Time-Optimal Control of T–S Fuzzy Models via Lie Algebra

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***Abstract*—This paper investigates a geometric property of time-optimal problem in the Takagi–Sugeno (T–S) fuzzy model via Lie algebra. We will focus on the existence of a time-optimal solution, singularity of switching function, and number of switching. These inherent problems are considered because of their rich geometric properties. The sufficient condition for the existence of a time-optimal solution reveals the controllability of T–S fuzzy model that can be found by the generalized rank condition. The time-optimal controller can be found as the bang–bang type with a finite number of switching by applying the maximum principle. In the study of the singularity problem, we will focus on the switching function whenever it vanishes over a finite time interval. Finally, we show that the bounded number of switching can be found if the T–S model (also a nonlinear system) is solvable.**

***Index Terms*—Controllability, fuzzy control, Lie algebras, Takagi–Sugeno (T–S) fuzzy model, time-optimal control.**

I. Introduction

In RECENT years, fuzzy logic control with human knowledge of the plant has witnessed an effective approach to the design of nonlinear control systems. Indeed, there have been many successful applications that are based on fuzzy control [1]–[8]. Takagi and Sugeno [9] proposed an approach to model nonlinear processes. This type of model is known as the T–S model that is further developed in [10]. The T–S fuzzy model blends the dynamics of each fuzzy implication by a linear consequence part [11]–[13]. In this type of fuzzy model, lots of important issues are addressed such as stability [2], [8], [11], performance [13]–[15], and robustness [16]–[18], etc. In [19], a fuzzy approach is used in the design of time-suboptimal feedback controllers.

The T–S fuzzy model has a strong connection with the polytopic linear differential inclusion (PLDI) [36], [37] that will lead to the relaxed version of T–S fuzzy model defined in this paper. The equivalence between the fuzzy model and the differential inclusion is revealed by the well-known Filippov's selection lemma [36], [37]. From Filippov's selection lemma, the set of solutions of T–S fuzzy model coincides with the set of solutions of the differential inclusion. By formulating the T–S fuzzy model as a relaxed version, we can perform some algebraic operations on it, such as linear combinations and the Lie bracket product.

The maximum principle has been extensively applied in many time-optimal control problems [20]–[35]. A series of results has been published on the applications of the maximum principle in time-optimal control of finite-dimensional linear systems and certain low-order nonlinear systems [21]–[23]. It is well known that Lie brackets play an essential role in the study of time-optimal control [31]–[35]. In general, the maximum principle can reduce the optimal control problem by the Hamiltonian. However, the Hamiltonian formulation contains no information about the existence of a time-optimal solution. It is better to convert the existence of a time-optimal solution to the study of reachable sets [25], [26], [28]. While the existence of a time-optimal solution is addressed as the compactness of a researchable set, we still have to generalize the analytical process, and this will lead us to the discussion of Lie algebra. An accessible Lie algebra spans a family of analytical vector fields that will imply the controllability of T–S fuzzy model.

Using the maximum principle, a time-optimal trajectory combined with the corresponding control is called an extremal. The bounded input is determined by the signs of the associated switching functions. The singularity of the system is a well-known problem in time-optimal control that was explored in [27] and [31]. An optimal trajectory may be singular, i.e., switching functions may vanish along the trajectory. The characterization of such trajectories will be investigated in this paper. The existence of extremal will imply that the time-optimal controller of the T–S fuzzy model has a finite number for switching, which can be found by Lie algebra in this paper.

This paper is organized as follows. In Section II, we will formulate the time-optimal problem in T–S fuzzy model. In Section III, the T–S fuzzy model is described as a polytopic linear differential inclusion and Lie algebra is adopted to find the controllability of T–S fuzzy model. It can also be shown that if the T–S fuzzy model is controllable, then the time-optimal does exist. Assuming the existence of a time-optimal solution, we will investigate the singular structure in fuzzy model in Section IV. The optimal trajectory is solved by the shooting method, and the numerical illustrations are provided in Section V. Finally, conclusions are included in Section VI.

II. Problem Formulation and Preliminaries

A. Problem Formulation

Consider a nonlinear control-affine system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

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where $x \in X$ is the system state, and u is the control input in an arbitrary set U. The state space X is a smooth differential manifold of dimension n. The vector fields f and g are assumed to be analytic.

In many situations, a fuzzy model with the human knowledge can provide a linguistic description of the nonlinear system in terms of IF–THEN rules. The ith rule of the T–S fuzzy model is described by the following form:

Rule i: IF $z_1(t)$ is $M_{i1} \cdots$ and $z_p(t)$ is M_{ip}, THEN

$$\dot{x} = A_i x + B_i u$$

where x denotes system states, taking values in an open subset X of $\mathbb{R}^n, u \in \mathbb{R}^m$ is a measurable bounded function on U, i is the number of IF–THEN rules, $z_i(t)$ are some fuzzy input variables, M_{ij} are fuzzy membership functions in the ith rule, and $\dot{x} = A_i x + B_i u$ is the output from the ith IF–THEN rule. The entire fuzzy model is formulated as follows:

$$\dot{x} = \sum_{i=1}^{r} \mu_i \left(z(t)\right) \left(A_i x + B_i u\right) \tag{2}$$

where r is the total number of rules, $\mu_i\left(z(t)\right)$ is the normalized membership function, and $\mu_i\left(z(t)\right) = \alpha_i / \sum_{i=1}^{r} \alpha_i$, where α_i is the firing strength of the ith rule such that $\alpha_i = \prod_{j=1}^{p} M_{ij}\left(z_j(t)\right)$.

The relaxed version of T–S fuzzy model is described by

$$\dot{x} \in \text{Co}\left\{[A_i x + B_i u] \mid i = 1, \ldots, r\right\} \tag{3}$$

where Co denotes a convex hull [36]. If the T–S fuzzy model is continuous and the control input U is compact, the set of solutions of (2) coincides with the set of solutions of (3) [36], [37], i.e.,

$$\text{Co}\{[A_i x + B_i u] \mid i = 1, \ldots, r\} \supseteq \sum_{i=1}^{r} \mu_i\left(z(t)\right)\left(A_i x + B_i u\right).$$

Therefore, we represent the T–S fuzzy model by (3) as

$$\dot{x} = \sum_{i=1}^{r} \mu_i(t)\left(A_i x + B_i u\right) \tag{4}$$

where $\mu_i(t) \in [0,1]$, and $\sum_{i=1}^{r} \mu_i(t) = 1$. To simplify the notion, we adopt $\sum A_i = \sum_{i=1}^{r} \mu_i(t) A_i, \sum B_i = \sum_{i=1}^{r} \mu_i(t) B_i$, and the jth column vector of $\sum B_i$ denoted as $\sum b_j = \sum_{i=1}^{r} \mu_i(t) B_{ij}, j = 1, \ldots, m$ and these are assumed to be linearly independent. Throughout the rest of this paper, the T–S fuzzy model is denoted as

$$\dot{x} = \sum A_i x + \sum B_i u. \tag{5}$$

In general, the variable $z(t)$ in (2) sometimes is chosen as the state variable $x(t)$, thus defuzzification $\mu_i\left(z(t)\right)$ causes (2) to become a class of nonlinear systems. This leads to difficultly in performing differential algebra on (2). To avoid this problem, such a T–S fuzzy model (5) is introduced to allow us to perform differential algebra on it.

In this paper, we will make the following assumptions on the control input.

Assumption 1: The control input is given by

$$U = \left\{u \in \mathbb{R}^m \mid a_j \le u_j \le b_j,\, j = 1, \ldots, m\right\}.$$

For a given control $u(t) \subset U$ on a time interval $[0, t_1]$ and any initial point $x(t_0) = x_0 \in X$, let $x(., x_0, u)$ denote the solution of the fuzzy model (5) with a measurable control u defined on an interval of $[0, t_1]$. For performing optimality on a segment $[0, t_1]$, we introduce a cost functional

$$J(u) = \int_0^{t_1} \varphi\left(x(t), u(t)\right) dt. \tag{6}$$

Let $x_0 \in X$ be an initial point and $x_1 \in X$ be a final point. We propose the following optimal control problem in terms of the cost functional J.

Problem 1: Find a control $u(t) \in U$ that minimizes (6) along the solution of (5) and satisfies the boundary condition

$$x(t_1, x_0, u) = x_1. \tag{7}$$

We note that this problem is well posed, i.e., an optimal control does exist. The intuitive interpretation of Problem 1 is clear: find a control that will push the initial state to a given final condition in a given amount of time.

B. Preliminaries

For the nonlinear control-affine system (1), the corresponding *Lie bracket* of two smooth vector fields f and g is denoted by $[f, g]$, and

$$[f, g](x) = \frac{\partial f}{\partial x} g(x) - \frac{\partial g}{\partial x} f(x)$$

where $\partial f / \partial x$ and $\partial g / \partial x$ denote the Jacobi matrices of their vector fields. The iterated Lie bracket of f and g is defined as

$$\text{ad}\,(f)^k\,(g)\,(x) = [f, \text{ad}\,(f)^{k-1}\, g]\,(x) \tag{8}$$

where $\text{ad}\,(f)^0\,(g) := g$, and $k \ge 1$. The Lie algebra generated by the vector fields can be expressed as

$$\begin{aligned}\mathcal{L} &= \{f, g_1, \ldots, g_m\}_{LA}\\ &= \text{span}\{[g_{i_1} \cdots [g_{i_{k-1}}, g_{i_k}] \cdots] \mid k \ge 1,\ 0 \le i_1, \ldots, i_k \le m\}\end{aligned}$$

where $g_0 = f$. Since the control-affine system can be represented by a family of vector fields, this will have direct applications to control systems. Considering a T–S fuzzy model with a compact set of control inputs U, the Lie bracket taken at a point of an analytic family of vector fields forms a complete set of its invariants. In particular, $\mathcal{L}(p_0)$ denotes the space of tangent vectors at p_0 defined by the Lie algebra. Due to the fact that $f = g_0 = \sum A_i x, g_1 = \sum b_1, \ldots, g_m = \sum b_m$, and that the Lie bracket of constant vector fields is zero, the iterated Lie bracket can be found as

$$\text{ad}\left(\sum A_i x\right)^k \sum b_j = \left[\sum A_i x, \text{ad}\left(\sum A_i x\right)^{k-1} \sum b_j\right]. \tag{9}$$

A Lie algebra $\mathcal{L}$ is recursively defined by

$$\mathcal{L}^{(1)} = [\mathcal{L}, \mathcal{L}]$$
$$\mathcal{L}^{(2)} = [\mathcal{L}^{(1)}, \mathcal{L}^{(1)}], \ldots, \mathcal{L}^{(k)} = [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}], \ldots$$

is called solvable if $\mathcal{L}^{(k)} = 0$ for large k, i.e., $\mathcal{L}^{(k)} \supset \mathcal{L}^{(k+1)}$. Furthermore, the Lie algebra $\mathcal{L}$ is called nilpotent if the sequence of $\mathcal{L}$ is always decreasing with respect to

$$\mathcal{L}^1 = \mathcal{L}, \mathcal{L}^2 = [\mathcal{L}, \mathcal{L}^1], \ldots, \mathcal{L}^k = [\mathcal{L}, \mathcal{L}^{k-1}], \ldots$$

and $\mathcal{L}^k = 0$. Any nilpotent Lie algebra is solvable. More details can be found in [38]. In the following, we will introduce notions and results that play a basic role in analyzing the structure of nonlinear control systems. They are directly related to controllability properties of nonlinear systems. In the following, we denote X as an n-dimensional C^∞ manifold.

Definition 1: Let $T_x X$ be a subspace of the tangent space at any point $x \in X$. A distribution Δ on X is a map that is

$$x \in X \to \Delta(x) \subset T_x X.$$

The distribution Δ is a smooth subspace of $\mathbb{R}^n$ to each point x. The dimension of Δ, in general, is not a constant. If the dimension is constant in a neighborhood of x, then x is said to be a regular point of the distribution. If any point of the distribution is regular with dimension k, the distribution is said to be regular and the dimension of the distribution is k.

Definition 2: A distribution $\Delta(x)$ is called *involutive* if for any two vector fields $f, g \in \Delta(x)$, their Lie bracket $[f, g] \in \Delta(x)$.

For convenience, the following Theorems 1–3 are listed here that are adapted from [35]–[39].

Theorem 1 (Chow's Theorem) [35]: Let $\mathcal{F}$ be a set of C^∞ vector fields on X and $\mathcal{L} = \{\lambda_0, \lambda_1, \ldots, \lambda_k\}_{LA}$ be the Lie algebra generated by $\mathcal{F}$. If $\dim(\mathcal{L}(x)) = n$ for all $x \in X$, then any point of X is reachable by trajectory of the vector fields $\mathcal{F}$. Thus

$$x_1 = e_{t_L}^{\lambda_L} \circ \cdots \circ e_{t_1}^{\lambda_1}(x_0)$$

for some $L \geq 1, \{\lambda_0, \lambda_1, \ldots, \lambda_k\} \in \mathcal{F}$ and $t_1, \ldots, t_L \in (0, \infty)$.

The following well-known theorem of Frobenius is characterized by the integrable distribution.

Theorem 2 (Frobenius' Theorem) [38]: If X is a C^ω (regular) manifold of dimension n and Δ is an involutive distribution, then around any point $x \in X$, there exists a largest integral manifold of Δ passing through x.

Remark 1: A distribution Δ is said to be integrable if there exists a submanifold S on X such that for any $x \in X$

$$\Delta(x) = T_x S$$

where S is passing through x.

Remark 2: Any analytic involutive distribution Δ is integrable [39].

Theorem 3 [39]: Let $\mathcal{F}$ be a set of C^ω vector fields on X and $\mathcal{L} = \{\lambda_0, \lambda_1, \ldots, \lambda_k\}_{LA}$ be the Lie algebra generated by $\mathcal{F}$. For all $x \in X$, there exists a largest integral manifold of $\mathcal{F}$ passing through x.

The proof of Theorem 3 can be found by using the Campbell–Baker–Hausdorff formula and Theorem 2.

In the following section, the T–S fuzzy model (5) associated with the Lie algebra is derived to show the controllability condition and imply the existence of optimal control.

III. Existence of Optimal Control in the T–S Fuzzy Model

We begin with the formal definition of reachability and controllability.

Definition 3: The *reachable set* $\mathcal{R}(x)$ of the T–S fuzzy model (5) for time $t \geq 0$, subject to the initial condition $x \in X$ is the set

$$\mathcal{R}_T(x) = \{x(t, u) : x \in X \text{ and } u : [0, T] \mapsto U\}.$$

Definition 4: The T–S fuzzy model (5) is *accessible* if its reachable set $\mathcal{R}_T(x), x \in X$ has a nonempty interior. Similarly, we will call this T–S fuzzy model *strongly accessible* if the reachable set $\mathcal{R}_T(x)$ has the nonempty interior for any $T > 0$.

Definition 5: The T–S fuzzy model (5) is controllable if $\forall x_0$ and $\forall x_1$ in the manifold of X, there exists a finite time T and an admissible control function $u : [0, T]$ such that $x(T; x_0, u) = x_1$.

In the following, we shall show that the existence of optimal solution of Problem 1 can be reduced to determine the accessibility of the *reachable set*. The qualitative properties of the reachable sets can be established. One of the basic properties can be shown in the following context.

Definition 6: For the T–S fuzzy model (5), the *accessibility Lie algebra* is defined as

$$\mathcal{L}_a := \left\{\sum A_i x, \sum b_j \mid \forall j = 1, \ldots, m\right\}_{LA}. \tag{10}$$

The $\mathcal{L}_a$ is a finite-dimensional Lie algebra of vector fields that contains the family $\{\sum A_i x, \sum b_j\}$. In fact, this *accessibility Lie algebra* plays basic role in the controllability of a T–S fuzzy model.

Theorem 4: If the *accessibility Lie algebra* of the T–S fuzzy model in (5) is full rank at x, that is

$$\operatorname{rank}(\mathcal{L}_a(x)) = n \qquad \forall x \in \mathbb{R}^n \tag{11}$$

then the reachable setup to any time $T > 0$ has the nonempty interior, and therefore, the fuzzy model is strongly accessible.

Proof: According to Chow's theorem [35], the reachable set $\mathcal{R}(x)$ is the largest integral manifold of $\mathcal{L}_a$ for $\forall x \in \mathbb{R}^n$. From (11), it contains an open neighborhood Ω of x. This implies that for any x_0, its reachable set is an open set. We can conclude that the reachable set $\mathcal{R}(x)$ is arcwise connected and spans into the $\mathbb{R}^n$ space. Q.E.D.

Remark 3: Since the T–S fuzzy model (5) is analytic, using Chow's theorem [35] and Frobenius' theorem [38], the manifold X represents the maximal-connected reachable manifolds. Each reachable manifold is the maximal integral manifold of $\mathcal{L}_a$.

Remark 4: By using Chow's theorem [35], the controllable manifolds can be spanned from $\{\sum A_i x, \sum b_j | \forall j = 1, \ldots, m\}$.

Remark 5: The $\mathcal{L}_a$ implies that the T–S fuzzy model (5) is accessible from x_0 if the same collection of vectors together with $\sum A_i x_0 + \sum B_i u$ span the whole space. This condition means that no vector $\sum B_i u$ belongs to a proper invariant subspace of $\sum A_i x_0$.

Theorem 5: If a T–S fuzzy model is strongly accessible, then it is also controllable.

Proof: Using Remark 3, for a T–S fuzzy model, the degree of largest integral manifold is related to rank of *accessibility Lie algebra* $\mathcal{L}_a$. As the fuzzy model is *strongly accessible*, there exists the nth-degree largest integral manifold. For a given point $x \in \mathbb{R}^n$, the fuzzy model is controllable. Q.E.D.

In the following, the generalized rank condition of accessible Lie algebra is derived to show the controllability of the T–S fuzzy model.

Corollary 1: The T–S fuzzy model (5) is controllable if and only if the following matrix

$$(W_0, W_1, \ldots, W_{n-1}) := \left(\sum b_j, \sum A_i \sum b_j, \ldots \times \left(\sum A_i\right)^{n-1} \sum b_j\right), \quad j = 1, \ldots, m \tag{12}$$

is of rank n for any $t > 0$.

Proof: First, we give the proof of sufficient part. Considering the T–S fuzzy model (5), let $f = g_0 = \sum A_i x$, and $g_1 = \sum b_j$ be a vector field. Then, we have the following iterated Lie brackets:

$$\left[\sum A_i x, \sum b_j\right] = -\sum A_i \sum b_j, \left[\sum A_i x, \left[\sum A_i x, \sum b_j\right]\right] = \sum A_i^2 \sum b_j, \ldots.$$

From (9), the iterated Lie brackets are rewritten as

$$\text{ad}\left(\sum A_i x\right)^l \sum b_j = \left((-1)\sum A_i\right)^l \sum b_j.$$

Therefore, the *accessibility Lie algebra* $\mathcal{L}_a$ consists of constant vector fields only

$$\mathcal{L}_a = \text{Span}\left\{\left(\sum A_i\right)^l \left(\sum b_j\right) \middle| l \geq 0, \quad j = 1, \ldots, m\right\}. \tag{13}$$

If (12) is satisfied, we can conclude that dim $(\mathcal{L}_a)$ is of full rank n for any $t > 0$, and then, the fuzzy model is controllable.

Next, we show the necessary condition. From the Frobenius' theorem [38] and Remark 3, it follows that if the T–S fuzzy model (5) is controllable, then there exits the *n*th-degree largest integral manifold for $x \in X$. If (12) is satisfied from Theorem 3 and Remark 1, there exists a largest integral *n*th-order submanifold S that is unique and contained in the largest integral manifold. Q.E.D.

Remark 6: In analyzing controllability properties of the fuzzy model (5), we can replace the set of $G(x) = \{A_i x + B_i u : u \in U, i = 1, \ldots, r\}$ by its convex hull, and the trajectories of convexified system can be approximated by the trajectories of the original fuzzy model (2). In particular, if $0 \in \text{intCo}\{G(x)\}$ for all $x \in X$, then the fuzzy model is controllable.

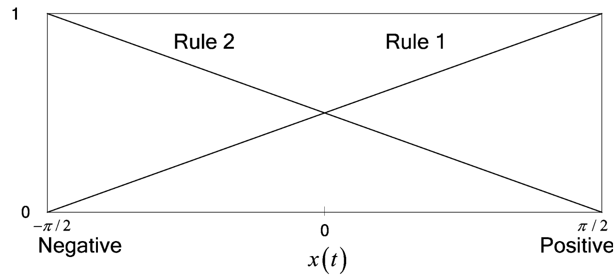

Fig. 1. Membership functions in Example 1.

Remark 7: Obviously, for the single-rule T–S fuzzy model, Corollary 1 degenerates to the Kalman controllability matrix of the linear system.

Remark 8: If all the subsystems are controllable, whereas the overall system cannot be concluded as controllable, then the overall system can be called *local controllable*.

The membership functions obviously play the critical roles in the controllability of the system. In the following examples, the local controllability and controllability of the T–S fuzzy model will be illustrated. The nonlinear system will be modeled with the distinct membership functions.

Example 1: Consider a nonlinear system

$$\dot{x} = \tan(u)$$
$$\dot{y} = 10\sin(x)\cos(x).$$

Assume that $x(t) \in [-\pi/2, \pi/2]$. Then, the T–S fuzzy model of the nonlinear system can be formulated as follows:

Rule i: IF $x(t)$ is about "Positive" and "Negative," THEN

$$\dot{X}(t) = A_i X(t) + B_i u, \qquad i = 1, 2 \tag{14}$$

where $X(t) = [x(t) \quad y(t)]^T$

$$A_1 = \begin{bmatrix} 0 & 0 \\ 10\beta & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ -10\beta & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $\beta = \cos(88°)$. The membership functions are shown in Fig. 1. According to Corollary 1, the corresponding rank of controllability matrix of the fuzzy model is

$$\text{Rank}\left(\sum b_j, \sum A_i \sum b_j\right)$$

where

$$W_0 = \sum b_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix} W_1 = \sum A_i \sum b_j$$
$$= \left(\mu_1 \begin{bmatrix} 0 & 0 \\ 0.349 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 \\ -0.349 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 \\ 0.349(\mu_1 - \mu_2) \end{bmatrix}.$$

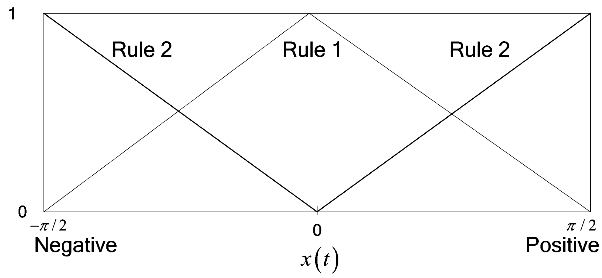

Fig. 2. Membership functions in Example 2.

The fuzzy model is controllable if Rank $([W_0, W_1]) = 2$. We can check the controllability by the following determinant:

$$\left|\begin{bmatrix} 1 & 0 \\ 0 & 0.349\,(\mu_1 - \mu_2) \end{bmatrix}\right| = 0.349\,(\mu_1 - \mu_2)\,.$$

Unfortunately, the rank of $[W_0, W_1]$ for $\mu_1 = \mu_2 = 0.5$ is 1. From the membership functions, we can observe that the fuzzy model is uncontrollable if $x(t) = 0$. Although $x(t) = 0$ is one of the equilibrium points; however, the fuzzy model is concluded to be uncontrollable when $x(t) = 0$ and $y(t) \neq 0$.

In following example, we redesign the nonlinear system with different membership functions.

Example 2: Consider the nonlinear system in Example 1. If the membership functions are chosen as Fig. 2, then the consequence parts of the fuzzy model can be formulated as

$$A_1 = \begin{bmatrix} 0 & 0 \\ 10 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 10\beta & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

By Corollary 1, the controllability matrix contains the vector fields

$$W_0 = \sum b_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$W_1 = \sum A_i \sum b_j = \left(\mu_1 \begin{bmatrix} 0 & 0 \\ 10 & 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 \\ 0.349 & 0 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 10\,(\mu_1 + 0.0349\mu_2) \end{bmatrix}.$$

If the fuzzy model is controllable, then the following condition is satisfied:

$$\left|\begin{bmatrix} 1 & 0 \\ 0 & 10\,(\mu_1 + 0.0349\mu_2) \end{bmatrix}\right| = 10\,(\mu_1 + 0.0349\mu_2) \neq 0.$$

Since the firing strengths $\mu_i \in [0, 1]$ and $\mu_1 + \mu_2 = 1$, then $10\,(\mu_1 + 0.0349\mu_2) \neq 0$ for $\forall t$. Then, we can conclude that the overall T–S fuzzy model is controllable.

Remark 9: An important and natural question arises in the design of a feedback controller using local controllability. The controllability of a physical system is a prerequisite for proceeding with the controller design.

The following theorem discusses the existence of the optimal solution for Problem 1.

Corollary 2: If the T–S fuzzy model in (5) is controllable, then there exists an optimal control for any bounded input.

Proof: Consider the T–S fuzzy model with bounded input $u(t) \in U \subseteq \mathbb{R}^m$. It is more convenient to consider the T–S fuzzy model in the form

$$\dot{x} = \sum A_i x + v, \qquad v \in V$$

where V is the image of U under the map $\sum b : \mathbb{R}^m \to \mathbb{R}^n$. Thus, the Lie brackets are

$$\left[\sum A_i x, v\right] = \sum A_i \cdot v, \qquad v \in V.$$

Let the set $W = \{v' - v'' |\, v', v'' \in V\}$. The Lie algebra of the T–S fuzzy model contains the vector fields

$$\sum A_i x + v' - \left(\sum A_i x + v''\right) = v' - v'' \in W.$$

Consider all constant vector fields$f = w, w \in W$. Thus, it contains the Lie brackets $[w, \sum A_i x + v] = \sum A_i w$. Since the fuzzy model is controllable, the *accessibility Lie algebra* $\mathcal{L}_a$ consists of constant vector fields if

$$\mathcal{L}_a = \text{dim span}\left\{\left(\sum A_i\right)^l w \middle| 0 \leq i \leq n-1, w \in W\right\} = n \tag{15}$$

for $l = 0, \ldots, n-1\ \forall t > 0$. This condition means that if the bounded input U is nonempty, then the controllability rank condition implies that the system can be spanned the whole space. Q.E.D.

The condition of Corollary 2 means that there exists no vector $v = v' - v'' \in U, j \neq k$, such that no image of U belongs to an invariant subspace of matrix $\sum A_i$. In the next section, we shall design the time-optimal controller for the T–S fuzzy model with the maximum principle.

IV. Designing Time-Optimal Controller for a Controllable T–S Fuzzy Model

In this section, we will study the properties of time-optimal control using the maximum principle [20], [27]. In general, Problem 1 can be formulated as a Hamiltonian by the maximum principle. The Hamiltonian for Problem 1 can be described as

$$H\,(x, \lambda, u) := \lambda^T \sum A_i x + \lambda^T \sum B_i u \tag{16}$$

where $\lambda : [0, t_1]$ is a *costate* satisfying the *adjoint equation* associated with (5)

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\lambda^T \sum A_i. \tag{17}$$

By using the maximum principle [20], Problem 1 becomes

$$H\,(x, \lambda, u) = \max_{v \in U} H\,(x, \lambda, v)\,. \tag{18}$$

Definition 7: Trajectories of (5), (16), and (17) that satisfy the maximum principle are called *extremal* $(x, \lambda, u) : [0, t_1] \mapsto R^n \times R^n \setminus \{0\} \times U$. When the constant λ_0 is zero, the extremal is said to be *abnormal* [31].

Definition 8: For $j = 1, \ldots, m$, the *switching functions* $\psi_j(\cdot)$ along an *extremal* (x, λ, u) are defined by

$$\psi_j : [0, t_1] \to \mathrm{R}, \quad \psi_j(t) := \lambda^T \sum b_j. \tag{19}$$

They are absolutely continuous functions [31].

The necessary condition for optimality provided by the maximum principle states that $u : [0, t_1]$ must pointwise maximize $H(x(t), \lambda(t), \cdot)$ for the costate λ associated with the optimal trajectory. Moreover, the Hamiltonian is constant along the solutions of (16) and must satisfy

$$H(x, \lambda, u) = \lambda_0, \qquad \lambda_0 \geq 0. \tag{20}$$

The maximum condition (18) is equivalent to the following:

$$u_j(t)\psi_j(t) = \max_{v_j \in U} v_j(t)\psi_j(t), \qquad j = 1, \ldots, m. \tag{21}$$

Obviously, the functions $\psi_j(t)$ play a crucial role in the study of time-optimal trajectories. Under Assumption 1, the time-optimal control must satisfy the following conditions almost everywhere:

$$\begin{aligned} u_j &= b_j, \quad \text{if } \psi_j(t) > 0 \\ u_j &= a_j, \quad \text{if } \psi_j(t) < 0 \end{aligned} \tag{22}$$

for $j = 1, \ldots, m$. In such a case, switching functions having zeros have to be carefully analyzed.

Remark 10: Determination of optimal control sequence of (22) is related to the trajectory of costates. This introduces other problems as the initial costates and final time are unknown. This kind of problem is called two-point boundary value problems (TPBVP). The shooting method [40], however, has been used to solve this problem. The optimal solution can be obtained by solving (5), (17), (20), and (22) simultaneously. For TPBVP, no practical method has been developed yet to compute the time-optimal feedback control.

Supposing that in the time interval $[0, t_1]$ there exists one nontrivial (or more) subinterval $[t_a, t_b] \subset [0, t_1]$ such that $\psi_j(t)$ is identically zero, then the corresponding extremal is called *singular*. If $\psi_j(t) \neq 0$ for almost all $t \in [0, t_1]$, the maximum principle implies that the control u_j corresponds to piecewise constant controls taking values in the set of m vertices of U called *bang–bang*. An extremal is said to be *normal* if control u_j is *bang–bang* with at most a finite number of switching.

If T–S fuzzy model is smooth and (x, λ, u) is an *extremal*, then the time derivative of the absolutely continuous function $\psi_j(t)$ is given by

$$\begin{aligned} \dot{\psi}_j(t) &= \lambda^T \left[-\sum A_i x(t), \sum b_j\right] + \lambda^T \left[\sum b_k, \sum b_j\right] u_j(t) \\ &= \lambda^T \left[-\sum A_i x(t), \sum b_j\right]. \end{aligned} \tag{23}$$

Since $\sum b_j$, $j = 1, \ldots, m$ and $j \neq k$ are constant terms, therefore, $[\sum b_k, \sum b_j] = 0$. It is obvious that the derivatives of the switching functions $\psi_j(t)$ are themselves absolutely continuous functions, and therefore, we can perform further derivatives of it. In the next theorem, Lie brackets will be crucial in establishing a bound on the number of switches when *bang–bang* controls are derived.

Theorem 6: If the T–S fuzzy model is controllable, then the *extremal* is *normal*.

Proof: Let (x, λ, u) be *extremal* in $t \in [0, t_1]$. We shall prove the theorem by contradiction. Suppose there exists a sequence of infinite distinct singular sets

$$S = \{s_0, \ldots, s_i, \ldots\}$$

where s_i is the *i*th time interval $[t_a, t_b]_i$ such that $\psi_j(t) = 0 \, \forall t \in [t_a, t_b]_i$, $j = 1, \ldots, m$. Assume $t_0 \in s_i$. Then, we have the following relation:

$$\psi_j(t) = \lambda^T(t_0) \sum b_j = 0, \qquad j = 1, \ldots, m. \tag{24}$$

From (24), we have the first derivation of $\psi_j(t)$

$$\dot{\psi}_j(t) = \lambda^T(t_0) \left[\sum A_i x(t), \sum b_j\right] = 0. \tag{25}$$

Indeed, the lth derivative of $\psi_j(t)$ can be expressed as

$$\psi_j^l(t) = \lambda^T(t_0)\, \mathrm{ad}\left(\sum A_i x(t)\right)^l \left(\sum b_j\right) = 0 \tag{26}$$

for $l = 1, \ldots, n-1$. By Corollary 2, we have

$$\mathrm{span}\left\{\mathrm{ad}\left(\sum A_i x(t)\right)^l \left(\sum b_j\right)\right\} \in \mathbb{R}^n, \quad l = 1, \ldots, n-1.$$

Hence, we have $\lambda(t_0) = 0$, which contradicts the necessary condition of the maximum principle. Therefore, we can conclude that the set S is finite. Outside the set S, the switching function $\lambda^T(t) \sum b_j$ attains the maximum on U at one vertex; thus, the optimal control $u(t)$ is *bang–bang* on $[0, t_1] \backslash t_0$.

Q.E.D.

If the T–S fuzzy model is extremal, then the system will also simultaneously establish a bounded number of switching for bang–bang optimal controls. Further, consider the trajectories for which m control vectors are simultaneously singular. From the proof of Corollary 2, we also know the the set of all vector fields $\{[\sum A_i x, \sum b_j]\}$ is linear independent, and therefore, we have the following result.

Theorem 7: If an extremal of the T–S fuzzy model in (5) is normal, then the switching function $\psi_j(t), j = 1, \ldots, m$ will not vanish for any t.

Proof: Assume that k is a fixed element of $\{1, \ldots, m\}$ and (x, λ, u) is extremal with a common accumulation point of zeros at $t = t_0$. From (24) and (25), we have

$$\psi_j(t) = \lambda^T(t_0) \sum b_j = 0$$

and its first derivative is

$$\dot{\psi}_j(t) = \lambda^T(t_0) \left[\sum A_i x(t), \sum b_j\right] = 0$$

for all $j = 1, \ldots, m, j \neq k$. If ψ_k and $\dot{\psi}_k$ vanish at $t = t_0$, then the vector field $\sum b_k, [\sum A_i x, \sum b_j]$ for $j = 1, \ldots, m$ is linear independent. This yields a contradiction with the nonvanishing condition for costate in the maximum principle. Q.E.D.

The solvable Lie algebra is defined for the T–S fuzzy model (5) as follows.

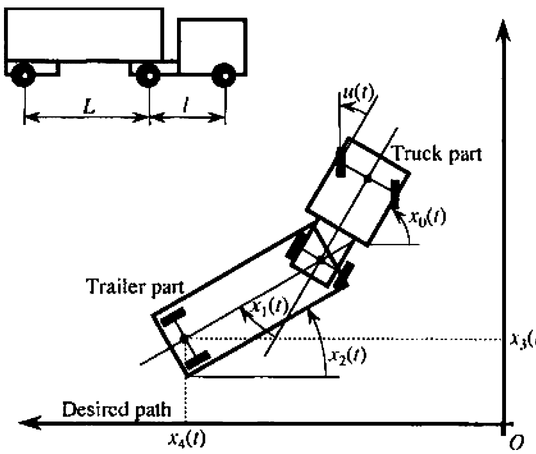

Fig. 3. Articulated vehicle model [1].

Definition 9: For T–S fuzzy model (5), the *solvable Lie algebra* is defined as

$$\mathcal{L}^{(k)} := \left\{\sum A_i x, \sum b_j \,|\forall j = 1, \ldots, m\right\}_{LA}. \quad (27)$$

If the derived series $\mathcal{L}^{(k)}$ vanishes for large k, then the T–S fuzzy model is called *solvable*.

In the next theorem, the *solvable Lie algebra*, which will be crucial in establishing a bound on the number of switching for bang–bang control will be derived.

Theorem 8: If the controllable T–S fuzzy model (5) is solvable, then the total number of switching is bounded.

Proof: The controllable T–S *fuzzy model* (5) will imply

$$\mathcal{L} = \text{span}\left\{\text{ad}\left(\sum A_i x\right)^k \sum b_j\right\}, \quad \text{for} \quad k = 1, \ldots, n-1.$$

If $\mathcal{L}$ is *solvable Lie algebra*, i.e., $\mathcal{L}^{(k)} = \text{ad}\left(\sum A_i x\right)^k \sum b_j = 0$, for $k \geq p \geq n-1$. From (26), we have

$$\psi_j^k(t) = \lambda^T(t_0)\,\text{ad}\left(\sum A_i x\right)^k \left(\sum b_j\right), \quad \text{for} \quad k \geq p \quad (28)$$

is identically zero as the T–S fuzzy model is solvable. In (28), if $\psi_j^k(t)$ vanishes for $k \geq p$, then the polynomial degree of switching function $\psi_j(t)$ does not exceed p. Q.E.D.

Remark 11: For $\sum b_j \neq 0$, the *solvable* condition (28) can be generalized as $\mathcal{L}^{(k)} = ad\left(\sum A_i x\right)^k = 0$.

V. Illustrative Examples

To utilize the time-optimal design techniques, two systems with single input and two inputs, respectively, will be illustrated.

Example 3: Consider an articulated vehicle [1] in Fig. 3. The kinematic model of the vehicle is the starting point to model the dynamics of the lateral and orientation motions. The dynamics of the articulated vehicle can be formulated as

$$\dot{x}_0 = \frac{v}{l}\tan(u(t))$$
$$x_1 = x_0 - x_2$$
$$\dot{x}_1 = \dot{x}_0 - \dot{x}_2 = \frac{v}{l}\tan(u(t)) - \frac{v}{L}\sin(x_1(t))$$

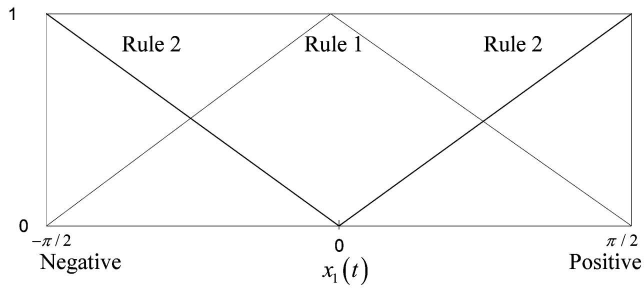

Fig. 4. Membership functions in Example 3.

$$\dot{x}_2 = \frac{v}{L}\sin(x_1(t))$$
$$\dot{x}_3 = v\cos(x_1(t))\cdot\sin(x_2(t))$$
$$\dot{x}_4 = -v\cos(x_1(t))\cdot\cos(x_2(t))$$

where

- $x_0(t)$ angle of truck;
- $x_1(t)$ angle difference between truck and trailer;
- $x_2(t)$ angle of trailer;
- $x_3(t)$ vertical position of rear end of trailer;
- $x_4(t)$ horizontal position of rear end of trailer;
- $u(t)$ steering angle.

l is the length of truck, L is the length of trailer, and v is the constant backward speed. In this example, let $l = 1$ m, $L = 2.5$ m, and $v = -5$ m/s.

The control purpose is to find the steering angle with constant backward speed so that the articulated vehicle will reach the straight line $x_3 = 0$, i.e.,

$$x_1(t) \to 0, x_2(t) \to 0, x_3(t) \to 0.$$

If the angle difference between the truck and trailer expands to 90°, i.e., $|x_1| = 90°$, this phenomenon is called "jackknife."

When a jackknife phenomenon happens, an articulated vehicle becomes uncontrollable and the backward motion cannot continue anymore. To avoid this problem, the analysis of the researchable set will be discussed in the following.

For constructing the T–S fuzzy model, assume that $u(t), x_2(t)$ are small and $x_1(t) \in (-\pi/2, \pi/2)$. Let $X(t) = [x_1(t)\;\; x_2(t)\;\; x_3(t)]^T$. The dynamics of the articulated vehicle can be formulated as:

Rule i: IF $x_1(t)$ is "Positive" and "Negative," THEN

$$\dot{X}(t) = A_i X(t) + B_i U(t), \qquad i = 1, 2 \quad (29)$$

where the membership functions are given in Fig. 4 and the consequent parts are chosen as

$$A_1 = \begin{bmatrix} -v/L & 0 & 0 \\ v/L & 0 & 0 \\ 0 & v & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -v/L & 0 & 0 \\ v/L & 0 & 0 \\ 0 & \beta\cdot v & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix}$$

and $\beta = \cos(88°)$.

From Corollary 1, we have

$$W_0 = \sum b_j = \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix}.$$

The matrix $\sum A_i \sum b_j$ is

$$W_1 = \left(\mu_1 \begin{bmatrix} -v/L & 0 & 0 \\ v/L & 0 & 0 \\ 0 & v & 0 \end{bmatrix} \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} -v/L & 0 & 0 \\ v/L & 0 & 0 \\ 0 & \beta v & 0 \end{bmatrix} \begin{bmatrix} v/l \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -v^2/(lL)(\mu_1+\mu_2) \\ v^2/(lL)(\mu_1+\mu_2) \\ 0 \end{bmatrix}.$$

The matrix $\sum A_i^2 \sum b_j$ is

$$W_2 = \begin{bmatrix} -v^3/\left(lL^2\right)(\mu_1+\mu_2) \\ v^3/\left(lL^2\right)(\mu_1+\mu_2) \\ -v^3/(lL)(\mu_1+\beta\mu_2) \end{bmatrix}.$$

The controllability of the fuzzy model can be reformulated by finding the determinant of $[W_0, W_1, W_2]$

$$\left| \begin{bmatrix} v/l & -v^2/(lL) & -v^3/\left(lL^2\right)(\mu_1+\mu_2) \\ 0 & -v^2/(lL) & v^3/\left(lL^2\right)(\mu_1+\mu_2) \\ 0 & 0 & -v^3/(lL)(\mu_1+\beta\mu_2) \end{bmatrix} \right|. \tag{30}$$

The determinant of (30) can be found as $(v/l) \cdot \left[-v^2/(lL)\right] \cdot \left\{-v^3/\left[(lL)(\mu_1+\beta\mu_2)\right]\right\}$. Since the determinant of (30) cannot be zero for $\forall \mu_i \in [0,1]$ with $\sum \mu_i = 1\ (i=1,2)$, therefore, we may conclude that the fuzzy model is controllable and a time-optimal solution does exist. To realize time-optimal control, we consider a control as $U = \tilde{u} + u^*$, where the control input $\tilde{u} = -kx$ can be designed by the pole assignment and time-optimal control u^* (steering angle) is constrained in $[5°, -5°]$. Choose the closed-loop eigenvalues as $[0 \quad 0 \quad 0]$, and we have $k = [-0.4 \quad 0 \quad 0]$. By closed-loop feedback, the consequent parts of the fuzzy model (29) can be reformulated as

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -5 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -0.1745 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}.$$

Due to $\sum B_i \neq 0$ for $\forall t \geq 0$, by using Remark 11, we have

$$\mathcal{L}^{(0)} = \mu_1 A_1 + \mu_2 A_2 = \mu_1 \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -5 & 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -0.1745 & 0 \end{bmatrix}$$

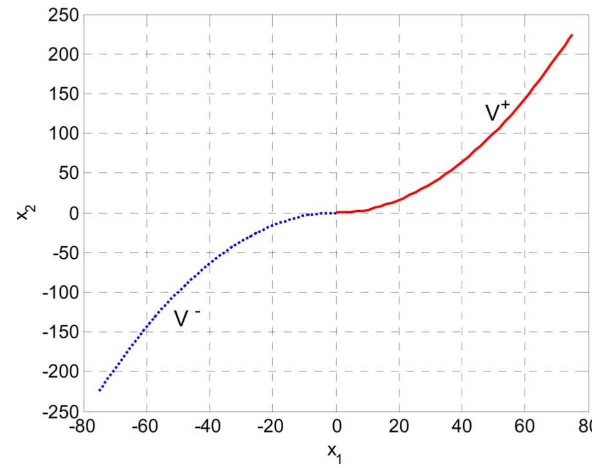

Fig. 5. Projection of the set V on the $x_1 - x_2$ plane.

$$\mathcal{L}^{(1)} = \mu_1 A_1 A_1 + \mu_2 A_2 A_2 = \mu_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10 & 0 & 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.3490 & 0 & 0 \end{bmatrix}$$

$$\mathcal{L}^{(2)} = \mu_1 A_1 A_1 A_1 + \mu_2 A_2 A_2 A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{L}^{(3)} = \mathcal{L}^{(4)} = \cdots = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $\forall \mu_i \in [0,1]$ with $\sum \mu_i = 1\ (i=1,2)$, and $k \geq 2$, $\mathcal{L}^{(k)}$ is identically zero; therefore, the fuzzy model is concluded to be solvable and the number of switching is at most 2. Assuming $u = 5°$, the bang–bang control does exist and the possible control sequence can be concluded as

$$\{u\},\ \{-u\},\ \{u,-u\},\ \{-u,u\},\ \{u,-u,u\},\ \{-u,u,-u\}.$$

The switching curves V are shown in Figs. 5 and 6. The dotted line is the set V^-, that is, the trajectory by control input $\{-u\}$, and the solid line shows the set V^+, that is, the trajectory by control input $\{u\}$. Let V_1 denote the set of states that can be forced to the origin by the control sequence $\{u,-u\}$ or $\{-u,u\}$. The transition from the control input u to $-u$ must occur on the set V. If the control sequence is from $-u$ to u, the transition must occur on the set V^+. The set $V_1{}^-$ is shown in Figs. 7 and 8. The dotted line is the set V_1^-, which is forced by the control sequence $\{-u,u\}$, and the solid line shows the set V_1^+, which is forced by the control sequence $\{u,-u\}$. The set V_2 is the trajectory that can be forced to the origin by the control sequence $\{u,-u,u\}$ or $\{-u,u,-u\}$. To prevent the jackknife phenomenon, the state x_1 should be constrained to be less than $90°$. In Figs. 9 and 10,

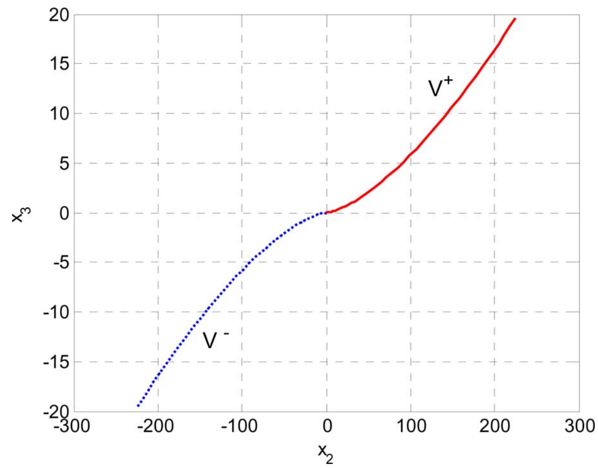

Fig. 6. Projection of the set *V* on the x_2–x_3 plane.

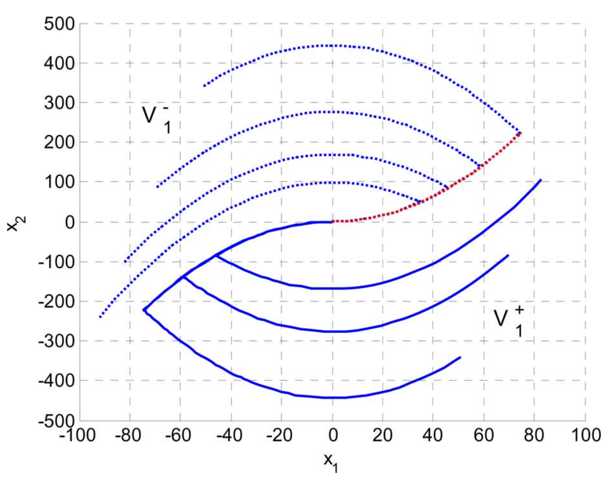

Fig. 7. Projection of the set V_1 on the x_1–x_2 plane.

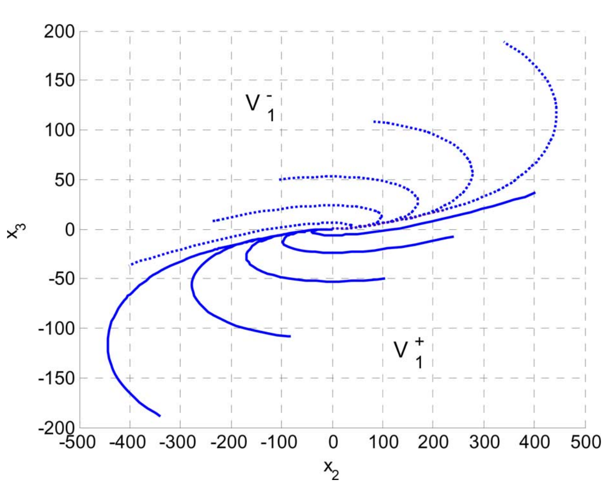

Fig. 8. Projection of the set V_1 on the x_2–x_3 plane.

the ellipses show the reachable set for $|x_1| \leq 90^\circ$, where the solid ellipses are the set V_1, and the dotted ellipses are the set V_2. In fact, $V \subseteq V_1 \subseteq V_2$. The maximal reasonable range of initial positions will be restricted on the reachable set V_2.

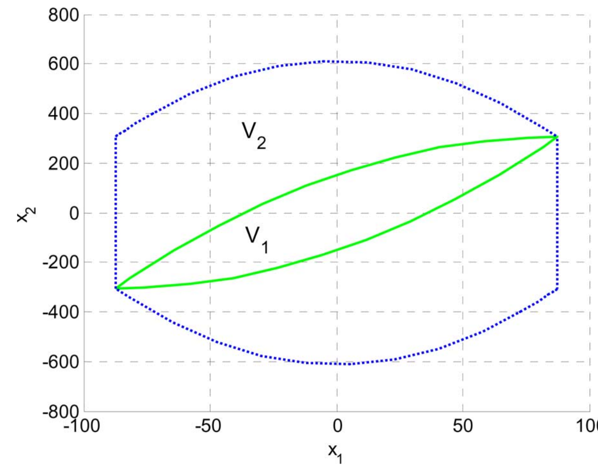

Fig. 9. Reachable set of V_1 and V_2 on the x_1–x_2 plane.

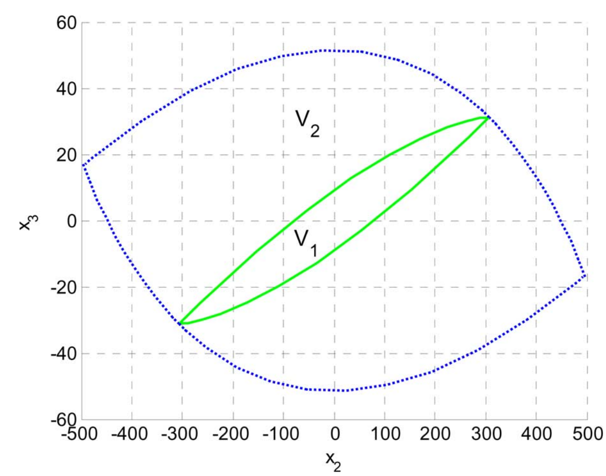

Fig. 10. Reachable set of V_1 and V_2 on the x_2–x_3 plane.

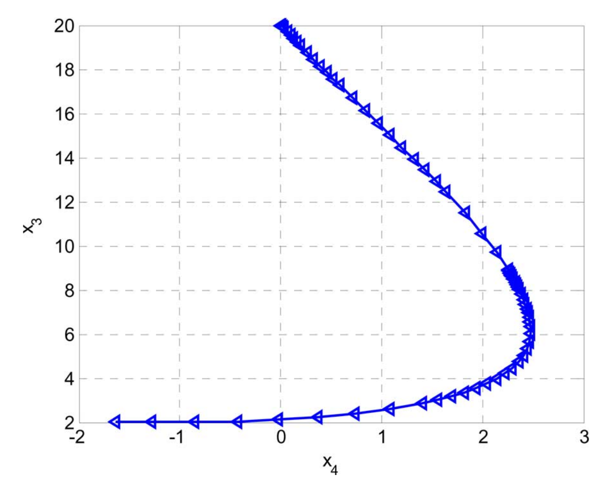

Fig. 11. Time-optimal trajectory in phase plane (Case I).

Case I: For the initial position, $x_0 = 240^\circ, x_1 = 200^\circ, x_2 = 40^\circ, x_3 = 20$ m, and $x_4 = 0$ m, the time-optimal trajectory of x_3 versus x_4 is depicted in Fig. 11. The corresponding time-optimal control $u^*(t)$ is shown in Fig. 12. The shortest time from the initial position to the origin is 2.4115 s.

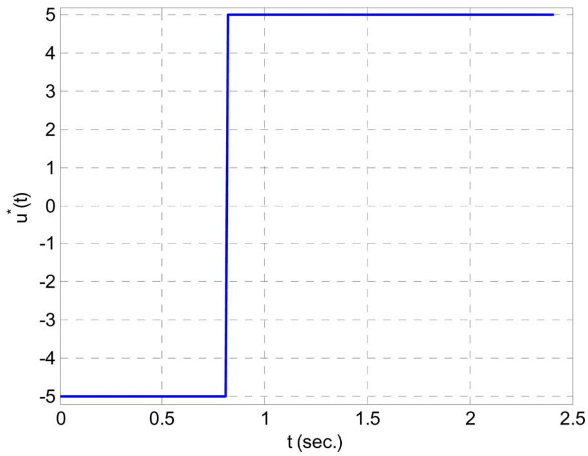

Fig. 12. Corresponded time-optimal control input (Case I).

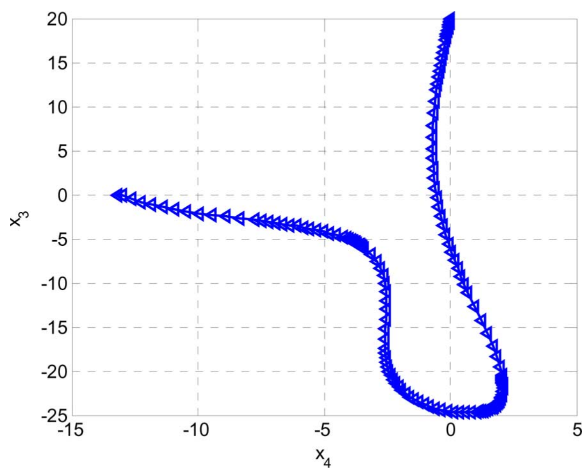

Fig. 13. Time-optimal trajectory in phase plane (Case II).

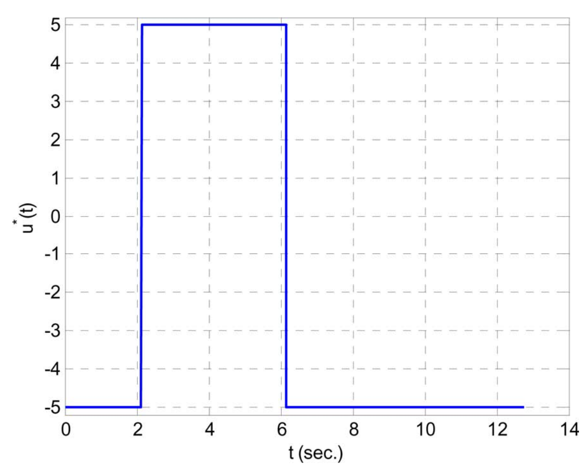

Fig. 14. Corresponded time-optimal control input (Case II).

Case II: For the initial position, $x_0 = 320°, x_1 = 20°, x_2 = 300°, x_3 = 20$ m, and $x_4 = 0$ m, the time-optimal trajectory of x_3 versus x_4 is depicted in Fig. 13. The corresponding time-optimal control $u^*(t)$ is shown in Fig. 14. The shortest time from the initial position to the origin is 13.6715 s.

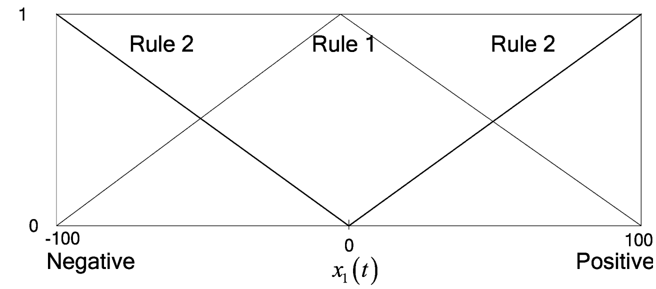

Fig. 15. Membership functions in Example 4.

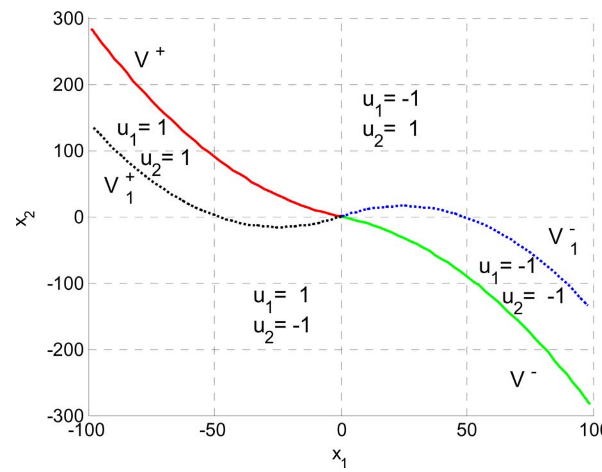

Fig. 16. Switching curve and time-optimal control input.

Example 4: The multiple inputs system is considered here. Consider the following T–S fuzzy model

Rule i: IF $x_1(t)$ is "Positive" and "Negative," THEN

$$\dot{X}(t) = A_i X(t) + B_i U(t), \qquad i = 1, 2 \tag{31}$$

where $X(t) = [x_1(t) \quad x_2(t)]^T$, $U(t) = [u_1(t) \quad u_2(t)]^T$, $|u_1(t)| \le 1, |u_2(t)| \le 1$, and the consequent parts are chosen as

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0.18 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 4 & 0.5 \\ 0.5 & -4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 & 0.5 \\ 0.5 & -4 \end{bmatrix}.$$

The membership functions of the fuzzy model are given in Fig. 15. The fuzzy model is found to be controllable by Corollary 1. The switching number is at most 2 that is obtained by using Remark 11. Therefore, the time-optimal sequences are

$$\{1, 1\}, \{-1, -1\}, \{1, -1\}, \{-1, 1\}.$$

Following the same analysis in Example 3, the switching curves are explained in the following. There are two possible switching curves in this example. Let the set of states V be forced by input $\{1, 1\}$ or $\{-1, -1\}$ and V_1 be forced by input $\{1, -1\}$ or $\{-1, 1\}$ to the origin. The switching curve V is depicted as a solid line in Fig. 16, the dotted line depicts switching curve V_1, and the time-optimal control inputs are also shown in Fig. 16.

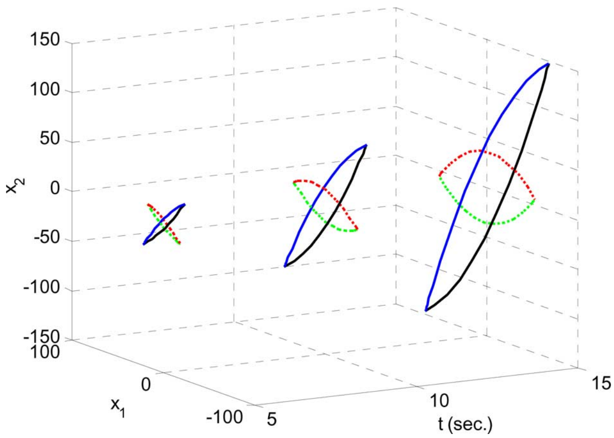

Fig. 17. Reachable sets in Example 4.

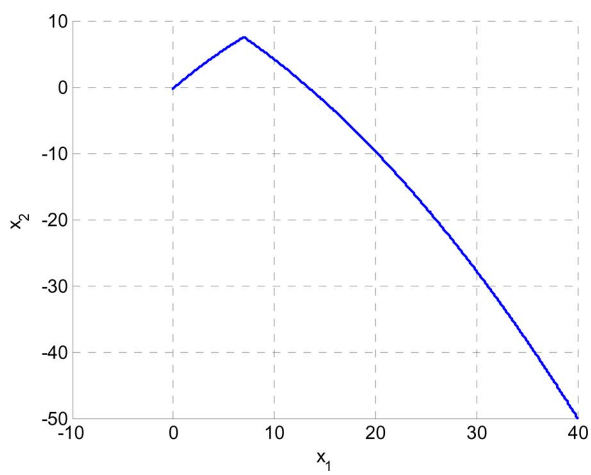

Fig. 18. Time-optimal trajectory in phase plane (Case I).

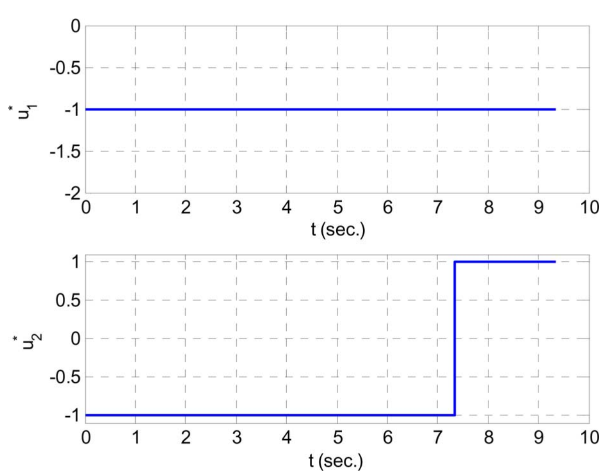

Fig. 19. Corresponded time-optimal control input (Case I).

Assume $\mathcal{R}(T)$ and $\mathcal{R}_1(T)$ are reachable sets for V and V_1, respectively, that can reach the origin at time T. Fig. 17 depicts a reachable set that is sampled from $T = 5$ to $T = 20$ every 5 s. The dotted line is the reachable set $\mathcal{R}_1(T)$, and the solid line is the reachable set $\mathcal{R}(T)$.

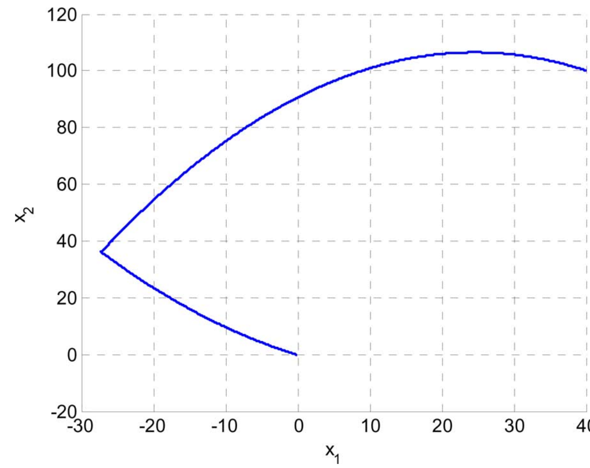

Fig. 20. Time-optimal trajectory in phase plane (Case II).

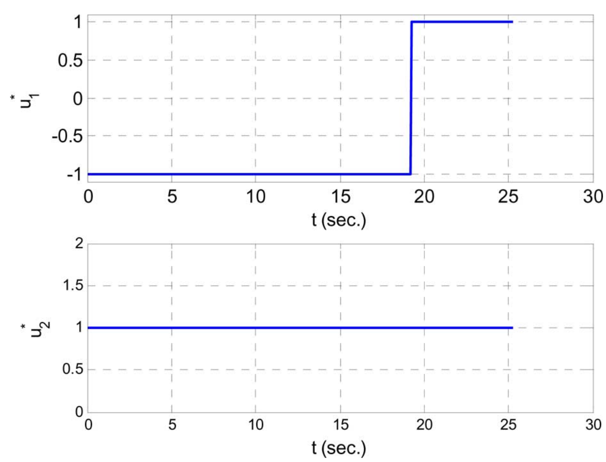

Fig. 21. Corresponded time-optimal control input (Case II).

Case I: For the initial state $X_0 = [40, -50]$, the time-optimal trajectory is shown in Fig. 18. The corresponding time-optimal control $u^*(t)$ is shown in Fig. 19. The shortest time from initial state to the origin is 9.350 s.

Case II: For the initial state $X_0 = [40, 100]$, the time-optimal trajectory is depicted in Fig. 20. The corresponding time-optimal control $u^*(t)$ is shown in Fig. 21. The shortest time from initial state to the origin is 25.249 s.

VI. Conclusion

This paper presents a new design of time-optimal controller for a controllable T–S fuzzy model in which the maximum principle is applied. In particular, the subsystems of T–S fuzzy model are blended by a set of firing strengths, which leads it to a class of nonlinear systems. First, we proposed the proof of the existence of optimal control in the T–S fuzzy model that can be addressed as the compactness of the reachable set. The generalized rank condition of the accessible Lie algebra is also applied in this paper for the proof of the existence of optimal controller for the T–S fuzzy model. This also results in the controllability of the T–S fuzzy model. According to the maximum principle, the time-optimal control of the T–S fuzzy model is *bang–bang* that is determined by the switching function. By investigating the singular structure of the switching functions of the controllable T–S fuzzy model, we can yield the conditions for the existence, i.e., *if the extremal* is *normal, then there exists the time-optimal controller for the T–S fuzzy model.* In other words, the time-optimal control of controllable T–S fuzzy model is *bang–bang* with a finite number of switchings over all trajectories for all t. The bounded number of switching is related to the polynomial degree of switching function that is obtained by introducing the solvable Lie algebra. Several examples are fully illustrated to show the conditions for the existence of a time-optimal controller with their optimal trajectories found by numerical simulation.

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REFERENCES

[1] K. Tanaka and M. Sano, "A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck-trailer," *IEEE Trans. Fuzzy Syst.*, vol. 2, no. 2, pp. 119–134, May 1994.

[2] K. Tanaka, T. Ikeda, and H. O. Wang, "Robust stabilization of a class of uncertain nonlinear system via fuzzy control: Quadratic stabilizability, H^{∞} control theory, and linear matrix inequalities," *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 1, pp. 1–13, Feb. 1996.

[3] A. E. Gegov and P. M. Frank, "Hierarchical fuzzy control of multivariable systems," *Fuzzy Sets Syst.*, vol. 72, no. 3, pp. 299–310, 1995.

[4] C. Lee, "Fuzzy logic in control systems: Fuzzy logic controller Part 1, Part 2," *IEEE Trans. Syst., Man, Cybern. B*, vol. 20, no. 2, pp. 404–435, Mar./Apr. 1990.

[5] J. I. Horiuchi and M. Kishimoto, "Application of fuzzy control to industrial bioprocesses in Japan," *Fuzzy Sets Syst.*, vol. 128, no. 1, pp. 117–124, 2002.

[6] C. C. Hsiao, S. F. Su, T. T. Lee, and C. C. Chuang, "Hybrid compensation control for affine TSK fuzzy control systems," *IEEE Trans. Syst., Man, Cybern. B*, vol. 34, no. 4, pp. 1865–1873, Aug. 2004.

[7] Y. G. Leu, T. T. Lee, and W. Y. Wang, "Observer-based adaptive fuzzy-neural control for unknown nonlinear dynamical systems," *IEEE Trans. Syst., Man, Cybern. B*, vol. 29, no. 5, pp. 583–591, Oct. 1999.

[8] H. O. Wang, K. Tanaka, and M. F. Griffin, "An approach to fuzzy control of nonlinear systems: Stability and design issues," *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 1, pp. 14–23, Feb. 1996.

[9] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-15, no. 1, pp. 116–132, Jan. 1985.

[10] S. Sugeno and G. T. Kang, "Structure identification of fuzzy model," *Fuzzy Sets Syst.*, vol. 28, no. 10, pp. 15–33, 1988.

[11] T. Takagi and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets Syst.*, vol. 45, no. 2, pp. 135–156, 1993.

[12] M. C. M. Teixerira and S. H. Zak, "Stabilization controller design for uncertain nonlinear systems using fuzzy models," *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 2, pp. 133–142, Apr. 1999.

[13] S. G. Cao, N. W. Rees, and G. Feng, "H-infinity control of nonlinear continuous-time system based on dynamical fuzzy models," *Int. J. Syst. Sci.*, vol. 27, pp. 821–830, 1996.

[14] K. Tanaka, T. Ikeda, and H. O. Wang, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs," *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 2, pp. 250–265, May 1998.

[15] B. S. Chen, C. S. Tseng, and H. J. Uang, "Mixed H_2/H_∞ fuzzy output feedback control design for nonlinear dynamic systems: An LMI approach," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 3, pp. 249–265, Jun. 2000.

[16] H. J. Lee, J. B. Park, and G. Chen, "Robust fuzzy control of nonlinear system with parametric uncertainties," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, pp. 369–379, Apr. 2001.

[17] G. Feng, "Approaches to quadratic stabilization of uncertain continuous time fuzzy dynamic systems," *IEEE Trans. Circuits Syst. I*, vol. 48, no. 2, pp. 369–379, Apr. 2001.

[18] K. R. Lee, E. T. Jeung, and H. B. Park, "Robust fuzzy H_∞ control for uncertain nonlinear systems via state feedback: An LMI approach," *Fuzzy Sets Syst.*, vol. 120, pp. 123–134, 2001.

[19] P. Kulczycki, "Fuzzy controller for mechanical systems," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 5, pp. 645–652, Oct. 2000.

[20] L. S. Pontryagin, V. Boltyanskii, T. Gamkrelideze, and E. Mishchenko, *The Mathematical Theory of Optimal Processes*. New York: Interscience, 1962.

[21] M. Athans and P. L. Palb, *Optimal Control*. New York: McGraw-Hill, 1966.

[22] H. Hermes and J. P. Lasalle, *Functional Analysis and Time Optimal Control*. San Diego, CA: Academic, 1969.

[23] J. Macki and A. Strauss, *Introduction to Optimal Control Theory*. New York: Springer-Verlag, 1982.

[24] D. E. Kirk, *Optimal Control Theory: An Introduction*. Englewood Cliffs, NJ: Prentice–Hall, 1970.

[25] B. Jakubczyk and W. Respondek, *Geometry of Feedback and Optimal Control*. New York: Marcel Dekker, 1998.

[26] H. J. Sussmann, *Differential Geometric Control Theory*. Boston, MA: Birkhäuser, 1983.

[27] H. J. Sussmann, "The maximum principle of optimal control theory," in *Mathematical Control Theory*, J. Baillieul and J. C. Willems, Eds. New York: Springer-Verlag, 1998, pp. 140–198.

[28] H. J. Sussmann and V. Jurdjevic, "Controllability of nonlinear system," *J. Diff. Equ.*, vol. 12, pp. 95–116, 1972.

[29] E. D. Sontag and H. J. Sussmann, "Time-optimal control of manipulators," in *Proc. IEEE Int. Conf. Robot. Autom.*, San Francisco, CA, 1986, pp. 1692–1697.

[30] E. D. Sontag, "Remarks on the time-optimal control of a class of Hamiltonian systems," in *Proc. IEEE Conf. Decis. Control*, 1989, pp. 317–221.

[31] M. Chyba, N. E. Leonard, and E. D. Sontag, "Singular trajectories in multi-input time-optimal problems: Application to controlled mechanical systems," *J. Dyn. Control Syst.*, vol. 9, pp. 73–88, 2003.

[32] A. D. Lewis, The geometry of the maximum principle for affine connection control systems. [Online]. Available: http://penelope.mast.queensu.ca/~andrew/

[33] A. A. Agrachev, "Mathematical control theory," in *Steklov Mathematical Institue*, A. A. Agrachev, Ed. Trieste, Italy: SISSA.

[34] R. Vinter, *Optimal Control*. Boston, MA: Birkhäuser, 2000.

[35] R. W. Brockett, "Nonlinear systems and differential geometry," *Proc. IEEE*, vol. 64, no. 1, pp. 61–71, Jan. 1976.

[36] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[37] J.-P. Aubin and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*. Berlin, Germany: Springer-Verlag, 1984.

[38] H. Samelson, *Note on Lie Algebras*. New York: Van Nostrand Reinhold, 1969.

[39] P. J. Olver, *Application of Lie Groups to Differential Equations*, 2nd ed. New York: Springer-Verlag, 1993.

[40] G. J. Lastman, "A shooting method for solving two-point boundary-value problems arising from nonsingular bang–bang optimal control processes," *Int. J. Control*, vol. 27, pp. 513–524, 1978.

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