Chapter 2

The mathematical model of a Brushless DC Motor

In this chapter, the mathematical model of a brushless DC (BLDC) motor will be described by a set of dynamic equations, including electrical equations and mechanical equations. Using space vectors, Section 2.1 derives the dynamic equations for a p-pole 3-phase BLDC motor, which generally possesses stator windings and a rotor with surface-mounted magnets. Since the mathematical model is often established in three-axis system, which is more complicated than a two-axis system, called d-q coordinate system. Thus, the coordinate transition is necessary and will be proposed in Section 2.2. Furthermore, the state-space equation will be presented in Section 2.3.

2.1 Dynamic equations of BLDC motors in vector space

The dynamic equation of the BLDC motor with Y-connected stator winding will be introduced in this section. Assume that the permeability of iron is infinite. Besides, the slot effect, the iron lost, and the end winding effect are negligible.

When the neutral point is isolated, the phase currents of the *p*-pole BLDC motor, i_{as}(t), i_{bs}(t), and i_{cs}(t) can be expressed as

$$i_{as}(t)+i_{bs}(t)+i_{cs}(t)=0 \tag{2-1}$$

Because the three windings are distributed with $\frac{2\pi}{3}$ in electrical degree apart to each other, the stator current in vector space are generally represented as

$$I_s(t) = i_{as}(t) + i_{bs}(t)e^{j\frac{2\pi}{3}} + i_{cs}(t)e^{j\frac{4\pi}{3}} \tag{2-2}$$

where $i_{as}(t)$, $i_{bs}(t)e^{j\frac{2\pi}{3}}$, and $i_{cs}(t)e^{j\frac{4\pi}{3}}$ are the corresponding three phase currents.

Let $\lambda_{as}(\theta_e, t)$, $\lambda_{bs}(\theta_e, t)$ and $\lambda_{cs}(\theta_e, t)$ be the fluxes related to the three phases of the stator and L_{ls}, L_{ss}, and L_{ms} correspondingly represent the magnetic leakage, the self-inductance and the mutual inductance of the stator. Hence, the stator flux can be expressed as

$$\lambda_{as}(\theta_e, t) = (L_{ls} + L_{ss})i_{as}(t) + L_{ms}i_{bs}(t)e^{j\frac{2\pi}{3}} + L_{ms}i_{cs}(t)e^{j\frac{4\pi}{3}} + \lambda_{pm}e^{j\theta_e} \tag{2-3}$$

$$\lambda_{bs}(\theta_e, t) = L_{ms}i_{as}(t)e^{-j\frac{2\pi}{3}} + (L_{ls} + L_{ss})i_{bs}(t) + L_{ms}i_{cs}(t)e^{j\frac{2\pi}{3}} + \lambda_{pm}e^{j\left(\theta_e - \frac{2\pi}{3}\right)} \tag{2-4}$$

$$\lambda_{cs}(\theta_e, t) = L_{ms}i_{as}(t)e^{-j\frac{4\pi}{3}} + L_{ms}i_{bs}(t)e^{-j\frac{2\pi}{3}} + (L_{ls} + L_{ss})i_{cs}(t) + \lambda_{pm}e^{j\left(\theta_e - \frac{4\pi}{3}\right)} \tag{2-5}$$

where θ_e is the permanent magnetic electrical angle. Besides, λ_{pm} is the flux magnitude produced by the permanent magnets, which are assumed sinusoidally distributed in the air-gap.

Based on the stator flux in (2-4)-(2-6), the stator voltages, v_{as}(t), v_{bs}(t), and v_{cs}(t), can be formulated as

$$v_{as}(t) = R_s i_{as}(t) + \frac{d}{dt}\lambda_{as}(\theta_e, t) \tag{2-6}$$

$$v_{bs}(t) = R_s i_{bs}(t) + \frac{d}{dt}\lambda_{bs}(\theta_e, t) \tag{2-7}$$

$$v_{cs}(t) = R_s i_{cs}(t) + \frac{d}{dt}\lambda_{cs}(\theta_e, t) \tag{2-8}$$

where R_s is the stator's resistance of each phase. Furthermore, the last term in each equation illustrates the back-EMF e_{emf}, which can be calculated as following

$$e_{emf} = \frac{d\lambda(\theta_e,t)}{dt} = \frac{\partial\lambda(\theta_e,t)}{\partial\theta_e}\omega_e + \frac{\partial\lambda(\theta_e,t)}{\partial t} \tag{2-9}$$

with $\omega_e = \frac{d\theta_e}{dt}$ as the electrical angular velocity. Actually, the back-EMF contains on the right-hand side of (2-11) corresponds to the motional voltage and the second to the transformer voltage. Rewriting the self-inductance as $L_s=L_{ls}+L_{ss}$ and employing (2-9) are shown the back-EMF of each phase as following

$$\dot{\lambda}_{as}(\theta_e,t) = L_s\dot{i}_{as}(t) - \frac{1}{2}L_{ms}\dot{i}_{bs}(t) - \frac{1}{2}L_{ms}\dot{i}_{cs}(t) - \omega_e\lambda_{pm}\sin\theta_e \tag{2-10}$$

$$\dot{\lambda}_{bs}(\theta_e,t) = -\frac{1}{2}L_{ms}\dot{i}_{as}(t) + L_s\dot{i}_{bs}(t) - \frac{1}{2}L_{ms}\dot{i}_{cs}(t) - \omega_e\lambda_{pm}\sin\left(\theta_e - \frac{2\pi}{3}\right) \tag{2-11}$$

$$\dot{\lambda}_{bs}(\theta_e,t) = -\frac{1}{2}L_{ms}\dot{i}_{as}(t) - \frac{1}{2}L_{ms}\dot{i}_{bs}(t) + L_s\dot{i}_{cs}(t) - \omega_e\lambda_{pm}\sin\left(\theta_e - \frac{4\pi}{3}\right) \tag{2-12}$$

Then the dynamic mathematical equation of the 3-phase stator is represented as

$$\dot{i}_{as}(t) = -\frac{R_s}{L}i_a(t) + \frac{1}{L}v_{as}(t) + \frac{\omega_e\lambda_{pm}}{L}sin\theta_e \tag{2-13}$$

$$\dot{i}_{bs}(t) = -\frac{R_s}{L}i_b(t) + \frac{1}{L}v_{bs}(t) + \frac{\omega_e\lambda_{pm}}{L}sin\left(\theta_e - \frac{2\pi}{3}\right) \tag{2-14}$$

$$\dot{i}_{cs}(t) = -\frac{R_s}{L}i_c(t) + \frac{1}{L}v_{cs}(t) + \frac{\omega_e\lambda_{pm}}{L}sin\left(\theta_e - \frac{4\pi}{3}\right) \tag{2-15}$$

where $L_s + \frac{1}{2}L_{ms}$ is replaced by L.

Besides, the two important things for a BLDC Motor are torque and rotational speed. The torque of a motor produces from the differential magnetic co-energy and the kinds of torque are separated by inductance, such as the reluctance torque and the cogging torque relate to the self-inductance, and the alignment torque relate to mutual inductance. In BLDC motor, the alignment torque is concerned as reciprocal effect between stator and rotor.

In order to investigate the electromagnetic torque of the BLDC motor, the synchronous AC motor should be considered first. It is possible to obtain an expression for the electromagnetic torque by putting the rate of change of the mechanical output energy equal to the mechanical power.

$$P_{mech} = \frac{dW_{mech}}{dt} = T_e \omega_e \tag{2-16}$$

Let W_e be the input electrical energy, W_{loss} be the stator and rotor losses energy, W_{field} be the magnetic energy stored in the field, and W_{mech} be the mechanical output energy, the total energy and the differential mechanical output energy can be represented as (2-18) and (2-19), respectively.

$$W_e = W_{loss} + W_{field} + W_{mech} \tag{2-17}$$

$$dW_{mech} = dW_e - dW_{loss} - dW_{field} \tag{2-18}$$

In general, for a p-pole 3-phase machine, the differential input electrical energies could be expressed as

$$dW_e = \frac{p}{2} Re\left(v_a i_{as}^* + v_b i_{bs}^* + v_c i_{cs}^*\right) dt \tag{2-19}$$

The loss energies are due to heat dissipation across the stator winding resistance, hysteretic and eddy current losses within the magnetic material, friction loss between moving parts and either their bearings or the surrounding air, and dielectric loss in the electric fields. However, if only the windings loss are considered, the differential loss energies related to the stator and rotor windings can be expressed as

$$dW_{loss} = \frac{p}{2}\left(R_s\left|i_{as}\right|^2 + R_s\left|i_{bs}\right|^2 + R_s\left|i_{cs}\right|^2\right)dt \tag{2-20}$$

The differential field energies can be obtained the induced stator and rotor transformer back-EMF, respectively.

$$dW_{field} = \frac{p}{2}\left[i_{as}\frac{d\left(Li_{as}\right)}{dt} + i_{bs}\frac{d\left(Li_{bs}\right)}{dt} + i_{cs}\frac{d\left(Li_{cs}\right)}{dt}\right]dt \tag{2-21}$$

Thus, differential mechanical energies can be calculated by using (2-6)-(2-8) and (2-19)-(2-21)

$$dW_{mech} = \frac{p}{2}\left[-\omega_e\lambda_m i_{as}\sin\theta_e - \omega_e\lambda_m i_{bs}\sin\left(\theta_e - \frac{2\pi}{3}\right) - \omega_e\lambda_m i_{cs}\sin\left(\theta_e - \frac{4\pi}{3}\right)\right]dt \tag{2-22}$$

Therefore, the electromagnetic torque can be formulated by multiplying each phase of the stator current and back-EMF, such as

$$T_e(t) = -\frac{p}{2}\lambda_{pm}\left[i_{as}\sin\theta_e + i_{bs}\sin\left(\theta_e - \frac{2\pi}{3}\right) + i_{cs}\sin\left(\theta_e - \frac{4\pi}{3}\right)\right] \tag{2-23}$$

According to the Newton law, the electromechanical equation can be expressed as

$$T_e - \frac{2}{p}\left(+B_m\omega_e + T_L\right) = \frac{2}{p}\left(J\dot{\omega}_e\right) \tag{2-24}$$

$$J\dot{\omega}_m + B_m\omega_m + T_L = \frac{2}{p}\left(J\dot{\omega}_e + B_m\omega_e + T_L\right) = T_e \tag{2-25}$$

where J is the motor's inertia, B_m is the viscous damping, T_L is the load torque, ω_m is the mechanical angular velocity.

Note that (2-13)-(2-15) and (2-25) are represented the whole dynamic equations of the BLDC motor. In these dynamic equations, there are four state variables, i_{as}, i_{bs}, i_{cs}, and ω_e, and three input voltages, v_{as}, v_{bs}, and v_{cs}, and one external load torque T_L. It is difficult to design a BLDC motor controller using the three-axis system. In order to simplify the design process, three-axis system should reduce to two-axis system, called d-q axis system, and will be proposed in next section.

2.2 The coordinate transition process

In this section, two transformations in the coordinate transition process will be proposed. First, the Clarke transformation transforms the three-axis system to stationary reference frame (α-β coordinate), shown in Figure. 2.1. Second, the Park transformation transforms stationary reference frame to rotating reference frame (d-q coordinate), shown in Figure. 2.2.

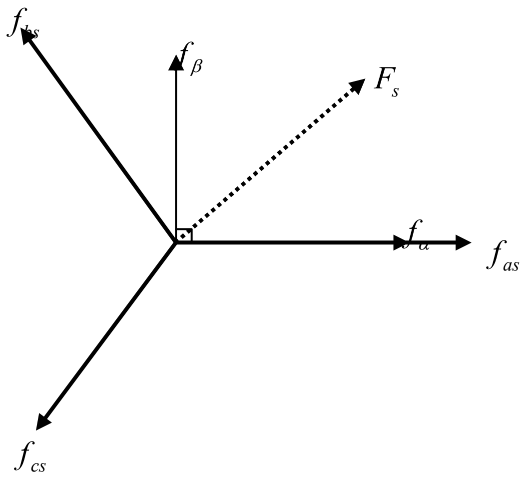

Figure. 2.1 The Clarke transformation

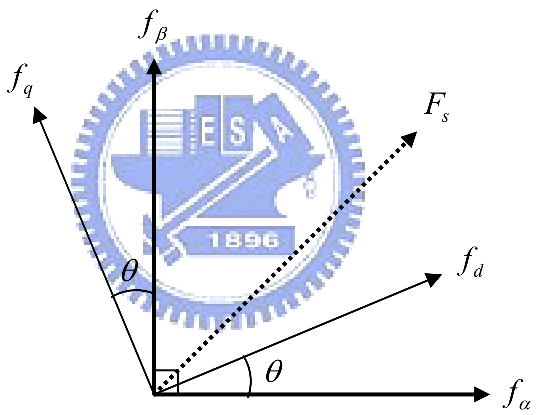

Figure. 2.2 The Park transformation

Before introducing the Clarke transformation, the transformed coefficient is existed and it can be deduced from power. The total instantaneous power in the three-axis system and the stationary reference frame are expressed as follows

$$P_{abcs} = Re\{v_{as}i_{as}^* + v_{bs}i_{bs}^* + v_{cs}i_{cs}^*\} \tag{2-26}$$

$$P_{\alpha\beta} = Re\{v_\alpha i_\alpha^* + v_\beta i_\beta^*\} \tag{2-27}$$

When the neutral point is isolated, the balanced current and voltage of the stator are shown as

$$i_{as} + i_{bs}e^{j\frac{2\pi}{3}} + i_{cs}e^{j\frac{4\pi}{3}} = 0 \quad (2\text{-}28)$$

$$v_{as} + v_{bs}e^{j\frac{2\pi}{3}} + v_{cs}e^{j\frac{4\pi}{3}} = 0 \quad (2\text{-}29)$$

After employing above two equations, the total instantaneous power will be equal with each other.

$$P_{\alpha\beta} = \frac{3}{2}k^2 P_{abcs} \quad (2\text{-}30)$$

There are two methods to decide the transfer constant k. First, called power invariant method, k is chosen as $\sqrt{\frac{2}{3}}$. Second, called non-power invariant, k is chosen as $\frac{2}{3}$.

In general case, the function $\boldsymbol{F}_s(t)$ can be the current of the stator $\boldsymbol{I}_s(t)$ or the voltage of the stator $\boldsymbol{V}_s(t)$, and the each phase of the stator in three-axis system can be expressed as

$$\boldsymbol{F}_s = \left(f_{as} + f_{bs}e^{j\frac{2\pi}{3}} + f_{cs}e^{j\frac{4\pi}{3}} \right) \quad (2\text{-}31)$$

Assume that the α-axis coincide with the a-axis and k is the transfer constant, can be represented as

$$\boldsymbol{F}_s = f_\alpha + jf_\beta = k\left(f_{as} + f_{bs}e^{j\frac{2\pi}{3}} + f_{cs}e^{j\frac{4\pi}{3}} \right) \quad (2\text{-}32)$$

Suppose that the non-power invariant is chosen; $k=\frac{2}{3}$, the Clarke transformation can be represented as

$$\begin{bmatrix} f_\alpha \\ f_\beta \\ f_0 \end{bmatrix} = \boldsymbol{T}_s \begin{bmatrix} f_{as} \\ f_{bs} \\ f_{cs} \end{bmatrix} \quad (33)$$

where $\boldsymbol{T}_s = \frac{2}{3}\begin{bmatrix} cos(0) & cos\left(\frac{2\pi}{3}\right) & cos\left(\frac{4\pi}{3}\right) \\ sin(0) & sin\left(\frac{2\pi}{3}\right) & sin\left(\frac{4\pi}{3}\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is the transformation matrix and f_0 is the zero-sequence component and it is equal to zero. Later, the inverse of $\boldsymbol{T}_s$, found as $(\boldsymbol{T}_s)^{-1} = \begin{bmatrix} cos(0) & -sin(0) & 1 \\ cos\left(\frac{2\pi}{3}\right) & -sin\left(\frac{2\pi}{3}\right) & 1 \\ cos\left(\frac{4\pi}{3}\right) & -sin\left(\frac{4\pi}{3}\right) & 1 \end{bmatrix}$, will be employed in calculating the differential of the flux.

Next, the Park transformation will be deduced from which the angle between the stationary reference frame and the rotating reference frame is θ. From Figure (2), the quantity of d-axis and q-axis are composed of stationary reference frame respectively.

$$f_d = f_\alpha \cos\theta + f_\beta \sin\theta \quad (2\text{-}34)$$

$$f_q = f_\alpha(-\sin\theta) + f_\beta \cos\theta \quad (2\text{-}35)$$

Then the matrix form of the Park transformation is denoted as

$$\begin{bmatrix} f_d \\ f_q \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} f_\alpha \\ f_\beta \end{bmatrix} \quad (2\text{-}36)$$

Integrating utilization of above two processing, the coordinate transformation from three-axis system to rotating reference frame is represented as

$$\begin{bmatrix} f_d \\ f_q \\ f_0 \end{bmatrix} = \boldsymbol{K}_s \begin{bmatrix} f_{as} \\ f_{bs} \\ f_{cs} \end{bmatrix} \tag{2-37}$$

where $\boldsymbol{K}_s = \dfrac{2}{3}\begin{bmatrix} \cos\theta & \cos\left(\theta - \dfrac{2\pi}{3}\right) & \cos\left(\theta + \dfrac{2\pi}{3}\right) \\ -\sin\theta & -\sin\left(\theta - \dfrac{2\pi}{3}\right) & -\sin\left(\theta + \dfrac{2\pi}{3}\right) \\ \dfrac{1}{2} & \dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix}$. Later, the inverse of $\boldsymbol{K}_s$, found as $\boldsymbol{K}_s^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ \cos\left(\theta - \dfrac{2\pi}{3}\right) & -\sin\left(\theta - \dfrac{2\pi}{3}\right) & 1 \\ \cos\left(\theta + \dfrac{2\pi}{3}\right) & -\sin\left(\theta + \dfrac{2\pi}{3}\right) & 1 \end{bmatrix}$, will be employed in calculating the differential of the flux.

2.3 The state space equation in d-q coordinate

In this section, the dynamic equations proposed in section 2.1 will be transformed into rotating reference frame. First, the voltage equations (2-6)-(2-8) can be re-expressed as

$$\boldsymbol{V}_{abcs} = \boldsymbol{R}\boldsymbol{I}_{abcs} + \frac{d}{dt}\left(\boldsymbol{\Lambda}_{abcs}\right) \tag{2-38}$$

where $\boldsymbol{V}_{abcs} = \begin{bmatrix} v_{as} & v_{bs} & v_{cs} \end{bmatrix}^T$, $\boldsymbol{I}_{abcs} = \begin{bmatrix} i_{as} & i_{bs} & i_{cs} \end{bmatrix}^T$, and $\boldsymbol{\Lambda}_{abcs} = \begin{bmatrix} \lambda_{as} & \lambda_{bs} & \lambda_{cs} \end{bmatrix}^T$.

By using the transformation matrix $\boldsymbol{T}_s$, (2-38) can be rearranged as

$$\boldsymbol{V}_{\alpha\beta 0} = \boldsymbol{R}\boldsymbol{I}_{\alpha\beta 0} + \frac{d}{dt}\left(\boldsymbol{\Lambda}_{\alpha\beta 0}\right) \tag{2-39}$$

where $\boldsymbol{V}_{\alpha\beta 0} = \boldsymbol{T}_s \boldsymbol{V}_{abcs} = \begin{bmatrix} v_\alpha & v_\beta & v_0 \end{bmatrix}^T$, $\boldsymbol{R} = \boldsymbol{T}_s \boldsymbol{R} \boldsymbol{T}_s^{-1}$, $\omega_e = \dfrac{d\theta_e}{dt}$,

$\boldsymbol{I}_{\alpha\beta 0} = \boldsymbol{T}_s \boldsymbol{I}_{abcs} = \begin{bmatrix} i_\alpha & i_\beta & i_0 \end{bmatrix}^T$, and $\boldsymbol{\Lambda}_{\alpha\beta 0} = \boldsymbol{T}_s \boldsymbol{\Lambda}_{abcs}$.

After arrange the equations, the stator voltage equations and dynamic equations of the stationary reference frame are expressed respectively as

$$\dot{i}_\alpha = -\frac{R_s}{L} i_\alpha + \frac{1}{L} v_\alpha + \frac{\omega_e}{L} \lambda_\beta \tag{2-40}$$

$$\dot{i}_\beta = -\frac{R_s}{L} i_\beta + \frac{1}{L} v_\beta - \frac{\omega_e}{L} \lambda_\alpha \tag{2-41}$$

where $\lambda_\alpha = \lambda_{pm} \cos\theta_e$ and $\lambda_\beta = \lambda_{pm} \sin\theta_e$ are the flux of the permanent magnet on the rotor decomposed into the stationary reference frame and the rotor position θ_e can be calculated by $\tan^{-1}\left(\dfrac{\lambda_\beta}{\lambda_\alpha}\right)$.

On the assumption that the d-axis is the magnet, θ is equal to θ_e. By using the transformation matrix $\boldsymbol{K}_s$, (2-38) can be rearranged as

$$\boldsymbol{V}_{dq0} = \boldsymbol{R}\boldsymbol{I}_{dq0} + \frac{d}{dt}\left(\boldsymbol{\Lambda}_{dq0}\right) \tag{2-42}$$

where $\boldsymbol{V}_{dq0} = \boldsymbol{K}_s \boldsymbol{V}_{abcs} = \begin{bmatrix} v_d & v_q & v_0 \end{bmatrix}^T$, $\boldsymbol{R} = \boldsymbol{K}_s \boldsymbol{R} \boldsymbol{K}_s^{-1}$, $\omega_e = \dfrac{d\theta_e}{dt}$,

$\boldsymbol{I}_{dq0} = \boldsymbol{K}_s \boldsymbol{I}_{abcs} = \begin{bmatrix} i_d & i_q & i_0 \end{bmatrix}^T$, and $\boldsymbol{\Lambda}_{dq0} = \boldsymbol{K}_s \boldsymbol{\Lambda}_{abcs}$.

After arrange the equations, the stator voltage equations and dynamic equations of the rotating reference frame are expressed respectively as

$$\dot{i}_d = -\frac{R_s}{L} i_d + \frac{1}{L} v_d + \omega_e i_q \tag{2-43}$$

$$\dot{i}_q = -\frac{R_s}{L} i_q + \frac{1}{L} v_q - \omega_e i_d - \frac{1}{L} \omega_e \lambda_{pm} \tag{2-44}$$

where $\lambda_d = \lambda_{pm}$ and $\lambda_q = 0$ are the flux of the permanent magnet decomposed into the rotating reference frame.

Furthermore, the electromagnet torque will be transformed to stationary reference frame and rotating reference frame through the process presented before and the calculated result can be shown respectively as

$$T_{\alpha\beta} = \frac{3}{2}\frac{p}{2}\lambda_m\left[-i_\alpha \sin\theta_e + i_\beta \cos\theta_e\right] \tag{2-45}$$

$$T_{dq} = \frac{3}{2}\frac{p}{2}\lambda_{pm} i_q \tag{2-46}$$

and the dynamic equation of the electrical angular velocity can be shown as

$$\dot{\omega}_e = \frac{3}{8}\frac{p^2}{J}\lambda_{pm} i_q - \frac{B}{J}\omega_e - \frac{1}{J}T_L \tag{2-47}$$