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Diversity of traveling wave solutions in FitzHugh–Nagumo type equations

Cheng-Hsiung Hsu [a,*,1], Ting-Hui Yang [b,2], Chi-Ru Yang [c]

[a] Department of Mathematics, National Central University, Chung-Li 32001, Taiwan
[b] Department of Mathematics, Tamkang University, Tamsui, Taipei County 25137, Taiwan
[c] Department of Applied Mathematics, National Chiao Tung University, Hsin-Chu 30010, Taiwan

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ABSTRACT

In this work we consider the diversity of traveling wave solutions of the FitzHugh–Nagumo type equations

$$u_t = u_{xx} + f(u, w), \qquad w_t = \varepsilon g(u, w),$$

where $f(u, w) = u(u - a(w))(1 - u)$ for some smooth function $a(w)$ and $g(u, w) = u - w$. When $a(w)$ crosses zero and one, the corresponding profile equation possesses special turning points which result in very rich dynamics. In [W. Liu, E. Van Vleck, Turning points and traveling waves in FitzHugh–Nagumo type equations, J. Differential Equations 225 (2006) 381–410], Liu and Van Vleck examined traveling waves whose slow orbits lie only on two portions of the slow manifold, and obtained the existence results by using the geometric singular perturbation theory. Based on the ideas of their work, we study the co-existence of different traveling waves whose slow orbits could involve all portions of the slow manifold. There are more complicated and richer dynamics of traveling waves than those of [W. Liu, E. Van Vleck, Turning points and traveling waves in FitzHugh–Nagumo type equations, J. Differential Equations 225 (2006) 381–410]. We give a complete classification of all different fronts of traveling waves, and provide an example to support our theoretical analysis.

* Corresponding author.
E-mail addresses: chhsu@math.ncu.edu.tw (C.-H. Hsu), thyang@mail.tku.edu.tw (T.-H. Yang), cryang.9322801@nctu.edu.tw (C.-R. Yang).

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1. Introduction

The purpose of this work is to investigate the existence of traveling wave solutions of FitzHugh–Nagumo type equations

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + f\big(u(x,t), w(x,t)\big), \\ w_t(x,t) = \varepsilon g\big(u(x,t), w(x,t)\big), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, $f(u,w) = u(u - a(w))(1-u)$ for some smooth function $a(w)$ and $g(u,w) = u - w$. The prototype of FitzHugh–Nagumo equation is of (1.1) but with $f(u,w) = u(1-u)(u-\alpha) - w$ and $g(u,w) = u - \gamma w$ for some constants α and γ, which can be considered as a simplification of the Hodgkin–Huxley equation that describes the propagation of action potentials in the nerve axon of the squid, cf. [6]. The dynamics of such specific equations, especially the traveling wave solutions, have been widely studied in the past, see [3,7,9,10,13,19] and the references therein.

Recently, Liu and Van Vleck [16] considered the co-existence of different traveling wave fronts of (1.1) by allowing $a(w)$ to cross 0 and 1, then the profile equations with respect to (1.1) can be reduced as a singularly perturbed system with turning points. Those special turning points exhibit the so-called delay of stability loss. Applying the geometric singular perturbation (GSP) theory (cf. [4, 5,12]) and the Exchange lemma for turning points (cf. [15]), Liu and Van Vleck show the existence of various types of traveling wave solutions which posses a special set of turning points. The slow manifold M for such singularly perturbed system consists of three parts, by $M = M_0 \cup M_a \cup M_1$ (see Section 2.1). They studied traveling wave solutions whose slow orbits lie only on the portions M_0 and M_1 of the slow manifold, and gave a complete classification of traveling wave solutions.

Motivated by the work of [16], in this paper we reexamine their results to the cases of traveling waves of (1.1) which involves all the portions M_0, M_a, M_1 of the slow manifold. The main difficulties in applying the GSP theory to our problem is to investigate the transversality of invariant manifolds by computing the Melnikov functions. In [16], the slow orbits lie only on the portions M_0 and M_1, then the Melnikov functions is not zero obviously. However, due to the consideration of M_a, the computation of Melnikov functions become more complicated. Using the exact formulas for the heteroclinic orbits of fast limiting dynamics (see Remark 2.1), we successfully derive the exact formula of Melnikov functions (first and second order) represented by Beta or Gamma functions. Thus we can apply the Exchange lemma to track the evolution of invariant manifolds as they pass the vicinity of the slow manifold. Under the consideration of M_a, there are more complicated and richer dynamics of traveling wave solutions than those of [16]. In this article, we give a complete classification of all different fronts of traveling waves.

This paper is organized as follows. In Section 2, we formulate the traveling profile equations of Eq. (1.1) from the viewpoint of dynamical systems, which can be treated as a singularly perturbed problem. Under some assumptions of $a(w)$, detailed analysis for the non-normal hyperbolicity of slow manifold (with turning points) are carried out. Then we establish the Exchange lemma of the slow manifold with (and without) turning points, and illustrate some admissible conditions to guarantee that the singular orbits can be shadowed by true orbits even in the presence of turning points. The main theorems are stated in Section 3. In Section 4, we first investigate the Melnikov function of connecting orbits to detect the transversality of invariant manifolds. Then we prove the main theorems by GSP theory. In the last section we provide an example to support our theoretical analysis.

2. Formulation of GSP problems

In this section, we consider the traveling wave solutions of system (1.1) by assuming $u(x,t) = u(x+ct) = u(\xi)$ and $w(x,t) = w(x+ct) = w(\xi)$ for some real constant $c > 0$, which is the speed of traveling waves. Under such assumptions, the profile equations of (1.1) yield to

$$\begin{cases} cu'(\xi) = u''(\xi) + f\big(u(\xi), w(\xi)\big), \\ cw'(\xi) = \varepsilon g\big(u(\xi), w(\xi)\big). \end{cases} \tag{2.2}$$

Introducing $v = u'$, then (2.2) can be rewritten as

$$\begin{cases} u'(\xi) = v(\xi), \\ v'(\xi) = cv(\xi) - f\big(u(\xi), w(\xi)\big), \\ cw'(\xi) = \varepsilon g\big(u(\xi), w(\xi)\big). \end{cases} \tag{2.3}$$

In terms of the slow variable $\eta := \varepsilon\xi$, we have

$$\begin{cases} \varepsilon\dot{u}(\eta) = v(\eta), \\ \varepsilon\dot{v}(\eta) = cv(\eta) - f\big(u(\eta), w(\eta)\big), \\ \dot{w}(\eta) = c^{-1}g\big(u(\eta), w(\eta)\big), \end{cases} \tag{2.4}$$

here "$\cdot$" means $\frac{d}{d\eta}$. Systems (2.3) and (2.4) are equivalent which give the standard singularly perturbed system in fast and slow scales respectively. Assume that $E := \{w \mid w = a(w)\}$ is a non-empty set, then system (2.3) or (2.4) has equilibria: $(0,0,0)$, $(1,0,1)$ and $(a(w_0), 0, w_0)$ with $w_0 \in E$. We are interest in traveling wave solutions related to such equilibria.

The main application of geometric singular perturbation theory to the problem is to lift limiting singular orbits to traveling wave solutions. In the following we examine the limiting slow and fast dynamics of (2.4) and (2.3) respectively.

2.1. Dynamics for the limiting slow system

The limiting slow dynamics is governed by

$$0 = v, \qquad 0 = cv - f(u,w), \qquad \dot{w} = c^{-1}g(u,w). \tag{2.5}$$

Thus the slow manifold M consists of three parts by $M := M_0 \cup M_a \cup M_1$, where

$$M_0 := \{u = v = 0\}, \qquad M_a := \big\{u = a(w),\ v = 0\big\}, \qquad M_1 := \{u = 1,\ v = 0\}.$$

It is easy to see that M_0 and M_1 are invariant with respect to the flow (2.3) for all ε, and equilibrium $(0,0,0)$ or $(1,0,1)$ attracts all solutions of (2.5) on M_0 or M_1 respectively. If we allow $a(w)$ crossing 0 and 1, then there exists a special type of turning points on M_0 and M_1. We will see that the invariance of M_0 and M_1 plays a crucial role when we consider the limiting slow orbits pass through the turning points.

2.2. Dynamics for the limiting fast system

The limiting fast dynamics is governed by

$$u' = v, \qquad v' = cv - f(u,w), \qquad w' = 0. \tag{2.6}$$

According to (2.5), the slow manifold M consists of equilibria of (2.6). From the above equations, we know that each plane $\{w = \text{const}\}$ is invariant, and there exist three equilibria of system (2.6):

$$E_0 := (0,0,w) \in M_0, \quad E_a(w) := \big(a(w), 0, w\big) \in M_a \quad \text{and} \quad E_1 := (1,0,w) \in M_1.$$

Let $\lambda_0^{\pm}(w,c)$, $\lambda_a^{\pm}(w,c)$ and $\lambda_1^{\pm}(w,c)$ be the linearized eigenvalues of system (2.6) with respect to E_0, E_a and E_1 respectively. Then we have

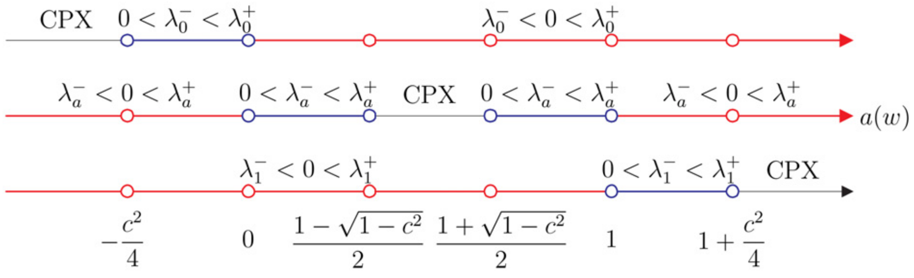

Fig. 1. Sign of linearized eigenvalues with respect to the range of $a(w)$, where CPX means that the eigenvalues are conjugate complex numbers in the range of $a(w)$.

$$\lambda_0^{\pm}(w,c)=\frac{c\pm\sqrt{c^2+4a(w)}}{2}, \tag{2.7}$$

$$\lambda_a^{\pm}(w,c)=\frac{c\pm\sqrt{c^2+4a(w)(a(w)-1)}}{2}, \tag{2.8}$$

$$\lambda_1^{\pm}(w,c)=\frac{c\pm\sqrt{c^2+4(1-a(w))}}{2}. \tag{2.9}$$

If $c\geqslant 1$, then $\lambda_a^{\pm}(w,c)$ are real. If $c<1$ then the sign of the above real eigenvalues with respect to the range of $a(w)$ can be classified in Fig. 1. Therefore, all the linearized eigenvalues are real in the region Ω defined by

$$\Omega:=\left\{(w,c)\in[0,1]\times R^+ \colon\ a(w)\in\left[-\frac{c^2}{4},1+\frac{c^2}{4}\right]\text{ for } c>1;\ \text{ or}\right.$$
$$\left. a(w)\in\left[-\frac{c^2}{4},\infty\right)\setminus\left(\frac{1-\sqrt{1-c^2}}{2},\frac{1+\sqrt{1-c^2}}{2}\right)\text{ for } c<1\right\}.$$

Now we consider the dynamics of (2.6). On each plane $\{w=\text{const}\}$, the limiting system is that for a prototype of Nagumo equations with specific cubic nonlinearity. The existence of heteroclinic orbits on the plane is well understood, cf. [1]. To classify all the possible heteroclinic orbits of (2.6), we first introduce the following notations:

$$a_1(c):=\max\left\{0,\frac{1-\sqrt{2}c}{2}\right\},\qquad a_2(c):=\min\left\{1,\frac{1+\sqrt{2}c}{2}\right\},\qquad a_3(c):=2+\sqrt{2}c,$$
$$a_4(c):=-1-\sqrt{2}c,\qquad a_5(c):=\max\{1,2-\sqrt{2}c\},\qquad a_6(c):=\min\{0,-1+\sqrt{2}c\},$$
$$H_i(c):=\left\{w\in(0,1)\colon\ a(w)=a_i(c),\ a'(w)\neq 0\right\},\quad i=1,\dots,6,$$
$$G_1(c):=\left\{w\in(0,1)\colon\ a(w)\leqslant a_6(c)\right\},\qquad G_2(c):=\left\{w\in(0,1)\colon\ a(w)\geqslant a_5(c)\right\},$$
$$G_3(c):=\left\{w\in(0,1)\colon\ 0>a(w)>a_4(c)\right\},$$
$$G_4(c):=\left\{w\in(0,1)\colon\ 1<a(w)<a_3(c)\right\},$$
$$G_5(c):=\left\{w\in(0,1)\colon\ 0<a(w)<a_2(c)\right\},$$
$$G_6(c):=\left\{w\in(0,1)\colon\ 1>a(w)>a_1(c)\right\}.$$

Furthermore, for any fixed $w\in[0,1]$ we denote $r\to s$ to be the heteroclinic orbit connecting $(r,0,w)$ to $(s,0,w)$, where $r\neq s$ and $r,s\in\{0,a(w),1\}$. According to the results of [1] and phase plane analysis,

Table 1
Classification of admissible heteroclinic orbits.

Type of orbit	Admissible parameter condition	Region
$0 \to 1$	$a(w) = a_1(c)$ or $a(w) \leqslant a_6(w)$	$w \in H_1 \cup G_1$
$1 \to 0$	$a(w) = a_2(c)$ or $a(w) \geqslant a_5(w)$	$w \in H_2 \cup G_2$
$0 \to a(w)$	$a(w) = a_3(c)$ or $a_4(c) < a(w) < 0$	$w \in H_3 \cup G_3$
$1 \to a(w)$	$a(w) = a_4(c)$ or $1 < a(w) < a_3(c)$	$w \in H_4 \cup G_4$
$a(w) \to 0$	$a(w) = a_5(c)$ or $0 < a(w) < a_2(c)$	$w \in H_5 \cup G_5$
$a(w) \to 1$	$a(w) = a_6(c)$ or $a_1(c) < a(w) < 1$	$w \in H_6 \cup G_6$

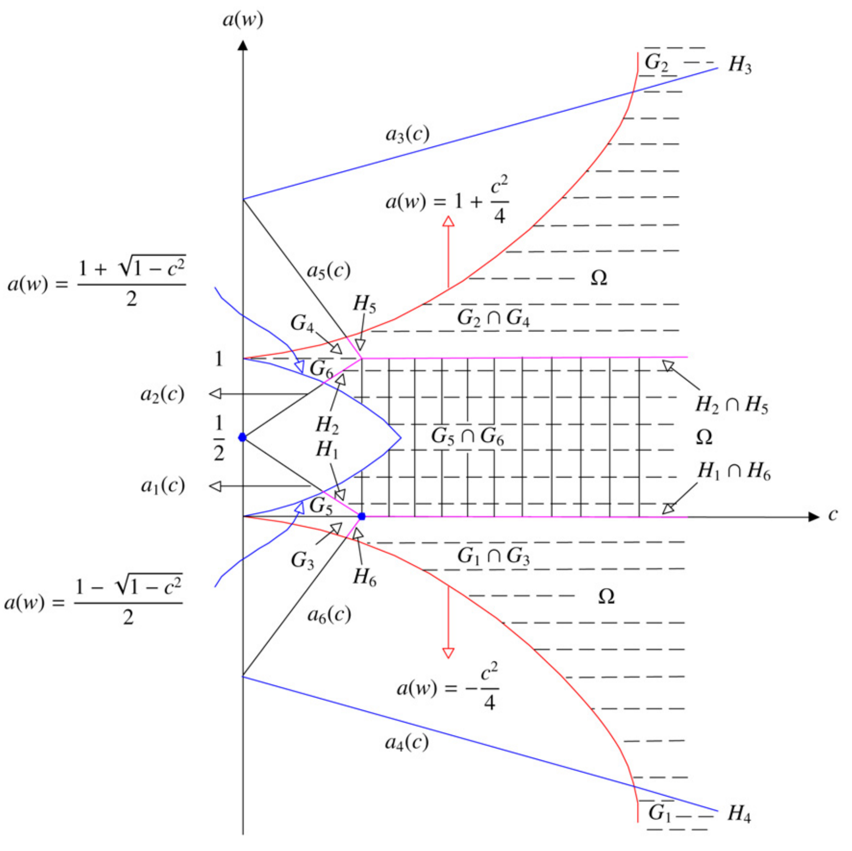

Fig. 2. Regions of Ω, G_i and H_i.

various types of heteroclinic orbits with respect to different regions of the parameters can be classified in Table 1. Note that the linearized eigenvalues are real in the region Ω. Throughout this work, we redefine sets H_i and G_i in Table 1 by $H_i \cap \Omega$ and $G_i \cap \Omega$. With a slight abusing the notation, we keep the same notations. The regions of Ω, H_i and G_i are illustrated in Fig. 2.

Remark 2.1.

(1) As shown in [1], if $w = w_0 \in H_i(c)$, $i = 3, 4, 5, 6$, then the exact formulas for the heteroclinic orbits $(u(t; w_0), v(t; w_0))$ of (2.6) can be expressed as follows:

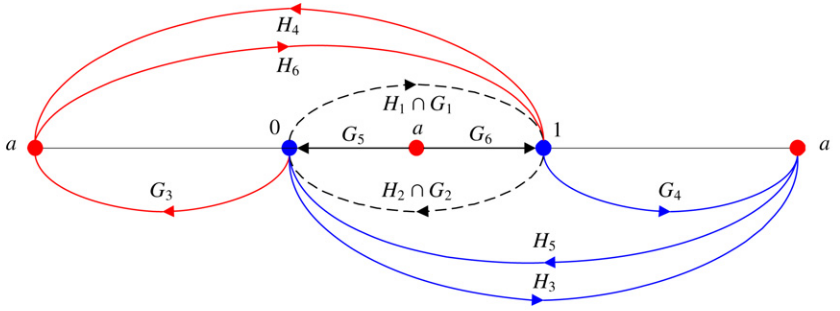

Fig. 3. Admissible heteroclinic orbits with respect to regions.

$$u(t; w_0) = \begin{cases} a(w_0) - a(w_0)(1 + e^{a(w_0)t/\sqrt{2}})^{-1}, & \text{if } w_0 \in H_3(c); \\ a(w_0) + (1 - a(w_0))(1 + e^{(1-a(w_0))t/\sqrt{2}})^{-1}, & \text{if } w_0 \in H_4(c); \\ a(w_0)(1 + e^{a(w_0)t/\sqrt{2}})^{-1}, & \text{if } w_0 \in H_5(c); \\ 1 - (1 - a(w_0))(1 + e^{(1-a(w_0))t/\sqrt{2}})^{-1}, & \text{if } w_0 \in H_6(c). \end{cases}$$

Based on the above formulas, the Melnikov functions (first and second order) for invariant manifolds of connecting orbits can be derived explicitly by Beta or Gamma functions, for details see Section 4.

(2) In [16], they examined traveling waves whose slow orbits lie only on the portions M_0 and M_1 of the slow manifold, thus only regions H_1, H_2, G_1, G_2 are considered (see dashed line paths of Fig. 3). To generalize their work to traveling waves whose slow orbits lie on all portions of M, we need to consider some additional regions than those of [16] (see the non-dash path of Fig. 3).

Next, we investigate the normal hyperbolicity of the slow manifolds. The normal hyperbolicity of the slow manifold of M_0 or M_1 is determined by the eigenvalues $\lambda_0^{\pm}(w,c)$ or $\lambda_1^{\pm}(w,c)$, respectively. If $(0,0,w) \in M_0$ at which $a(w) = 0$, then $\lambda_0^{-}(w,c) = 0$ and the slow manifold M_0 loses normal hyperbolicity at this point. Similarly, the slow manifold M_0 loses normal hyperbolicity at points $(1,0,w) \in M_1$ satisfying $a(w) = 1$. All such points are called *turning points*. Since M_0 and M_1 are invariant, the existence of turning points on them can cause the phenomena of delay of stability loss, see [15]. To describe the results for delay of stability loss, Exchange lemma with turning points and our main theorems, in this article we assume that the curve $u = a(w)$ crosses $u = 0$ and $u = 1$, and satisfies the following assumption:

(H) There exist (increasing) ordered sets $\{T_0^i\}_{i=1}^p, \{T_1^j\}_{j=1}^q \subseteq [0,1]$ such that

$$a(T_0^i) = 0, \qquad a(T_1^j) = 1, \qquad a'(T_0^i) \neq 0, \qquad a'(T_1^j) \neq 0,$$

for all $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$.

By (H), the sets of points $\{(0,0,T_0^i)\}_{i=1}^p$ and $\{(1,0,T_1^j)\}_{j=1}^q$ are turning points on the slow manifold M_0 and M_1 respectively. For the position of equilibria and turning points, dynamics on the slow manifold and heteroclinic orbit for fast dynamics, see Fig. 4.

From Table 1 and the hyperbolicity of slow manifold for the limiting system (2.4), we plan to construct singular orbits (unions of slow and fast orbits) as candidates for limits of traveling wave solutions. Then we can obtain the existence of traveling wave solutions of (2.2) by applying the geometric singular perturbation theorem to lift singular orbits to the true orbits.

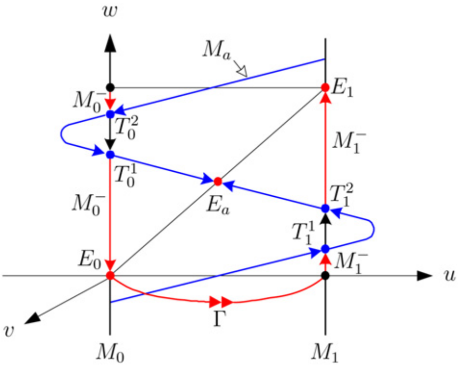

Fig. 4. Equilibria, turning points, dynamics on the slow manifold and heteroclinic orbit Γ of the fast dynamics which connects E_0 and $(1,0,0)$ when $a(0)=a_1(c)$ or $a(0)\leqslant a_6(0)$. The red segments $M_{0,1}^-$ on $M_{0,1}$ are defined by $M_0^-=\{(0,0,w)\in M_0 \mid \lambda_0^-(w,c)<0\}$ and $M_1^-=\{(1,0,w)\in M_1 \mid \lambda_1^-(w,c)<0\}$. For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.

2.3. Delay of stability loss and Exchange lemmas

In this section, we recall and reformulate the results in [15,16] about the delay of stability loss and Exchange lemma for turning points. For any fixed $c>0$, let us denote

$$M_0^- := \{(0,0,w)\in M_0 \mid \lambda_0^-(w,c)<0\} \quad \text{and} \quad M_1^- := \{(1,0,w)\in M_1 \mid \lambda_1^-(w,c)<0\}.$$

If the above sets are non-empty, then we define two maps P_0 and P_1 on such sets as follows.

(P_0) Let $P_0 : M_0^- \to M_0$ be defined by

$$P_0(0,0,w) = \begin{cases} (0,0,\overline{w}), & \text{if } \overline{w} \text{ exists,} \\ (0,0,0), & \text{otherwise,} \end{cases}$$

where $\overline{w}\in(0,w)$ is the first value such that

$$\int_{\overline{w}}^{w} \frac{\lambda_0^-(\eta,c)}{g(0,\eta)}\,d\eta = 0.$$

(P_1) Let $P_1 : M_1^- \to M_1$ be defined by

$$P_1(1,0,w) = \begin{cases} (1,0,\overline{w}), & \text{if } \overline{w} \text{ exists,} \\ (1,0,1), & \text{otherwise,} \end{cases}$$

where $\overline{w}\in(w,1)$ is the first value such that

$$\int_{w}^{\overline{w}} \frac{\lambda_1^-(\eta,c)}{g(1,\eta)}\,d\eta = 0.$$

Based on the above two maps, Liu and Van Vleck [16] reformulated the Exchange lemma on M_0 and M_1 for system (2.3) with an extra equation $c'=0$, that is

$$u'(\xi)=v, \qquad v'(\xi)=cv-f(u,w), \qquad w'(\xi)=\varepsilon c^{-1}g(u,w), \qquad c'=0. \tag{2.10}$$

To guarantee the existence of unstable manifold $W_0^u(K)$ and center manifold $W_0^c(K)$ for any set $K \subset M_0 \cup M_1$, we restrict c belonging to the following set

$$S := \left\{c > 0\colon\ a(w) \in \left[-\frac{c^2}{4}, 1 + \frac{c^2}{4}\right] \text{ for all } w \in [0,1]\right\}.$$

Denote

$$\begin{aligned} M_1^\delta(w) &:= \bigl\{(1, 0, \overline{w}) \in M_1\colon\ \overline{w} \in (w - \delta, w + \delta)\bigr\}, \\ M_0^\delta(w) &:= \bigl\{(0, 0, \overline{w}) \in M_0\colon\ \overline{w} \in (w - \delta, w + \delta)\bigr\}, \\ M_a^\delta(w) &:= \bigl\{\bigl(a(\overline{w}), 0, \overline{w}\bigr) \in M_a\colon\ \overline{w} \in (w - \delta, w + \delta)\bigr\}, \end{aligned}$$

for any small $\delta > 0$ and any $w \in [0,1]$. The Exchange lemma for M_1 with turning points is stated as follows.

Proposition 2.2 *(Exchange lemma with turning point). (Cf. [15,16].) Let M^ε be a two-dimensional invariant manifold of system* (2.10) *which is smooth in ε. For $\varepsilon = 0$, suppose that M^0 intersects $W_0^c(M_1 \times (c_1, c_2))$ transversally. Let N be the intersection. Then* $\dim N = 1$. *Suppose that $\omega(N) = \{(1, 0, w_1, c^*)\}$ and let $w_2 \in (w_1, 1)$ be any number. We have*:

(1) *If $w_2 < P_1(w_1)$, then for $\varepsilon > 0$ small, a portion of M^ε will approach $(1, 0, w_1, c^*)$, follow the slow orbit from $(1, 0, w_1, c^*)$ to $(1, 0, w_2, c^*)$, leave the vicinity of $M_1 \times (c_1, c_2)$, and upon leaving, it is C^1 $O(\varepsilon)$-close to the unstable manifold $W^u(M_1^\delta(w_2) \times \{c^*\})$ for some $\delta > 0$ independent of ε (see Fig.* 5*).*
(2) *If $w_2 = P_1(w_1) \not\subseteq \{T_1^1, T_1^2, \ldots, T_1^q\}$, then for $\varepsilon > 0$ small, a portion of M^ε will approach $(1, 0, w_1, c^*)$, follow the slow orbit from $(1, 0, w_1, c^*)$ to $(1, 0, w_2, c^*)$, leave the vicinity of $M_1 \times (c_1, c_2)$, and upon leaving, it is C^1 $O(\varepsilon)$-close to the center–unstable manifold $W^{cu}(1, 0, w_2, c^*)$ (see Fig.* 6*).*
(3) *If $w_2 > P_1(w_1)$, then for $\varepsilon > 0$ small, there is no portion of M^ε that approaches $(1, 0, w_1, c^*)$, follows the slow orbit from $(1, 0, w_1, c^*)$, leave the vicinity of M_1 in a neighborhood of $(1, 0, w_2, c^*)$.*

For singular orbits passing no turning point, we use the following Exchange lemma without turning points.

Proposition 2.3 *(Exchange lemma without turning point). (Cf. [11,14,18].) Let M^ε be a two-dimensional invariant manifold of system* (2.10) *which is smooth in ε. For $\varepsilon = 0$, suppose that M^0 intersects $W_0^c(M_a \times \{c^*\})$ transversally. Let N be the intersection. Then* $\dim N = 1$. *Suppose that $\omega(N) = \{(a(w_1), 0, w_1, c^*)\}$. Let w_2 be any number such that $a(w) \neq 0$ or 1, for all w between w_1 and w_2. then for $\varepsilon > 0$ small, a portion of M^ε will approach $(a(w_1), 0, w_1, c^*)$, follow the slow orbit from $(a(w_1), 0, w_1, c^*)$ to $(a(w_2), 0, w_2, c^*)$, leave the vicinity of $M_a \times \{c^*\}$, and upon leaving, it is C^1 $O(\varepsilon)$-close to the unstable manifold $W^u(M_a^\delta(w_2) \times \{c^*\})$ for some $\delta > 0$ independent of ε.*

2.4. Admissible conditions for singular orbits

In view of the results of Exchange lemma with turning points, not all singular orbits are shadowed by true orbits. To guarantee the shadowing property, we introduce some admissible conditions for the construction of singular orbits.

Let $w = (w_1, w_2, \ldots, w_n)$ with $w_i \in [0,1]$ and $s = (s_1, s_2, \ldots, s_{n+1})$ with $s_1 = 1$, $s_i \in \{0, a, 1\}$, $s_i \neq s_{i+1}$ and $s_{n+1} \in \{0,1\}$. For any two words w and s, we denote the singular orbit starting from 0 to s_{n+1} by $0 \to s_1 \to \cdots \to s_{n+1}$ such that the local path $s_i \to s_{i+1}$ (part of the orbit from s_i to s_{i+1}) occurring at the plane $w = w_i$. Since the manifold M_a does not persist for all $\varepsilon > 0$, the Exchange lemma cannot be applied to our problem directly. Therefore, we only focus on the cases with

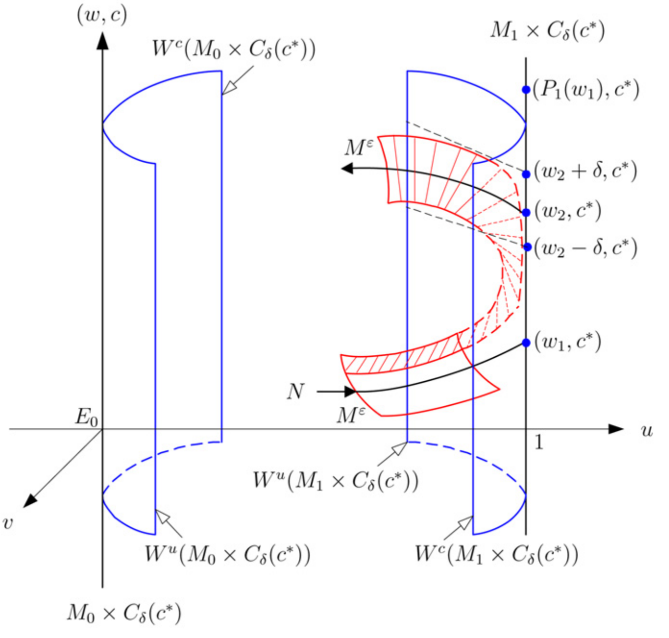

Fig. 5. Part (1) of Proposition 2.2. In this graph, we denote $C_\delta(c) := (c - \delta, c + \delta)$ for some $\delta > 0$ and identify the (w, c)-plane with the vertical axis.

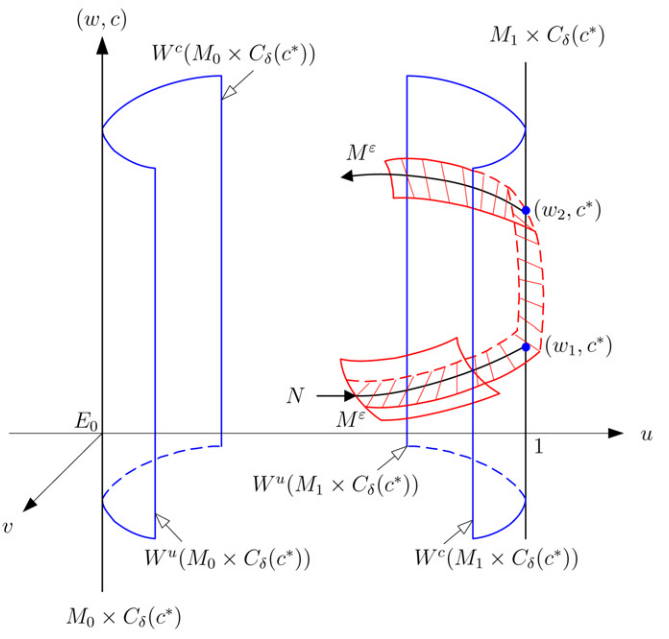

Fig. 6. Part (2) of Proposition 2.2. In this graph, we denote $C_\delta(c) := (c - \delta, c + \delta)$ for some $\delta > 0$ and identify the (w, c)-plane with the vertical axis.

$w \in (w_{i-1}, w_i)$, $a(w) > 1$ if $w_i \in H_5(c^*)$, and $a(w) < 0$ if $w_i \in H_6(c^*)$. It could be seen that the singular orbit on the manifold M_a will not pass the turning point.

In the following we say that w is *s-admissible with respect to some $c^* > 0$* if

$$\begin{cases} w_i \in H_1(c^*) \cup G_1(c^*) \setminus \{T_0^1, T_0^2, \dots, T_0^p\}, & \text{when } s_i s_{i+1} = 01, \\ w_i \in H_2(c^*) \cup G_2(c^*) \setminus \{T_1^1, T_1^2, \dots, T_1^q\}, & \text{when } s_i s_{i+1} = 10, \\ w_i \in H_3(c^*) \cup G_3(c^*) \setminus \{T_0^1, T_0^2, \dots, T_0^p\}, & \text{when } s_i s_{i+1} = 0a, \\ w_i \in H_4(c^*) \cup G_4(c^*) \setminus \{T_1^1, T_1^2, \dots, T_1^q\}, & \text{when } s_i s_{i+1} = 1a, \\ w_i \in H_5(c^*), & \text{when } s_i s_{i+1} = a0, \\ w_i \in H_6(c^*), & \text{when } s_i s_{i+1} = a1, \end{cases} \tag{2.11}$$

for $i = 1, \dots, n$ and the following conditions (A1)–(A3) hold:

(A1) $P_1(0) > w_1$ and

$$\begin{cases} a(w) < 1, & \forall w \in [w_n, 1], \text{ if } s_{n+1} = 1, \\ a(w) > 0, & \forall w \in [0, w_n], \text{ if } s_{n+1} = 0. \end{cases}$$

(A2) For $s_i = 0$, $w_{i-1} > w_i$ and

$$P_0(w_{i-1}) \begin{cases} < w_i, & \text{when } w_i \in H_1(c^*) \cup H_3(c^*), \\ = w_i, & \text{when } w_i \in G_1(c^*) \cup G_3(c^*) \setminus \{T_0^1, T_0^2, \dots, T_0^p\}. \end{cases}$$

(A3) For $s_i = 1$, $w_{i-1} < w_i$ and

$$P_1(w_{i-1}) \begin{cases} > w_i, & \text{when } w_i \in H_2(c^*) \cup H_4(c^*), \\ = w_i, & \text{when } w_i \in G_2(c^*) \cup G_4(c^*) \setminus \{T_1^1, T_1^2, \dots, T_1^q\}. \end{cases}$$

Furthermore, we say that a word $w = (w_1, w_2, \dots, w_n)$ is *admissible with respect to c^** if there is a word $s = (s_1, s_2, \dots, s_{n+1})$ with $s_1 = 1$, $s_i \in \{0, a, 1\}$, $s_i \neq s_{i+1}$ and $s_{n+1} \in \{0, 1\}$, such that w is s-admissible with respect to c^*.

3. Main results

According to the Exchange lemma and the admissible conditions defined in previous section, we state the main theorems in this section and prove them in next section. For a description of the statement of our main results, we give the following definition.

Definition 3.1. Let $\mathcal{O}$ be a singular orbit for some fixed $c^* > 0$. The singular orbit $\mathcal{O}$ is "weakly shadowed" if for any neighborhood $\mathcal{U}$ of the singular orbit, $\mathcal{O}$, there is an $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \leqslant \varepsilon_0$, there is a true orbit $\mathcal{O}(\varepsilon) \in \mathcal{U}$ of system (2.3) with $c = c(\varepsilon)$ and $(\mathcal{O}(\varepsilon), c(\varepsilon)) \to (\mathcal{O}, c^*)$ as $\varepsilon \to 0$ with respect to the Hausdorff distance of sets. Furthermore, if $c(\varepsilon) = c^*$ for all $0 < \varepsilon \leqslant \varepsilon_0$, then we say the singular orbit $\mathcal{O}$ is "strongly shadowed".

First, we consider the traveling wave solutions connecting $(0, 0, 0)$ to $(1, 0, 1)$. From Table 1, we know that such kind of traveling wave solutions exists only if $s_1 = 1$ or $s_1 = a(0)$. If $s_1 = 1$ then it is required that $\lambda_1^-(0; c) < 0$ and $\lambda_1^-(1; c) < 0$ to guarantee the first and last connection. It is easy to see that these two conditions are equivalent to $a(0) < 1$ and $a(1) < 1$ respectively. In addition, it could be seen that the structures of traveling wave solutions are dramatically different for different sign of $a(0)$. Therefore, we will consider two situations $a(0) > 0$ and $a(0) < 0$ separately.

If $a(0) > 0$, then there exists a unique c^* with $a_1(c^*) = a(0)$ (in fact $c^* = (1 - 2a(0))/\sqrt{2}$) such that system (2.6) has a heteroclinic orbit connecting from $(0,0,0)$ to $(1,0,0)$ approaching $(0,0,0)$ backward along the eigenvector associated to $\lambda_0^+(0, c^*)$.

Theorem 3.2. *Assume that* $0 < a(0) < 1$, $a(1) < 1$ *and* $c^* \in S$ *is the unique value such that* $a_1(c^*) = a(0)$.

(1) *If* $w = (w_1, w_2, \ldots, w_n)$ *is admissible with respect to* c^**, then the associated singular orbit is weakly shadowed.*
(2) *If* $w = (w_1, w_2, \ldots, w_n)$ *is not admissible with respect to* c^**, then the associated singular orbit is not weakly shadowed.*

If $a(0) \leqslant 0$, from Table 1, system (2.6) possess a heteroclinic orbit connecting from $(0,0,0)$ to $(1,0,0)$ only if $a(0) \leqslant a_6(c)$, or equivalent to $c \in \Lambda := \{c\colon\ c \geqslant (1 + a(0))/\sqrt{2}\}$.

Theorem 3.3. *Assume that* $a(0) < 0$, $a(1) < 1$ *and* $c^* \in \Lambda \cap S$.

(1) *If* $w = (w_1, w_2, \ldots, w_n)$ *is admissible with respect to* c^**, then the associated singular orbit is strongly shadowed.*
(2) *If* $w = (w_1, w_2, \ldots, w_n)$ *is not admissible with respect to* c^**, then the associated singular orbit is not weakly shadowed.*

Next, we consider the traveling wave solutions connecting $(0,0,0)$ to $(0,0,0)$, i.e. traveling pulse solutions. By Table 1, we know that such kind of traveling wave solutions exists only if $s_1 = 1$ or $s_1 = a(0)$. If $s_1 = 1$, then it is required that $\lambda_1^-(0;c) < 0$ and $\lambda_0^-(0;c) < 0$ to guarantee the first and last connection. Both conditions are equivalent to $0 < a(0) < 1$. If $0 < a(0) < 1$ then there exists a unique c^* with $a_1(c^*) = a(0)$ (in fact $c^* = (1 - 2a(0))/\sqrt{2}$) such that system (2.6) has a heteroclinic orbit from $(0,0,0)$ to $(1,0,0)$ approaching $(0,0,0)$ backward along the eigenvector associated to $\lambda_0^+(0, c^*)$.

Theorem 3.4. *Assume that* $0 < a(0) < 1$ *and* $c^* \in S$ *is the unique value such that* $a_1(c^*) = a(0)$.

(1) *If* $w = (w_1, w_2, \ldots, w_n)$ *is admissible with respect to* c^**, then the associated singular orbit is weakly shadowed.*
(2) *If* $w = (w_1, w_2, \ldots, w_n)$ *is not admissible with respect to* c^**, then the associated singular orbit is not weakly shadowed.*

Remark 3.5. Theorems 3.2–3.4 present results on traveling fronts from $(0,0,0)$ to $(1,0,1)$, and traveling pulses to $(0,0,0)$. Following the similar arguments, the existence of traveling waves involving the equilibria $(a(w_0), 0, w_0)$ for $w_0 \in E$ can also be investigated in the same way.

4. Proof of the main results

To prove the main results in this section, we first detect the transversality of invariant manifolds for connecting orbits by investigating the Melnikov function.

4.1. Melnikov function and transversality of manifolds

First, we recall the results for the formula of Melnikov function [2,8,11,17].

Lemma 4.1. *Consider the plane system*

$$y' = R_0(y) + \bar{\varepsilon} R_1(y, \bar{\varepsilon}), \tag{4.12}$$

where $\bar{\varepsilon} \geqslant 0$ and $R_0, R_1 \in C^r$ with $r \geqslant 2$. Suppose that y_0^1 and y_0^2 are two different hyperbolic saddle points of (4.12)$|_{\bar{\varepsilon}=0}$, and there exists a heteroclinic orbit $y_0(t)$ of (4.12)$|_{\bar{\varepsilon}=0}$ connecting from y_0^1 to y_0^2. Then the Melnikov function of (4.12) is

$$\mathcal{M}(y_0) = \int_{-\infty}^{\infty} e^{-\int_0^t \sigma(s)\,ds} D(t)\,dt, \tag{4.13}$$

where $\sigma(t) = tr\frac{\partial R_0}{\partial y}(y_0(t))$ and $D(t) = R_0(y_0(t)) \wedge R_1(y_0(t), 0)$.

According to formula (4.13), we can compute the Melnikov function of system (2.6) in the following.

Lemma 4.2. *Suppose, for some c_0 and w_0, system (2.6) has a heteroclinic orbit $\Gamma := r(t; w_0) = (u_0(t; w_0), v_0(t; w_0))$.*

(1) *For fixed $w = w_0$ and varying c, the Melnikov function with respect to the heteroclinic orbit Γ is given by*

$$\mathcal{M}(c_0) = \int_{-\infty}^{\infty} e^{-c_0 t} v_0^2(t; w_0)\,dt.$$

In particular, $\mathcal{M}(c_0) \neq 0$.

(2) *For fixed $c = c_0$ and varying w, the Melnikov function is given by*

$$\mathcal{M}(w_0) = a'(w_0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0) u_0(t; w_0)\big(1 - u_0(t; w_0)\big)\,dt. \tag{4.14}$$

Proof. (1) For fixed $w = w_0$, let $F(u, v; c)$ be the vector field of system (2.6), i.e., $F(u, v; c) = (v, cv - u(u - a(w_0)(1 - u)))$ and define $F_c(u, v; c) := (0, v)$. Applying Lemma 4.1 by taking $\bar{\varepsilon} = c$, the Melnikov function is

$$\begin{aligned}\mathcal{M}(c_0) &= \int_{-\infty}^{\infty} e^{-\int_0^t trDF(r(s;w_0);c_0)\,ds}\Big(F\big(r(t; w_0); c_0\big) \wedge F_c\big(r(t; w_0); c_0\big)\Big)\,dt \\ &= \int_{-\infty}^{\infty} e^{-c_0 t} v_0^2(t; w_0)\,dt \neq 0.\end{aligned}$$

(2) For fixed $c = c_0$, we have $F(u, v; w) = (v, c_0 v - u(u - a(w)(1 - u)))$. Denote $F_w(u, v; w) := (0, u(1 - u)a'(w))$. Applying Lemma 4.1 by taking $\bar{\varepsilon} = w$, the Melnikov function is

$$\begin{aligned}\mathcal{M}(w_0) &= \int_{-\infty}^{\infty} e^{-\int_0^t trDF(r(s;w_0);w_0)\,ds}\Big(F\big(r(t; w_0); w_0\big) \wedge F_w\big(r(t; w_0); w_0\big)\Big)\,dt \\ &= a'(w_0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0) u_0(t; w_0)\big(1 - u_0(t; w_0)\big)\,dt.\end{aligned}$$

The proof is complete. □

Based on the results of Lemma 4.2, we now compute the first-order Melnikov function of system (2.6) when w varies in different parameter regions.

Lemma 4.3. *Assume that $a'(w_0) \neq 0$, then $\mathcal{M}(w_0) \neq 0$ for any $c_0 \in (0, 1/\sqrt{2})$ and $w_0 \in H_i(c_0)$, $i = 1, \dots, 6$.*

Proof. (1) If $w_0 \in H_1(c_0) \cup H_2(c_0)$ then $u_0(t; w_0) \in (0, 1)$ for all t. By Lemma 4.2, we have

$$\mathcal{M}(w_0) = a'(w_0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0) u_0(t; w_0) \big(1 - u_0(t; w_0)\big)\, dt \neq 0.$$

(2) If $w_0 \in H_5(c_0)$ then $a(w_0) = 2 - \sqrt{2} c_0 \in (1, 2)$. According to Remark 2.1, the heteroclinic orbit $(u_0(t; w_0), v_0(t; w_0))$ can be represented explicitly in the following:

$$u_0(t; w_0) = a(w_0)\big(1 + e^{a(w_0)t/\sqrt{2}}\big)^{-1} \quad \text{and} \quad v_0(t; w_0) = u_0'(t; w_0).$$

Thus

$$e^{-c_0 t} = \big(a(w_0) - u_0(t; w_0)\big)^{\ell} u_0^{-\ell}(t; w_0), \quad \text{where } 0 < \ell := 1 - \frac{2}{a(w_0)} < 1/2.$$

We can compute Eq. (4.14) by

$$\begin{aligned}
\frac{\mathcal{M}(w_0)}{a'(w_0)} &= \int_{a(w_0)}^{0} u^{1-\ell}\big(a(w_0) - u\big)^{\ell}(1 - u)\, du \\
&= \int_{0}^{a(w_0)} u^{2-\ell}\big(a(w_0) - u\big)^{\ell}\, du - \int_{0}^{a(w_0)} u^{1-\ell}\big(a(w_0) - u\big)^{\ell}\, du \\
&= a^3(w_0) \int_0^1 t^{2-\ell}(1-t)^{\ell}\, dt - a^2(w_0) \int_0^1 t^{1-\ell}(1-t)^{\ell}\, dt \\
&= a^3(w_0) B(1+\ell, 3-\ell) - a^2(w_0) B(1+\ell, 2-\ell) \\
&= a^3(w_0)\big(\Gamma(1+\ell)\Gamma(3-\ell)/\Gamma(4)\big) - a^2(w_0)\big(\Gamma(1+\ell)\Gamma(2-\ell)/\Gamma(3)\big) \\
&= a^2(w_0)\big(a(w_0) - 1\big)\Gamma(1+\ell)\Gamma(2-\ell)/\Gamma(4) > 0,
\end{aligned}$$

where $B(x, y)$ and $\Gamma(x)$ are the Beta function and the Gamma function respectively. Note that $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$.

(3) If $w_0 \in H_3(c_0)$ then $a(w_0) = 2 + \sqrt{2} c_0 \in (2, 3)$. Similar to the proof of part (2), the heteroclinic orbit $(u_0(t; w_0), v_0(t; w_0))$ satisfies

$$u_0(t; w_0) = a(w_0) - a(w_0)\big(1 + e^{a(w_0)t/\sqrt{2}}\big)^{-1},$$
$$e^{-c_0 t} = \big(a(w_0) - u_0(t; w_0)\big)^{\ell} u_0(t; w_0)^{-\ell}, \quad \text{where } 0 < \ell < 1/3.$$

After simple computation, we can obtain

$$\mathcal{M}(w_0) = a'(w_0)a^2(w_0)\big(1 - a(w_0)\big)\Gamma(1+\ell)\Gamma(2-\ell)/\Gamma(4) < 0.$$

(4) Similarly, if $w_0 \in H_4(c_0)$ then $a(w_0) = -1 - \sqrt{2}c_0 \in (-2, -1)$ and the heteroclinic orbit $(u_0(t; w_0), v_0(t; w_0))$ satisfies

$$u_0(t; w_0) = a(w_0) + \big(1 - a(w_0)\big)\big(1 + e^{(1-a(w_0))t/\sqrt{2}}\big)^{-1},$$
$$e^{-c_0 t} = \big(u_0(t; w_0) - a(w_0)\big)^{\gamma}\big(1 - u_0(t; w_0)\big)^{-\gamma},$$

where $\gamma := 1 - 2(1 - a(w_0))^{-1}$. Therefore $0 < \gamma < 1/3$ and

$$\mathcal{M}(w_0) = -a'(w_0)a(w_0)\big(1 - a(w_0)\big)^2\Gamma(1+\gamma)\Gamma(2-\gamma)/\Gamma(4) > 0.$$

(5) Finally, if $w_0 \in H_6(c_0)$ then $a(w_0) = -1 + \sqrt{2}c_0 \in (-1, 0)$ and the heteroclinic orbit $(u_0(t; w_0), v_0(t; w_0))$ satisfies

$$u_0(t; w_0) = 1 - \big(1 - a(w_0)\big)\big(1 + e^{(1-a(w_0))t/\sqrt{2}}\big)^{-1},$$
$$e^{-c_0 t} = \big(u_0(t; w_0) - a(w_0)\big)^{\gamma}\big(1 - u_0(t; w_0)\big)^{-\gamma},$$

where $-1 < \gamma < 0$. Then we have

$$\mathcal{M}(w_0) = a'(w_0)a(w_0)\big(1 - a(w_0)\big)^2\Gamma(1+\gamma)\Gamma(2-\gamma)/\Gamma(4) < 0.$$

The proof is complete. □

However, if $a'(w_0) = 0$ in Lemma 4.3 then $\mathcal{M}(w_0) = 0$. Therefore we need to compute the higher-order term of Melnikov function to detect the transversality of the invariant manifolds. In the following we only investigate the second-order term of Melnikov function $\mathcal{M}_2(w_0)$.

Lemma 4.4. *Suppose, for some small c_0 and w_0, $a'(w_0) = 0$ and system* (2.6) *has a heteroclinic orbit $\Gamma_0 := (u_0(t; w_0), v_0(t; w_0))$. For fixed $c = c_0$ and varying parameter w, the second-order Melnikov function is given by*

$$\mathcal{M}_2(w_0) = a''(w_0)\int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0)u_0(t; w_0)\big(1 - u_0(t; w_0)\big)\,dt.$$

Proof. Without lost of generality, we may assume $w_0 = 0$. For such fixed w near w_0, let us write the system (2.6) in the following vector form

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} u(t; w) \\ v(t; w)\end{pmatrix} &= \begin{pmatrix} v(t; w) \\ c_0 v(t; w) - u(t; w)(1 - u(t; w))(u(t; w) - a(w))\end{pmatrix} \\ &= R_0\big(r(t; w)\big) + R_1\big(r(t; w)\big)w + R_2\big(r(t; w)\big)w^2 + O\big(w^3\big),\end{aligned} \tag{4.15}$$

where $r(t; w) := (u(t; w), v(t; w))^T$,

$$R_0\big(r(t;w)\big)=\begin{pmatrix} v(t;w) \\ c_0 v(t;w)+u(t;w)^3-(a(0)+1)u(t;w)^2+a(0)u(t;w)\end{pmatrix},$$
$$R_1\big(r(t;w)\big)=\begin{pmatrix} 0 \\ a'(0)(u(t;w)-u^2(t;w))\end{pmatrix},$$
$$R_2\big(r(t;w)\big)=\begin{pmatrix} 0 \\ a''(0)(u(t;w)-u^2(t;w))/2\end{pmatrix}.$$

As $w=0$, system (2.6) has a heteroclinic orbit $r(t;w)$ connecting two equilibria, E_0^1 and E_0^2. Let L be a line segment transversal to $r(t;0)$ at $r(0;0)$. For sufficiently small w, there exists a unique bounded solution $r^u(t;w)$ for $t\leqslant 0$ such that $r^u(t;w)$ in the unstable manifold of one equilibrium E_w^1 and $r^u(0;w)\in L$. For $t\leqslant 0$, let us define

$$z^u(t):=\frac{\partial}{\partial w}r^u(t;w)\Big|_{w=0},\qquad \triangle^u(t):=z^u(t)\wedge R_0\big(r^u(t;0)\big),$$
$$y^u(t):=\frac{\partial^2}{\partial^2 w}r^u(t;w)\Big|_{w=0},\qquad \Box^u(t):=y^u(t)\wedge R_0\big(r^u(t;0)\big).$$

Differentiating Eq. (4.15) with respect to w, we have

$$\begin{aligned}\frac{d}{dt}\frac{\partial}{\partial w}r^u(t;w)&=\frac{\partial R_0}{\partial r}\big(r^u(t;w)\big)\frac{\partial}{\partial w}r^u(t;w)+\frac{\partial R_1}{\partial r}\big(r^u(t;w)\big)\frac{\partial}{\partial w}r^u(t;w)w+R_1\big(r^u(t;w)\big)\\&\quad+\frac{\partial R_2}{\partial r}\big(r^u(t;w)\big)\frac{\partial}{\partial w}r^u(t;w)w^2+2R_2\big(r^u(t;w)\big)w+O\big(w^2\big).\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}y^u(t)&=\frac{\partial R_0}{\partial r}\big(r^u(t;0)\big)y^u(t)+\left(\frac{\partial}{\partial w}\frac{\partial R_0}{\partial r}\big(r^u(t;w)\big)\Big|_{w=0}\right)z^u(t)\\&\quad+2\frac{\partial R_1}{\partial r}\big(r^u(t;0)\big)z^u(t)+2R_2\big(r^u(t;0)\big),\\ \frac{d}{dt}\Box^u(t)&=\left(\frac{d}{dt}y^u(t)\right)\wedge R_0\big(r^u(t;0)\big)+y^u(t)\wedge\left(\frac{\partial R_0}{\partial r}\big(r^u(t;0)\big)R_0\big(r^u(t;0)\big)\right)\\&=tr\frac{\partial R_0}{\partial r}\Box^u(t)+2R_2\wedge R_0+\left(\frac{\partial}{\partial w}\frac{\partial R_0}{\partial r}\big(r^u(t;w)\big)\Big|_{w=0}\right)z^u(t)\wedge R_0\big(r^u(t;0)\big)\\&\quad+2\frac{\partial R_1}{\partial r}\big(r^u(t;0)\big)z^u(t)\wedge R_0\big(r^u(t;0)\big)\\&=\sigma(t)\Box^u(t)+D_2(t),\end{aligned}$$

where $\sigma(t)=tr\frac{\partial R_0}{\partial r}$ and

$$D_2(t)=v_0u_0(1-u_0)a''(0)+2v_0\big[3u_0-a(0)-1\big]\left(\frac{\partial u}{\partial w}\Big|_{w=0}\right)^2=v_0u_0(1-u_0)a''(0),$$

since $\frac{\partial u}{\partial w}|_{w=0}=\frac{\partial u}{\partial a}a'(0)=0$. Our purpose is to compute $\Box^u(0)$. By the variation of constant formula,

$$\Box^u(t)=e^{\int_0^t\sigma(\tau)\,d\tau}\left\{\Box^u(0)+\int_0^t e^{-\int_0^\tau\sigma(s)\,ds}D_2(\tau)\,d\tau\right\},\quad\text{for }t<0.$$

Since

$$\lim_{t \to -\infty} e^{-\int_0^t \sigma(\tau)\, d\tau} \square^u(t) = 0,$$

we have

$$\square^u(0) = \int_{-\infty}^{0} e^{-\int_0^\tau \sigma(s)\, ds} D_2(\tau)\, d\tau.$$

Similarly, there exists a unique bounded solution $r^s(t; w)$ for $t \geqslant 0$ such that $r^s(t; w)$ in the unstable manifold of the other equilibrium E_w^2 and $r^s(0; w) \in L$. For $t \geqslant 0$, let us define

$$y^s(t) := \frac{\partial^2}{\partial^2 w} r^s(t; w)\bigg|_{w=0} \quad \text{and} \quad \square^s(t) := y^s(t) \wedge R_0\big(r^s(t; 0)\big).$$

By the similar computation, we have

$$\square^s(0) = \int_{\infty}^{0} e^{-\int_0^\tau \sigma(s)\, ds} D_2(\tau)\, d\tau.$$

Hence

$$\mathcal{M}_2(0) = \square^u(0) - \square^s(0) = a''(0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t) u_0(t)\big(1 - u_0(t)\big)\, dt.$$

The proof is complete. $\square$

By the proof of Lemmas 4.3 and 4.4, we have the following corollary.

Corollary 4.5. *Under the same assumptions as stated in Lemma* 4.4, *if* $a''(w_0) \neq 0$, *then* $\mathcal{M}_2(w_0) \neq 0$ *for any* $c_0 \in (0, 1/\sqrt{2})$ *and* $w_0 \in H_i(c_0)$, $i = 1, \dots, 6$.

For more higher-order terms of Melnikov function, the computation is similar but more complicated. In the following we only state the general result, and skip the proof.

Lemma 4.6. *Suppose, for some small* c_0 *and* w_0, $a^{(i)}(w_0) = 0$ *for all* $1 \leqslant i < k$, *where* k *is a given positive integer, and system* (2.6) *has a heteroclinic orbit* $\Gamma_0 := (u_0(t; w_0), v_0(t; w_0))$. *For fixed* $c = c_0$ *and varying parameter* w, *the kth-order Melnikov function is given by*

$$\mathcal{M}_k(w_0) = a^{(k)}(w_0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0) u_0(t; w_0)\big(1 - u_0(t; w_0)\big)\, dt.$$

As a consequence of previous lemmas and corollary, we have the following conclusions for the transversality of invariant manifolds.

Lemma 4.7. *Let* $M \pitchfork N$ *be in the sense that manifolds* M *and* N *intersect transversally, and* $C_\delta(c) := (c - \delta, c + \delta)$.

Table 2
Transversalities of manifolds $W_0^c(M^\delta_{0,1,a}(w))$, $W_0^u(M^\delta_{0,1,a}(w))$, $W_0^{cu}(0,0,w)$ and $W_0^{cu}(1,0,w)$.

Region of w	Transversality of manifolds along $\Gamma(w)$
$w \in H_1(c_0)$	$W_0^u(M_0^\delta(w)) \pitchfork W_0^c(M_1^\delta(w))$
$w \in H_2(c_0)$	$W_0^u(M_1^\delta(w)) \pitchfork W_0^c(M_0^\delta(w))$
$w \in H_3(c_0)$	$W_0^u(M_a^\delta(w)) \pitchfork W_0^c(M_0^\delta(w))$
$w \in H_4(c_0)$	$W_0^u(M_1^\delta(w)) \pitchfork W_0^c(M_a^\delta(w))$
$w \in H_5(c_0)$	$W_0^u(M_a^\delta(w)) \pitchfork W_0^c(M_0^\delta(w))$
$w \in H_6(c_0)$	$W_0^u(M_a^\delta(w)) \pitchfork W_0^c(M_1^\delta(w))$
$w \in G_1(c_0)$	$W_0^{cu}(0,0,w) \pitchfork W_0^c(M_1^\delta(w))$
$w \in G_2(c_0)$	$W_0^{cu}(1,0,w) \pitchfork W_0^c(M_0^\delta(w))$
$w \in G_3(c_0)$	$W_0^{cu}(0,0,w) \pitchfork W_0^c(M_a^\delta(w))$
$w \in G_4(c_0)$	$W_0^{cu}(1,0,w) \pitchfork W_0^c(M_a^\delta(w))$

Table 3
Transversalities of manifolds $W_0^c(M^\delta_{0,1,a}(w) \times C_\delta(c_0))$, $W_0^u(M^\delta_{0,1,a}(w)) \times \{c_0\}$, $W_0^{cu}(0,0,w,c_0)$ and $W_0^{cu}(1,0,w,c_0)$.

Region of w	Transversality of manifolds along $\Gamma(w) \times \{c_0\}$
$w \in H_1(c_0)$	$W_0^u(M_0^\delta(w)) \times \{c_0\} \pitchfork W_0^c(M_1^\delta(w) \times C_\delta(c_0))$
$w \in H_2(c_0)$	$W_0^u(M_1^\delta(w)) \times \{c_0\} \pitchfork W_0^c(M_0^\delta(w) \times C_\delta(c_0))$
$w \in H_3(c_0)$	$W_0^u(M_0^\delta(w) \times \{c_0\}) \pitchfork W_0^c(M_a^\delta(w) \times C_\delta(c_0))$
$w \in H_4(c_0)$	$W_0^u(M_1^\delta(w)) \times \{c_0\} \pitchfork W_0^c(M_a^\delta(w) \times C_\delta(c_0))$
$w \in H_5(c_0)$	$W_0^u(M_a^\delta(w) \times \{c_0\}) \pitchfork W_0^c(M_0^\delta(w) \times C_\delta(c_0))$
$w \in H_6(c_0)$	$W_0^u(M_a^\delta(w)) \times \{c_0\} \pitchfork W_0^c(M_1^\delta(w) \times C_\delta(c_0))$
$w \in G_1(c_0)$	$W_0^{cu}(0,0,w,c_0) \pitchfork W_0^c(M_1^\delta(w) \times C_\delta(c_0))$
$w \in G_2(c_0)$	$W_0^{cu}(1,0,w,c_0) \pitchfork W_0^c(M_0^\delta(w) \times C_\delta(c_0))$
$w \in G_3(c_0)$	$W_0^{cu}(0,0,w,c_0) \pitchfork W_0^c(M_a^\delta(w) \times C_\delta(c_0))$
$w \in G_4(c_0)$	$W_0^{cu}(1,0,w,c_0) \pitchfork W_0^c(M_a^\delta(w) \times C_\delta(c_0))$

(1) *Consider system* (2.6) *with* $c = c_0 \in (0, 1/\sqrt{2})$. *The transversality of various invariant manifolds of* (2.6) *along* $\Gamma(w)$ *are illustrated in Table* 2.
(2) *Consider* (2.10) *with* $c = c_0 \in (0, 1/\sqrt{2})$. *The transversality of various invariant manifolds of* (2.10) *along* $\Gamma(w) \times \{c_0\}$ *are illustrated in Table* 3.

4.2. *Proof of the theorems*

Now we begin the proof of the main theorems.

Proof of Theorem 3.2. We only prove the first part of the theorem. The proof for the second part of the theorem is similar to the proof of Theorem 2.3 of [16] and is omitted.

Assume that $w = (w_1, \ldots, w_n)$ is s-admissible with respect to c^*, where $s = (s_1, \ldots, s_{n+1})$. We first claim that the singular local orbit $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2$ is weakly shadowed by a true local orbit.

According to Table 1 and the assumption $a(0) = a_1(c^*)$, we know that $0 \in H_1(c^*)$ and there exists a singular local orbit $0 \to 1$ at $w = 0$. For the following local path $1 \to s_2$, the admissible conditions lead to $s_2 = 0$ or a. Therefore, there exists a singular local orbit $1 \to s_2$ at $w = w_1$ if $w_1 \in H_2(c^*) \cup G_2(c^*)$ or $H_4(c^*) \cup G_4(c^*)$ (in fact, $w_1 \in H_2(c^*)$ or $H_4(c^*)$). Take $M^0 = W_0^u((0,0,0) \times C_\delta(c^*))$. According to Lemma 4.7, we have

$$M^0 \pitchfork W_0^c\big(M_1^\delta(0) \times C_\delta(c^*)\big).$$

Let N_0 be their intersection. Since the phase space of system (2.10) is $\mathbb{R}^4$ and dimensions of M^0 and $W_0^c(M_1^\delta(0) \times C_\delta(c^*))$ are 2 and 3 respectively, then $\dim N_0 = 2+3-4 = 1$. We now apply the Exchange lemma to the vicinity of the slow manifold $M_1 \times C_\delta(c^*)$ along the slow orbit from $(1,0,0,c^*)$ to $(1,0,w_1,c^*)$. Taking $M^\varepsilon = W^u_\varepsilon((0,0,0) \times C_\delta(c^*))$ be such that $M^\varepsilon \to M^0$ as $\varepsilon \to 0$. By condition (A1) and part (1) of Proposition 2.2, a portion $M^\varepsilon_{p_0}$ of M^ε will proceed near the singular orbit and leave the vicinity of slow orbit close to $W^u_\varepsilon(M^\delta_1(w_1) \times C_\delta(c^*))$. Note that $\dim W^u_0(M^\delta_1(w_1) \times C_\delta(c^*)) = 3$. Thus, the singular orbit $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2$ is weakly shadowed by a true orbit.

Next, we claim that the singular local orbit $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2 \xrightarrow{w=w_2} s_3$ is weakly shadowed by a true local orbit. Two cases for $s_2 = 0$ or a are considered. For the case $s_2 = 0$, a portion $M^\varepsilon_{p_1}$ of $W^u_\varepsilon((a(w_1),0,w_1) \times C_\delta(c^*))$ will proceed near the singular orbit and leave the vicinity of slow orbit close to $W^u_\varepsilon((s_3,0) \times (w_2-\delta, w_2+\delta) \times C_\delta(c^*))$ or $W^{cu}_\varepsilon((s_3,0) \times \{w_2\} \times \{c^*\})$. The details of proof can be found in [16] by using the Exchange lemma with turning point (Proposition 2.2) and are omitted. For the case $s_2 = a$, the admissible conditions imply $s_3 = 1$ or 0. Thus there exists a singular local orbit $s_2 \to s_3$ at $w = w_2$ if $w_2 \in H_5(c^*)$ or $H_6(c^*)$. From the admissible condition (A4), there is no turning point between w_1 and w_2. It is also easy to see that $W^u_0((1,0) \times (w_1-\delta, w_1+\delta) \times C_\delta(c^*))$ and $W^c_0((a(w_1),0) \times (w_1-\delta, w_1+\delta) \times C_\delta(c^*))$ intersect transversally. Then, by Proposition 2.3, a portion $M^\varepsilon_{p_1}$ of $W^u_\varepsilon((a(w_1),0) \times (w_1-\delta, w_1+\delta) \times C_\delta(c^*))$ will proceed near the singular orbit and leave the vicinity of slow orbit close to $W^u_\varepsilon((s_3,0) \times (w_2-\delta, w_2+\delta) \times C_\delta(c^*))$. Note that $\dim W^u_0(M^\delta_1(w_1) \times C_\delta(c^*)) = \dim W^{cu}_0((s_3,0) \times \{w_2\} \times C_\delta(c^*)) = 3$.

According to the above discussions, we can conclude inductively that the singular local orbit $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} \cdots \xrightarrow{w=w_{n-1}} s_n$ is weakly shadowed by a true local orbit. Generally, for $2 < i < (n+1)$, we consider the following two cases.

(1) Assume that there exists a turning point between w_{i-1} and w_i. By Proposition 2.2, a portion $M^\varepsilon_{p_{i-1}}$ of $W^u_\varepsilon((s_i,0,w_{i-1}) \times C_\delta(c^*))$ will proceed near the singular orbit and leave the vicinity of slow orbit close to $W^u_\varepsilon((s_{i+1},0) \times (w_i-\delta, w_i+\delta) \times C_\delta(c^*))$ or $W^{cu}_\varepsilon((s_{i+1},0,w_i) \times C_\delta(c^*))$.

(2) Assume that there is no turning point between w_{i-1} and w_i. By Proposition 2.3, a portion $M^\varepsilon_{p_{i-1}}$ of $W^u_\varepsilon((s_i,0,w_{i-1}) \times C_\delta(c^*))$ will leave the vicinity of slow orbit close to $W^u_\varepsilon((s_{i+1},0) \times (w_i-\delta, w_i+\delta) \times C_\delta(c^*))$.

Furthermore, we have $\dim W^u_0((s_{i+1},0) \times (w_i-\delta, w_i+\delta) \times C_\delta(c^*)) = 3$ and $\dim W^{cu}_0((s_{i+1},0,w_i) \times C_\delta(c^*)) = 3$.

Finally, we prove that the true orbits obtained by the above arguments are C^1 $O(\varepsilon)$-close to the unstable manifold $W^u_0((s_{n+1},0,w_n) \times C_\delta(c^*))$. Since $s_{n+1} = 1$, the admissible conditions lead to $w_n \in H_1(c^*) \cup G_1(c^*)$ or $H_6(c^*)$. Thus, there exists a singular local orbit $s_n \to 1$ at $w = w_n$. By condition (A1), we have $a(w) < 1$ for all $w \in [w_n, 1]$ and the singular orbit will approach to $(1,0,1)$ as time goes infinity. Moreover, $W^u_0((s_n,0,w_n) \times \{c^*\})$ intersects $W^c_0(M^\delta_1(w_n) \times C_\delta(c^*))$ transversally. As a result, the true orbit will approach a neighborhood of $(1,0,1,c^*)$, near the singular orbit and C^1 $O(\varepsilon)$-close to the unstable manifold $W^u_0((s_{n+1},0,w_n) \times C_\delta(c^*))$. The proof of the theorem is complete. □

Proof of Theorem 3.3. Let $w = (w_1,\dots,w_n)$ be s-admissible for $c = c^*$ and $s = (s_1,\dots,s_{n+1})$. We first claim that the singular local orbit $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2$ is strongly shadowed by a true orbit.

According to Table 1 and the assumption $a(0) \leqslant a_6(c^*)$, we know that $0 \in G_1(c^*)$ and there exists a singular local orbit $0 \to 1$ at $w = 0$. For the following local path $1 \to s_2$, the admissible conditions lead to $s_2 = 0$ or a. Therefore, there exists a singular local orbit $1 \to s_2$ at $w = w_1$ if $w_1 \in H_2(c^*) \cup G_2(c^*)$ or $H_4(c^*) \cup G_4(c^*)$ (in fact, $w_1 \in H_2(c^*)$ or $H_4(c^*)$). Take $M^0 = W^u_0(0,0,0)$. According to Lemma 4.7, we have

$$M^0 \pitchfork W^c_0\{M^\delta_1(0)\}.$$

Let N_0 be their intersection. Since the phase space of system (2.3) is $\mathbb{R}^3$ and both dimensions of M^0 and $W^c_0(M^\delta_1(0))$ are 2, then $\dim N_0 = 2+2-3 = 1$. We now apply Exchange lemma to the vicinity of the slow manifold M_1 along the slow orbit from $(1,0,0)$ to $(1,0,w_1)$. Taking $M^\varepsilon = W^u_\varepsilon((0,0,0))$ be such that $M^\varepsilon \to M^0$ as $\varepsilon \to 0$. By condition (A1) and part (1) of Proposition 2.2, a portion $M^\varepsilon_{p_0}$ of M^ε will proceed near the singular orbit and leave the vicinity of slow orbit close to $W^u_\varepsilon(M^\delta_1(w_1))$. Note that $\dim W_0^u(M_1^\delta(w_1)) = 2$. Thus, the singular orbit $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2$ is strongly shadowed by a true orbit.

Next, we claim that the singular local orbit $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2 \xrightarrow{w=w_2} s_3$ is strongly shadowed by a true local orbit. Two cases for $s_2 = 0$ or a are considered. For the case $s_2 = 0$, a portion $M_{p_1}^\varepsilon$ of $W_\varepsilon^u((a(w_1), 0, w_1))$ will proceed near the singular orbit and leave the vicinity of slow orbit close to $W_\varepsilon^u((s_3, 0) \times (w_2 - \delta, w_2 + \delta))$ or $W_\varepsilon^{cu}((s_3, 0) \times \{w_2\})$. The details of proof can be found in [16] by using the Exchange lemma with turning point (Proposition 2.2) and are omitted. For the case $s_2 = a$, the admissible conditions imply $s_3 = 1$ or 0. Thus there exists a singular local orbit $s_2 \to s_3$ at $w = w_2$ if $w_2 \in H_5(c^*)$ or $H_6(c^*)$. From the admissible condition (A4), there is no turning point between w_1 and w_2. It is also easy to see that $W_0^u((1, 0) \times (w_1 - \delta, w_1 + \delta))$ and $W_0^c((a(w_1), 0) \times (w_1 - \delta, w_1 + \delta))$ intersect transversally. Then, by Proposition 2.3, a portion $M_{p_1}^\varepsilon$ of $W_\varepsilon^u((a(w_1), 0) \times (w_1 - \delta, w_1 + \delta))$ will proceed near the singular orbit and leave the vicinity of slow orbit close to $W_\varepsilon^u((s_3, 0) \times (w_2 - \delta, w_2 + \delta))$. Note that $\dim W_0^u(M_1^\delta(w_1)) = \dim W_0^{cu}((s_3, 0) \times \{w_2\}) = 2$.

According to the above discussions, we can conclude inductively that the singular local orbit $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} \cdots \xrightarrow{w=w_{n-1}} s_n$ is strongly shadowed by a true local orbit. Generally, for $2 < i < (n + 1)$, we consider the following two cases.

(1) Assume that there exists a turning point between w_{i-1} and w_i. By Proposition 2.2, a portion $M_{p_{i-1}}^\varepsilon$ of $W_\varepsilon^u((s_i, 0, w_{i-1}))$ will proceed near the singular orbit and leave the vicinity of slow orbit close to $W_\varepsilon^u((s_{i+1}, 0) \times (w_i - \delta, w_i + \delta))$ or $W_\varepsilon^{cu}((s_{i+1}, 0, w_i))$.

(2) Assume that there is no turning point between w_{i-1} and w_i. By Proposition 2.3, a portion $M_{p_{i-1}}^\varepsilon$ of $W_\varepsilon^u((s_i, 0, w_{i-1}))$ will leave the vicinity of slow orbit close to $W_\varepsilon^u((s_{i+1}, 0) \times (w_i - \delta, w_i + \delta))$.

Furthermore, we have $\dim W_0^u((s_{i+1}, 0) \times (w_i - \delta, w_i + \delta)) = \dim W_0^{cu}((s_{i+1}, 0, w_i)) = 2$.

Finally, we prove that the true orbits obtained by the above arguments are C^1 $O(\varepsilon)$-close to the unstable manifold $W_0^u((s_{n+1}, 0, w_n))$. Since $s_{n+1} = 1$, the admissible conditions lead to $w_n \in H_1(c^*) \cup G_1(c^*)$ or $H_6(c^*)$. Thus, there exists a singular local orbit $s_n \to 1$ at $w = w_n$. By condition (A1), we have $a(w) < 1$ for all $w \in [w_n, 1]$ and the singular orbit will approach to $(1, 0, 1)$ as time goes infinity. Moreover, $W_0^u((s_n, 0, w_n))$ intersects $W_0^c(M_1^\delta(w_n))$ transversally. As a result, the true orbit will approach a neighborhood of $(1, 0, 1)$, near the singular orbit and C^1 $O(\varepsilon)$-close to the unstable manifold $W_0^u((s_{n+1}, 0, w_n))$. The proof of the theorem is complete. □

The results of Theorem 3.4 can also be proved in the same way and omitted.

5. Examples

In this section we provide an example to support our main results. Note that $H_6 \cap \partial\Omega$ at $c = 2\sqrt{3} - 2\sqrt{2}$. Assume that $c \in S \cap (2\sqrt{3} - 2\sqrt{2}, 1/\sqrt{2})$, $0 < \varepsilon < 1/4$ and $\delta \geqslant 0$ be fixed numbers. Then $a_6(c) < 0 < a_1(c) < a_2(c) < 1$. Let us define $a(w)$ on $[0, 1]$ by

$$a(w) = \begin{cases} (\alpha - a_2(c)) \exp\{\frac{(w-1)^4}{(w-1)^4 - (1/4)^4}\} + a_2(c) + \delta, & \text{if } w \in [\frac{3}{4}, 1], \\ (a_2(c) - \varepsilon) \exp\{\frac{(w-(3/4))^4}{(w-(3/4))^4 - \varepsilon^4}\} + \varepsilon + \delta, & \text{if } w \in [\frac{3}{4} - \varepsilon, \frac{3}{4}], \\ \varepsilon \exp\{\frac{(w+\varepsilon-(3/4))^4}{(w+\varepsilon-(3/4))^4 - ((1/4)-\varepsilon)^4}\} + \delta, & \text{if } w \in [\frac{1}{2}, \frac{3}{4} - \varepsilon], \\ \beta(1 - \exp\{\frac{(w-(1/2))^4}{(w-(1/2))^4 - (1/2)^4}\}) + \delta, & \text{if } w \in [0, \frac{1}{2}], \end{cases}$$

where $\alpha \in (a_2(c), 1)$ and $\beta \in (-c^2/4, a_6(c))$. It is not difficult to verify that $a(w)$ is a monotonic increasing C^2 function on $[0, 1]$.

If $\delta = 0$, then $a(3/4) = a_2(c)$, $a(1/2) = 0$ and there exist $w^L < 1/2 < w^R < 3/4$ such that $a(w^L) = a_6(c)$ and $a(w^R) = a_1(c)$, see the left part of Fig. 7. According to Table 1, there exist orbits of (2.5) connecting from a to 1, 1 to 0 and 0 to 1 at levels $w = w^L$, $w = 3/4$, and $w = w^R$ respectively. Now we estimate the following integrals:

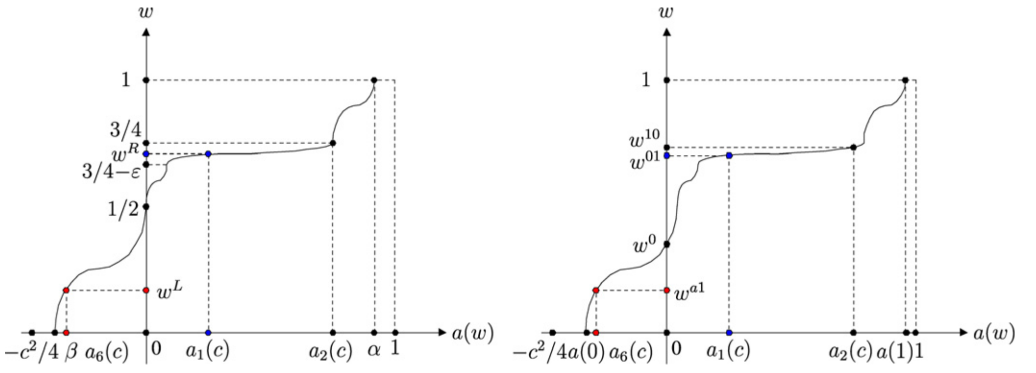

Fig. 7. Graph of $a(w)$ with $\delta = 0$ (left part) and $\delta > 0$ (right part).

$$I_1 := \int_{\frac{1}{2}}^{w^L} \frac{\lambda_0^-(\eta; c)}{g(0,\eta)}\, d\eta = \int_{\frac{1}{2}}^{w^L} \frac{\sqrt{c^2 - 4a(\eta)} + c}{2\eta}\, d\eta > 0,$$

$$I_2 := \int_{\frac{3}{4}-\varepsilon}^{\frac{1}{2}} \frac{\lambda_0^-(\eta; c)}{g(0,\eta)}\, d\eta = \int_{\frac{3}{4}-\varepsilon}^{\frac{1}{2}} \frac{\sqrt{c^2 + 4a(\eta)} - c}{2\eta}\, d\eta > K_1\varepsilon,$$

$$I_3 := \int_{\frac{3}{4}}^{\frac{3}{4}-\varepsilon} \frac{\lambda_0^-(\eta; c)}{g(0,\eta)}\, d\eta = \int_{\frac{3}{4}}^{\frac{3}{4}-\varepsilon} \frac{\sqrt{c^2 + 4a(\eta)} - c}{2\eta}\, d\eta > K_2\varepsilon,$$

where K_1 and K_2 are negative constants. Since I_1 is independent of ε, if ε is small enough then

$$\int_{\frac{3}{4}}^{w^L} \frac{\lambda_0^-(\eta; c)}{g(0,\eta)}\, d\eta = I_1 + I_2 + I_3 > 0.$$

Thus there exists $\bar{w} \in (w^L, 1/2)$ such that

$$\int_{\frac{3}{4}}^{\bar{w}} \frac{\lambda_0^-(\eta; c)}{g(0,\eta)}\, d\eta = 0,$$

and orbit of (2.5) connecting from 0 to a at level $w = \bar{w}$. Since $a(1/2) = 0$ and $a'(1/2) = 0$, $w = 1/2$ is a degenerate turning point such that Theorem 3.3 cannot be applied directly to obtain the traveling wave solutions. To avoid the degeneracy of turning point, in the following we consider the case of $a(w)$ but with $\delta > 0$.

For $\delta > 0$, it is obvious that the graph of $a(w)$ is a shift of left part of Fig. 7, see the right part of Fig. 7. By continuity, if δ is small enough then there exists $w^{a1} < w^{0a} < w^0 < w^{01} < w^{10}$ such that

(1) $a(1) \in (a_2(c), 1)$ and $a(0) \in (-c^2/4, a_6(c))$;
(2) $a(w^{a1}) = a_6(c)$, $a(w^0) = 0$, $a'(w^0) \neq 0$, $a(w^{01}) = a_1(c)$ and $a(w^{10}) = a_2(c)$;

$$\int_{w^{10}}^{w^{a1}} \frac{\lambda_0^-(\eta;c)}{g(0,\eta)}\,d\eta > 0 \quad \text{and} \quad \int_{w^{10}}^{w^{0a}} \frac{\lambda_0^-(\eta;c)}{g(0,\eta)}\,d\eta = 0. \tag{3}$$

Thus there exist orbits of (2.5) connecting from a to 1, 0 to a, 0 to 1 and 1 to 0 at levels $w = w^{a1}$, $w = w^{0a}$, $w = w^{01}$, and $w = w^{10}$ respectively. Since $a'(w^0) \neq 0$, by the admissible conditions, the word $w = (w^{10}, w^{0a}, w^{a1}, w^{10}, w^{01})$ is *s-admissible* with respect to c with $s = (1, 0, a, 1, 0, 1)$. Furthermore, by repeating the local paths, the singular orbits along the path

$$01 \xrightarrow{H_2} \underbrace{01\cdots01}_{n_1\ 01s} \xrightarrow{H_2} \underbrace{0a1\cdots0a1}_{n_2\ 0a1s} \xrightarrow{H_2} \underbrace{01\cdots01}_{n_3\ 01s}, \quad n_1, n_2, n_3 \in \mathbb{Z}^+ \cup \{0\},$$

or any copy of the such path can be weakly shadowed by true orbits. Since all n_i are arbitrary, by Theorem 3.3, such kind of $a(w)$ with small $\varepsilon > 0$ and $\delta > 0$ provide us the multiplicity of traveling wave solutions.

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