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Chaos control of new Ikeda–Lorenz systems by GYC partial region stability theory

Zheng-Ming Ge*,†, Chun-Yen Ho, Shih-Yu Li and Ching-Ming Chang

Department of Mechanical Engineering, National Chiao Tung University, Hsinchu, Taiwan

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SUMMARY

A new strategy to achieve chaos control by GYC partial region stability theory is proposed. By using the GYC partial region stability theory, the Lyapunov function is a simple linear homogeneous function of error states, the controllers are more simple and have less simulation error because they are in lower degree than that of traditional controllers. Simulation results for a new Ikeda–Lorenz system show the effectiveness of this strategy. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: chaos control; partial region stability theory; Ikeda–Lorenz system; Genesio system

1. INTRODUCTION

Chaos, as an interesting nonlinear phenomenon, has been intensively investigated. It is well known that chaotic systems have sensitive dependence on initial conditions. A chaotic system is a nonlinear deterministic system that displays complex dynamical behaviors [1].

The theory of chaos control has developed since 1990 [2–4] and today is at the forefront of research in the field of nonlinear dynamics. Techniques have been experimentally implemented in mechanical [5], chemical [6], electronic [7], laser [8], communication [9] and biological [10] systems. Though there are now many different algorithms developed for the control of chaos for specific cases, in general all make use of typical properties of chaotic systems, namely, multiple coexisting solutions, sensitivity and ergodicity.

In this paper, a new chaos control strategy by GYC (Ge-Yao-Chen) partial region stability theory is proposed [11–13]. By using the GYC partial region stability theory, the Lyapunov function is a simple linear homogeneous function of error states and the controllers are more simple and have

*Correspondence to: Zheng-Ming Ge, Department of Mechanical Engineering, National Chiao Tung University, Hsinchu, Taiwan.
†E-mail: zmg@cc.nctu.edu.tw

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less simulation error because they are in lower degree than that of traditional controllers, for which the Lyapunov function is a quadratic form of error states.

This paper is organized as follows. In Section 2, chaos control strategy by GYC partial region stability theory is proposed. In Section 3, chaos of a new Ikeda–Lorenz system is given. In Section 4, simulation results are given for three examples. In Section 5, the superiority of new strategy is presented by the comparison between the simulation results of new strategy and that of traditional method. In Section 6, conclusions are drawn. In Appendix, GYC partial region stability theory is presented.

2. CHAOS CONTROL STRATEGY BY GYC PARTIAL REGION STABILITY THEORY

Consider the following chaotic systems:

$$\dot{\mathbf{x}}=\mathbf{f}(t,\mathbf{x}) \tag{1}$$

where $\mathbf{x}=[x_1,x_2,\ldots,x_n]^{\mathrm{T}}\in R^n$ is a state vector, $\mathbf{f}: R_+\times R^n\to R^n$ is a vector function.

The goal system, which can be either chaotic or regular, is

$$\dot{\mathbf{y}}=\mathbf{g}(t,\mathbf{y}) \tag{2}$$

where $\mathbf{y}=[y_1,y_2,\ldots,y_n]^{\mathrm{T}}\in R^n$ is a state vector and $\mathbf{g}: R_+\times R^n\to R^n$ is a vector function.

In order to make the chaos state $\mathbf{x}$ approach the goal state $\mathbf{y}$, define $\mathbf{e}=\mathbf{x}-\mathbf{y}$ as the state error. The chaos control is accomplished in the sense that [14–20]

$$\lim_{t\to\infty}\mathbf{e}=\lim_{t\to\infty}(\mathbf{x}-\mathbf{y})=0 \tag{3}$$

i.e. the controlled system of chaotic state $\mathbf{x}$ is synchronized with the goal system of chaotic or nonchaotic state $\mathbf{y}$.

By using GYC partial region stability theory, the positive definite Lyapunov function is a homogeneous linear function of error states and the controllers can be designed in lower order than that of traditional controllers, for which the Lyapunov function is a quadratic form of error states.

3. CHAOS OF A NEW IKEDA–LORENZ SYSTEM

A new Ikeda–Lorenz system is described as follows:

$$\begin{aligned}\dot{x}_1&=-a_1x_1-b_1\sin x_1+\sigma(x_2-x_1)\\ \dot{x}_2&=-a_2x_1-b_2\sin x_1+rx_1-x_1x_3-x_2\\ \dot{x}_3&=-a_3x_1-b_3\sin x_1+x_1x_2-cx_3\end{aligned} \tag{4}$$

which is a combination of Ikeda system [21–23] without time delay and Lorenz system. The parameters $a_1=0.1$, $b_1=1$, $\sigma=16$, $a_2=0.2$, $b_2=0.3$, $r=45.92$, $a_3=0.05$, $b_3=1.8$, and $c=4$ are used. The phase portrait and Lyapunov exponents of the new Ikeda–Lorenz system are shown in Figures 1 and 2.

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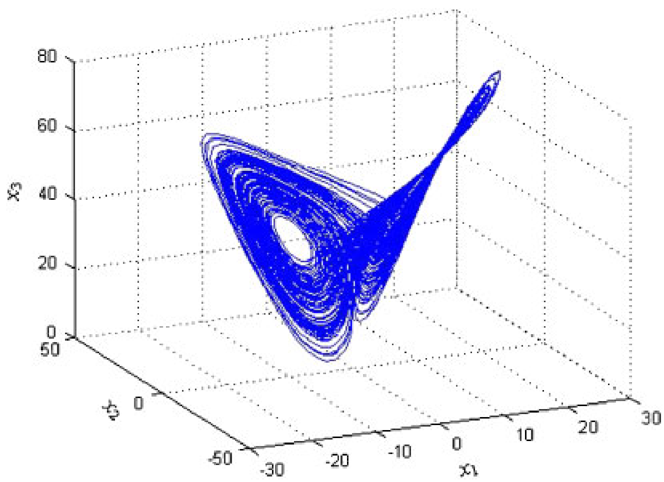

Figure 1. The phase portrait of new Ikeda–Lorenz system with parameters $a_1=0.1$, $b_1=1$, $\sigma=16$, $a_2=0.2$, $b_2=0.3$, $r=45.92$, $a_3=0.05$, $b_3=1.8$, $c=4$ and initial conditions $x_1(0)=1$, $x_2(0)=2$, $x_3(0)=3$.

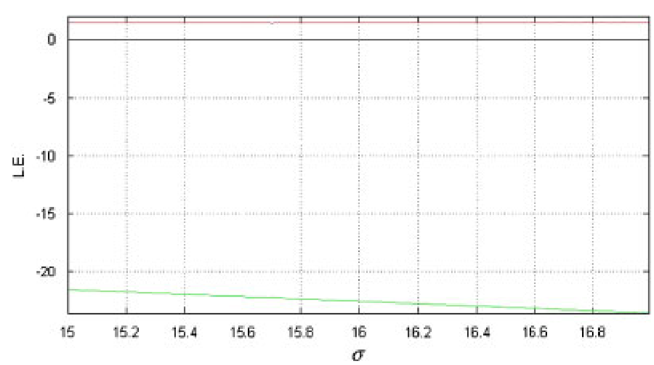

Figure 2. Lyapunov exponents of new Ikeda–Lorenz system with parameters $a_1=0.1$, $b_1=1$, $\sigma=16$, $a_2=0.2$, $b_2=0.3$, $r=45.92$, $a_3=0.05$, $b_3=1.8$, $c=4$ and initial conditions $x_1(0)=1$, $x_2(0)=2$, $x_3(0)=3$.

4. SIMULATION RESULTS

The following chaotic system is the Ikeda–Lorenz system of which the old origin is translated to $(x_1, x_2, x_3)=(150, 150, 150)$ and the chaotic motion always happens in the first quadrant of coordinate system (x_1, x_2, x_3):

$$\begin{aligned}
\dot{x}_1 &= -a_1(x_1-150)-b_1(\sin(x_1-150))+\sigma((x_2-150)-(x_1-150))\\
\dot{x}_2 &= -a_2(x_1-150)-b_2(\sin(x_1-150))+r(x_1-150)-(x_1-150)(x_3-150)-(x_2-150)\\
\dot{x}_3 &= -a_3(x_1-150)-b_3(\sin(x_1-150))+(x_1-150)(x_2-150)-c(x_3-150)
\end{aligned}\tag{5}$$

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This Ikeda–Lorenz system is presented as a simulated example where the initial conditions are $x_1(0)=1$, $x_2(0)=2$, $x_3(0)=3$, $a_1=0.1$, $b_1=1$, $\sigma=16$, $a_2=0.2$, $b_2=0.3$, $r=45.92$, $a_3=0.05$, $b_3=1.8$ and $c=4$.

In order to lead (x_1, x_2, x_3) to the goal, we add control terms u_1, u_2 and u_3 to each equation of Equation (5), respectively.

$$\begin{aligned}\dot{x}_1 &= -a_1(x_1-150)-b_1(\sin(x_1-150))+\sigma((x_2-150)-(x_1-150))+u_1\\ \dot{x}_2 &= -a_2(x_1-150)-b_2(\sin(x_1-150))+r(x_1-150)-(x_1-150)(x_3-150)-(x_2-150)+u_2\\ \dot{x}_3 &= -a_3(x_1-150)-b_3(\sin(x_1-150))+(x_1-150)(x_2-150)-c(x_3-150)+u_3\end{aligned} \tag{6}$$

Case I: *Control the chaotic motion to zero.*

In this case we will control the chaotic motion of the Ikeda–Lorenz system (5) to zero. The goal is $\mathbf{y}=0$. The state error is $\mathbf{e}=\mathbf{x}-\mathbf{y}=\mathbf{x}$ and error dynamics becomes

$$\begin{aligned}\dot{e}_1 &= \dot{x}_1 = -a_1(x_1-150)-b_1(\sin(x_1-150))+\sigma((x_2-150)-(x_1-150))+u_1\\ \dot{e}_2 &= \dot{x}_2 = -a_2(x_1-150)-b_2(\sin(x_1-150))+r(x_1-150)\\ &\qquad -(x_1-150)(x_3-150)-(x_2-150)+u_2\\ \dot{e}_3 &= \dot{x}_3 = -a_3(x_1-150)-b_3(\sin(x_1-150))+(x_1-150)(x_2-150)-c(x_3-150)+u_3\end{aligned} \tag{7}$$

In Figure 3, we see that the error dynamics always exists in first quadrant.

By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant as

$$V=e_1+e_2+e_3 \tag{8}$$

Its time derivative is

$$\begin{aligned}\dot{V} &= \dot{e}_1+\dot{e}_2+\dot{e}_3\\ &= -a_1(x_1-150)-b_1(\sin(x_1-150))+\sigma((x_2-150)-(x_1-150))+u_1\\ &\quad -a_2(x_1-150)-b_2(\sin(x_1-150))+r(x_1-150)-(x_1-150)(x_3-150)-(x_2-150)\\ &\quad +u_2-a_3(x_1-150)-b_3(\sin(x_1-150))+(x_1-150)(x_2-150)-c(x_3-150)+u_3\end{aligned} \tag{9}$$

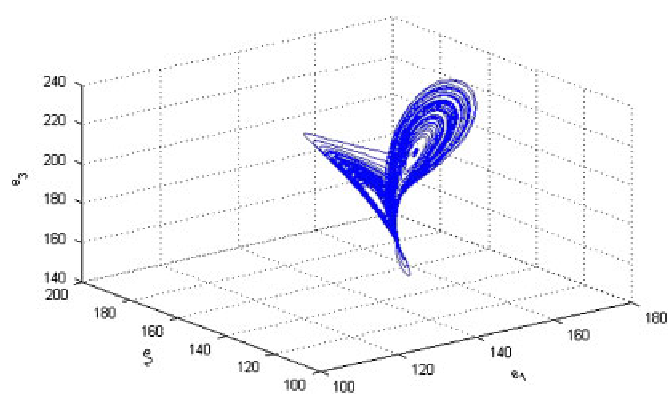

Figure 3. Phase portrait of error dynamics for *Case I*.

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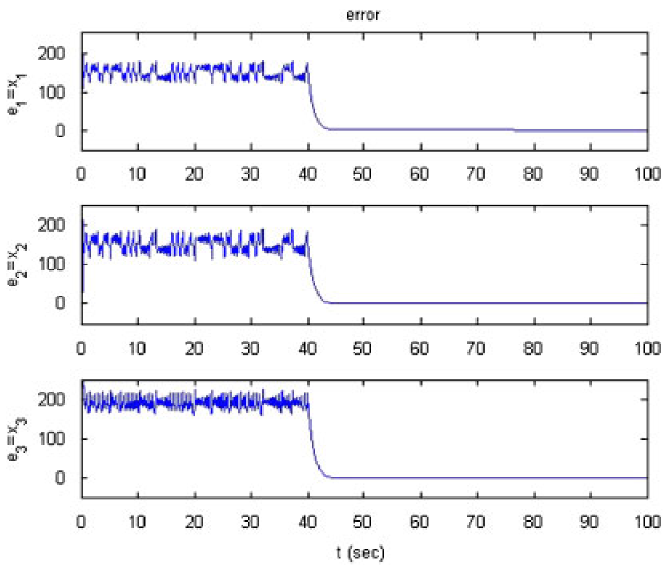

Figure 4. Time histories of errors for *Case I*.

Choose

$$\begin{aligned} u_1 &= -[-a_1(x_1-150)-b_1(\sin(x_1-150))+\sigma((x_2-150)-(x_1-150))]-e_1 \\ u_2 &= -[-a_2(x_1-150)-b_2(\sin(x_1-150))+r(x_1-150) \\ &\quad -(x_1-150)(x_3-150)-(x_2-150)]-e_2 \\ u_3 &= -[-a_3(x_1-150)-b_3(\sin(x_1-150))+(x_1-150)(x_2-150)-c(x_3-150)]-e_3 \end{aligned} \tag{10}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 < 0 \tag{11}$$

which is a negative definite function in first quadrant. Simulation results are shown in Figure 4. The motion trajectories approach the origin after 40 s.

If the traditional quadratic Lyapunov function

$$V = \tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2) \tag{12}$$

is used. Its time derivative is

$$\dot{V} = \dot{e}_1 e_1 + \dot{e}_2 e_2 + \dot{e}_3 e_3 \tag{13}$$

The three controllers u_1', u_2', u_3' are chosen to make

$$\dot{V} = -e_1^2 - e_2^2 - e_3^2 \tag{14}$$

Comparing with Equation (10), we see that every term of u_1', u_2', u_3' is of higher degree than the corresponding term of u_1, u_2, u_3. Therefore, u_1', u_2', u_3' are more complex than u_1, u_2, u_3. The situation is the same for the following *Cases II* and *III*.

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Case II: *Control the chaotic motion to a periodic function*.

In this case, we will control the chaotic motion of the Ikeda–Lorenz system (5) to a periodic function of time. The goal is $y = F\sin^2\omega t$. Error e becomes

$$e = x - F\sin^2\omega t \tag{15}$$

It is demanded that

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x - F_i\sin^2\omega_i t) = 0, \quad i = 1,2,3 \tag{16}$$

and $\dot{e}_i = \dot{x}_i - F_i\omega_i\sin 2\omega_i t$, $(i = 1,2,3)$, where $F_1 = F_2 = F_3 = 10$, $\omega_1 = 0.5$, $\omega_2 = 2$, $\omega_3 = 0.2$.

The error dynamics is

$$\begin{aligned}
\dot{e}_1 = \dot{x}_1 - F\omega_1\sin 2\omega_1 t &= -a_1(x_1-150) - b_1(\sin(x_1-150)) + \sigma((x_2-150)\\
&\quad -(x_1-150)) - F\omega_1\sin 2\omega_1 t + u_1\\
\dot{e}_2 = \dot{x}_2 - F\omega_2\sin 2\omega_2 t &= -a_2(x_1-150) - b_2(\sin(x_1-150)) + r(x_1-150)\\
&\quad -(x_1-150)(x_3-150) - (x_2-150) - F\omega_2\sin 2\omega_2 t + u_2\\
\dot{e}_3 = \dot{x}_3 - F\omega_3\sin 2\omega_3 t &= -a_3(x_1-150) - b_3(\sin(x_1-150)) + (x_1-150)(x_2-150)\\
&\quad -c(x_3-150) - F\omega_3\sin 2\omega_3 t + u_3
\end{aligned} \tag{17}$$

Let the initial states be $(x_1(0), x_2(0), x_3(0)) = (1,2,3)$ and system parameters be $a_1 = 0.1$, $b_1 = 1$, $\sigma = 16$, $a_2 = 0.2$, $b_2 = 0.3$, $r = 45.92$, $a_3 = 0.05$, $b_3 = 1.8$ and $c = 4$. We find that the error dynamics always exists in first quadrant as shown in Figure 5. By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant as

$$V = e_1 + e_2 + e_3 \tag{18}$$

Its time derivative is

$$\begin{aligned}
\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3\\
&= [-a_1(x_1-150) - b_1(\sin(x_1-150)) + \sigma((x_2-150)-(x_1-150)) - F\cdot\omega_1\cdot\sin 2\omega_1 t + u_1]\\
&\quad + [-a_2(x_1-150) - b_2(\sin(x_1-150)) + r(x_1-150) - (x_1-150)(x_3-150)\\
&\quad - (x_2-150) - F\cdot\omega_2\cdot\sin 2\omega_2 t + u_2] + [-a_3(x_1-150) - b_3(\sin(x_1-150))\\
&\quad + (x_1-150)(x_2-150) - c(x_3-150) - F\cdot\omega_3\cdot\sin 2\omega_3 t + u_3]
\end{aligned} \tag{19}$$

Choose

$$\begin{aligned}
u_1 &= -[-a_1(x_1-150) - b_1(\sin(x_1-150)) + \sigma((x_2-150)-(x_1-150)) - F\omega_1\sin 2\omega_1 t] - e_1\\
u_2 &= -[-a_2(x_1-150) - b_2(\sin(x_1-150)) + r(x_1-150) - (x_1-150)(x_3-150)\\
&\quad - (x_2-150) - F\omega_2\sin 2\omega_2 t] - e_2\\
u_3 &= -[-a_3(x_1-150) - b_3(\sin(x_1-150)) + (x_1-150)(x_2-150) - c(x_3-150)\\
&\quad - F\omega_3\sin 2\omega_3 t] - e_3
\end{aligned} \tag{20}$$

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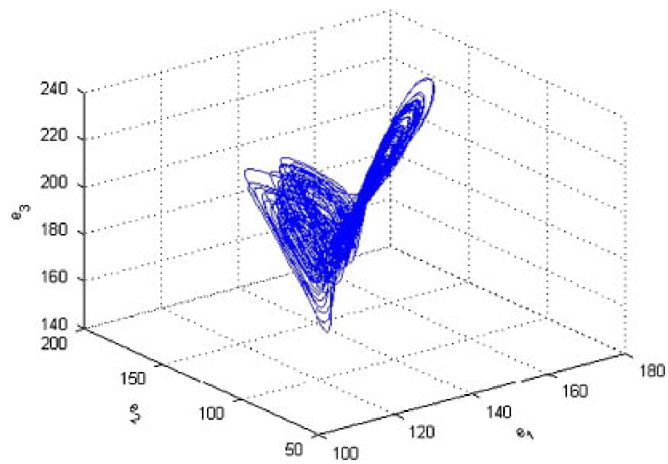

Figure 5. Phase portrait of error dynamics for *Case II*.

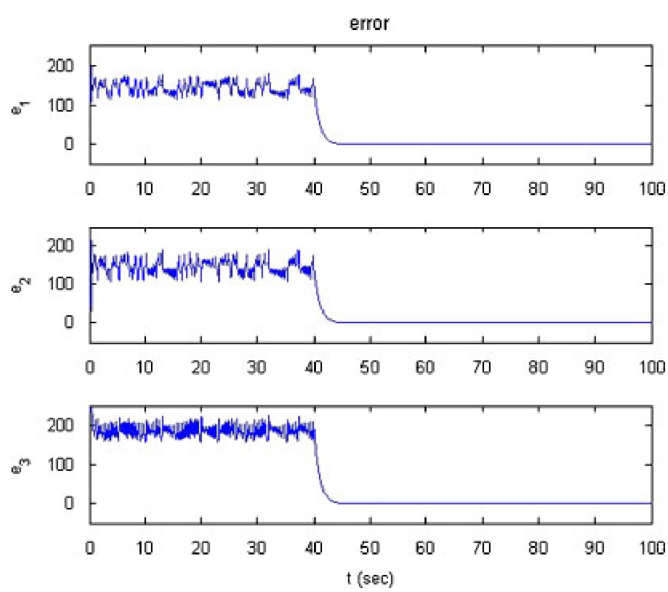

Figure 6. Time histories of errors for *Case II*.

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 < 0 \tag{21}$$

which is a negative definite function in first quadrant. The numerical results are shown in Figures 6 and 7. After 40 s, the errors approach zero and the motion trajectories approach to the periodic functions.

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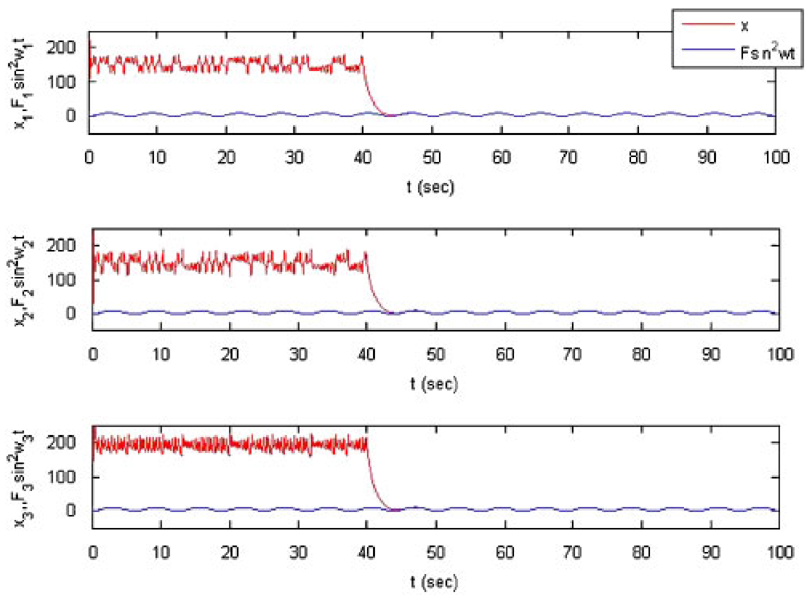

Figure 7. Time histories of errors for *Case II*.

Case III: *Control the chaotic motion of Ikeda–Lorenz system to chaotic motion of a chaotic Genesio system* [24].

In this case we will control chaotic motion of Ikeda–Lorenz system (3) to that of a chaotic Genesio system. The goal system is Genesio system

$$\begin{aligned}
\dot{z}_1 &= z_2\\
\dot{z}_2 &= z_3\\
\dot{z}_3 &= z_1^2 - a_4 z_1 - b_4 z_2 - c_4 z_3
\end{aligned} \tag{22}$$

For $z_1(0)=1$, $z_2(0)=1$, $z_3(0)=1$, $a_4=6$, $b_4=2.92$, $c_4=1.2$, the system is chaotic [21].

The error equation is $e=x-z$; our aim is $\lim_{t\to\infty} \mathbf{e}=0$.

The error dynamics become

$$\begin{aligned}
\dot{e}_1 = \dot{x}_1-\dot{z}_1 &= -a_1(x_1-150)-b_1(\sin(x_1-150))+\sigma((x_2-150)-(x_1-150))-z_2+u_1\\
\dot{e}_2 = \dot{x}_2-\dot{z}_2 &= -a_2(x_1-150)-b_2(\sin(x_1-150))+r(x_1-150)-(x_1-150)(x_3-150)\\
&\quad -(x_2-150)-z_3+u_2\\
\dot{e}_3 = \dot{x}_3-\dot{z}_3 &= -a_3(x_1-150)-b_3(\sin(x_1-150))+(x_1-150)(x_2-150)-c(x_3-150)\\
&\quad -(z_1^2-a_4 z_1-b_4 z_2-c_4 z_3)+u_3
\end{aligned} \tag{23}$$

Let the initial states be $(x_1(0), x_2(0), x_3(0))=(1,2,3)$, $(z_1(0), z_2(0), z_3(0))=(1,1,1)$ and system parameters $a_1=0.1$, $b_1=1$, $\sigma=16$, $a_2=0.2$, $b_2=0.3$, $r=45.92$, $a_3=0.05$, $b_3=1.8$, $c=4$, $a_4=6$,

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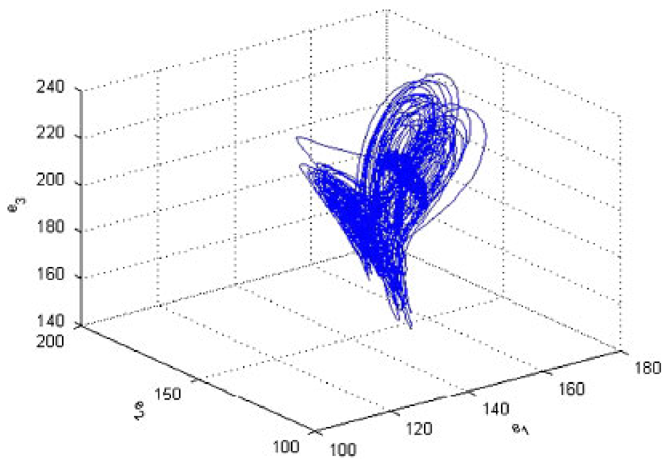

Figure 8. Phase portrait of error dynamics for *Case III*.

$b_4 = 2.92$ and $c_4 = 1.2$. We find that the error dynamics always exists in first quadrant as shown in Figure 8.

By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant

$$V = e_1 + e_2 + e_3 \tag{24}$$

Its time derivative is

$$\begin{aligned}\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3\\ &= [-a_1(x_1-150) - b_1(\sin(x_1-150)) + \sigma((x_2-150)-(x_1-150)) - z_2 + u_1]\\ &\quad +[-a_2(x_1-150) - b_2(\sin(x_1-150)) + r(x_1-150) - (x_1-150)(x_3-150)\\ &\quad -(x_2-150) - z_3 + u_2] + [-a_3(x_1-150) - b_3(\sin(x_1-150)) + (x_1-150)(x_2-150)\\ &\quad -c(x_3-150) - (z_1^2 - a_4 z_1 - b_4 z_2 - c_4 z_3) + u_3]\end{aligned} \tag{25}$$

Choose

$$\begin{aligned}u_1 &= -[-a_1(x_1-150) - b_1(\sin(x_1-150)) + \sigma((x_2-150)-(x_1-150)) - z_2] - e_1\\ u_2 &= -[-a_2(x_1-150) - b_2(\sin(x_1-150)) + r(x_1-150) - (x_1-150)(x_3-150)\\ &\quad -(x_2-150) - z_3] - e_2\\ u_3 &= -[-a_3(x_1-150) - b_3(\sin(x_1-150)) + (x_1-150)(x_2-150) - c(x_3-150)\\ &\quad -(z_1^2 - a_4 z_1 - b_4 z_2 - c_4 z_3)] - e_3\end{aligned} \tag{26}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 < 0 \tag{27}$$

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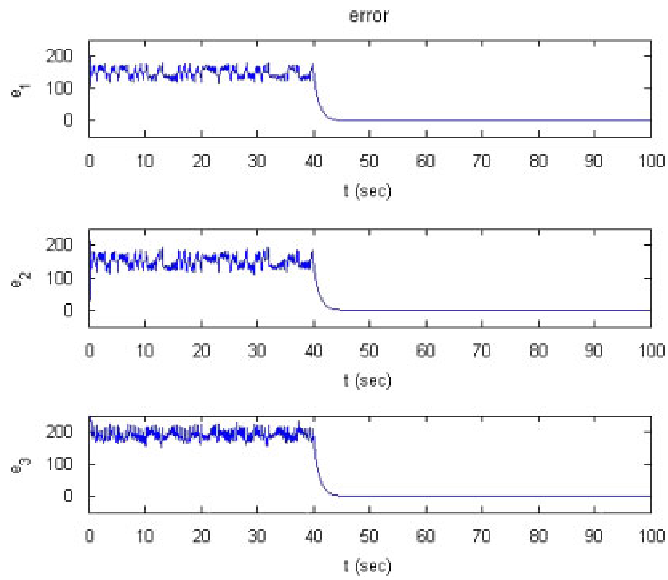

Figure 9. Time histories of errors for *Case III*.

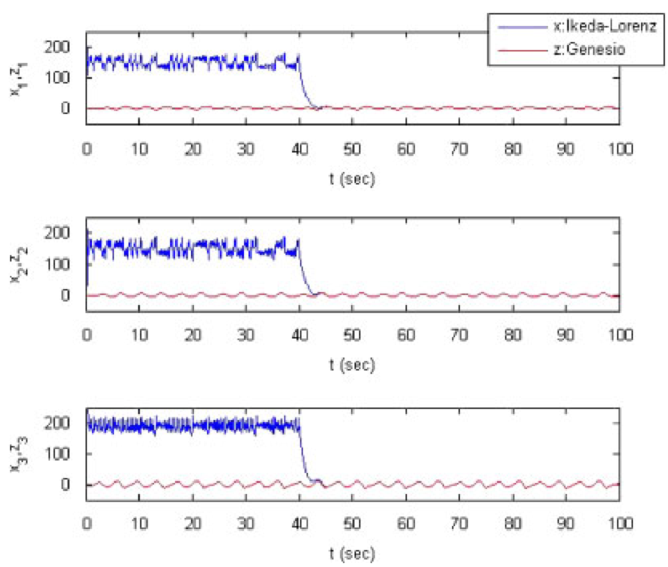

Figure 10. Time histories of errors for *Case III*.

which is a negative definite function in first quadrant. The numerical results are shown in Figures 9 and 10. After 40 s, the errors approach zero and the chaotic trajectories of Ikeda–Lorenz system approach to the chaotic trajectories of the Genesio system.

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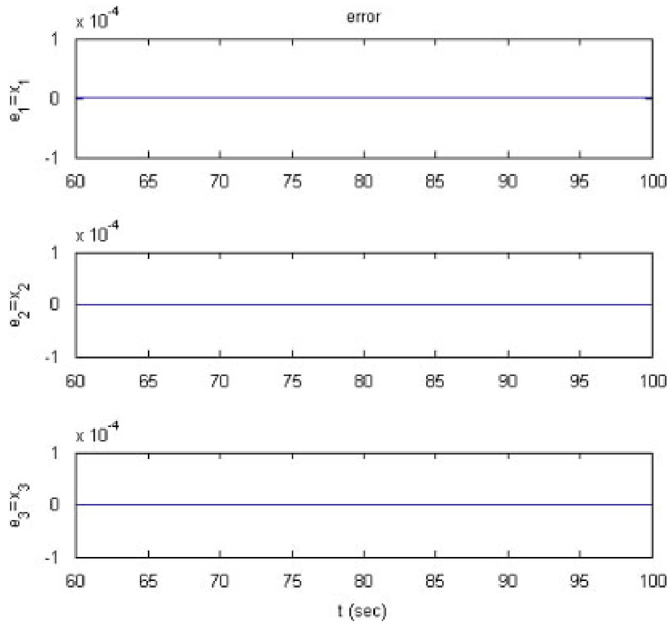

Figure 11. Time histories of errors of using GYC partial region stability theory for *Case I*.

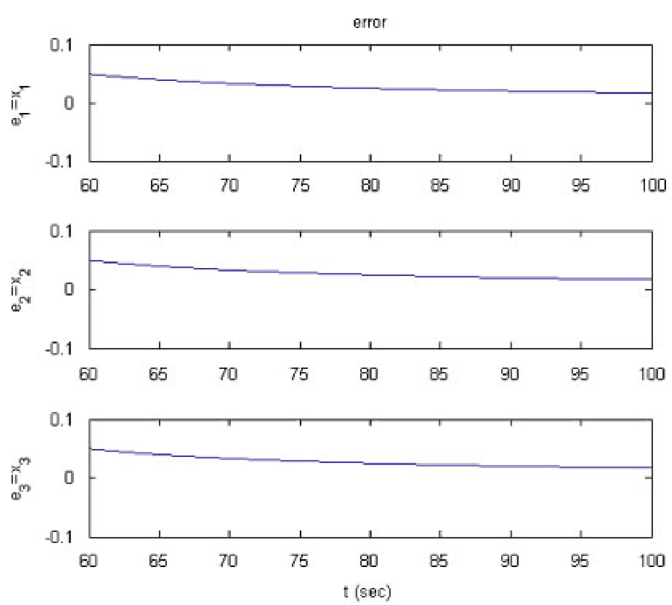

Figure 12. Time histories of errors of using traditional controller design method for *Case I*.

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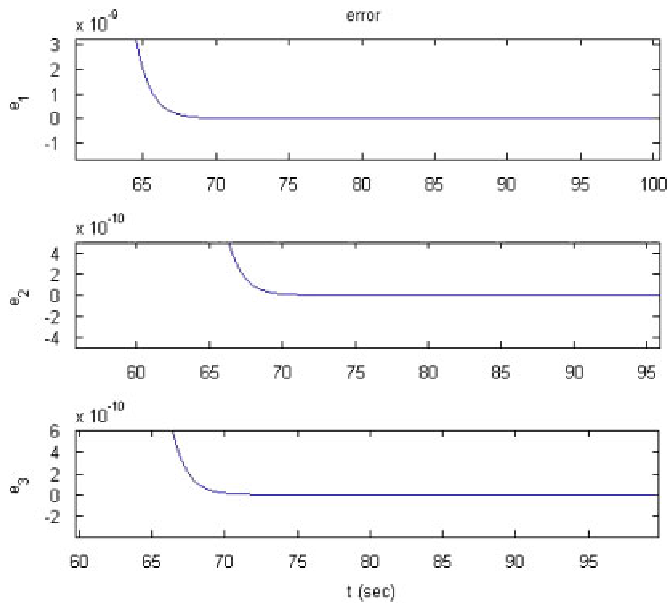

Figure 13. Time histories of errors of using GYC partial region stability theory for *Case II*.

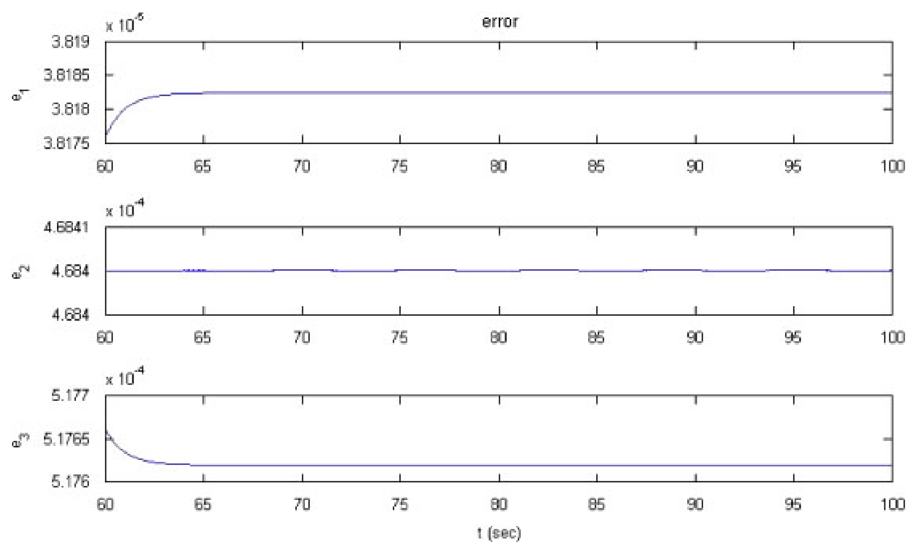

Figure 14. Time histories of errors of using traditional controller design method for *Case II*.

5. COMPARISON BETWEEN NEW STRATEGY AND TRADITIONAL METHOD

In the last part of *Case I* in Section 4, it is shown that in new strategy, $V=e_1+e_2+e_3$ and $\dot{V}=-e_1-e_2-e_3$, while in traditional method, $V=\frac{1}{2}(e_1^2+e_2^2-e_3^2)$ and $\dot{V}=-e_1^2-e_2^2-e_3^2$; therefore,

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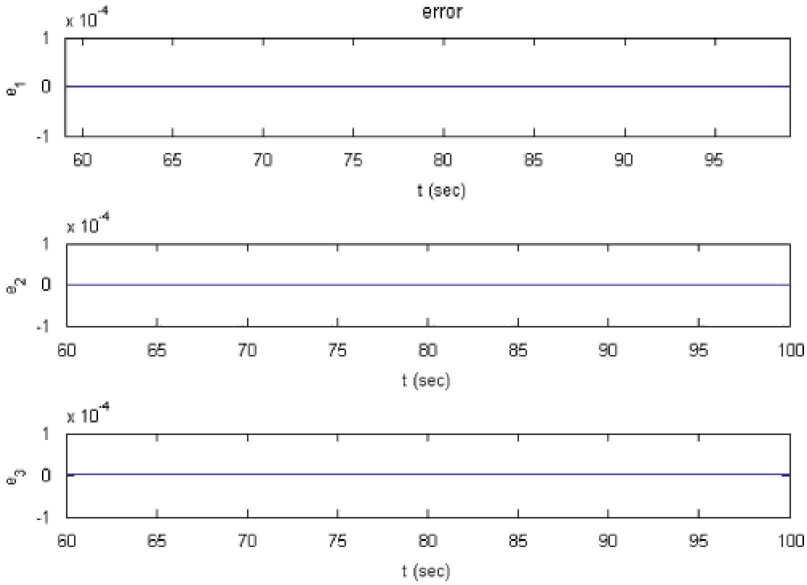

Figure 15. Time histories of errors of using GYC partial region stability theory for *Case III*.

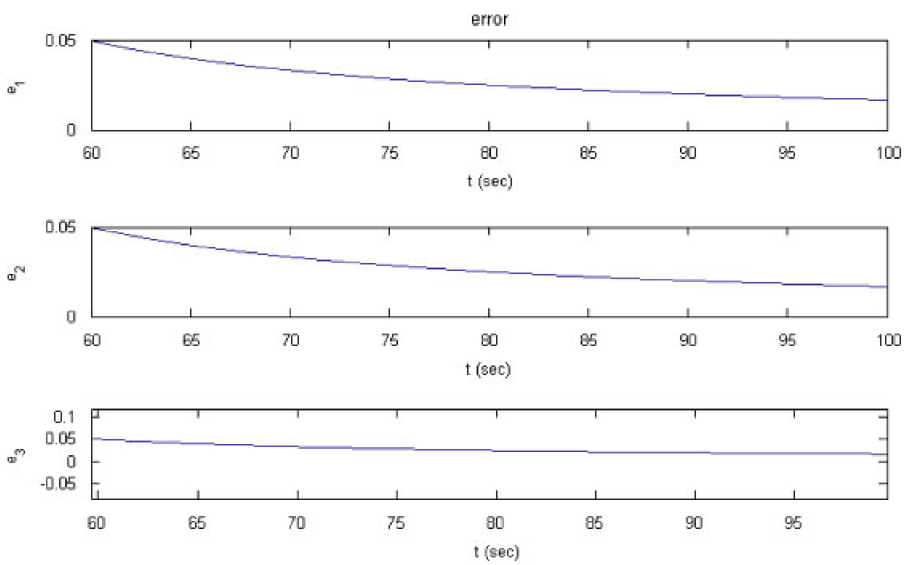

Figure 16. Time histories of errors of using traditional controller design method for *Case III*.

the controllers u_1, u_2, u_3 are more simple than the traditional controllers u_1', u_2', u_3', and will give less simulation errors. This conclusion can be proved by the following simulation results.

In Figures 11 and 12, it is presented clearly that the errors from traditional method are several thousand times of that from new strategy. In Figures 13–16, the results are similar.

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Table I. Comparison between error data at 99.66, 99.67, 99.68, 99.69 s after the action of controllers.

Error for new strategy	Error for traditional method
Case I: e_1	
0.000000000000001	0.016710430555974
0.000000000000001	0.016707638637622
0.000000000000001	0.016704847652041
0.000000000000001	0.016702057598765
Case I: e_2	
0.000000000001818	0.016730794379602
0.000000000001818	0.016727995653045
0.000000000001818	0.016725197862674
0.000000000001818	0.016722401008018
Case I: e_3	
0.000000000001807	0.016754609605322
0.000000000001807	0.016751802906143
0.000000000001807	0.016748997147152
0.000000000001807	0.016746192327877
Case II: e_1	
−0.000000000000022	0.000038182369977
−0.000000000000022	0.000038182369978
−0.00000000000002	0.000038182369979
−0.000000000000019	0.00003818236998
Case II: e_2	
0.00000000000001	0.000468399999968
0.000000000000004	0.000468399999969
−0.000000000000007	0.00046839999997
0.000000000000012	0.000468399999971
Case II: e_3	
0.00000000000001	0.000517619999975
0.000000000000004	0.000517619999975
−0.000000000000007	0.000517619999977
0.000000000000012	0.000517619999978
Case III: e_1	
0.000000000000001	0.016721740924575
0.000000000000001	0.016718945225869
0.000000000000001	0.016716150461829
0.000000000000001	0.016713356631987
Case III: e_2	
0.000000000001761	0.016733574425884
0.000000000001758	0.016730774769236
0.000000000001761	0.01672797604924
0.000000000001776	0.016725178265425
Case III: e_3	
0.000000000001761	0.016751501937327
0.000000000001759	0.016748696279146
0.000000000001768	0.016745891560629
0.000000000001755	0.016743087781305

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Furthermore, in Table I, comparison between error data is given. All data are picked from 99.66 to 99.69 s with sampling time 0.01 s. From these data, the superiority of new strategy is obvious.

6. CONCLUSIONS

In this paper, a new strategy to achieve chaos control by GYC partial region stability is proposed. By using the GYC partial region stability theory, the Lyapunov function is a simple linear homogeneous function of error states, the controllers are more simple and have less simulation error because they are in lower degree than that of traditional controllers. The new Ikeda–Lorenz system and Genesio system are used as simulation examples that effectively confirm the chaos control scheme. Comparison between the errors from new strategy and that from traditional method shows the superiority of new strategy.

APPENDIX A: GYC PARTIAL REGION STABILITY THEORY [11–13]

Consider the differential equations of disturbed motion [25, 26] of a nonautonomous system in the normal form

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) \quad (s = 1, \ldots, n) \tag{A1}$$

where the function X_s is defined on the intersection of the partial region Ω (shown in Figure A1) and

$$\sum_s x_s^2 \leqslant H \tag{A2}$$

and $t > t_0$, where t_0 and H are certain positive constants. X_s, which vanishes when the variables x_s are all zero, is a real-valued function of $t, x_1, \ldots, x_n$. It is assumed that X_s is smooth enough

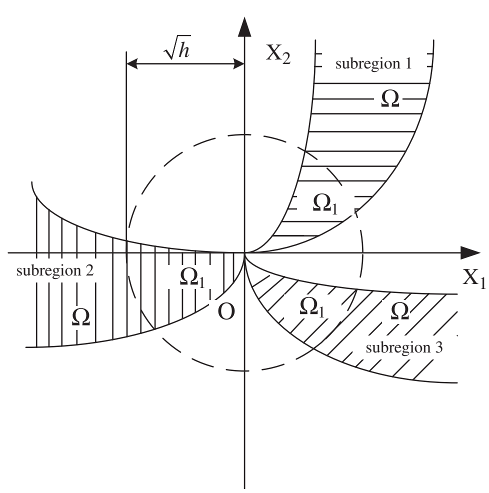

Figure A1. Partial regions Ω and Ω_1.

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to ensure the existence, uniqueness of the solution of the initial value problem. When X_s does not contain t explicitly, the system is autonomous.

Obviously, $x_s = 0$ $(s = 1, \dots, n)$ is a solution of Equation (A1). We are interested in the stability and asymptotical stability of this zero solution on partial region Ω (including the boundary) of the neighborhood of the origin, which in general may consist of several subregions (Figure A1).

Definition A1
For any given number $\varepsilon > 0$, if there exists a $\delta > 0$ such that on the closed given partial region Ω when

$$\sum_s x_{s0}^2 \leqslant \delta \quad (s = 1, \dots, n) \tag{A3}$$

where x_{s0} $(s = 1, \dots, n)$ are initial values of x_s, for all $t \geqslant t_0$, the inequality

$$\sum_s x_s^2 < \varepsilon \quad (s = 1, \dots, n) \tag{A4}$$

is satisfied for the solutions of Equation (A1) on Ω, then the undisturbed motion $x_s = 0$ $(s = 1, \dots, n)$ is stable on the partial region Ω.

Definition A2
If the undisturbed motion is stable on the partial region Ω, and there exists a $\delta' > 0$, so that on the given partial region Ω when

$$\sum_s x_{s0}^2 \leqslant \delta' \quad (s = 1, \dots, n) \tag{A5}$$

the equality

$$\lim_{t \to \infty} \left(\sum_s x_s^2 \right) = 0 \tag{A6}$$

is satisfied for the solutions of Equation (A1) on Ω, then the undisturbed motion $x_s = 0$ $(s = 1, \dots, n)$ is asymptotically stable on the partial region Ω.

The intersection of Ω and region defined by Equation (A5) is called the region of attraction.

Definition of functions $V(t, x_1, \dots, x_n)$
Let us consider the functions $V(t, x_1, \dots, x_n)$ given on the intersection Ω_1 of the partial region Ω and the region

$$\sum_s x_s^2 \leqslant h \quad (s = 1, \dots, n) \tag{A7}$$

for $t \geqslant t_0 > 0$, where t_0 and h are positive constants. We suppose that the functions are single-valued and have continuous partial derivatives and become zero when $x_1 = \cdots = x_n = 0$.

Definition A3
If there exists $t_0 > 0$ and a sufficiently small $h > 0$, so that on partial region Ω_1 and $t \geqslant t_0$, $V \geqslant 0$ (or $\leqslant 0$), then V is a positive (or negative) semidefinite, in general semidefinite, function on Ω_1 and $t \geqslant t_0$.

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Definition A4
If there exists a positive (negative) definitive function $W(x_1, \ldots, x_n)$ on Ω_1, so that on the partial region Ω_1 and $t \geqslant t_0$

$$V - W \geqslant 0 \quad (\text{or } -V - W \geqslant 0) \tag{A8}$$

then $V(t, x_1, \ldots, x_n)$ is a positive definite function on the partial region Ω_1 and $t \geqslant t_0$.

Definition A5
If $V(t, x_1, \ldots, x_n)$ is neither definite nor semidefinite on Ω_1 and $t \geqslant t_0$, then $V(t, x_1, \ldots, x_n)$ is an indefinite function on the partial region Ω_1 and $t \geqslant t_0$. That is, for any small $h > 0$ and any large $t_0 > 0$, $V(t, x_1, \ldots, x_n)$ can take either positive or negative value on the partial region Ω_1 and $t \geqslant t_0$.

Definition A6 (*Bounded function* V)
If there exist $t_0 > 0$, $h > 0$, so that on the partial region Ω_1, we have

$$|V(t, x_1, \ldots, x_n)| < L$$

where L is a positive constant, then V is said to be bounded on Ω_1.

Definition A7 (*Function with infinitesimal upper bound*)
If V is bounded and for any $\lambda > 0$, there exists $\mu > 0$, so that on Ω_1 when $\sum_s x_s^2 \leqslant \mu$ and $t \geqslant t_0$, we have

$$|V(t, x_1, \ldots, x_n)| \leqslant \lambda$$

then V admits an infinitesimal upper bound on Ω_1.

Theorem A1
If there can be found for the differential equations of the disturbed motion (Equation (A1)) a definite function $V(t, x_1, \ldots, x_n)$ on the partial region, and its derivative with respect to time based on these equations as given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} X_s \tag{A9}$$

is a semidefinite function on the partial region whose sense is opposite to that of V, or it becomes zero identically, then the undisturbed motion is stable on the partial region.

Proof
Let us assume for the sake of definiteness that V is a positive definite function. Consequently, there exists a sufficiently large number t_0 and a sufficiently small number $h < H$, such that on the intersection Ω_1 of partial region Ω and

$$\sum_s x_s^2 \leqslant h \quad (s = 1, \ldots, n)$$

and $t \geqslant t_0$, the following inequality is satisfied:

$$V(t, x_1, \ldots, x_n) \geqslant W(x_1, \ldots, x_n)$$

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where W is a certain positive definite function, which does not depend on t. Besides this, Equation (A9) may assume only negative or zero value in this region.

Let ε be an arbitrarily small positive number. We shall suppose that in any case $\varepsilon<h$. Let us consider the aggregation of all possible values of the quantities $x_1, \ldots, x_n$, which are on the intersection ω_2 of Ω_1 and

$$\sum_s x_s^2 = \varepsilon \tag{A10}$$

and let us designate by $l>0$, the precise lower limit of the function W under this condition. By virtue of Equation (A8), we shall have

$$V(t, x_1, \ldots, x_n) \geqslant l \quad \text{for } (x_1, \ldots, x_n) \text{ on } \omega_2 \tag{A11}$$

We shall now consider the quantities x_s as functions of time, which satisfy the differential equations of disturbed motion. We shall assume that the initial values x_{s0} of these functions for $t=t_0$ lie on the intersection Ω_2 of Ω_1 and the region

$$\sum_s x_s^2 \leqslant \delta \tag{A12}$$

where δ is so small that

$$V(t_0, x_{10}, \ldots, x_{n0}) < l \tag{A13}$$

By virtue of the fact that $V(t_0, 0, \ldots, 0) = 0$, such a selection of the number δ is obviously possible. We shall suppose that in any case the number δ is smaller than ε. Then the inequality

$$\sum_s x_s^2 < \varepsilon \tag{A14}$$

being satisfied at the initial instant will be satisfied, in the very least, for a sufficiently small $t-t_0$, since the functions $x_s(t)$ vary continuously with time. We shall show that these inequalities will be satisfied for all values $t>t_0$. Indeed, if these inequalities were not satisfied at some time, there would have to exist such an instant $t=T$ for which this inequality would become an equality. In other words, we would have

$$\sum_s x_s^2(T) = \varepsilon$$

and consequently, on the basis of Equation (A11)

$$V(T, x_1(T), \ldots, x_n(T)) \geqslant l \tag{A15}$$

On the other hand, since $\varepsilon<h$, the inequality (Equation (A7)) is satisfied in the entire interval of time $[t_0, T]$ and, consequently, in this entire time interval $\mathrm{d}V/\mathrm{d}t \leqslant 0$. This yields

$$V(T, x_1(T), \ldots, x_n(T)) \leqslant V(t_0, x_{10}, \ldots, x_{n0})$$

which contradicts Equation (A14) on the basis of Equation (A13). Thus, the inequality Equation (A4) must be satisfied for all values of $t>t_0$; hence, follows that the motion is stable. □

Finally, we must point out that from the view-point of mathematics, the stability on partial region in general should not be related logically to the stability on whole region. If an undisturbed

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solution is stable on a partial region, it may be either stable or unstable on the whole region. In specific practical problems, we do not study the solution starting from Ω_2 and going out of Ω.

Theorem A2
If in satisfying the conditions of Theorem A1, the derivative $\mathrm{d}V/\mathrm{d}t$ is a definite function on the partial region with opposite sign to that of V and the function V itself permits an infinitesimal upper bound, then the undisturbed motion is asymptotically stable on the partial region.

Proof
Let us suppose that V is a positive definite function on the partial region and that consequently, $\mathrm{d}V/\mathrm{d}t$ is negative definite. Thus, on the intersection Ω_1 of Ω and the region defined by Equation (A7) and $t \geqslant t_0$, there will be satisfied not only the inequality (Equation (A8)), but the following inequality as well:

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant -W_1 \quad (x_1, \ldots, x_n) \tag{A16}$$

where W_1 is a positive definite function on the partial region independent of t.

Let us consider the quantities x_s as functions of time that satisfy the differential equations of disturbed motion assuming that the initial values $x_{s0} = x_s(t_0)$ of these quantities satisfy the inequalities (Equation (A12)). Since the undisturbed motion is stable in any case, the magnitude δ may be selected so small that for all values of $t \geqslant t_0$, the quantities x_s remain within Ω_1. Then, on the basis of Equation (A16), the derivative of function $V(t, x_1(t), \ldots, x_n(t))$ will be negative at all times and, consequently, this function will approach a certain limit, as t increases without limit, remaining larger than this limit at all times. We shall show that this limit is equal to some positive quantity different from zero. Then for all values of $t \geqslant t_0$, the following inequality will be satisfied:

$$V(t, x_1(t), \ldots, x_n(t)) > \alpha \tag{A17}$$

where $\alpha > 0$.

Since V permits an infinitesimal upper bound, it follows from this inequality that

$$\sum_s x_s^2(t) \geqslant \lambda \quad (s = 1, \ldots, n) \tag{A18}$$

where λ is a certain sufficiently small positive number. Indeed, if such a number λ did not exist, that is, if the quantity $\sum_s x_s^2(t)$ were smaller than any preassigned number no matter how small, then the magnitude $V(t, x_1(t), \ldots, x_n(t))$, as follows from the definition of an infinitesimal upper bound, would also be arbitrarily small, which contradicts Equation (A17).

If for all values of $t \geqslant t_0$ the inequality (Equation (A18)) is satisfied, then Equation (A16) shows that the following inequality will be satisfied at all times:

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant -l_1$$

where l_1 is a positive number different from zero, which constitutes the precise lower limit of the function $W_1(t, x_1(t), \ldots, x_n(t))$ under condition (Equation (A18)). Consequently, for all values of $t \geqslant t_0$ we shall have

$$V(t, x_1(t), \ldots, x_n(t)) = V(t_0, x_{10}, \ldots, x_{n0}) + \int_{t_0}^{t} \frac{\mathrm{d}V}{\mathrm{d}t} \mathrm{d}t \leqslant V(t_0, x_{10}, \ldots, x_{n0}) - l_1(t - t_0)$$

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which is, obviously, in contradiction with Equation (A17). The contradiction thus obtained shows that the function $V(t, x_1(t), \ldots, x_n(t))$ approaches zero as t increases without limit. Consequently, the same will be true for the function $W(x_1(t), \ldots, x_n(t))$ as well, from which it follows directly that

$$\lim_{t \to \infty} x_s(t) = 0 \quad (s = 1, \ldots, n)$$

which proves the theorem. □

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