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Embedding paths of variable lengths into hypercubes with conditional link-faults

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ABSTRACT

Faults in a network may take various forms such as hardware failures while a node or a link stops functioning, software errors, or even missing of transmitted packets. In this paper, we study the link-fault-tolerant capability of an n-dimensional hypercube (n-cube for short) with respect to path embedding of variable lengths in the range from the shortest to the longest. Let F be a set consisting of faulty links in a wounded n-cube Q_n, in which every node is still incident to at least two fault-free links. Then we show that $Q_n - F$ has a path of any odd (resp. even) length in the range from the distance to $2^n - 1$ (resp. $2^n - 2$) between two arbitrary nodes even if $|F| = 2n - 5$. In order to tackle this problem, we also investigate the fault diameter of an n-cube with hybrid node and link faults.

1. Introduction

In many parallel computer systems, processors are connected on the basis of interconnection networks. Such networks usually have a regular degree, i.e., every node is incident to the same number of links. Popular instances of interconnection networks include hypercubes, star graphs, meshes, bubble-sort networks, etc.

The hypercube is one of the most versatile interconnection networks yet discovered for parallel computation. It can efficiently simulate many other networks of various sizes [14]. Because nodes and/or links in a network may fail accidentally, it is demanded to consider fault tolerance of a network. Hence, the issue of faulty hypercubes has been widely addressed in researches [2,4,11,16,20–24]. For example, Latifi et al. [11] proved that an n-dimensional hypercube (n-cube for short) has a hamiltonian cycle even if it has $n-2$ faulty links. Furthermore, Li et al. [16] showed that an n-cube is bipancyclic even if it has up to $n-2$ faulty links; Tsai et al. [20] showed that a faulty n-cube is both hamiltonian laceable and strongly hamiltonian laceable if it has $n-2$ faulty links. Recently, Xu et al. [24] showed that an n-cube, with $n-2$ faulty links, contains a path of length l between any two nodes of distance d^* for each integer l satisfying $d^* \leqslant l \leqslant 2^n - 1$ and $2|(l-d^*)$, where expression $2|(l-d^*)$ means that $l - d^* \equiv 0 \pmod 2$. Moreover, Fu [4] proved that a fault-free path of length at least $2^n - 2f - 1$ (or $2^n - 2f - 2$) can be embedded to join two arbitrary nodes of odd (or even) distance in an n-cube with $f \leqslant n-2$ faulty nodes.

Since linear array and rings are two of the most fundamental structures for parallel and distributed computation, a variety of efficient algorithms were developed on these two topologies [14]. In particular, embedding of linear array and rings in a

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faulty interconnection network is of great significance. For example, path embedding in a faulty n-cube was addressed in [16,20,24]. However, one should notice that each component of a network may have different reliability. Thus, the probability that all faulty components would be close to one another seems low. With this observation, Harary [7] first introduced the concept of conditional connectivity. Later, Latifi et al. [13] defined the *conditional node-faults*, which require each node of a network to have at least g fault-free neighbors. It is intuitive to extend this concept by defining *conditional link-faults*, which require that every node will be incident to at least g fault-free links. In this paper, we only concern $g = 2$. For convenience, we say a network is *conditionally faulty* if and only if every node is incident to at least two fault-free links. Under this assumption, Chan and Lee [2] discussed the existence of hamiltonian cycles in an n-cube with $2n - 5$ conditional link-faults. In addition, Tsai [21] showed that an injured n-cube contains a fault-free cycle of every even length from 4 to 2^n inclusive even if it has up to $2n - 5$ conditional link-faults. It was also proved in [21] that an n-cube with $2n - 5$ conditional link-faults is hamiltonian laceable and strongly hamiltonian laceable.

As Shih et al. [18] showed, any fault-free link of a faulty n-cube lies on a cycle of even length in the range from 6 to 2^n when up to $2n - 5$ conditional link-faults may occur. In other words, there exists a path of odd length from 1 to $2^n - 1$, excluding 3, between any two adjacent nodes in a faulty n-cube with $2n - 5$ conditional link-faults. In this paper, we are curious whether paths of variable lengths still can be constructed to join two arbitrary nodes of distance greater than one. More precisely, we will show that a conditionally faulty n-cube, with $2n - 5$ faulty links, contains a fault-free path of length l between any two nodes u and v of distance $d^* \geqslant 2$ for each l satisfying $d^* \leqslant l \leqslant 2^n - 1$ and $2|(l - d^*)$.

The rest of this paper is organized as follows. In Section 2, basic definitions and notations are introduced. In Section 3, the fault diameter of the n-cube is investigated. The partition of a conditionally faulty n-cube is presented in Section 4. Fault-tolerant path embedding is shown in Section 5. Finally, the conclusion is presented in Section 6.

2. Preliminaries

Throughout this paper, we concentrate on loopless undirected graphs. For the graph definitions, we follow the ones given by Bondy and Murty [1]. A *graph* G consists of a *node set* $V(G)$ and a *link set* $E(G)$ that is a subset of $\{(u, v)|(u, v)$ is an unordered pair of $V(G)\}$. Two nodes, u and v, of G are adjacent if $(u, v) \in E(G)$. Then u is a *neighbor* of v, and vice versa. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is *bipartite* if its node set can be partitioned into two disjoint partite sets, $V_0(G)$ and $V_1(G)$, such that every link joins a node of $V_0(G)$ and a node of $V_1(G)$.

A *path* P of length k from node x to node y in a graph G is a sequence of distinct nodes $\langle v_1, v_2, \dots, v_{k+1}\rangle$ such that $v_1 = x$, $v_{k+1} = y$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \leqslant i \leqslant k$ if $k \geqslant 1$. Moreover, a path of length zero consisting of a single node x is denoted by $\langle x\rangle$. For convenience, we write P as $\langle v_1, \dots, v_i, Q, v_j, \dots, v_{k+1}\rangle$, where $Q = \langle v_i, \dots, v_j\rangle$. The ith node of P is denoted by $P(i)$; i.e., $P(i) = v_i$. We use $\ell(P)$ to denote the length of P. The *distance* between any two nodes, u and v, of G, denoted by $d_G(u, v)$, is the length of the shortest path joining u and v in G. The *diameter* of G, denoted by $D(G)$, is defined to be $\max\{d_G(u, v) \mid u, v \in V(G)\}$. A *cycle* is a path with at least three nodes such that the last node is adjacent to the first one. For clarity, a cycle of length k is represented by $\langle v_1, v_2, \dots, v_k, v_1\rangle$. A path (or cycle) in a graph G is a *hamiltonian path* (or *hamiltonian cycle*) if it spans G. A bipartite graph is *hamiltonian laceable* [19] if there exists a hamiltonian path between any two nodes that are in different partite sets. Moreover, a hamiltonian laceable graph G is *hyper-hamiltonian laceable* [15] if, for any node $v \in V_i(G)$ and $i \in \{0, 1\}$, there exists a hamiltonian path of $G - \{v\}$ between two arbitrary nodes of $V_{1-i}(G)$. Later Hsieh et al. [9] introduced *strongly hamiltonian laceability*. A hamiltonian laceable graph G is strongly hamiltonian laceable if there exists a path of length $|V(G)| - 2$ between any two nodes in the same partite set.

Let $u = b_{n-1} \dots b_i \dots b_0$ be an n-bit binary string. For any j, $0 \leqslant j \leqslant n - 1$, we use $(u)^j$ to denote the binary string $b_{n-1} \dots \bar{b}_j \dots b_0$. Moreover, we use $(u)_j$ to denote the bit b_j of u. The *Hamming weight* of u, denoted by $w_H(u)$, is $|\{0 \leqslant i \leqslant n - 1 \mid (u)_i = 1\}|$. The n-cube Q_n consists of 2^n nodes and $n2^{n-1}$ links. Each node corresponds to an n-bit binary string. Two nodes, u and v, are adjacent if and only if $v = (u)^j$ for some j and we call the link $(u, (u)^j)$ *j-dimensional*. We define $dim((u, v)) = j$ if $v = (u)^j$. The *Hamming distance* between u and v, denoted by $h(u, v)$, is defined to be $|\{0 \leqslant i \leqslant n - 1 \mid (u)_i \neq (v)_i\}|$. Hence two nodes, u and v, are adjacent if and only if $h(u, v) = 1$. It is well known that Q_n is a bipartite graph with partite sets $V_0(Q_n) = \{u \in V(Q_n)|w_H(u)$ is even$\}$ and $V_1(Q_n) = \{u \in V(Q_n)|w_H(u)$ is odd$\}$. Moreover, Q_n is both node-transitive and link-transitive [14].

Let $Q_n^{j,i}$ be a subgraph of Q_n induced by $\{u \in V(Q_n) \mid (u)_j = i\}$ for $0 \leqslant j \leqslant n - 1$ and $i \in \{0, 1\}$. Clearly, $Q_n^{j,i}$ is isomorphic to Q_{n-1}. Then the node partition of Q_n into subgraphs $Q_n^{j,0}$ and $Q_n^{j,1}$ is called *j-partition*. The set of crossing links between $Q_n^{j,0}$ and $Q_n^{j,1}$, denoted by $E_c^j = \{(u, v) \in E(Q_n) \mid u \in V(Q_n^{j,0}), v \in V(Q_n^{j,1})\}$, consists of all j-dimensional links of Q_n. In order to clearly indicate the faulty elements in graph G, we use $F(G)$ to denote the set of all faulty elements in G.

3. Fault diameter of the *n*-cube

Let G be a graph. A faulty link (or faulty node) of G is a link (or node) that can be deleted from G. To be precise, the deletion of a subset F_e of $E(G)$, denoted by $G - F_e$, is the spanning subgraph of G obtained by deleting the links in F_e from G; the deletion of a proper subset F_v of $V(G)$, denoted by $G - F_v$, is the subgraph containing the nodes of G not in F_v and the links of G not incident with any node in F_v. By such definition, if a node is deleted from G, then all links incident with this node are deleted. Moreover, we define that $G - (F_e \cup F_v) = (G - F_e) - F_v$. Suppose that u is an arbitrary node of G and v is a neighbor of u. We say that v is a *reachable* neighbor of u if both v and (u, v) are fault-free; otherwise, v is an *unreachable* neighbor of u. The following lemma is a basic property of Q_n.

Lemma 1 [17]. *For any two nodes, u and v, of Q_n, there exist n internally node-disjoint paths joining u and v, $h(u, v)$ of which are of length $h(u, v)$ and the other $n - h(u, v)$ of which are of length $h(u, v) + 2$.*

The next corollary directly follows from Lemma 1.

Corollary 1. *Let F be a set of $n - 1$ node-faults and/or link-faults in Q_n. For any pair u, v of distinct nodes in $Q_n - F$, then $d_{Q_n-F}(u, v) \leqslant h(u, v) + 2$.*

Latifi [12] investigated the fault diameter of Q_n under the assumption that every node has at least one fault-free neighbor. The following theorem was proved in [12].

Theorem 1 [12]. *Let F be a set of $2n - 3$ faulty nodes in Q_n such that every node of Q_n has at least one fault-free neighbor. For any pair u, v of distinct nodes in $Q_n - F$, then $d_{Q_n-F}(u, v) \leqslant h(u, v) + 4$.*

Although only node-faults are admitted by Latifi [12], it is noticed that a similar result can be obtained when both node-faults and link-faults are involved. To be precise, we improve Theorem 1 by proving the next corollary.

Corollary 2. *Suppose that u and v are any two distinct nodes of $Q_n, n \geqslant 2$. Let F be a set of utmost $2n - 3$ hybrid node-faults and/or link-faults in Q_n such that both u and v are fault-free with at least one reachable neighbor. Then*

$$d_{Q_n-F}(u, v)\begin{cases} = n & \text{if } |F| \leqslant 2n-3,\ h(u, v) = n, \text{ and } n \geqslant 2, \\ \leqslant n+1 & \text{if } |F| \leqslant 2n-3,\ h(u, v) = n-1, \text{ and } n \geqslant 2, \\ \leqslant h(u, v)+4 & \text{if } |F| \leqslant 2n-3,\ h(u, v) \leqslant n-2, \text{ and } n \geqslant 3, \\ \leqslant n & \text{if } |F| = 2n-4,\ h(u, v) = n-2, \text{ and } n \neq 4. \end{cases}$$

For clarity, we prove the the first part of Corollary 2 in advance.

Proposition 1. *Suppose that u and v are any two distinct nodes of Q_n with $h(u, v) = n$. Let F be a set of $2n - 3$ hybrid node-faults and/or link-faults in Q_n such that both u and v are fault-free with at least one reachable neighbor. Then $d_{Q_n-F}(u, v) = n$.*

Proof. It is not difficult to verify that this proposition holds for $n = 2$. Hence, we only concern the case that $n \geqslant 3$. Let $I_u = \{i_1, \ldots, i_p\}$ be a set of p distinct integers of $\{0, 1, \ldots, n - 1\}$ such that $(u)^{i_1}, \ldots, (u)^{i_p}$ are reachable neighbors of u. Similarly, let $I_v = \{i'_1, \ldots, i'_q\} \subseteq \{0, 1, \ldots, n - 1\}$ be a set of q distinct integers such that $(v)^{i'_1}, \ldots, (v)^{i'_q}$ are reachable neighbors of v. We distinguish the following two cases.

Case 1: Suppose that $I_u \cap I_v \neq \emptyset$. Let $j \in I_u \cap I_v$. Then we partition Q_n into $Q_n^{j,0}$ and $Q_n^{j,1}$. For convenience, let $F_0 = F(Q_n^{j,0})$ and $F_1 = F(Q_n^{j,1})$. Since $h(u, v) = n$, nodes u and v are located in different subcubes. Moreover, we have $h(u, (v)^j) = n - 1$. By the pigeonhole principle, we have $|F_0| \leqslant n - 2$ or $|F_1| \leqslant n - 2$. Without loss of generality, we assume that $|F_0| \leqslant n - 2$. Moreover, we assume that $u \in V(Q_n^{j,0})$. By Lemma 1, $Q_n^{j,0}$ has at least one fault-free path L of length $n - 1$ between u and $(v)^j$. Hence, $\langle u, L, (v)^j, v\rangle$ forms a fault-free path of length n between u and v.

Case 2: Suppose that $I_u \cap I_v = \emptyset$. Since $|F| = 2n - 3$, we can conclude that $3 \leqslant p + q \leqslant n$. Without loss of generality, we assume that $p \geqslant q$. Thus, we have $p \geqslant 2$.

Suppose that $n = 3$. We have $p = 2$ and $q = 1$. Let $j \in I_v$. Without loss of generality, we assume that $u \in V(Q_n^{j,0})$. Obviously $Q_n^{j,0}$ is fault-free and it has a fault-free path L of length two between u and $(v)^j$. Then $\langle u, L, (v)^j, v\rangle$ is a fault-free path of length three.

Suppose that $n \geqslant 4$. Let $j \in I_u$. Since $I_u \cap I_v = \emptyset, (u)^j$ is a reachable neighbor of u whereas $(v)^j$ is an unreachable neighbor of v. Again, we assume that $u \in V(Q_n^{j,0})$. Let $F_0 = F(Q_n^{j,0})$ and $F_1 = F(Q_n^{j,1})$. If $|F_1| \leqslant n - 2$, Lemma 1 ensures that $Q_n^{j,1}$ has a fault-free path R of length $n - 1$ between $(u)^j$ and v. Hence, $\langle u, (u)^j, R, v\rangle$ is a fault-free path of length n between u and v.

Suppose that $|F_1| \geqslant n - 1$. Thus, we have $|F_0| + |F \cap E_c^j| \leqslant n - 2$. Let $\widetilde{I}_v = \{k \in I_v | ((v)^k)^j \in N_{Q_n-F}((v)^k)\}$, where $N_{Q_n-F}((v)^k)$ is the set of all reachable neighbors of $(v)^k$.

Subcase 2.1: Suppose that $\widetilde{I}_v \neq \emptyset$. Let $k \in \widetilde{I}_v$ and Θ be a subgraph of Q_n induced by $\{x \in V(Q_n) | (x)_j = (u)_j, (x)_k = (u)_k\}$. Then Θ is an $(n - 2)$-cube inside $Q_n^{j,0}$. Because $(v)^j$ is an unreachable neighbor of v and it is outside Θ, there are utmost $n - 3$ faulty elements in Θ. By Lemma 1, Θ has a fault-free path L of length $n - 2$ between u and $((v)^k)^j$. So $\langle u, L, ((v)^k)^j, (v)^k, v\rangle$ is a fault-free path of length n.

Subcase 2.2: Suppose that $\widetilde{I}_v = \emptyset$. Let $k_1 \in I_v$. Since $|F| \leqslant 2n - 3$ and $p + q \leqslant n$, there exists an integer $k_2 \in \{0, 1, \ldots, n - 1\} - \{j, k_1\}$ such that $((v)^{k_1})^{k_2}$ is a reachable neighbor of $(v)^{k_1}$ and $(((v)^{k_1})^{k_2})^j$ is a reachable neighbor of $((v)^{k_1})^{k_2}$. Let $w = ((v)^{k_1})^{k_2}$ and Ω be a subgraph of Q_n induced by $\{x \in V(Q_n) \mid (x)_j = (u)_j, (x)_{k_1} = (u)_{k_1}, (x)_{k_2} = (u)_{k_2}\}$. Then Ω is an $(n - 3)$-cube inside $Q_n^{j,0}$. Obviously, $(u)^{k_1}$, $(v)^j$, and $((v)^{k_1})^j$ are unreachable neighbors of u, v, and $(v)^{k_1}$, respectively. Since $(u)^{k_1}, (v)^j$, and $((v)^{k_1})^j$ are outside Ω, there are utmost $n - 4$ faulty elements in Ω. It follows from Lemma 1 that Ω has a fault-free path L of length $n - 3$ between u and $(w)^j$. So $\langle u, L, (w)^j, w, (w)^{k_2} = (v)^{k_1}, v\rangle$ is a fault-free path of length n between u and v.

In summary, we conclude that $d_{Q_n-F}(u, v) = n$ and the proof is completed. □

Proof of Corollary 2. Now we concern that $h(u,v) \leqslant n-1$. The proof is by the induction on n. The result is true for $n=2$. As our inductive hypothesis, we assume that the result holds for Q_{n-1} with $n \geqslant 3$. Since $h(u,v) \leqslant n-1$, we partition Q_n along some dimension j such that both u and v are in the same subcube. By transitivity, we assume that $j=0$ and $u,v \in V(Q_n^{0,1})$. Let $F_i = F(Q_n^{0,i})$ for $i \in \{0,1\}$.

Case 1: Suppose that $|F_1| \leqslant 2n-5 = 2(n-1)-3$. First, we consider the case that both u and v have at least one reachable neighbor in $Q_n^{0,1}$. Then it follows from the inductive hypothesis that $d_{Q_n-F}(u,v) = d_{Q_n^{0,1}-F_1}(u,v) = n-1$ if $h(u,v) = n-1$, $d_{Q_n-F}(u,v) \leqslant d_{Q_n^{0,1}-F_1}(u,v) \leqslant n$ if $h(u,v) = n-2$, and $d_{Q_n-F}(u,v) \leqslant d_{Q_n^{0,1}-F_1}(u,v) \leqslant h(u,v)+4$ if $h(u,v) \leqslant n-3$ for $n \geqslant 4$.

Now we consider the case that either u or v has no reachable neighbors in $Q_n^{0,1}$. Thus, we have $|F_1| \geqslant n-1$ and $|F_0| + |F \cap E_c^0| \leqslant n-2$. Since $n-1 \leqslant |F_1| \leqslant 2n-5$, we have $n \geqslant 4$. Without loss of generality, we assume that u has no reachable neighbors in $Q_n^{0,1}$. Accordingly, $(u)^0$ is the unique reachable neighbor of u.

Suppose that $h(u,v) = n-1$. Since $h((u)^0, v) = n$, it follows from Proposition 1 that $d_{Q_n-F}((u)^0, v) = n$. Let P be a fault-free path of length n between $(u)^0$ and v. Obviously, we have $u \notin V(P)$. Hence $\langle u, (u)^0, P, v\rangle$ forms a fault-free path of length $n+1$.

Suppose that $h(u,v) \leqslant n-2$. If $(v)^0$ is a reachable neighbor of v, then it follows from Corollary 1 that $d_{Q_n^{0,0}-F_0}((u)^0,(v)^0) \leqslant h((u)^0,(v)^0)+2 = h(u,v)+2$ since $|F_0| \leqslant n-2$. Let H be a shortest path between $(u)^0$ and $(v)^0$ in $Q_n^{0,0} - F_0$. Then $\langle u,(u)^0,H,(v)^0,v\rangle$ forms a fault-free path of length at most $h(u,v)+4$. When $|F| = 2n-4$, we have $|F_0| \leqslant n-3$. Therefore, $Q_n^{0,0}-F_0$ has a path H of length $n-2$ between $(u)^0$ and $(v)^0$ if $h(u,v) = n-2$. Thus $\langle u,(u)^0,H,(v)^0,v\rangle$ is a fault-free path of length n. On the other hand, if $(v)^0$ is an unreachable neighbor of v, then we have $(v)^0 \in F$ or $(v,(v)^0) \in F$. By Lemma 1, $Q_n^{0,0}$ has $n-1$ internally node-disjoint paths $L_1,\dots,L_{n-1}$ between $(u)^0$ and $(v)^0$. For clarity, L_i can be written as $\langle (u)^0, L_i', ((v)^0)^i, (v)^0\rangle$ for $1 \leqslant i \leqslant n-1$. Let $T_i = \langle (u)^0, L_i', ((v)^0)^i, (v)^i, v\rangle$ with $1 \leqslant i \leqslant n-1$. Then $\{T_1,\dots,T_{n-1}\}$ is a set of $n-1$ internally node-disjoint paths between $(u)^0$ and v. We distinguish two subcases.

Subcase 1.1: One of $\{T_1,\dots,T_{n-1}\}$, say T_i, is fault-free. Hence, $\langle u,(u)^0,T_i,v\rangle$ is a path of length at most $h(u,v)+4$ between u and v. In particular, we consider the case that $h(u,v) = n-2$. Clearly, $n-2$ paths of $\{T_1,\dots,T_{n-1}\}$ are of length $n-1$. When $n \geqslant 5$, u and v have no common neighbors. Since $(\{(v)^0,(v,(v)^0)\} \cup \bigcup_{i=1}^{n-1}\{(u)^i,(u,(u)^i)\}) \cap (\bigcup_{i=1}^{n-1} V(T_i) \cup E(T_i)) = \emptyset$, at most $n-3$ faults may appear on $T_1,\dots,T_{n-1}$. Hence there exists a fault-free path T_k of $\{T_1,\dots,T_{n-1}\}$ such that $\ell(T_k) = n-1$ if $n \geqslant 5$. Then $\langle u,(u)^0,T_k,v\rangle$ is a fault-free path of length n.

Subcase 1.2: None of $\{T_1,\dots,T_{n-1}\}$ is fault-free. It is noticed that $|F| = 2n-3$ in this subcase. Moreover, we claim that $h(u,v) = 2$. Because $T_1,\dots,T_{n-1}$ are internally node-disjoint and u has no reachable neighbors in $Q_n^{0,1}$, every of $\{T_1,\dots,T_{n-1}\}$ contains exactly one faulty element. Since $V(T_i) \cap V(Q_n^{0,1}) = \{v,(v)^i\}$ for $1 \leqslant i \leqslant n-1$, there exist two distinct integers t_1 and t_2, $1 \leqslant t_1, t_2 \leqslant n-1$, such that $F(T_{t_1}) = \{(v)^{t_1}\} = \{(u)^{t_2}\}$ and $F(T_{t_2}) = \{(v)^{t_2}\} = \{(u)^{t_1}\}$. By transitivity, we assume that $t_1 = n-1$ and $t_2 = n-2$. Again, Lemma 1 ensures that $Q_n^{0,1}$ has $n-1$ internally node-disjoint paths $R_1,\dots,R_{n-1}$ of length at most four between u and v. For clarity, we can write R_i as $\langle u, R_i', (v)^i, v\rangle$ for $1 \leqslant i \leqslant n-1$. Thus, we have $\ell(R_{n-2}) = \ell(R_{n-1}) = 2$ and $\ell(R_i) = 4$ for $1 \leqslant i \leqslant n-3$. Because $(v)^0$ is an unreachable neighbor of v, v has a reachable neighbor in $Q_n^{0,1}$, say $(v)^k$ with some $k \in \{1,\dots,n-3\}$. To be precise, we write $R_k = \langle u, x_k, y_k, (v)^k, v\rangle$ and $L_k = \langle (u)^0, (x_k)^0, (y_k)^0, ((v)^k)^0, (v)^0\rangle$, where x_k is some neighbor of u and y_k is a common neighbor of x_k and $(v)^k$.

Subcase 1.2.1: Suppose that $((v)^k)^0$ is an unreachable neighbor of $(v)^k$. Let $S_k^{(1)} = \langle (u)^0, (x_k)^0, (y_k)^0\rangle$ and $S_k^{(2)} = \langle (y_k)^0, y_k, (v)^k\rangle$. Because T_k has only one faulty element, $S_k^{(1)}$ is fault-free. Since $(V(S_k^{(2)}) \cup E(S_k^{(2)})) \cap (\bigcup_{i \neq k} V(T_i) \cup E(T_i)) = \emptyset$, $S_k^{(2)}$ is also fault-free. Then $\langle u,(u)^0,S_k^{(1)},(y_k)^0,S_k^{(2)},(v)^k,v\rangle$ is a fault-free path of length six.

Subcase 1.2.2: Suppose that $((v)^k)^0$ is a reachable neighbor of $(v)^k$. Let Θ be the subgraph of $Q_n^{0,0}$ induced by $\{x \in V(Q_n^{0,0}) \mid (x)_p = (u)_p, p \in \{1,\dots,n-3\} - \{k\}\}$. Obviously, Θ is isomorphic to Q_3. Then we claim that $|F(\Theta)| \leqslant 2$. Since $|F_0| \leqslant n-2$, this claim holds for $n=4$. In what follows, we concern that $n \geqslant 5$. It is easy to see that L_k, L_{n-2}, and L_{n-1} are inside Θ. Moreover, we have $(V(T_i) \cup E(T_i)) \cap (V(\Theta) \cup E(\Theta)) = \{(u)^0\}$ for $i \in \{1,\dots,n-3\} - \{k\}$. Since T_i contains one faulty element for each $1 \leqslant i \leqslant n-1$, at least $n-4$ faulty elements are outside Θ; i.e., $|F(\Theta)| \leqslant 2$. Since $h((u)^0,((v)^k)^0) = 3$, it follows from Lemma 1 that Θ has a fault-free path S of length three between $(u)^0$ and $((v)^k)^0$. As a result, $\langle u,(u)^0,S,((v)^k)^0,(v)^k,v\rangle$ is a fault-free path of length six.

Case 2: Suppose that $|F_1| \geqslant 2n-4$. Thus, we have $|F_0| + |F \cap E_c^0| \leqslant 1$.

Subcase 2.1: Suppose that $(u)^0$ and $(v)^0$ are reachable neighbors of u and v, respectively. Since $|F_0| \leqslant 1$, it follows from Lemma 1 that $Q_n^{0,0}$ has a fault-free path L of length at most $h(u,v)+2$ between $(u)^0$ and $(v)^0$. Then $\langle u,(u)^0,L,(v)^0,v\rangle$ is a fault-free path of length at most $h(u,v)+4$ between u and v. When $|F| = 2n-4$, we have $|F_0| + |F \cap E_c^0| = 0$. Hence $Q_n^{0,0}$ has a path L of length $h(u,v)$ between $(u)^0$ and $(v)^0$. Then $\langle u,(u)^0,L,(v)^0,v\rangle$ is a fault-free path of length $h(u,v)+2$ between u and v.

Subcase 2.2: Suppose that $(u)^0$ or $(v)^0$ is an unreachable neighbor of u or v, respectively. It is noticed that $|F| = 2n-3$ in this subcase. Since $|F_0| + |F \cap E_c^0| \leqslant 1$, we assume that $(u)^0$ is an unreachable neighbor of u. If v is a reachable neighbor of u, then $d_{Q_n-F}(u,v) = 1$. Otherwise, let $(u)^k$ be a reachable neighbor of u with some $k \in \{1,\dots,n-1\}$. Since

$|F_0| + |F \cap E_c^0| \leqslant 1$, $((u)^k)^0$ is a reachable neighbor of $(u)^k$. If $(u)_k \neq (v)_k$, then $h((u)^k, v) = h(u, v) - 1$. Obviously, $(u)^0$ is not on any shortest path between $((u)^k)^0$ and $(v)^0$. Thus, $Q_n^{0,0}$ has a fault-free path L of length $h(((u)^k)^0, (v)^0) = h(u, v) - 1$ between $((u)^k)^0$ and $(v)^0$. Then $\langle u, (u)^k, ((u)^k)^0, L, (v)^0, v\rangle$ is a fault-free path of length $h(u, v) + 2$. If $(u)_k = (v)_k$, then $h((u)^k, v) = h(u, v) + 1$. By Lemma 1, $Q_n^{0,0}$ has a fault-free path L of length $h(u, v) + 1$ between $((u)^k)^0$ and $(v)^0$. Then $\langle u, (u)^k, ((u)^k)^0, L, (v)^0, v\rangle$ is a fault-free path of length $h(u, v) + 4$.

The proof is completed. □

The following theorem characterizes a property of shortest paths in a faulty n-cube.

Theorem 2. *Let F be a set of $2n-5$ faulty links in Q_n such that every node of $Q_n - F$ has at least two neighbors. Moreover, let j be an integer of $\{0, 1, \ldots, n-1\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty links. Suppose that u is a node of $Q_n^{j,0}$ and v is a node of $Q_n^{j,1}$. Then there exists a shortest path P^* between u and v in $Q_n - F$ such that P^* crosses the dimension j exactly once.*

Proof. Since $|F(Q_n^{j,0})| + |F(Q_n^{j,1})| \leqslant |F| = 2n-5$, we assume that $|F(Q_n^{j,1})| \leqslant n-3$. Since $(u)_j \neq (v)_j$, every shortest path between u and v crosses the dimension j an odd number of times. If there is a shortest path between u and v crossing the dimension j exactly once, the proof is done. Thus, we assume that one shortest path between u and v, namely P, crosses the dimension j more than once. Accordingly, the shortest path P can be represented as $\langle u, P_0, x_1, (x_1)^j, P_1, (x_2)^j, x_2, P_2, x_3, (x_3)^j, \ldots, x_r, (x_r)^j, P_r, v\rangle$ with odd integer $r \geqslant 3$. For convenience, let $H = \langle (x_1)^j, P_1, (x_2)^j, x_2, P_2, x_3, (x_3)^j, \ldots, x_r, (x_r)^j, P_r, v\rangle$. By Corollary 1, we have $d_{Q_n^{j,1} - F(Q_n^{j,1})}((x_1)^j, v) \leqslant h((x_1)^j, v) + 2$. Suppose that R is a shortest path between $(x_1)^j$ and v in $Q_n^{j,1} - F(Q_n^{j,1})$. Then we have $\ell(H) \leqslant \ell(R)$. Since $r \geqslant 3$, we have $\ell(H) \geqslant h((x_1)^j, v) + 2 \geqslant \ell(R)$. As a result, $P^* = \langle u, P_0, x_1, (x_1)^j, R, v\rangle$ happens to be a shortest path between u and v and it crosses the dimension j exactly once. □

The fault diameter of Q_n is computed as follows.

Theorem 3. [12] *Let F be a set of faulty nodes in Q_n such that every node of Q_n has at least one fault-free neighbor. Then the diameter of $Q_n - F$ is computed as follows:*

$$D(Q_n - F) = \begin{cases} n & \text{if } |F| \leqslant n-2, \\ n+1 & \text{if } |F| = n-1, \\ n+2 & \text{if } |F| = 2n-3. \end{cases}$$

We improve Theorem 3 by proving the next corollary.

Corollary 3. *Let F be a set of hybrid node-faults and/or link-faults in Q_n, $n \geqslant 3$, such that every node of Q_n has at least one reachable neighbor. Then $D(Q_4 - F) = 4$ if $|F| \leqslant 2$; $D(Q_4 - F) = 5$ if $|F| = 3$; $D(Q_4 - F) = 6$ if $|F| \in \{4, 5\}$. When $n \neq 4$,*

$$D(Q_n - F) = \begin{cases} n & \text{if } |F| \leqslant n-2, \\ n+1 & \text{if } n-1 \leqslant |F| \leqslant 2n-4, \\ n+2 & \text{if } |F| = 2n-3. \end{cases}$$

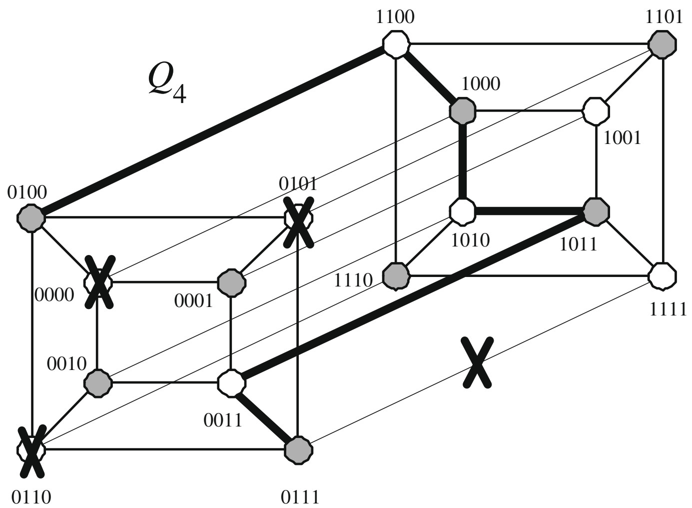

Fig. 1. An example that the distance between 0100 and 0111 is 6.

Proof. Suppose that $n \neq 4$. The result follows from Lemma 1, Corollary 2, and Theorem 3. Suppose that $n = 4$. Applying Lemma 1, Corollary 2, and Theorem 3, we also have $D(Q_4 - F) = 4$ if $|F| \leqslant 2, D(Q_4 - F) = 5$ if $|F| = 3, D(Q_4 - F) \leqslant 6$ if $|F| = 4$, and $D(Q_4 - F) = 6$ if $|F| = 5$. Let $F = \{0000, 0101, 0110, (0111, 1111)\}$. Then $d_{Q_4-F}(0100, 0111) = 6$. See Fig. 1. Therefore, $D(Q_4 - F) = 6$ if $|F| = 4$. □

4. Partition of an *n*-cube with conditional link-faults

In this section, we propose a procedure to partition Q_n with $2n - 5$ conditional link-faults. Recall that a network is said to be conditionally faulty if every node of this network is incident to at least two fault-free links. Suppose that $Q_n, n \geqslant 4$, is conditionally faulty with $2n - 5$ faulty links. For convenience, let $F = F(Q_n)$ and F_i denote the set of faulty i-dimensional links. Since $|F| = 2n - 5$, there are utmost two nodes of Q_n incident to $n - 2$ faulty links. For any two distinct nodes, u and v, of Q_n, the procedure $\mathsf{Partition}(Q_n, F, u, v)$ determines a dimension j according to the following rules:

(1) Suppose that there are exactly two nodes incident to $n - 2$ faulty links. Then the two nodes must be connected by a faulty link $(w, (w)^j)$ with some $j \in \{0, 1, \dots, n-1\}$. Obviously, both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $n - 3$ faulty links.

(2) Suppose that there is only one node, namely z, incident to $n - 2$ faulty links. Let $S = \{0 \leqslant i \leqslant n-1 \mid (z, (z)^i) \in F\} = \{k_3, \dots, k_n\}$ and $\{0, 1, \dots, n-1\} - S = \{k_1, k_2\}$. Then both $Q_n^{i,0}$ and $Q_n^{i,1}$ are conditionally faulty for each $i \in S$.

(2.1) If there exists a dimension j of S such that $|F_j| > 1$, then we partition Q_n along dimension j. Otherwise, if there exists a dimension j of S such that $|F(Q_n^{j,0})| \cdot |F(Q_n^{j,1})| > 0$, then we partition Q_n along dimension j. Obviously, both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n - 7$ or less faulty links.

(2.2) Suppose that $|F_i| = 1$ and $|F(Q_n^{i,0})| \cdot |F(Q_n^{i,1})| = 0$ for every $i \in S$. That is, for any $i \in S$, either $|F(Q_n^{i,0})|$ or $|F(Q_n^{i,1})|$ remains $2n - 6$. Hence, for any $(x, y) \in F - \{(z, (z)^i) \mid i \in S\}$, we have $(x)_i = (y)_i = (z)_i$ for every $i \in S$. That is, for $(x, y) \in F - \{(z, (z)^i) \mid i \in S\}$, we have $x, y \in \{z, (z)^{k_1}, (z)^{k_2}, ((z)^{k_1})^{k_2}\}$. Because both $(z, (z)^{k_1})$ and $(z, (z)^{k_2})$ are fault-free, it follows that $F - \{(z, (z)^i) \mid i \in S\} \subseteq \{((z)^{k_1}, ((z)^{k_1})^{k_2}), ((z)^{k_2}, ((z)^{k_1})^{k_2})\}$. Since $|F - \{(z, (z)^i) \mid i \in S\}| = n - 3 \leqslant 2$, we obtain $n \in \{4, 5\}$. The faulty links are distributed as illustrated in Fig. 2.

(2.2.1) If there exists a dimension j of S such that $(z)^j$ is neither u nor v, then we partition Q_n along dimension j.

(2.2.2) Otherwise, $\{u, v\}$ equals to $\{(z)^i \mid i \in S\}$; thus, we have $n = 4$. In this case, we partition Q_4 along any dimension $j \in S$. Clearly, u and v belong to the same partite set of Q_4.

(3) Suppose that every node is incident to utmost $n - 3$ faulty links. Obviously, every $(n-1)$-cube in Q_n is conditionally faulty. Let $S = \{0 \leqslant i \leqslant n-1 \mid F_i \neq \emptyset\}$.

(3.1) Suppose that $|F_j| \geqslant 2$ with some $j \in S$. Then both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n - 7$ or less faulty links.

(3.2) Suppose that $|F_i| \leqslant 1$ for each $i \in S$. Clearly we have $2n - 5 = |F| = |\bigcup_{i \in S} F_i| = \sum_{i \in S} |F_i| \leqslant n$; i.e., $n \leqslant 5$. Then a dimension j of S can be chosen so that both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n - 7$ or less faulty links.

(3.2.1) When $n = 5$, we claim that $|F(Q_n^{j,0})| \cdot |F(Q_n^{j,1})| > 0$ for some $j \in S$. Let $e_i = (b_{i4} \dots b_{ii} \dots b_{i0}, b_{i4} \dots \bar{b}_{ii} \dots b_{i0})$ be an i-dimensional link of Q_5 for $i \in \{0, 1, 2, 3, 4\}$. Suppose that $F = \{e_0, e_1, e_2, e_3, e_4\}$ is a faulty set of Q_5 such that $|F(Q_5^{i,0})| \cdot |F(Q_5^{i,1})| = 0$ for each $i \in \{0, 1, 2, 3, 4\}$. Then we have $b_{0i} = b_{1i} = b_{2i} = b_{3i} = b_{4i}$ for each $i \in \{0, 1, 2, 3, 4\}$; i.e., all faulty links are incident with an identical node. This contradicts the assumption that every node is incident to utmost $n - 3$ faulty links.

(3.2.2) Similarly, there exists an integer $j \in S$ such that $|F(Q_4^{j,0})| \cdot |F(Q_4^{j,1})| > 0$.

In summary, the proposed procedure determines a j-partition of Q_n such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $|F(Q_n^{j,0})| + |F(Q_n^{j,1})| \leqslant 2n - 6$.

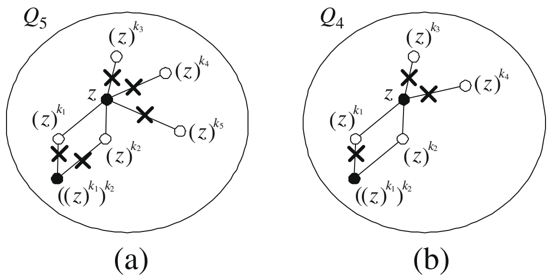

Fig. 2. The distributions of faulty links indicated in (2.2).

5. Path embedding in hypercubes

The following theorems were proved by Tsai [20] and Xu [24].

Theorem 4 [20]. *Let $n \geqslant 3$. Suppose that $F \subseteq E(Q_n)$ is a set of utmost $n-2$ faulty links. Then $Q_n - F$ is hamiltonian laceable and strongly hamiltonian laceable.*

Theorem 5 [20]. *Let $n \geqslant 3$. Suppose that $F \subseteq E(Q_n)$ is a set of utmost $n-3$ faulty links. Then $Q_n - F$ is hyper-hamiltonian laceable.*

Theorem 6 [24]. *Let F be a set of $n-2$ faulty links in Q_n ($n \geqslant 2$). Suppose that u and v are any two different nodes of $Q_n - F$. Then $Q_n - F$ contains a path of length l between u and v for every l satisfying $d_{Q_n-F}(u,v) \leqslant l \leqslant 2^n - 1$ and $2|(l - d_{Q_n-F}(u,v))$.*

As Tsai [21] showed, an n-cube with $2n-5$ conditional link-faults is hamiltonian laceable and strongly hamiltonian laceable.

Theorem 7 [21]. *Let F be a set of faulty links in Q_n ($n \geqslant 3$) such that every node of $Q_n - F$ has at least two neighbors. Then $Q_n - F$ is hamiltonian laceable and strongly hamiltonian laceable if $|F| \leqslant 2n-5$.*

To prove our main result, we need the next two lemmas.

Lemma 2 [21]. *Assume that $n \geqslant 2$. Let x and u be two distinct nodes of $V_0(Q_n)$; let y and v be two distinct nodes of $V_1(Q_n)$. Then there exist two node-disjoint paths P_1 and P_2 such that the following conditions are satisfied: (1) P_1 joins x to y, (2) P_2 joins u to v, and (3) $V(P_1) \cup V(P_2) = V(Q_n)$.*

Lemma 3. *Let v be any node of Q_n ($n \geqslant 3$) and let (w,b) be any link of $Q_n - \{v\}$. For every odd integer l in the range from 1 to $2^n - 3$, $Q_n - \{v\}$ has a path of length l between w and b.*

Proof. Since Q_n is node-transitive, we assume that $v = 0^n$. We prove this lemma by the induction on n. The induction base depends on Q_3. With the link-transitivity, the required paths are listed in Table 1.

When $n \geqslant 4$, we assume that the result is true for Q_{n-1}. Then we partition Q_n along dimension p other than $dim((w,b))$. Obviously, v is located in $Q_n^{p,0}$.

Case 1: Suppose that (w,b) is in $Q_n^{p,0}$. By the inductive hypothesis, $Q_n^{p,0} - \{v\}$ has a path of odd length l_0 between w and b for any odd integer l_0 from 1 to $2^{n-1} - 3$. Let H be a path of length $2^{n-1} - 3$ between w and b in $Q_n^{p,0} - \{v\}$. Since $2^{n-1} - 3 > 1$, we can represent H as $\langle w, u, H_0, b \rangle$. By Theorem 6, $Q_n^{p,1}$ has a path H_1 of odd length l_1 between $(w)^p$ and $(u)^p$ for any odd integer l_1 from 1 to $2^{n-1} - 1$. As a result, $\langle w, (w)^p, H_1, (u)^p, u, H_0, b \rangle$ is a path of odd length $2^{n-1} - 2 + l_1$, in the range from $2^{n-1} - 1$ to $2^n - 3$.

Case 2: Suppose that (w,b) is in $Q_n^{p,1}$. By Theorem 6, $Q_n^{p,1}$ has a path of odd length l_1 between w and b for any odd integer l_1 from 1 to $2^{n-1} - 1$. Let H be a path of length $2^{n-1} - 1$ between w and b in $Q_n^{p,1}$. Then we can choose a link (x,y) on H such that $v \notin \{(x)^p, (y)^p\}$. Hence, we can represent H as $\langle w, H_1', x, y, H_1'', b \rangle$. By the inductive hypothesis, $Q_n^{p,0} - \{v\}$ has a path H_0 of odd length l_0 between $(x)^p$ and $(y)^p$ for any odd integer l_0 from 1 to $2^{n-1} - 3$. As a result, $\langle w, H_1', x, (x)^p, H_0, (y)^p, y, H_1'', b \rangle$ is a path of odd length $2^{n-1} + l_0$, in the range from $2^{n-1} + 1$ to $2^n - 3$. □

As Shih et al. [18] showed, any fault-free link of Q_n lies on a cycle of even length from 6 to 2^n when up to $2n-5$ conditional link-faults may occur.

Theorem 8 [18]. *Let F be a set of $2n-5$ faulty links in Q_n such that every node of $Q_n - F$ has at least two neighbors. Suppose that u and v are any two adjacent nodes of $Q_n - F$. Then $Q_n - F$ contains a path of odd length l between u and v if l is in the range from 1 to $2^n - 1$ excluding 3.*

In the following discussion, we focus on constructing paths between any two nodes with distance greater than one.

Theorem 9. *Let F be a set of $2n-5$ faulty links in $Q_n (n \geqslant 3)$ such that every node of $Q_n - F$ has at least two neighbors. Suppose that u and v are two arbitrary nodes of $Q_n - F$ with distance $d^* = d_{Q_n-F}(u,v) \geqslant 2$. Then $Q_n - F$ contains a path of length l between*

Table 1
The paths of variable lengths between w and b in $Q_3 - \{000\}$.

$(w,b) = (011,001)$	$\langle 011,111,101,001 \rangle, \langle 011,111,110,100,101,001 \rangle$
$(w,b) = (011,111)$	$\langle 011,001,101,111 \rangle, \langle 011,001,101,100,110,111 \rangle$
$(w,b) = (101,001)$	$\langle 101,111,011,001 \rangle, \langle 101,100,110,111,011,001 \rangle$
$(w,b) = (101,100)$	$\langle 101,111,110,100 \rangle, \langle 101,111,011,010,110,100 \rangle$
$(w,b) = (101,111)$	$\langle 101,100,110,111 \rangle, \langle 101,100,110,010,011,111 \rangle$

u and v for every integer l satisfying both $d^* \leqslant l \leqslant 2^n - 1$ *and* $2|(l - d^*)$, *where expression* $2|(l - d^*)$ *means that* $l - d^* \equiv 0 \pmod 2$.

Proof. Applying procedure Partition(Q_n, F, u, v), we can determine a j-partition of Q_n such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $|F(Q_n^{j,0})| + |F(Q_n^{j,1})| \leqslant 2n - 6$. As a result, the proof can proceed by the induction on n. The induction base, depending upon Q_3, follows from Theorem 6. As our inductive hypothesis, we assume that the result holds for Q_{n-1} when $n \geqslant 4$.

Case I: Suppose that u and v are in the different partite sets of Q_n. Without loss of generality, we assume that $u \in V_0(Q_n)$ and $v \in V_1(Q_n)$. By Theorem 7, $Q_n - F$ is hamiltonian laceable. Moreover, a shortest path between u and v can be easily obtained by a simple breadth-first search. Therefore, we mainly concentrate on the paths of odd lengths in the range from $d^* + 2$ to $2^n - 3$.

Subcase I.1: Suppose that $|F(Q_n^{j,0})| \leqslant 2n - 7$ and $|F(Q_n^{j,1})| \leqslant 2n - 7$. Without loss of generality, we assume that $|F(Q_n^{j,0})| \geqslant |F(Q_n^{j,1})|$; thus, $|F(Q_n^{j,1})| \leqslant n - 3$.

Subcase I.1.1: Suppose that both u and v are in $Q_n^{j,0}$. By the inductive hypothesis, $Q_n^{j,0} - F(Q_n^{j,0})$ contains a path H_0 of length $2^{n-1} - 1$ between u and v. Let $A = \{(H_0(i), H_0(i+1)) \mid 1 \leqslant i \leqslant 2^{n-1}, i \equiv 1 \pmod 2\}$ be a set of disjoint links on H_0. Since $|A| = \lceil \frac{2^{n-1}-1}{2} \rceil > 2n - 5$ for any $n \geqslant 4$, there exists a link (w, b) of A such that $(w, (w)^j), (b, (b)^j)$, and $((w)^j, (b)^j)$ are all fault-free. Hence, H_0 can be written as $\langle u, H_0', w, b, H_0'', v\rangle$. Since $|F(Q_n^{j,1})| \leqslant n - 3$, it follows from Theorem 6 that $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path H_1 of odd length l_1 between $(w)^j$ and $(b)^j$ for any odd integer l_1 from 1 to $2^{n-1} - 1$. As a result, $\langle u, H_0', w, (w)^j, H_1, (b)^j, b, H_0'', v\rangle$ is a path of odd length $2^{n-1} + l_1$, in the range from $2^{n-1} + 1$ to $2^n - 1$. See Fig. 3a for illustration.

The paths of lengths less than $2^{n-1} + 1$ can be obtained as follows. By Corollary 2, we have $d^* = d_{Q_n - F}(u, v) \leqslant h(u, v) + 4$ and $d_{Q_n^{j,0} - F(Q_n^{j,0})}(u, v) \leqslant h(u, v) + 4$. By the inductive hypothesis, $Q_n^{j,0} - F(Q_n^{j,0})$ has a path T_0 of length l_0 between u and v for any odd integer l_0 in the range from $d_{Q_n^{j,0} - F(Q_n^{j,0})}(u, v)$ to $2^{n-1} - 1$. If $d^* = h(u, v)$ or $d^* = h(u, v) + 4$, then $d_{Q_n^{j,0} - F(Q_n^{j,0})}(u, v) = d^*$. Otherwise, if $d^* = h(u, v) + 2$, then $d_{Q_n^{j,0} - F(Q_n^{j,0})}(u, v) \leqslant d^* + 2$.

Subcase I.1.2: Suppose that both u and v are in $Q_n^{j,1}$. Since $|F(Q_n^{j,1})| \leqslant n - 3$, it follows from Corollary 1 that $d^* \leqslant d_{Q_n^{j,1} - F(Q_n^{j,1})}(u, v) \leqslant h(u, v) + 2$. Thus, there exists a shortest path between u and v in $Q_n - F$ such that it does not cross the dimension j. By inductive hypothesis, $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path T_1 of odd length l_1 between u and v for each odd integer l_1 from d^* to $2^{n-1} - 1$. Let $\overline{T}_1$ be a path of length $2^{n-1} - 1$ between u and v in $Q_n^{j,1} - F(Q_n^{j,1})$. Moreover, let $A = \{(\overline{T}_1(i), \overline{T}_1(i+1)) \mid 1 \leqslant i \leqslant 2^{n-1}, i \equiv 1 \pmod 2\}$ be a set of disjoint links on $\overline{T}_1$. Since $|A| = \lceil \frac{2^{n-1}-1}{2} \rceil > 2n - 5$ for $n \geqslant 4$, there exists a link (w, b) of A such that $(w, (w)^j), (b, (b)^j)$, and $((w)^j, (b)^j)$ are all fault-free. Hence, $\overline{T}_1$ can be written as $\langle u, T_1', w, b, T_1'', v\rangle$. Since $|F(Q_n^{j,0})| \leqslant 2n - 7$, it follows from Theorem 8 that $Q_n^{j,0} - F(Q_n^{j,0})$ contains a path T_0 of odd length l_0 between $(w)^j$ and $(b)^j$ for any odd integer l_0 in the range from 1 to $2^{n-1} - 1$ excluding 3. As a result, $\langle u, T_1', w, (w)^j, T_0, (b)^j, b, T_1'', v\rangle$ is a path of odd length $2^{n-1} + l_0$, in the range from $2^{n-1} + 1$ to $2^n - 1$ excluding $2^{n-1} + 3$. See Fig. 3b for illustration.

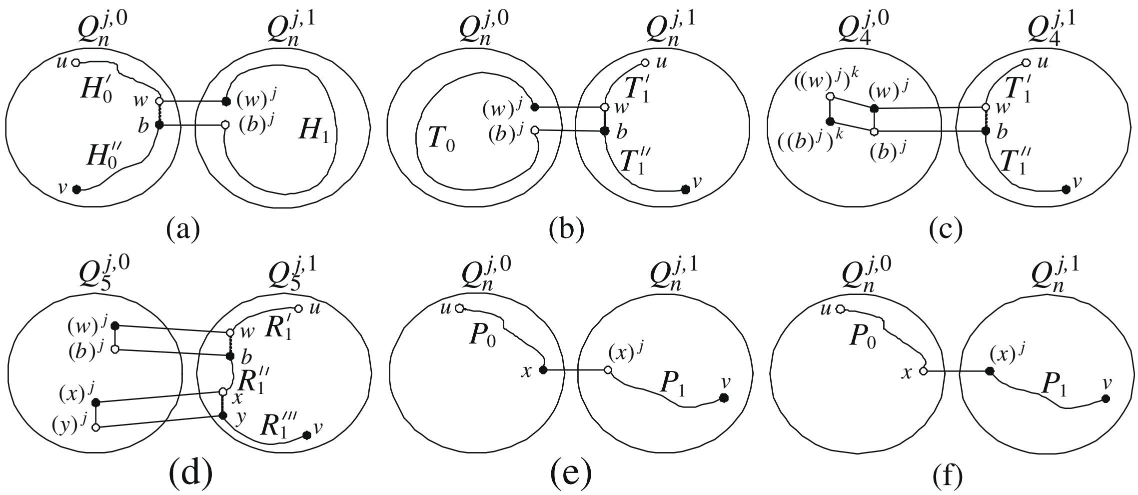

Fig. 3. Illustration for Subcase I.1.

The path of length $2^{n-1}+3$ is discussed as follows. When $n=4$, we have $|F(Q_n^{j,0})| \leqslant 1$. Thus, there exists an integer k of $\{0,1,2,3\} - \{j, dim((w,b))\}$ such that $((w)^j, ((w)^j)^k)$, $((b)^j, ((b)^j)^k)$, and $(((w)^j)^k, ((b)^j)^k)$ are all fault-free. Hence, $\langle u, T'_1, w, (w)^j, ((w)^j)^k, ((b)^j)^k, (b)^j, b, T''_1, v\rangle$ is a path of length 11. See Fig. 3c for illustration. When $n \geqslant 5$, we have $|A| - |F| \geqslant |A| - (2n-5) = \lceil \frac{2^{n-1}-1}{2} \rceil - (2n-5) \geqslant 2$. Thus, there is a link (x,y) of A, other than (w,b), such that (x,y) and (w,b) have no shared endpoints and $(x,(x)^j)$, $(y,(y)^j)$, and $((x)^j,(y)^j)$ are all fault-free. Without loss of generality, $\overline{T}_1$ can be written as $\langle u, R'_1, w, b, R''_1, x, y, R'''_1, v\rangle$. Hence, $\langle u, R'_1, w, (w)^j, (b)^j, b, R''_1, x, (x)^j, (y)^j, y, R'''_1, v\rangle$ is a path of length $2^{n-1}+3$. See Fig. 3d.

Subcase I.1.3: Suppose that u is in $Q_n^{j,0}$ and v is in $Q_n^{j,1}$. By Theorem 2, we have a shortest path P^* between u and v in $Q_n - F$ such that P^* crosses the dimension j exactly once. Thus, P^* can be represented as $\langle u, P_0, x, (x)^j, P_1, v\rangle$, where P_0 is a shortest path joining u to some node x in $Q_n^{j,0} - F(Q_n^{j,0})$ and P_1 is a shortest path joining $(x)^j$ to v in $Q_n^{j,1} - F(Q_n^{j,1})$. See Fig. 3e and f for illustration.

Subcase I.1.3.1: Suppose that $\ell(P_0) > 0$ and $\ell(P_1) > 0$. By Theorem 6, $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path T_1 of length l_1 between $(x)^j$ and v for each l_1 satisfying $\ell(P_1) \leqslant l_1 \leqslant 2^{n-1}-1$ and $2|(l_1 - \ell(P_1))$. Suppose that $\ell(P_0) = 1$. It follows from Theorem 8 that $Q_n^{j,0} - F(Q_n^{j,0})$ contains a path T_0 of odd length l_0 between u and x for any odd integer l_0 in the range from 1 to $2^{n-1}-1$ excluding 3. Suppose that $\ell(P_0) > 1$. By the inductive hypothesis, $Q_n^{j,0} - F(Q_n^{j,0})$ contains a path T_0 of length l_0 between u and x for each l_0 satisfying $\ell(P_0) \leqslant l_0 \leqslant 2^{n-1}-1$ and $2|(l_0 - \ell(P_0))$. As a result, $\langle u, T_0, x, (x)^j, T_1, v\rangle$ is a path of odd length $l_0 + l_1 + 1$, in the range from d^* to $2^n - 3$.

Subcase I.1.3.2: Suppose that $\ell(P_0) = 0$ or $\ell(P_1) = 0$. Since $d^* = d_{Q_n - F}(u,v) > 1$, we have $u \neq x$ or $v \neq (x)^j$. With symmetry, we assume that $\ell(P_0) = 0$. By the inductive hypothesis, $Q_n^{j,1} - F(Q_n^{j,1})$ contains a path T_1 of even length l_1 between $(x)^j$ and v for each even integer l_1 from $\ell(P_1)$ to $2^{n-1}-2$. As a result, $\langle u = x, (x)^j, T_1, v\rangle$ is a path of odd length $l_1 + 1$ in the range from $\ell(P_1) + 1 = d^*$ to $2^{n-1}-1$.

The paths of odd lengths in the range from $2^{n-1}+1$ to $2^n - 1$ are constructed as follows. Since $|V_1(Q_n^{j,0})| = 2^{n-2} > 2n-5$ for $n \geqslant 4$, we can choose a node y from $V_1(Q_n^{j,0})$ such that $(y,(y)^j)$ is fault-free. Let R_0 be a path joining u to y in $Q_n^{j,0} - F(Q_n^{j,0})$ and R_1 be a path joining $(y)^j$ to v in $Q_n^{j,1} - F(Q_n^{j,1})$. Similar to **Subcase I.1.3.1**, $H = \langle u, R_0, y, (y)^j, R_1, v\rangle$ is a path of any odd length in the range from $d' = d_{Q_n^{j,0} - F(Q_n^{j,0})}(u,y) + d_{Q_n^{j,1} - F(Q_n^{j,1})}((y)^j, v) + 1$ to $2^n - 1$. By Corollary 3, we have $d' \leqslant (n+1) + (n-1) + 1 \leqslant 2^{n-1}+1$ for $n \geqslant 4$. That is, H can be a path of any odd length in the range from $2^{n-1}+1$ to $2^n - 1$.

Subcase I.2: Suppose that $|F(Q_n^{j,0})| = 2n-6$ or $|F(Q_n^{j,1})| = 2n-6$. Without loss of generality, we assume that $|F(Q_n^{j,0})| = 2n-6$. Thus, $Q_n^{j,1}$ is fault-free. By procedure $\mathrm{Partition}(Q_n, F, u, v)$, the faulty links are distributed as shown in Fig. 2.

Subcase I.2.1: Suppose that both u and v are in $Q_n^{j,0}$. Let (w,b) be a faulty link of $Q_n^{j,0}$ such that both $(w,(w)^j)$ and $(b,(b)^j)$ are fault-free. For convenience, let $F_0 = F(Q_n^{j,0}) - \{(w,b)\}$. By the inductive hypothesis, $Q_n^{j,0} - F_0$ has a path P_l of odd length l between u and v for any odd integer l in the range from $d_{Q_n^{j,0} - F_0}(u,v)$ to $2^{n-1}-1$. If (w,b) is on P_l, we write P_l as $\langle u, P'_l, w, b, P''_l, v\rangle$ and define $\widetilde{P}_l = \langle u, P'_l, w, (w)^j, (b)^j, b, P''_l, v\rangle$. Otherwise, P_l can be written as $\langle u, P'_l, x, y, P''_l, v\rangle$, where (x,y) is a link on P_l such that both $(x,(x)^j)$ and $(y,(y)^j)$ are fault-free. Similarly, we define $\widetilde{P}_l = \langle u, P'_l, x, (x)^j, (y)^j, y, P''_l, v\rangle$. Then $\widetilde{P}_l$ is a path of length $l+2$. By Corollary 2, we have $d^* = d_{Q_n - F}(u,v) \leqslant h(u,v) + 4$ and $d_{Q_n^{j,0} - F_0}(u,v) \leqslant h(u,v) + 4$. First, if $d^* = h(u,v)$ or $d^* = h(u,v) + 4$, then we have $d^* = d_{Q_n^{j,0} - F_0}(u,v)$ and thus l ranges from d^* to $2^{n-1}-1$. Next, if $d^* = h(u,v) + 2 = d_{Q_n^{j,0} - F_0}(u,v)$, then l ranges from d^* to $2^{n-1}-1$. Finally, if $d^* = h(u,v) + 2$ and $d_{Q_n^{j,0} - F_0}(u,v) = h(u,v) + 4$, then l ranges from $d^* + 2$ to $2^{n-1}-1$. For the final case, a shortest path between u and v in $Q_n - F$ can be constructed by a breadth-first search. In summary, the paths of odd lengths from $d^* + 2$ to $2^{n-1}+1$ are constructed.

By Theorem 6, $Q_n^{j,1}$ contains a path T_1 of length l_1 between $(w)^j$ and $(b)^j$ for each odd integer l_1 from 1 to $2^{n-1}-1$. Similarly, $Q_n^{j,1}$ contains a path R_1 of length l_1 between $(x)^j$ and $(y)^j$ for each odd integer l_1 from 1 to $2^{n-1}-1$. Thus, $\langle u, P'_{2^{n-1}-1}, w, (w)^j, T_1, (b)^j, b, P''_{2^{n-1}-1}, v\rangle$ (or $\langle u, P'_{2^{n-1}-1}, x, (x)^j, R_1, (y)^j, y, P''_{2^{n-1}-1}, v\rangle$) is a path of length $2^{n-1} + l_1$, in the range from $2^{n-1}+1$ to $2^n - 1$.

Subcase I.2.2: Suppose that both u and v are in $Q_n^{j,1}$. Let $(w,(w)^i)$ be a faulty link in $Q_n^{j,0}$ such that both $(w,(w)^j)$ and $((w)^i, ((w)^i)^j)$ are fault-free. Since $d^* = d_{Q_n - F}(u,v) > 1$, we assume that $(w)^j$ is different from u and v. Moreover, since $n \geqslant 4$, we assume that $t \in \{0,1,\ldots,n-1\} - \{j,i\}$. Let $X = \{((w)^j, ((w)^j)^k) \mid k \notin \{i,j,t\}\}$. Since $|X| = n-3$, our inductive hypothesis ensures that $Q_n^{j,1} - X$ contains a path T_1 of odd length l_1 between u and v for any odd integer l_1 satisfying $d^* \leqslant l_1 \leqslant 2^{n-1}-1$. Let $\overline{T}_1$ denote a path of length $2^{n-1}-1$ between u and v in $Q_n^{j,1} - X$. It is noted that $((w)^j, ((w)^j)^i)$ is on $\overline{T}_1$. Hence, $\overline{T}_1$ can be represented as $\langle u, T'_1, (w)^j, ((w)^j)^i, T''_1, v\rangle$. By Theorem 8, $Q_n^{j,0} - (F(Q_n^{j,0}) - \{(w,(w)^i)\})$ contains a path T_0 of odd length l_0 between w and $(w)^i$ for $5 \leqslant l_0 \leqslant 2^{n-1}-1$. As a result, $\langle u, T'_1, (w)^j, w, T_0, (w)^i, ((w)^j)^i, T''_1, v\rangle$ is a path of odd length $2^{n-1} + l_0$, in the range from $2^{n-1}+5$ to $2^n - 1$. See Fig. 4a for illustration.

Let $\overline{T}_0$ denote the longest path between w and $(w)^i$ in $Q_n^{j,0} - (F(Q_n^{j,0}) - \{(w,(w)^i)\})$. Moreover, let $A = \{(\overline{T}_0(k), \overline{T}_0(k+1)) \mid 1 \leqslant k \leqslant 2^{n-1}, k \equiv 1 \pmod 2\}$ be a set of disjoint links on $\overline{T}_0$. The paths of lengths $2^{n-1}+1$ and $2^{n-1}+3$ can be obtained as follows:

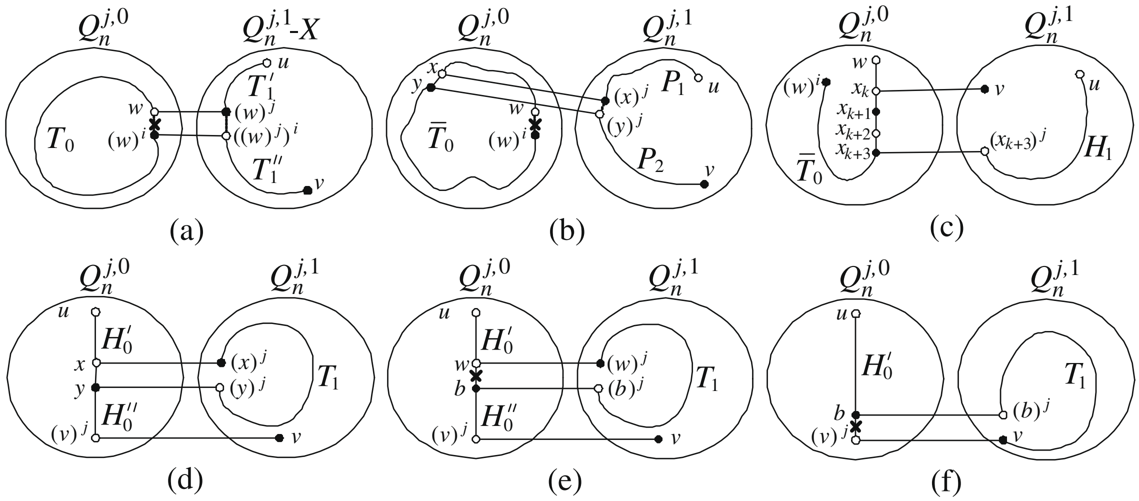

Fig. 4. Illustration for Subcase I.2.

(a) Since $|A| = \lceil \frac{2^{n-1}-1}{2} \rceil > 3$ for $n \geqslant 4$, there exists a link (x, y) of A such that both $F \cap \{(x, (x)^j), (y, (y)^j)\} = \emptyset$ and $\{(x)^j, (y)^j\} \cap \{u, v\} = \emptyset$ are satisfied. Without loss of generality, we assume that $x \in V_0(Q_n)$. By Lemma 2, there exist two node-disjoint paths P_1 and P_2 in $Q_n^{j,1}$ such that (i) P_1 joins u to $(x)^j$, (ii) P_2 joins $(y)^j$ to v, and (iii) $V(P_1) \cup V(P_2) = V(Q_n^{j,1})$. As a result, $\langle u, P_1, (x)^j, x, y, (y)^j, P_2, v\rangle$ is a path of length $2^{n-1}+1$. See Fig. 4b for illustration.

(b) We write $\overline{T}_0$ as $\langle w = x_0, x_1, \ldots, x_{2^{n-1}-1} = (w)^i\rangle$. Then we can choose a pair of nodes from $\{\{x_0, x_3\}, \{x_1, x_4\}, \{x_2, x_5\}\}$, namely $\{x_k, x_{k+3}\}$, such that both $F \cap \{(x_k, (x_k)^j), (x_{k+3}, (x_{k+3})^j)\} = \emptyset$ and $|\{(x_k)^j, (x_{k+3})^j\} \cap \{u, v\}| \leqslant 1$ are satisfied.

(b.1) Suppose that $x_k \in V_0(Q_n)$. If $|\{(x_k)^j, (x_{k+3})^j\} \cap \{u, v\}| = 0$, Lemma 2 ensures that $Q_n^{j,1}$ has two node-disjoint paths P_1 and P_2 such that (i) P_1 joins u to $(x_k)^j$, (ii) P_2 joins $(x_{k+3})^j$ to v, and (iii) $V(P_1) \cup V(P_2) = V(Q_n^{j,1})$. Hence, $\langle u, P_1, (x_k)^j, x_k, x_{k+1}, x_{k+2}, x_{k+3}, (x_{k+3})^j, P_2, v\rangle$ is a path of length $2^{n-1}+3$. If $|\{(x_k)^j, (x_{k+3})^j\} \cap \{u, v\}| = 1$, we assume that $(x_k)^j = v$. By Theorem 5, $Q_n^{j,1} - \{v\}$ has a hamiltonian path H_1 joining u to $(x_{k+3})^j$. Then $\langle u, H_1, (x_{k+3})^j, x_{k+3}, x_{k+2}, x_{k+1}, x_k, (x_k)^j = v\rangle$ is a path of length $2^{n-1}+3$. See Fig. 4c.

(b.2) Suppose that $x_k \in V_1(Q_n)$. The required paths can be obtained similarly.

Subcase I.2.3: Suppose that u is in $Q_n^{j,0}$ and v is in $Q_n^{j,1}$. If $(u, (u)^j)$ is fault-free, the shortest path between u and v can be of the form $\langle u, (u)^j, P_1, v\rangle$, where P_1 is a shortest path joining $(u)^j$ to v in $Q_n^{j,1}$. By the inductive hypothesis, $Q_n^{j,1}$ contains a path T_1 of even length l_1 between $(u)^j$ and v for any even integer l_1 from $d_{Q_n^{j,1}}((u)^j, v) = d^* - 1$ to $2^{n-1}-2$. Then $\langle u, (u)^j, T_1, v\rangle$ is a path of odd length $l_1 + 1$ in the range from d^* to $2^{n-1}-1$. On the other hand, if $(u, (u)^j)$ is faulty, we choose a neighbor of u, namely x, in $Q_n^{j,0} - F(Q_n^{j,0})$. Obviously, we have either $h((x)^j, v) = h(u, v) - 2$ or $h((x)^j, v) = h(u, v)$. Let R_1 be a shortest path joining $(x)^j$ to v in $Q_n^{j,1}$. Then $\langle u, x, (x)^j, R_1, v\rangle$ is a path of length $h(u, v)$ or $h(u, v) + 2$. Thus, we have $d^* \leqslant h(u, v) + 2$. By Theorem 6, $Q_n^{j,1}$ has a path T_1 of length l_1 between $(x)^j$ and v for any odd integer l_1 from $h((x)^j, v)$ to $2^{n-1}-1$. Then $\langle u, x, (x)^j, T_1, v\rangle$ is a path of odd length $l_1 + 2$ in the range from $d^* + 2$ to $2^{n-1}+1$.

The paths of lengths greater than $2^{n-1}-1$ can be obtained as follows. Since $|F(Q_n^{j,0})| = 2n - 6$, the j-partition determined by Partition (Q_n, F, u, v) guarantees that link $(v, (v)^j)$ is fault-free if $h(u, v)$ is odd. (See (2.2) in Section 4). Let (w, b) be a faulty link in $Q_n^{j,0}$ such that both $(w, (w)^j)$ and $(b, (b)^j)$ are fault-free. By the inductive hypothesis, $Q_n^{j,0} - (F(Q_n^{j,0}) - \{(w, b)\})$ contains a path H_0 of length $2^{n-1}-2$ between u to $(v)^j$. Three subcases are distinguished.

Subcase I.2.3.1: Suppose that (w, b) is not located on H_0. See Fig. 4d. We choose a link (x, y) on H_0 such that $(x, (x)^j)$ and $(y, (y)^j)$ are fault-free and $((x)^j, (y)^j)$ is not incident with v. Thus, H_0 can be represented as $\langle u, H_0', x, y, H_0'', (v)^j\rangle$. By Lemma 3, $Q_n^{j,1} - \{v\}$ contains a path T_1 of odd length l_1 between $(x)^j$ and $(y)^j$ for any odd integer l_1 from 1 to $2^{n-1}-3$. Consequently, $\langle u, H_0', x, (x)^j, T_1, (y)^j, y, H_0'', (v)^j, v\rangle$ is a path of odd length $2^{n-1}+l_1$, in the range from $2^{n-1}+1$ to 2^n-3.

Subcase I.2.3.2: Suppose that (w, b) is located on H_0 and (w, b) is not incident with $(v)^j$. See Fig. 4e. Thus, H_0 can be represented as $\langle u, H_0', w, b, H_0'', (v)^j\rangle$. By Lemma 3, $Q_n^{j,1} - \{v\}$ contains a path T_1 of odd length l_1 between $(w)^j$ and $(b)^j$ for $1 \leqslant l_1 \leqslant 2^{n-1}-3$. Hence, $\langle u, H_0', w, (w)^j, T_1, (b)^j, b, H_0'', (v)^j, v\rangle$ is a path of odd length $2^{n-1}+l_1$, in the range $2^{n-1}+1$ to 2^n-3.

Subcase I.2.3.3: Suppose that (w, b) is located on H_0 and (w, b) is incident with $(v)^j$. See Fig. 4f. Let $w = (v)^j$. Thus, H_0 can be represented as $\langle u, H_0', b, w = (v)^j\rangle$. By Theorem 6, $Q_n^{j,1}$ contains a path T_1 of odd length l_1 between $(b)^j$ and v for any odd integer l_1 satisfying $1 \leqslant l_1 \leqslant 2^{n-1}-1$. Then $\langle u, H_0', b, (b)^j, T_1, v\rangle$ is a path of odd length $2^{n-1}+l_1-2$, in the range from $2^{n-1}-1$ to 2^n-3.

Case II: Suppose that u and v belong to the same partite set of Q_n. This case is similar to **Case I** and the details are described in Appendix A. □

6. Conclusion

Fault tolerance is an important research issue in the area of interconnection networks. Since linear array and rings are two of the most fundamental structures, the node-fault and link-fault tolerance are widely investigated for path embedding in various kinds of network topologies. By induction, we show that a conditionally faulty Q_n, with $2n-5$ faulty links, has a fault-free path of odd (resp. even) length in the range from d^* to 2^n-1 between two arbitrary nodes of odd (resp. even) distance d^*.

Let $Pr(n)$ denote the probability that every node of an n-cube containing $2n-5$ faulty links is incident to at least two fault-free links. Then $Pr(n)$ is computed as follows: $Pr(n)=1$ if $n=3$; $Pr(n)=1-\frac{2^n\times\binom{n}{2n-5}}{\binom{n\times 2^{n-1}}{2n-5}}$ if $n=4$; $Pr(n)=1-\frac{2^n\times\binom{n\times 2^{n-1}-n}{n-5}+2^n\times\binom{n}{n-1}\binom{n\times 2^{n-1}-n}{n-4}}{\binom{n\times 2^{n-1}}{2n-5}}$ if $n\geqslant 5$. One can verify that $Pr(n)$ approaches to 1 as n increases. Thus, the assumption of conditional link-faults is probabilistically reasonable.

Let u be any node of Q_n and let $v=((u)^0)^1$. Suppose that $F=\{(u,(u)^i)\mid 2\leqslant i\leqslant n-1\}\cup\{(v,(v)^i)\mid 2\leqslant i\leqslant n-1\}$ is a set of $2n-4$ faulty links in Q_n. Obviously, Q_n-F has no hamiltonian paths joining u and $(u)^1$. That is, an n-cube with $2n-4$ or more conditional link-faults is likely to have no paths of some specific lengths. In this sense, our result is optimal. A number of researchers [5,8,10,22,23] addressed the fault-tolerant hamiltonicity (or hamiltonian connectivity) in some special classes of network topologies under the consideration of conditional fault model. For example, the crossed cube [3], which is a variation of hypercubes, possesses some properties superior to the hypercube. Fu [6] showed that a conditionally faulty n-dimensional crossed cube contains a fault-free hamiltonian cycle even if it has $2n-5$ faulty links. Hence, it is intriguing to study fault-tolerant path embedding on crossed cubes under the assumption of conditional faults.

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Appendix A. Case II in proof of Theorem 9

Case II: Suppose that u and v belong to the same partite set of Q_n. Thus, the distance d^* between u and v is even. Without loss of generality, we assume that $u,v\in V_0(Q_n)$. By Theorem 7, Q_n-F is strongly hamiltonian laceable. Moreover, a shortest path between u and v can be obtained by a breadth-first search. Hence, we concentrate on the paths of even lengths in the range from d^*+2 to 2^n-4.

Subcase II.1: Suppose that $|F(Q_n^{j,0})|\leqslant 2n-7$ and $|F(Q_n^{j,1})|\leqslant 2n-7$. Without loss of generality, we assume that $|F(Q_n^{j,0})|\geqslant|F(Q_n^{j,1})|$. Thus, $|F(Q_n^{j,1})|\leqslant n-3$.

Subcase II.1.1: Suppose that both u and v are in $Q_n^{j,0}$. By the inductive hypothesis, $Q_n^{j,0}-F(Q_n^{j,0})$ has a path H_0 of length $2^{n-1}-2$ between u and v. Let $A=\{(H_0(i),H_0(i+1))\mid 1\leqslant i\leqslant 2^{n-1}-1, i\equiv 1 \pmod 2\}$ be a set of disjoint links on H_0. First, suppose that $|F(Q_n^{j,0})|>0$. Since $|A|=\lceil\frac{2^{n-1}-2}{2}\rceil>2n-5-|F(Q_n^{j,0})|$ for $n\geqslant 4$, there exists a link (w,b) of A such that $(w,(w)^j),(b,(b)^j)$, and $((w)^j,(b)^j)$ are all fault-free. Next, suppose that $|F(Q_n^{j,0})|=0$ and $n\geqslant 5$. Since $|A|=\lceil\frac{2^{n-1}-2}{2}\rceil>2n-5$, there still exists a link (w,b) of A such that $(w,(w)^j),(b,(b)^j)$, and $((w)^j,(b)^j)$ are all fault-free. Finally, suppose that $|F(Q_n^{j,0})|=0$ and $n=4$. If there does not exist any node z of $V_1(Q_4^{j,0})$ such that $(z,(z)^j)$ is faulty, there must exist a link (w,b) on H_0 such that $(w,(w)^j)$, $(b,(b)^j)$, and $((w)^j,(b)^j)$ are all fault-free. If there exists a node z of $V_1(Q_4^{j,0})$ such that $(z,(z)^j)$ is faulty, then it follows from Theorem 5 that $Q_4^{j,0}-\{z\}$ has a hamiltonian path, still namely H_0, between u and v. Obviously, there also exists a link (w,b) on H_0 such that $(w,(w)^j),(b,(b)^j)$, and $((w)^j,(b)^j)$ are all fault-free. In summary, H_0 can be written as $\langle u,H_0',w,b,H_0'',v\rangle$. Since $|F(Q_n^{j,1})|\leqslant n-3$, it follows from Theorem 6 that $Q_n^{j,1}-F(Q_n^{j,1})$ contains a path H_1 of odd length l_1 between $(w)^j$ and $(b)^j$ for any odd integer l_1 satisfying $1\leqslant l_1\leqslant 2^{n-1}-1$. As a result, $\langle u,H_0',w,(w)^j,H_1,(b)^j,b,H_0'',v\rangle$ is a path of even length in the range from 2^{n-1} to 2^n-2.

The paths of lengths less than 2^{n-1} are obtained as follows. By Corollary 2, we have $d^*=d_{Q_n-F}(u,v)\leqslant h(u,v)+4$ and $d_{Q_n^{j,0}-F(Q_n^{j,0})}(u,v)\leqslant h(u,v)+4$. By inductive hypothesis, $Q_n^{j,0}-F(Q_n^{j,0})$ has a path T_0 of length l_0 between u and v for any even length from $d_{Q_n^{j,0}-F(Q_n^{j,0})}(u,v)$ to $2^{n-1}-2$. If $d^*=h(u,v)$ or $d^*=h(u,v)+4$, then $d_{Q_n^{j,0}-F(Q_n^{j,0})}(u,v)=d^*$. If $d^*=h(u,v)+2$, then $d_{Q_n^{j,0}-F(Q_n^{j,0})}(u,v)\leqslant d^*+2$.

Subcase II.1.2: Suppose that both u and v are in $Q_n^{j,1}$. Since $|F(Q_n^{j,1})|\leqslant n-3$, it follows from Lemma 1 that $d^*\leqslant h(u,v)+2$. Thus, Q_n-F has a shortest path between u and v that does not cross the dimension j. By the inductive hypothesis, $Q_n^{j,1}-F(Q_n^{j,1})$ contains a path T_1 of length l_1 between u and v for any even integer l_1 satisfying $d^*\leqslant l_1\leqslant 2^{n-1}-2$. Let $\overline{T}_1$ be a path of length $2^{n-1}-2$ between u and v in $Q_n^{j,1}-F(Q_n^{j,1})$. Moreover, let $A=\{(\overline{T}_1(i),\overline{T}_1(i+1))\mid 1\leqslant i\leqslant 2^{n-1}-1,\ i\equiv 1 \pmod 2\}$ be a set of disjoint links on $\overline{T}_1$. First, suppose that $|F(Q_n^{j,1})|>0$. Since $|A|=\lceil\frac{2^{n-1}-2}{2}\rceil>2n-5-|F(Q_n^{j,1})|$ for $n\geqslant 4$, there exists a link $(w,b)\in A$ such that $(w,(w)^j)$, $(b,(b)^j)$, and $((w)^j,(b)^j)$ are all fault-free. Next, suppose that $|F(Q_n^{j,1})|=0$ and $n\geqslant 5$. Since $|A|=\lceil\frac{2^{n-1}-2}{2}\rceil>2n-5$, there still exists a link $(w,b)\in A$ such that $(w,(w)^j),(b,(b)^j)$ and $((w)^j,(b)^j)$ are all fault-free. Finally, suppose that $|F(Q_n^{j,1})|=0$ and $n=4$. If there does not exist any node z of $V_1(Q_4^{j,1})$ such that $(z,(z)^j)$ is faulty, there exists a link (w,b) on $\overline{T}_1$ such that $(w,(w)^j)$, $(b,(b)^j)$ and $((w)^j,(b)^j)$ are all fault-free. If there exists a node z of $V_1(Q_4^{j,1})$ such that $(z,(z)^j)$ is faulty, Theorem 5 ensures that $Q_4^{j,1}-\{z\}$ has a hamiltonian path, still namely $\overline{T}_1$, between u and v. Obviously, there also exists a link (w,b) on $\overline{T}_1$ such that $(w,(w)^j),(b,(b)^j)$ and $((w)^j,(b)^j)$ are all fault-free. In summary, $\overline{T}_1$ can be written as $\langle u,T_1',w,b,T_1'',v\rangle$. Since $|F(Q_n^{j,0})|\leqslant 2n-7$, it follows from Theorem 8 that $Q_n^{j,0}-F(Q_n^{j,0})$ contains a path T_0 of length l_0 between $(w)^j$ and $(b)^j$ for any odd integer l_0 from 1 to $2^{n-1}-1$ excluding 3. As a result, $\langle u,T_1',w,(w)^j,T_0,(b)^j,b,T_1'',v\rangle$ is a path of any even length in the range from 2^{n-1} to 2^n-2, excluding $2^{n-1}+2$.

The path of length $2^{n-1}+2$ is discussed as follows. When $n=4$, $|F(Q_n^{j,0})|\leqslant 1$. Thus, there exists an integer k of $\{0,1,2,3\}-\{j,dim((w,b))\}$ such that $((w)^j,((w)^j)^k)$, $((b)^j,((b)^j)^k)$, and $(((w)^j)^k,((b)^j)^k)$ are all fault-free. Hence, $\langle u,T_1',w,(w)^j,((w)^j)^k,((b)^j)^k,(b)^j,b,T_1'',v\rangle$ is a path of length 10. When $n\geqslant 5$, we have $|A|-|F|=\lceil\frac{2^{n-1}-2}{2}\rceil-(2n-5)\geqslant 2$. Thus, there is another link (x,y) of A, other than (w,b), such that $(x,(x)^j),(y,(y)^j)$, and $((x)^j,(y)^j)$ are all fault-free. Without loss of generality, $\overline{T}_1$ can be written as $\langle u,R_1',w,b,R_1'',x,y,R_1''',v\rangle$. Hence, $\langle u,R_1',w,(w)^j,(b)^j,b,R_1'',x,(x)^j,(y)^j,R_1''',v\rangle$ is a path of length $2^{n-1}+2$.

Subcase II.1.3: Suppose that u is in $Q_n^{j,0}$ and v is in $Q_n^{j,1}$. By Theorem 2, there exists a shortest path P^* between u and v in Q_n-F such that P^* crosses the dimension j exactly once. Thus, P^* can be written as $\langle u,P_0,x,(x)^j,P_1,v\rangle$, where P_0 is a shortest path joining u to some node x in $Q_n^{j,0}-F(Q_n^{j,0})$ and P_1 is a shortest path joining $(x)^j$ to v in $Q_n^{j,1}-F(Q_n^{j,1})$.

Subcase II.1.3.1: Suppose that $\ell(P_0)>0$ and $\ell(P_1)>0$. By Theorem 6, $Q_n^{j,1}-F(Q_n^{j,1})$ has a path T_1 of length l_1 between $(x)^j$ and v for each l_1 satisfying $\ell(P_1)\leqslant l_1\leqslant 2^{n-1}-1$ and $2|(l_1-\ell(P_1))$. Suppose that $\ell(P_0)=1$. By Theorem 8, $Q_n^{j,0}-F(Q_n^{j,0})$ has a path T_0 of length l_0 between u and x for any odd integer l_0 from 1 to $2^{n-1}-1$ excluding 3. Suppose that $\ell(P_0)>1$. By the inductive hypothesis, $Q_n^{j,0}-F(Q_n^{j,0})$ has a path T_0 of length l_0 between u and x for each l_0 satisfying $\ell(P_0)\leqslant l_0\leqslant 2^{n-1}-1$ and $2|(l_0-\ell(P_0))$. Hence, $\langle u,T_0,x,(x)^j,T_1,v\rangle$ is a path of even length l_0+l_1+1 in the range from d^* to 2^n-2.

Subcase II.1.3.2: Suppose that $\ell(P_0)=0$ or $\ell(P_1)=0$. With symmetry, we assume $u=x$. By the inductive hypothesis, $Q_n^{j,1}-F(Q_n^{j,1})$ contains a path T_1 of length l_1 between $(u)^j$ and v for any odd integer l_1 form $\ell(P_1)$ to $2^{n-1}-1$. Then $\langle u,(u)^j,T_1,v\rangle$ is a path of even length l_1+1 in the range from $\ell(P_1)+1=d^*$ to 2^{n-1}.

The paths of lengths greater than 2^{n-1} are constructed as follows. Since $|V(Q_n^{j,0})-\{u\}|-(2n-5)>1$ for $n\geqslant 4$, we can choose a node y from $V(Q_n^{j,0})-\{u\}$ such that $(y,(y)^j)$ is fault-free and $(y)^j$ is not v. Let R_0 be a path joining u to y in $Q_n^{j,0}-F(Q_n^{j,0})$ and R_1 be a path joining $(y)^j$ to v in $Q_n^{j,1}-F(Q_n^{j,1})$. Similar to **Subcase II.1.3.1**, $H=\langle u,R_0,y,(y)^j,R_1,v\rangle$ is a path of even length in the range from $d'=d_{Q_n^{j,0}-F(Q_n^{j,0})}(u,y)+d_{Q_n^{j,1}-F(Q_n^{j,1})}((y)^j,v)+1$ to 2^n-2. By Corollary 3, we have $d'\leqslant(n+1)+(n-1)+1\leqslant 2^{n-1}+2$ for $n\geqslant 4$. Therefore, H is a path of even length in the range from $2^{n-1}+2$ to 2^n-2.

Subcase II.2: Suppose that $|F(Q_n^{j,0})|\leqslant 2n-6$ or $|F(Q_n^{j,1})|\leqslant 2n-6$. Without loss of generality, we assume that $|F(Q_n^{j,0})|=2n-6$. Thus, $Q_n^{j,1}$ is fault-free. It is noticed that the faulty links are distributed as shown in Fig. 2.

Subcase II.2.1: Suppose that both u and v are in $Q_n^{j,0}$. Let (w,b) be a faulty link of $Q_n^{j,0}$ such that both $(w,(w)^j)$ and $(b,(b)^j)$ are fault-free. Let $F_0=F(Q_n^{j,0})-\{(w,b)\}$. By the inductive hypothesis, $Q_n^{j,0}-F_0$ has a path P_l of length l between u and v for any even integer l from $d_{Q_n^{j,0}-F_0}(u,v)$ to $2^{n-1}-2$. If (w,b) is on P_l, we write P_l as $\langle u,P_l',w,b,P_l'',v\rangle$ and define $\widetilde{P}_l=\langle u,P_l',w,(w)^j,(b)^j,b,P_l'',v\rangle$. Otherwise, P_l can be written as $\langle u,P_l',x,y,P_l'',v\rangle$, where (x,y) is a link on P_l such that both $(x,(x)^j)$ and $(y,(y)^j)$ are fault-free. Similarly, we define $\widetilde{P}_l=\langle u,P_l',x,(x)^j,(y)^j,y,P_l'',v\rangle$. Then $\widetilde{P}_l$ is a path of length $l+2$. By Corollary 2, we have $d^*=d_{Q_n-F}(u,v)\leqslant h(u,v)+4$ and $d_{Q_n^{j,0}-F_0}(u,v)\leqslant h(u,v)+4$. If $d_{Q_n^{j,0}-F_0}(u,v)=d^*$, then path $\widetilde{P}_l$ is the desired path. Otherwise, if $d_{Q_n^{j,0}-F_0}(u,v)=d^*+2$, then $\widetilde{P}_l$ is a path of even length in the range from d^*+4 to 2^{n-1}. It is noticed that a shortest path between u and v in Q_n-F can be constructed based on a breadth-first search.

By Theorem 6, $Q_n^{j,1}$ contains a path T_1 of length l_1 between $(w)^j$ and $(b)^j$ or a path R_1 of odd length l_1 between $(x)^j$ and $(y)^j$ for any odd integer l_1 from 1 to $2^{n-1}-1$. Thus, $\langle u,P_{2^{n-1}-2}',w,(w)^j,T_1,(b)^j,b,P_{2^{n-1}-2}'',v\rangle$ (or $\langle u,P_{2^{n-1}-2}',x,(x)^j,R_1,(y)^j,y,P_{2^{n-1}-2}'',v\rangle$) is a path of even length in the range from 2^{n-1} to 2^n-2.

Subcase II.2.2: Suppose that both u and v are in $Q_n^{j,1}$. Let $(w,(w)^i)$ be a faulty link of $Q_n^{j,0}$ such that both $(w,(w)^j)$ and $((w)^i,((w)^i)^j)$ are fault-free. Since $n\geqslant 4$, we assume that $t\in\{0,1,\ldots,n-1\}-\{j,i\}$. Moreover, we assume that $w\in V_0(Q_n^{j,0})$. Let $X=\{((w)^j,((w)^j)^k)\mid k\notin\{i,j,t\}\}$. Since $|X|=n-3$, our inductive hypothesis ensures that $Q_n^{j,1}-X$ contains a path T_1 of even length l_1 between u and v for $d^*\leqslant l_1\leqslant 2^{n-1}-2$. Let $\overline{T}_1$ denote the longest path between u and v in $Q_n^{j,1}-X$. It is noted that $((w)^j,((w)^j)^i)$ is on $\overline{T}_1$. Hence, $\overline{T}_1$ can be represented as $\langle u,T_1',(w)^j,((w)^j)^i,T_1'',v\rangle$. By the inductive hypothesis, $Q_n^{j,0}-(F(Q_n^{j,0})-\{(w,(w)^i)\})$ contains a path T_0 of odd length l_0 between w to $(w)^i$ for $5\leqslant l_0\leqslant 2^{n-1}-1$. As a result, $\langle u,T_1',(w)^j,w,T_0,(w)^i,((w)^j)^i,T_1'',v\rangle$ is a path of even length $2^{n-1}+l_0-1$, in the range from $2^{n-1}+4$ to 2^n-2.

Let $A=\{(\overline{T}_1(k),\overline{T}_1(k+1))\mid 1\leqslant k\leqslant 2^{n-1}-1, k\equiv 1 \pmod 2\}$ be a set of disjoint links on $\overline{T}_1$. Then the paths of lengths 2^{n-1} and $2^{n-1}+2$ can be obtained as follows. When $n=4$, we suppose that $\{p,q,j,i\}=\{0,1,2,3\}$. Since $(w,(w)^i)$ is faulty, we have either $\{(w,(w)^p),((w)^p,((w)^p)^i),((w)^p)^i,(w)^i)\}\cap F=\emptyset$ or $\{(w,(w)^q),((w)^q,((w)^q)^i),((w)^q)^i,(w)^i,(w)^q)^i)\}\cap F=\emptyset$. Without loss

Table 2
The paths of lengths 10, 12, and 14 between $u = 0101$ and $v = 1001$ in $Q_4 - \{e_f, (0001, 0101), (0001, 1001)\}$.

$e_f \in \{(0000, 0010), (0010, 0011)\}$	$\langle u = 0101, 0100, 0110, 0111, 0011, 0001, 0000, 1000, 1100, 1101, 1001 = v\rangle$ $\langle u = 0101, 0100, 0110, 0111, 0011, 0001, 0000, 1000, 1100, 1110, 1111, 1101, 1001 = v\rangle$ $\langle u = 0101, 0100, 0110, 0111, 0011, 0001, 0000, 1000, 1100, 1110, 1010, 1011, 1111, 1101, 1001 = v\rangle$
$e_f = (0100, 0110)$	$\langle u = 0101, 0111, 0110, 0010, 0011, 0001, 0000, 1000, 1100, 1101, 1001 = v\rangle$ $\langle u = 0101, 0111, 0110, 0010, 0011, 0001, 0000, 1000, 1100, 1110, 1111, 1101, 1001 = v\rangle$ $\langle u = 0101, 0111, 0110, 0010, 0011, 0001, 0000, 1000, 1100, 1110, 1010, 1011, 1111, 1101, 1001 = v\rangle$
$e_f = (0110, 0111)$	$\langle u = 0101, 0111, 0011, 0010, 0110, 0100, 0000, 1000, 1100, 1101, 1001 = v\rangle$ $\langle u = 0101, 0111, 0011, 0010, 0110, 0100, 0000, 1000, 1100, 1110, 1111, 1101, 1001 = v\rangle$ $\langle u = 0101, 0111, 0011, 0010, 0110, 0100, 0000, 1000, 1100, 1110, 1010, 1011, 1111, 1101, 1001 = v\rangle$

of generality, we assume $\{(w, (w)^p), ((w)^p, ((w)^p)^i), ((w)^i, (w)^p)^i)\} \cap F = \emptyset$. Obviously, $\langle u, T_1', (w)^j, w, (w)^p, ((w)^p)^i, (w)^i, ((w)^j)^i, T_1'', v\rangle$ is a path of length $2^{n-1} + 2$. Moreover, since $|A| - |F| = \lceil \frac{2^{n-1}-1}{2} \rceil - (2n - 5) = 1$ for $n = 4$, there exists one link $(x, y) \in A$ such that $(x, (x)^j), (y, (y)^j)$, and $((x)^j, (y)^j)$ is fault-free. Hence, $\overline{T}_1$ can be represented as $\langle u, R_1, x, y, R_2, v\rangle$. Obviously, $\langle u, R_1, x, (x)^j, (y)^j, y, R_2, v\rangle$ is a path of length 2^{n-1}. When $n \geqslant 5$, we have $|A| - |F| = \lceil \frac{2^{n-1}-2}{2} \rceil - (2n - 5) \geqslant 2$. Thus, there are two links $(x_1, y_1), (x_2, y_2) \in A$ such that $\{(x_k, (x_k)^j), (y_k, (y_k)^j), ((x_k)^j, (y_k)^j) \mid k = 1, 2\} \cap F = \emptyset$. Hence, $\overline{T}_1$ can be represented as $\langle u, R_1, x_1, y_1, R_2, x_2, y_2, R_3, v\rangle$. Obviously, $\langle u, R_1, x_1, (x_1)^j, (y_1)^j, y_1, R_2, x_2, y_2, R_3, v\rangle$ and $\langle u, R_1, x_1, (x_1)^j, (y_1)^j, y_1, R_2, x_2, (x_2)^j, (y_2)^j, y_2, R_3, v\rangle$ are paths of length 2^{n-1} and of length $2^{n-1} + 2$, respectively.

Subcase II.2.3: Suppose that u is in $Q_n^{j,0}$ and v is in $Q_n^{j,1}$. If $(u, (u)^j)$ is fault-free, the shortest path between u and v can be of the form $\langle u, (u)^j, P_1, v\rangle$, where P_1 is a shortest path joining $(u)^j$ to v in $Q_n^{j,1}$. By the inductive hypothesis, $Q_n^{j,1}$ contains a path T_1 of odd length l_1 between $(u)^j$ and v for $d^* - 1 \leqslant l_1 \leqslant 2^{n-1} - 1$. Then $\langle u, (u)^j, T_1, v\rangle$ is a path of even length in the range from d^* to 2^{n-1}. If $(u, (u)^j)$ is faulty, we choose a neighbor of u in $Q_n^{j,0} - F(Q_n^{j,0})$, namely x, such that $(x)^j \neq v$. Obviously, we have either $h((x)^j, v) = h(u, v) - 2$ or $h((x)^j, v) = h(u, v)$. Let R_1 be a shortest path joining $(x)^j$ to v in $Q_n^{j,1}$. Then $\langle u, x, (x)^j, R_1, v\rangle$ is a path of length $h(u, v)$ or $h(u, v) + 2$. By Theorem 6, $Q_n^{j,1}$ contains a path T_1 of even length l_1 between $(x)^j$ and v for any even integer l_1 from $h((x)^j, v)$ to $2^{n-1} - 2$. Then $\langle u, x, (x)^j, T_1, v\rangle$ is a path of even length in the range from $d^* + 2$ to 2^{n-1}.

The paths of lengths greater than 2^{n-1} are obtained as follows. Let (w, b) be a faulty link in $Q_n^{j,0}$ such that both $(w, (w)^j)$ and $(b, (b)^j)$ are fault-free. Depending on whether $(v, (v)^j)$ is faulty, we distinguish two subcases.

Subcase II.2.3.1: Suppose that $(v, (v)^j)$ is fault-free. By the inductive hypothesis, $Q_n^{j,0} - (F(Q_n^{j,0}) - \{(w, b)\})$ contains a path H_0 of length $2^{n-1} - 1$ between u to $(v)^j$.

Subcase II.2.3.1.a: Suppose that (w, b) is not located on H_0. We choose a link (x, y) on H_0 such that $(x, (x)^j)$ and $(y, (y)^j)$ are fault-free and $((x)^j, (y)^j)$ is not incident with v. Thus, H_0 can be represented as $\langle u, H_0', x, y, H_0'', (v)^j\rangle$. By Lemma 3, $Q_n^{j,1} - \{v\}$ contains a path T_1 of odd length l_1 between $(x)^j$ and $(y)^j$ for any odd integer l_1 from 1 to $2^{n-1} - 3$. Consequently, $\langle u, H_0', x, (x)^j, T_1, (y)^j, y, H_0'', (v)^j, v\rangle$ is a path of even length $2^{n-1} + l_1 + 1$, in the range from $2^{n-1} + 2$ to $2^n - 2$.

Subcase II.2.3.1.b: Suppose that (w, b) is located on H_0 and (w, b) is not incident with $(v)^j$. Thus, H_0 can be represented as $\langle u, H_0', w, b, H_0'', (v)^j\rangle$. By Lemma 3, $Q_n^{j,1} - \{v\}$ contains a path T_1 of odd length l_1 between $(w)^j$ and $(b)^j$ for any odd integer l_1 from 1 to $2^{n-1} - 3$. Then $\langle u, H_0', w, (w)^j, T_1, (b)^j, b, H_0'', (v)^j, v\rangle$ is a path of even length $2^{n-1} + l_1 + 1$, in the range from $2^{n-1} + 2$ to $2^n - 2$.

Subcase II.2.3.1.c: Suppose that (w, b) is on H_0 and (w, b) is incident with $(v)^j$. Let $b = (v)^j$. Thus, H_0 can be written as $\langle u, H_0', w, b = (v)^j\rangle$. By Theorem 6, $Q_n^{j,1}$ has a path T_1 of odd length l_1 between $(w)^j$ and v for $1 \leqslant l_1 \leqslant 2^{n-1} - 1$. Thus, $\langle u, H_0', w, (w)^j, T_1, v\rangle$ is a path of even length $2^{n-1} + l_1 - 1$, in the range from 2^{n-1} to $2^n - 2$.

Subcase II.2.3.2: Suppose that $(v, (v)^j)$ is faulty. According to procedure Partition(Q_n, F, u, v), this subcase occurs only when $n = 4$ and there is a unique node z of $V_1(Q_4)$ such that both (z, u) and (z, v) are faulty links. In addition, each faulty link corresponds to a unique dimension. By transitivity, we assume that $z = 0001, u = 0101$, and $v = 1001$. Then the paths obtained by brute force are listed in Table 2.

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