bar{\boldsymbol{G}}_{C_1})>0$ (see Table I). And so

$$\begin{aligned}\mu_2(\boldsymbol{I}_{m-1}\otimes\boldsymbol{A}+\bar{\boldsymbol{G}}_{C_1}\otimes\boldsymbol{I}_n) &\ge \mu_2(\bar{\boldsymbol{G}}_{C_1})-\mu_2(-\boldsymbol{A}) \\ &= \mu_2(\bar{\boldsymbol{G}}_{C_1})+\min\{a_1,\ldots,a_n\} \\ &> 0.\end{aligned}$$

VI. CONCLUSIONS

A general framework for determining the global stability of synchronous chaotic oscillations in coupled oscillator systems with complex networks has been discussed. This framework allows one to address very large array of oscillators with complex topology including time-varying networks, networks with asymmetric positive and negative coupling, and networks with some randomness. Furthermore, the verification of our framework can be easily checked. The vehicle for providing such general synchronization theory is the matrix measure as well as the newly introduced coordinate transformation.

Theoretical studies of globally synchronous chaos have been conducted with the coupled Lorentz oscillators. The x, y, and z-component couplings of the system have been used as illustrations on how to apply the main theorem. The networks such as the star type, the wavelet transformed type, the pristine world joining with some randomness, the generalized wheel type, and the prism type have been discussed. The comparisons with the existing methods have also been provided.

We would like to conclude our paper with the following remarks. To prove global synchronization of the coupled chaotic systems, one needs to assume the bounded dissipation of the systems, which plays the role of an *a priori* estimate. Such an assumption is also implicitly required in both CGS and MMA. Unfortunately, there have not many general theorems been provided for the bounded dissipation of coupled chaotic systems with complex topology. Note that the bounded dissipation of the individual oscillator does not necessarily imply that the coupled systems with complex networks would share the same property. Therefore, it would certainly be of great interest to develop a theory of the bounded dissipation of the coupled chaotic systems with complex network topologies.

ACKNOWLEDGMENTS

We thank referees for suggesting numerous improvements to the original draft.

APPENDIX A: CONCEPTS OF MATRIX MEASURE

In our derivation of synchronization of system (2), we need the concept of matrix measures. For completeness and ease of references, we also recall the following definition of matrix measures and their properties (see, e.g., Ref. 43).

Definition 3: [Vidyasagar (Ref. 43)] Let $\|\cdot\|_i$ be an induced matrix norm on $\mathbb{R}^{n\times n}$. The matrix measure of matrix $\boldsymbol{K}$ on $\mathbb{R}^{n\times n}$ is defined to be $\mu_i(\boldsymbol{K})=\lim_{\epsilon\to 0^+}\|\boldsymbol{I}+\epsilon\boldsymbol{K}\|_i-1/\epsilon$.

Lemma 1: [Vidyasagar (Ref. 43)] Let $\|\cdot\|_k$ be an induced k-norm on $\mathbb{R}^{n\times n}$, where $k=1,2,\infty$. Then each of matrix measure $\mu_k(\boldsymbol{K})$, $k=1,2,\infty$, of matrix $\boldsymbol{K}=(k_{ij})$ on $\mathbb{R}^{n\times n}$ is, respectively,

$$\mu_\infty(\boldsymbol{K}) = \max_i\left\{k_{ii}+\sum_{j\neq i}|k_{ij}|\right\},$$

$$\mu_1(\boldsymbol{K}) = \max_j\left\{k_{jj}+\sum_{i\neq j}|k_{ij}|\right\},$$

and

$$\mu_2(\boldsymbol{K}) = \lambda_{\max}(\boldsymbol{K}^T+\boldsymbol{K})/2. \tag{A1}$$

Here $\lambda_{\max}(\boldsymbol{K})$ is the maximum eigenvalue of $\boldsymbol{K}$.

Theorem 5: *[Vidyasagar (Ref. 43)] Consider the differential equation $\dot{\boldsymbol{x}}(t)=\boldsymbol{K}(t)\boldsymbol{x}(t)+\boldsymbol{v}(t)$, $t\geq 0$, where $\boldsymbol{x}(t)\in\mathbb{R}^n$, $\boldsymbol{K}(t)\in\mathbb{R}^{n\times n}$, and $\boldsymbol{K}(t)$, $\boldsymbol{v}(t)$ are piecewise continuous. Let $\|\cdot\|_i$ be a norm on $\mathbb{R}^n$ and $\|\cdot\|_i$, μ_i denote, respectively, the corresponding induced norm and matrix measure on $\mathbb{R}^{n\times n}$. Then whenever $t\geq t_0\geq 0$, we have*

$$\begin{aligned}
&\|\boldsymbol{x}(t_0)\|_i\exp\left\{\int_{t_0}^t -\mu_i(-\boldsymbol{K}(s))ds\right\} - \int_{t_0}^t \exp\left\{\int_s^t -\mu_i(-\boldsymbol{K}(\tau))d\tau\right\}\|\boldsymbol{v}(s)\|_i ds\\
&\leq \|\boldsymbol{x}(t)\|_i \leq \|\boldsymbol{x}(t_0)\|_i\exp\left\{\int_{t_0}^t \mu_i(\boldsymbol{K}(s))ds\right\} + \int_{t_0}^t \exp\left\{\int_s^t \mu_i(\mathbf{K}(\tau))d\tau\right\}\|\boldsymbol{v}(s)\|_i ds.
\end{aligned} \tag{A2}$$

APPENDIX B: PROOFS OF MAIN RESULTS

The following notation is needed. Let $\boldsymbol{u}=(u_1,\ldots,u_i,u_{i+1},\ldots,u_j,\ldots,u_n)^T$. Denote by $[\boldsymbol{u}]_i^j=(u_i,u_{i+1},\ldots,u_j)^T$. Write the difference of $\boldsymbol{f}(\cdot,t)$ at $\boldsymbol{u}$ and $\boldsymbol{v}$ in the form

$$\begin{aligned}
\boldsymbol{f}(\boldsymbol{u},t)-\boldsymbol{f}(\boldsymbol{v},t) &= \begin{pmatrix} f_1(\boldsymbol{u},t)-f_1(\boldsymbol{v},t)\\ \vdots \\ f_n(\boldsymbol{u},t)-f_n(\boldsymbol{v},t)\end{pmatrix}\\
&=: \begin{pmatrix} \boldsymbol{f}_c(\boldsymbol{u},t)-\boldsymbol{f}_c(\boldsymbol{v},t)\\ \boldsymbol{f}_u(\boldsymbol{u},t)-\boldsymbol{f}_u(\boldsymbol{v},t)\end{pmatrix}\\
&=: \begin{pmatrix} \boldsymbol{f}_c(\boldsymbol{u},t)-\boldsymbol{f}_c(\boldsymbol{v},t)\\ \boldsymbol{Q}(\boldsymbol{v},t)[\boldsymbol{u}-\boldsymbol{v}]_{k+1}^n+\boldsymbol{r}(\boldsymbol{u},\boldsymbol{v},t)\end{pmatrix},
\end{aligned} \tag{B1}$$

where $\boldsymbol{f}_c(\cdot,t)\in\mathbb{R}^k$, $\boldsymbol{f}_u(\cdot,t)\in\mathbb{R}^{n-k}$, and matrix $\boldsymbol{Q}(\boldsymbol{v},t)$ is of the size $(n-k)\times(n-k)$. Since $\boldsymbol{r}(\boldsymbol{u},\boldsymbol{v},t)$ could depend on all components of $\boldsymbol{u}$ and $\boldsymbol{v}$, such a decomposition in Eq. (B1) can always be achieved.

Proposition 4: Suppose $f_i(\cdot,t)$, $i=1,\ldots,k$ are uniformly Lipschitz, i.e., there exists a positive constant $r>0$ such that

$$|f_i(\boldsymbol{u},t)-f_i(\boldsymbol{v},t)|\leq r\|\boldsymbol{u}-\boldsymbol{v}\| \tag{B2}$$

for all $i=1,\ldots,k$. Then the inequality in (H3†) is satisfied with $b_1=r\sqrt{k}\,\mathrm{cond}(\boldsymbol{C}_1\boldsymbol{C}^T)$. Here $\boldsymbol{C}_1$ is given as in Eq. (8) and $\mathrm{cond}(\boldsymbol{C}_1\boldsymbol{C}^T)=\|\boldsymbol{C}_1\boldsymbol{C}^T\|\|(\boldsymbol{C}_1\boldsymbol{C}^T)^{-1}\|$ is the condition number of $(\boldsymbol{C}_1\boldsymbol{C}^T)$.

Proof: Note first that $\boldsymbol{C}_1=\boldsymbol{C}_1\boldsymbol{C}^T\boldsymbol{C}$ and $\boldsymbol{C}=(\boldsymbol{C}_1\boldsymbol{C}^T)^{-1}\boldsymbol{C}_1$. Now,

$$\begin{aligned}
\|\bar{\boldsymbol{F}}_c(\bar{\boldsymbol{y}},t)\| &= \left\|\begin{pmatrix}\boldsymbol{C}\tilde{\boldsymbol{f}}_1(\tilde{\boldsymbol{x}},t)\\ \vdots\\ \boldsymbol{C}\tilde{\boldsymbol{f}}_k(\tilde{\boldsymbol{x}},t)\end{pmatrix}\right\|\\
&= \left\|(\boldsymbol{I}_k\otimes(\boldsymbol{C}_1\boldsymbol{C}^T)^{-1})\begin{pmatrix}\boldsymbol{C}_1\tilde{\boldsymbol{f}}_1(\tilde{\boldsymbol{x}},t)\\ \vdots\\ \boldsymbol{C}_1\tilde{\boldsymbol{f}}_k(\tilde{\boldsymbol{x}},t)\end{pmatrix}\right\|\\
&\leq \|(\boldsymbol{C}_1\boldsymbol{C}^T)^{-1}\|\left\|\begin{pmatrix}\boldsymbol{C}_1\tilde{\boldsymbol{f}}_1(\tilde{\boldsymbol{x}},t)\\ \vdots\\ \boldsymbol{C}_1\tilde{\boldsymbol{f}}_k(\tilde{\boldsymbol{x}},t)\end{pmatrix}\right\|.
\end{aligned}$$

Since

$$\|C_1\tilde{f}_1(\tilde{x},t)\|^2 = \left\| \begin{pmatrix} f_i(x_1,t) - f_i(x_2,t) \\ \vdots \\ f_i(x_1,t) - f_i(x_m,t) \end{pmatrix} \right\|^2 \le r^2 \left\| \begin{pmatrix} x_1 - x_2 \\ \vdots \\ x_1 - x_m \end{pmatrix} \right\|^2$$

for all $i=1,\ldots,k$, we have that

$$\begin{aligned} \|\overline{F}_c(\bar{y},t)\| &\le \sqrt{k}r\|(C_1C^T)^{-1}\| \left\| \begin{pmatrix} x_1 - x_2 \\ \vdots \\ x_1 - x_m \end{pmatrix} \right\| \\ &= \sqrt{k}r\|(C_1C^T)^{-1}\| \left\| (C_1 \otimes I_n) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \right\| \\ &= \sqrt{k}r\|(C_1C^T)^{-1}\|\|(C_1C^T \otimes I_n)(C \otimes I_n)x\| \\ &\le \sqrt{k}r \operatorname{cond}(C_1C^T)\|\bar{y}\|. \end{aligned} \tag{B3}$$

The proof of the proposition is completed.

The above Proposition 4 amounts to saying that if f_i, $i=1,\ldots,k$, the coupled parts of the vector field of the individual oscillator are uniformly Lipschitz, then hypothesis (H3†) holds.

We next turn our attention to the structure of the vector field of the uncoupled parts.

Proposition 5: (i) Suppose matrix $Q(v,t)$ can be written as the block diagonal form

$$Q(v,t) = \operatorname{diag}(Q_1(v,t),\ldots,Q_l(v,t)),$$

where the size of matrices $Q_j(v,t)$ is $k_j \times k_j$, $\forall j=1,\ldots,l$ and indices l, k_j are given as in (H4†)*. Moreover, there is some $\gamma > 0$ such that*

$$\mu_2(Q_j(v,t)) \le -\gamma. \tag{B4a}$$

Here γ is independent of v,t. Then the inequality in (H4†) *is fulfilled. (ii) Denoted by $s_1=k$ and $s_j=k+\Sigma_{i=1}^{j-1}k_i$, $j=2,\ldots,l$, where k_i and l are defined in* (H4†)*. Let $C=(c_{i,j})_{(m-1)\times m}$. Suppose, for any $1\le j\le l$, there is $\delta>0$ such that*

$$\|[r(u,v,t)]_{s_j+1}^{s_j+k_j}\| \le \delta\|[u-v]_1^{s_j}\|. \tag{B4b}$$

Then the inequality in (H5†) *is satisfied with $b_2 = \delta\|\tilde{C}\|\|C_1C^T\|$. Here*

$$\tilde{C} = (c_{i,j+1}) \in \mathbb{R}^{(m-1)\times(m-1)}, \quad 1 \le i,j \le m-1. \tag{B4c}$$

Proof: Write $\overline{F}_u(\bar{y},t)$ as $(F_{u1}(\bar{y},t),\ldots,F_{ul}(\bar{y},t))^T$, which is in consistence with the block diagonal form of $U(t)$. Now, for $1\le j\le l$,

$$F_{uj}(\bar{y},t) = \begin{pmatrix} C\tilde{f}_{s_j+1}(\tilde{x},t) \\ \vdots \\ C\tilde{f}_{s_j+k_j}(\tilde{x},t) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^m c_{1,k}f_{s_j+1}(x_k,t) \\ \wr \\ \sum_{k=1}^m c_{m-1,k}f_{s_j+1}(x_k,t) \\ \vdots \\ \sum_{k=1}^m c_{1,k}f_{s_j+k_j}(x_k,t) \\ \wr \\ \sum_{k=1}^m c_{m-1,k}f_{s_j+k_j}(x_k,t) \end{pmatrix} = P \begin{pmatrix} \sum_{k=1}^m c_{1,k}f_{s_j+1}(x_k,t) \\ \wr \\ \sum_{k=1}^m c_{1,k}f_{s_j+k_j}(x_k,t) \\ \vdots \\ \sum_{k=1}^m c_{m-1,k}f_{s_j+1}(x_k,t) \\ \wr \\ \sum_{k=1}^m c_{m-1,k}f_{s_j+k_j}(x_k,t) \end{pmatrix} =: Ph.$$

Here P is a permutation matrix. That is, we exchange certain rows of $\overline{F}_{uj}(\bar{y},t)$ to obtain F. Using the fact that the row sums of C are all zeros, we have that for $1\le i\le m-1$, $s_j+1\le l\le s_j+k_j$,

$$\sum_{k=1}^m c_{i,k}f_l(x_k,t) = \sum_{k=2}^m c_{i,k}(f_l(x_k,t) - f_l(x_1,t)). \tag{B5}$$

To save notations, $\forall i=1,\ldots,k_j$, we denote by $[r_{s_j+i}(x_l,x_1,t)]_{l=2}^m$ the vector $(r_{s_j+i}(x_2,x_1,t),r_{s_j+i}(x_3,x_1,t),\ldots,r_{s_j+i}(x_m,x_1,t))^T$.

Applying Eq. (B1) and Eqs. (B4a)–(B4c), we shall be able to rewrite h as

$$\begin{pmatrix} \sum_{k=2}^{m} c_{1,k}\boldsymbol{Q}_j(\boldsymbol{x}_1,t)[\boldsymbol{x}_k-\boldsymbol{x}_1]_{s_j+1}^{s_j+k_j} \\ \vdots \\ \sum_{k=2}^{m} c_{m-1,k}\boldsymbol{Q}_j(\boldsymbol{x}_1,t)[\boldsymbol{x}_k-\boldsymbol{x}_1]_{s_j+1}^{s_j+k_j} \end{pmatrix} + \begin{pmatrix} \sum_{k=2}^{m} c_{1,k}[\boldsymbol{r}(\boldsymbol{x}_k,\boldsymbol{x}_1,t)]_{s_j+1}^{s_j+k_j} \\ \vdots \\ \sum_{k=2}^{m} c_{m-1,k}[\boldsymbol{r}(\boldsymbol{x}_k,\boldsymbol{x}_1,t)]_{s_j+1}^{s_j+k_j} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{Q}_j(\boldsymbol{x}_1,t)\left[\sum_{k=1}^{m} c_{1,k}\boldsymbol{x}_k\right]_{s_j+1}^{s_j+k_j} \\ \vdots \\ \boldsymbol{Q}_j(\boldsymbol{x}_1,t)\left[\sum_{k=1}^{m} c_{m-1,k}\boldsymbol{x}_k\right]_{s_j+1}^{s_j+k_j} \end{pmatrix} + \boldsymbol{P}^T \begin{pmatrix} \tilde{\boldsymbol{C}}[\boldsymbol{r}_{s_j+1}(\boldsymbol{x}_l,\boldsymbol{x}_1,t)]_{l=2}^{m} \\ \vdots \\ \tilde{\boldsymbol{C}}[\boldsymbol{r}_{s_j+k_j}(\boldsymbol{x}_l,\boldsymbol{x}_1,t)]_{l=2}^{m} \end{pmatrix}$$

$$= (\boldsymbol{I}_{m-1}\otimes\boldsymbol{Q}_j(\boldsymbol{x}_1,t))\boldsymbol{P}^T \begin{pmatrix} \boldsymbol{C}\tilde{\boldsymbol{x}}_{s_j+1} \\ \vdots \\ \boldsymbol{C}\tilde{\boldsymbol{x}}_{s_j+k_j} \end{pmatrix} + \boldsymbol{P}^T(\boldsymbol{I}_{k_j}\otimes\tilde{\boldsymbol{C}}) \begin{pmatrix} [\boldsymbol{r}_{s_j+1}(\boldsymbol{x}_l,\boldsymbol{x}_1,t)]_{l=2}^{m} \\ \vdots \\ [\boldsymbol{r}_{s_j+k_j}(\boldsymbol{x}_l,\boldsymbol{x}_1,t)]_{l=2}^{m} \end{pmatrix}$$

$$= (\boldsymbol{I}_{m-1}\otimes\boldsymbol{Q}_j(\boldsymbol{x}_1,t))\boldsymbol{P}^T\bar{\boldsymbol{y}}_{uj} + \boldsymbol{P}^T(\boldsymbol{I}_{k_j}\otimes\tilde{\boldsymbol{C}})\boldsymbol{P} \begin{pmatrix} [\boldsymbol{r}(\boldsymbol{x}_2,\boldsymbol{x}_1,t)]_{s_j+1}^{s_j+k_j} \\ \vdots \\ [\boldsymbol{r}(\boldsymbol{x}_m,\boldsymbol{x}_1,t)]_{s_j+1}^{s_j+k_j} \end{pmatrix}.$$

Therefore,

$$\boldsymbol{U}_j(t) = \boldsymbol{P}(\boldsymbol{I}_{m-1}\otimes\boldsymbol{Q}_j(\boldsymbol{x}_1,t))\boldsymbol{P}^T \tag{B6a}$$

and

$$\boldsymbol{R}_{uj}(\bar{\boldsymbol{y}},t) = (\boldsymbol{I}_{k_j}\otimes\tilde{\boldsymbol{C}})\boldsymbol{P} \begin{pmatrix} [\boldsymbol{r}(\boldsymbol{x}_2,\boldsymbol{x}_1,t)]_{s_j+1}^{s_j+k_j} \\ \vdots \\ [\boldsymbol{r}(\boldsymbol{x}_m,\boldsymbol{x}_1,t)]_{s_j+1}^{s_j+k_j} \end{pmatrix}. \tag{B6b}$$

The first assertion of the position now follows from Eq. (B6a), $\mu_2(\boldsymbol{U}_j(t))=\mu_2(\boldsymbol{Q}_j(\boldsymbol{x}_1,t))\leq-\gamma$. Upon using the similar techniques as those in establishing the inequality in Eq. (B3), we conclude that Eqs. (B4a)–(B4c) hold as asserted.

[1]A. Pikovsky, M. G. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2003).
[2]E. Mosekilde, Y. Maistrenko, and D. Postnov, *Chaotic Synchronization: Applications to Living Systems* (World Scientific, Singapore, 2002).
[3]R. Albert and A. L. Barabasi, Rev. Mod. Phys. **74**, 47 (2002).
[4]S. Boccaletti, V. Latora, Y. Moreon, M. Chavez, and D. U. Hwang, Phys. Rep. **424**, 175 (2006).
[5]D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[6]L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[7]M. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[8]J. Lu, X. Yu, G. Chen, and D. Cheng, IEEE Trans. Circuits Syst., I: Regul. Pap. **51**, 787 (2004).
[9]J. Lü and G. Chen, IEEE Trans. Automat. Contr. **50**, 841 (2005).
[10]Z. Li and G. Chen, IEEE Trans. Circuits Syst., II: Express Briefs **53**, 28 (2006).
[11]C. W. Wu, Phys. Lett. A **319**, 495 (2003).
[12]C. W. Wu and L. O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **42**, 430 (1995).
[13]C. W. Wu, Nonlinearity **18**, 1057 (2005).
[14]C. W. Wu, *Synchronization in Coupled Chaotic Circuits and Systems* (World Scientific, Singapore, 2002).
[15]J. Fan and X. Wang, Europhys. Lett. **349**, 443 (2005).
[16]M. Chen and D. Zhou, Chaos **16**, 013101 (2006).
[17]M. Chen, IEEE Trans. Circuits Syst., II: Express Briefs **53**, 1185 (2006).
[18]Y. Chen, G. Rangarajan, and M. Ding, Phys. Rev. E **67**, 026209 (2003).
[19]R. Amritkar and C. K. Hu, Chaos **16**, 015117 (2006).
[20]J. D. Skufca and E. Bollt, Math. Biosci. **1**, 347 (2004).
[21]J. D. Skufca, E. M. Bollt, and D. G. Roberson, SIAM J. Appl. Dyn. Syst. **5**, 140 (2006).
[22]I. V. Belykh, V. N. Belykh, and M. Hasler, Physica D **195**, 188 (2004).
[23]I. V. Belykh, V. N. Belykh, and M. Hasler, Physica D **195**, 159 (2004).
[24]I. V. Belykh, V. N. Belykh, and M. Hasler, Chaos **16**, 015102 (2006).
[25]I. V. Belykh, V. N. Belykh, and M. Hasler, Physica D **224**, 42 (2006).
[26]J. Juang, C. L. Li, and Y. H. Liang, Chaos **17**, 033111 (2007).
[27]J. Juang and Y. H. Liang, SIAM J. Appl. Dyn. Syst. **7**, 755 (2008).
[28]Y. C. Chang and J. Juang, SIAM J. Appl. Dyn. Syst. **7**, 1445 (2008).
[29]W. Wang and J. E. Slotine, Biol. Cybern. **92**, 38 (2005).
[30]A. Arenas, D. G. Albert, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep. **469**, 93 (2008).
[31]T. Nishikawa, A. E. Motter, Y. C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).
[32]A. E. Motter, C. Zhou, and J. Kurths, Phys. Rev. E **71**, 016116 (2005).
[33]M. Chen, IEEE Trans. Circuits Syst., I: Regul. Pap. **55**, 1335 (2008).
[34]J. F. Heagy, L. M. Pecora, and T. L. Carroll, Phys. Rev. Lett. **74**, 4185 (1995).
[35]V. N. Belykh, I. V. Belykh, and M. Hasler, Phys. Rev. E **62**, 6332 (2000).
[36]V. N. Belykh, I. V. Belykh, K. V. Nevidin, and M. Hasler, Chaos **13**, 165 (2003).
[37]V. N. Belykh, I. V. Belykh, K. V. Nevidin, and M. Hasler, Int. J. Bifurcation Chaos Appl. Sci. Eng. **13**, 755 (2003).
[38]A. Pogromsky and H. Nijmeijer, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **48**, 152 (2001).
[39]M. Chen, Phys. Rev. E **76**, 016104 (2007).
[40]C. Juang, T. M. Hwang, J. Juang, and W. W. Lin, IEEE J. Quantum Electron. **36**, 300 (2000).
[41]G. W. Wei, M. Zhan, and C. H. Lai, Phys. Rev. Lett. **89**, 284103 (2002).
[42]J. Juang, C. L. Li, and J. W. Chang, J. Math. Phys. **47**, 122702 (2006).
[43]M. Vidyasagar, *Nonlinear Systems Analysis*, 1st ed. (Prentice-Hall, Englewood Cliffs, 1978).