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Error-correcting pooling designs associated with some distance-regular graphs

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ABSTRACT

Motivated by the pooling designs over the incidence matrices of matchings with various sizes of the complete graph K_{2n} considered by Ngo and Du [Ngo and Du, Discrete Math. 243 (2003) 167–170], two families of pooling designs over the incidence matrices of t-cliques (resp. strongly t-cliques) with various sizes of the Johnson graph $J(n, t)$ (resp. the Grassmann graph $J_q(n, t)$) are considered. Their performances as pooling designs are better than those given by Ngo and Du. Moreover, pooling designs associated with other special distance-regular graphs are also considered.

1. Introduction

The basic problem of group testing is to identify the set of defective items in a large population of items. Suppose we have n items to be tested and that there are at most d defective items among them. Each *test* (or *pool*) is (or contains) a subset of items. We assume that some testing mechanism exists which, if applied to an arbitrary subset of the population, gives a *negative outcome* if the subset contains no positive and a *positive outcome* otherwise. Objectives of group testing vary from minimizing the number of tests, limiting number of pools, limiting pool sizes, to tolerating a few errors. It is conceivable that these objectives are often contradictory, thus testing strategies are application-dependent. A group testing algorithm is *non-adaptive* if all tests must be specified without knowing the outcomes of other tests. A non-adaptive testing algorithm is useful in many areas such as DNA library screening. (See [3]).

A group testing algorithm is *error tolerant* if it can detect some errors in test outcomes. A mathematical model of error-tolerance designs is a d^e-*disjunct* matrix. A binary matrix M is said to be d^e-disjunct if, given any $d+1$ columns of M with one designated, there are $e+1$ rows with a 1 in the designated column and 0 in each of the other d columns. A d^e-disjunct matrix with $e = 0$ is said to be d-*disjunct*. Macula [12] proposed a novel way of constructing d-disjunct matrices by the containment relation of subsets in a finite set, while in [13] he constructed d^e-disjunct matrices for certain values of e. Ngo and Du [14] extended the construction to some geometric structures, such as simplicial complexes, and some graph properties, such as matchings. Huang and Weng [9] generalized the constructions to pooling spaces, while they proved that a d^{2e}-disjunct matrix is e-error-correcting in [10].

Du and Ngo [15] pointed out that the subject of pooling designs is a young and interesting field with deep connections to coding theory and design theory, and they strongly believe that the theory of association schemes – in particular distance

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regular graphs – should play an important role in improving pooling designs. For more information about pooling designs, see [2,6–8].

Let $\Gamma = (X, R)$ be a connected graph of diameter D, and let $\partial(x, y)$ denote the distance of the vertices x and y. Γ is said to be *distance-regular* whenever for all non-negative integers h, i, j, and for any two vertices x and y at distance h, the number

$$p^h_{i,j} = |\{z \in X \mid \partial(x, z) = i,\ \partial(z, y) = j\}|$$

is independent of the choice of x and y. For more information, the reader may consult [1].

For any positive integer n we shall use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. Also, given any set X and any vector space V over a finite field $\mathbb{F}_q$, $\binom{X}{k}$ denotes the collection of all k-subsets of X, and $\left[{V \atop k}\right]$ denotes the collection of all k-subspaces of V.

The *Johnson graph* $J(n, t)$ is defined on $\binom{[n]}{t}$ such that two vertices A and B are adjacent if and only if $|A \cap B| = t - 1$. Similarly, the *Grassmann graph* $J_q(n, t)$ is defined on $\left[{\mathbb{F}_q^n \atop t}\right]$ such that two vertices A and B are adjacent if and only if $\dim(A \cap B) = t - 1$. Johnson graphs and Grassmann graphs are two families of well-known distance-regular graphs.

Let $\Gamma = (X, R)$ be a connected graph. An l-subset Δ of X is said to be a *t-clique* of Γ with size l if any two distinct vertices in Δ are at distance t. 1-clique is the clique in traditional use. A *strongly t-clique* of $J_q(n, t)$ with size l is a subfamily $\{A_1, A_2, \ldots, A_l\} \subseteq \left[{\mathbb{F}_q^n \atop t}\right]$ such that $\dim(A_1 + A_2 + \cdots + A_l) = tl$. Note that an l-matching on K_{2n} is a 2-clique of $J(n, 2)$ with size l. Hence a t-clique of $J(n, t)$ with size l is a generalization of an l-matching.

A class of pooling designs over the incidence matrices of matchings with various sizes of the complete graph K_{2n} is considered by Ngo and Du [14]. In this paper, we try to generalize Ngo and Du's construction. The rest of the article is organized as follows. In Section 2, we review some results on pooling designs over Johnson graphs and Grassmann graphs, and then compute one important parameter for these pooling designs. (See Theorems 2.9 and 2.10). In Section 3, with an interpretation of matchings as 2-cliques of the Johnson graph $J(n, 2)$, the pooling designs by Ngo and Du are generalized to the incidence matrices of t-cliques with various sizes of the Johnson graph $J(n, t)$ and strongly t-cliques with various sizes of the Grassmann graph $J_q(n, t)$, respectively. We show that our pooling designs have the same capability of error-detecting and error-correcting as Ngo and Du's. However, the test-to-item ratio of ours is much smaller. In Section 4, we construct pooling designs associated with some special distance-regular graphs.

2. Disjunctness over Johnson graphs and Grassmann graphs

For a binary matrix M of order $N \times T$, let $B(D)$ denote the Boolean sum of those columns indexed by elements of $D \subseteq [T]$, and let $d_H(B(D), B(D'))$ denote the Hamming distance between $B(D)$ and $B(D')$ whenever D and D' are two distinct subsets of $[T]$.

Let

$$e_s = \min_{|D|=|D'|=s} d_H(B(D), B(D')).$$

The larger the parameter e_s, the better its capacity of error correcting.

In this section, we first review some results on pooling designs over Johnson graphs and Grassmann graphs, and then compute the parameter e_s for those s^e-disjunct matrices.

2.1. Some known results

D'yachkov et al. [5] proposed the concept of fully s^e-disjunct matrices. An s^e-disjunct matrix is *fully s^e-disjunct* if it is not $d^{e'}$-disjunct whenever $d > s$ or $e' > e$. D'yachkov et al. [4] gave the lower bounds of e_s for a fully s^e-disjunct matrix.

Proposition 2.1 (*[4, Lemma 3.4]*). *Let M be a fully s^e-disjunct matrix. Then $e_s \geq 2(e + 1)$.*

Macula [12] constructed d-disjunct matrices using the containment relation in a structure. D' yachkov et al. [5] discussed the error-correcting property of Macula's construction.

Definition 2.1 (*[12]*). For positive integers $d < k < n$, let $J(n, d, k)$ be the binary matrix with row-indexed (resp. column-indexed) by $\binom{[n]}{d}$ $\left(\text{resp. } \binom{[n]}{k}\right)$ such that $M(A, B) = 1$ if and only if $A \subseteq B$ and 0 otherwise.

Theorem 2.2 (*[5, Proposition 2]*). *Suppose $1 \leq s \leq d < k < n$ and $e = e(s) = \binom{k-s}{k-d} - 1$. Then $J(n, d, k)$ is fully s^e-disjunct.*

Ngo and Du [14] gave a q-analogue of Macula's construction. The error-correcting property of Ngo and Du's construction was discussed in [5,4], respectively.

Definition 2.2 (*[14]*). For positive integers $d < k < n$, let $J_q(n, d, k)$ be the binary matrix with row-indexed (resp. column-indexed) by $\begin{bmatrix} \mathbb{F}_q^n \\ d \end{bmatrix}$ $\left(\text{resp. } \begin{bmatrix} \mathbb{F}_q^n \\ k \end{bmatrix}\right)$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise.

Let $\mathbb{F}_q$ be the finite field with q elements, where q is a prime power. Let $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. For a positive integer n, the *Gaussian binomial coefficients with basis q* is defined by

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \begin{cases} \displaystyle\prod_{j=0}^{i-1} \frac{n-j}{i-j}, & \text{if } q = 1, \\ \displaystyle\prod_{j=0}^{i-1} \frac{q^n - q^j}{q^i - q^j}, & \text{if } q \neq 1. \end{cases}$$

Naturally, $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ i \end{bmatrix}_q = 0$ if $i > n$. In the case $q = 1$, we shall write $\binom{n}{i}$ instead of $\begin{bmatrix} n \\ i \end{bmatrix}_1$ for convenience.

In the rest of this paper, for positive integers $d < k$ and $k - d \geq 2r$, we always assume that

$$p_q(r) = \left\lfloor \frac{\begin{bmatrix} k \\ d \end{bmatrix}_q - \begin{bmatrix} k-r \\ d \end{bmatrix}_q}{\begin{bmatrix} k-r \\ d \end{bmatrix}_q - \begin{bmatrix} k-2r \\ d \end{bmatrix}_q} \right\rfloor$$

and

$$e_q(s, r) = \begin{bmatrix} k \\ d \end{bmatrix}_q - \begin{bmatrix} k-r \\ d \end{bmatrix}_q - (s-1)\left(\begin{bmatrix} k-r \\ d \end{bmatrix}_q - \begin{bmatrix} k-2r \\ d \end{bmatrix}_q\right) - 1.$$

Theorem 2.3 (*[5, Proposition 4],[4, Theorem 4.4 and Corollary 4.5]). Let q be a prime power. Suppose $k - d \geq 2$ and $e = e_q(s, 1)$.*

(i) *If $s \in [p_q(1)]$, then $J_q(n, d, k)$ is s^e-disjunct.*
(ii) *If $s \in [q + 1]$, then $J_q(n, d, k)$ is fully s^e-disjunct.*

Based on $J(n, d, k)$, Macula [13] proposed another family of s^e-disjunct matrices. D'yachkov et al. [5] also discussed their error-correcting property.

Definition 2.3. (i) A family $\mathcal{K} \subseteq \binom{[n]}{k}$ is called an $\{r, r+1, \ldots, k\}$-*clique* of $J(n, k)$ if $|K \cap K'| \leq k - r$ for any two distinct $K, K' \in \mathcal{K}$.
(ii) A family $\mathcal{F} \subseteq \begin{bmatrix} \mathbb{F}_q^n \\ k \end{bmatrix}$ is called an $\{r, r+1, \ldots, k\}$-*clique* of $J_q(n, k)$ if $\dim(K \cap K') \leq k - r$ for any two distinct $K, K' \in \mathcal{F}$.

Definition 2.4 (*[13]*). For positive integers $d < k < n$, let $\mathcal{K} \subseteq \binom{[n]}{k}$. Suppose $J(n, d, \mathcal{K})$ denotes the binary matrix with row-indexed (resp. column-indexed) by $\binom{[n]}{d}$ (resp. $\mathcal{K}$) such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise.

Theorem 2.4 (*[5, Proposition 3], [13, Theorem 2]). Let $\mathcal{K}$ be an $\{r, r+1, \ldots, k\}$-clique of $J(n, k)$.*

(i) *Let $d \geq 1$ with $1 + \frac{r}{k-d} \leq r$ and $\alpha_d = \min(r^d, k - d)$. Then $J(n, d, \mathcal{K})$ is $d^{\alpha_d - 1}$-disjunct.*
(ii) *$J(n, d, \mathcal{K})$ is s^e-disjunct where $s \in [p_1(r)]$ and $e = e_1(s, r)$.*

As the q-analogue of $J(n, d, \mathcal{K})$, we propose the following definition.

Definition 2.5. For positive integers $d < k < n$, let $\mathcal{F} \subseteq \begin{bmatrix} \mathbb{F}_q^n \\ k \end{bmatrix}$. Suppose $J_q(n, d, \mathcal{F})$ denotes the binary matrix with row-indexed (resp. column-indexed) by $\begin{bmatrix} \mathbb{F}_q^n \\ d \end{bmatrix}$ (resp. $\mathcal{F}$) such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise.

Similar to Theorem 2.4, we have the following result.

Corollary 2.5. *Let $\mathcal{F}$ be an $\{r, r+1, \ldots, k\}$-clique of $J_q(n, k)$. Then $J_q(n, d, \mathcal{F})$ is s^e-disjunct where $s \in [p_q(r)]$ and $e = e_q(s, r)$.*

Let $X = \{x = (x_1, x_2, \ldots, x_n) \mid x_i \in F\}$ where $F = \{0, 1, \ldots, q\}$, and let P_i denote the set of elements with weight i of X. For positive integers $1 \leq d \leq k \leq n$, define $H(n, q, d, k)$ to be the binary matrix with row-indexed (resp. column-indexed) by P_d (resp. P_k) such that $M(x, y) = 1$ if $x_i = 0$ or $x_i = y_i$ and $x_i \neq 0$.

D'yachkov et al. [5] proposed the above matrix and discussed its disjunctness.

Theorem 2.6 (*[5]). For $1 \leq s \leq d \leq k \leq n$, $H(n, q, d, k)$ is fully s^e-disjunct, where $e = \binom{k-s}{k-d} - 1$.*

Let V be the vector space of dimension $n + r$ over $\mathbb{F}_q$, and let W be a fixed r-subspace of V. Let $P_i = \{A \mid A \in \begin{bmatrix} V \\ i \end{bmatrix}$ and $A \cap W = 0\}$. For positive integers $1 \leq d \leq k \leq n$, define $B_q(n, r, d, k)$ to be the binary matrix with row-indexed (resp. column-indexed) by P_d (resp. P_k) such that $M(A, B) = 1$ if $A \subseteq B$.

Huang and Weng [9] proved that $B_q(n, r, d, k)$ is a d^e-disjunct matrix for some e. Similar to Theorem 2.3, we may obtain the following results.

Theorem 2.7. *Let q be a prime power. Suppose $k - d \geq 2$ and $e = e_q(s, 1)$.*

(i) *If $s \in [p_q(1)]$, then $B_q(n, r, d, k)$ is s^e-disjunct.*
(ii) *If $s \in [q + 1]$, then $B_q(n, r, d, k)$ is fully s^e-disjunct.*

2.2. Parameter e_s for error tolerance

The complement M^c of a binary matrix M is the matrix that results when one interchanges the 0's and 1's in M. Let $\mathcal{K}$ be any subset of $\binom{[n]}{k}$. Macula [13] considered the matrix $J^*(n, d, \mathcal{K})$ as that which results by row augmenting the matrix $J(n, d, \mathcal{K})$ with $J^c(n, 1, \mathcal{K})$. He claimed that $e_1 \geq 4$ for $J^*(n, d, \mathcal{K})$. Hwang [11] gave a proof.

Theorem 2.8 ([11, Theorem 2]). *Given $J^*(n, d, \mathcal{K})$ with $k - d \geq 3$. Then $e_1 \geq 4$.*

In the rest of this subsection, we shall compute the parameter e_s for $J(n, d, k)$ and $J_q(n, d, k)$, respectively. We begin with an example.

Example 2.1. Given a matrix $J(n, d, k)$ with $1 \leq d < k < n$. For $s \in [d]$, let

$$D_0 = \{\widehat{1}, \widehat{2}, \ldots, \widehat{s-1}, \widehat{k+1}\} \quad \text{and} \quad D'_0 = \{\widehat{1}, \widehat{2}, \ldots, \widehat{s-1}, \widehat{k}\}$$

where $\widehat{i} = [k+1] - \{i\}$ and $i \in [k+1]$. Then

$$\left| \left\{ R | R \in \binom{[k]}{d}, R \not\subseteq \widehat{1}, \widehat{2}, \ldots, \widehat{s-1}, \widehat{k} \right\} \right| = \binom{k-s}{k-d}.$$

By the symmetry, $d_H(B(D_0), B(D'_0)) = 2\binom{k-s}{k-d}$.

Theorem 2.9. *Given a matrix $J(n, d, k)$ with $1 \leq s \leq d < k < n$. Then $e_s = 2\binom{k-s}{k-d}$.*

Proof. The upper bound for e_s is derived from Example 2.1. By Theorem 2.2 and Proposition 2.1, it is also a lower bound for e_s. Hence the desired result follows. □

Similar to the case for Johnson graphs, we consider the following example. □

Example 2.2. Given a matrix $J_q(n, k, d)$ with $k - d \geq 2$. For each $i \in [k + 1]$, let e_i be the row vector of V whose i-th coordinate is 1 and all other coordinates are 0. Suppose $\mathbb{F}_q = \{a_1, a_2, \ldots, a_q\}$ and $s \leq q + 1$.

For $i \in [s + 1]$, let

$$D_0 = \{C_1, \ldots, C_{s-1}, C_s\} \quad \text{and} \quad D'_0 = \{C_1, \ldots, C_{s-1}, C_{s+1}\},$$

where

$$C_s = \langle e_1, e_2, \ldots, e_k \rangle, \; C_{s+1} = \langle e_2, e_3, \ldots, e_k, e_{k+1} \rangle, \; C_i = \langle e_1 + a_i e_2, e_3, \ldots, e_k, e_{k+1} \rangle.$$

By the principle of inclusion and exclusion,

$$\left| \left\{ R | R \in \begin{bmatrix} C_s \\ d \end{bmatrix}, R \not\subseteq C_1, \ldots, C_{s-1}, C_{s+1} \right\} \right| = \begin{bmatrix} k \\ d \end{bmatrix}_q - \binom{s}{1} \begin{bmatrix} k-1 \\ d \end{bmatrix}_q + \begin{bmatrix} k-2 \\ d \end{bmatrix}_q \sum_{j=2}^{s} (-1)^j \binom{s}{j}$$
$$= e_q(s, 1) + 1.$$

By the symmetry, $d_H(B(D_0), B(D'_0)) = 2e_q(s, 1) + 2$.

Theorem 2.10. *Given a matrix $J_q(n, d, k)$ with $k - d \geq 2$. If $s \in [q + 1]$, then $e_s = 2e_q(s, 1) + 2$.*

Proof. The upper bound for e_s is derived from Example 2.2. By Theorem 2.3 and Proposition 2.1, it is also a lower bound. Therefore the desired result follows. □

3. Disjunctness over matchings on K_{2m} and its extensions

For positive integers $d < k \leq m$, let M be a binary matrix with row-indexed (resp. column-indexed) by d-matchings (resp. k-matchings) of K_{2m} such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M(2m, d, k)$. In [14], Ngo and Du proposed the matrix and discussed its disjunctness.

Theorem 3.1 (*[14, Theorem 11,Corollary 12]*). *Let $1 \leq d < k \leq m$. Then*

(i) *$M(2m, d, k)$ is a d-disjunct matrix.*
(ii) *$M(2m, d, m)$ is d-error-detecting and $\lfloor d/2 \rfloor$-error-correcting.*
(iii) *Moreover, if the number of positives is known to be exactly d, then $M(2m, d, m)$ is $(2d + 1)$-error-detecting and d-error-correcting.*

With an interpretation of matchings as 2-cliques of Johnson graph $J(n, 2)$, we shall generalize Ngo and Du's designs to the incidence matrices of t-cliques with various sizes of the Johnson graph $J(n, t)$ and strongly t-cliques with various sizes of the Grassmann graph $J_q(n, t)$, respectively. We show that our pooling designs have the same capability of error-detecting and error-correcting as Ngo and Du's. However, the test-to-item ratio of ours is much smaller.

Definition 3.1. Given positive integers $d < k$ and $kt \leq n$.

(i) Let $J(n, t, d, k)$ be the binary matrix with row-indexed (resp. column-indexed) by t-cliques with size d (resp. k) of $J(n, t)$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise.
(ii) Let $J_q(n, t, d, k)$ be the binary matrix with row-indexed (resp. column-indexed) by strongly t-cliques with size d (resp. k) of $J_q(n, t)$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise.

Since $J(2m, 2, d, k) = M(m, d, k)$, $J(n, t, d, k)$ is a generalization of $M(m, d, k)$.

Lemma 3.2. *Let W be a given k-subspace of $\mathbb{F}_q^n$. Then the number of d-subspaces of $\mathbb{F}_q^n$ intersecting trivially with W is $\begin{bmatrix} n-k \\ d \end{bmatrix}_q q^{dk}$.*

Proof. Let $D = \{A \mid A \in \begin{bmatrix} \mathbb{F}_q^n \\ d \end{bmatrix}, A \cap W = 0\}$. Counting the set $\{(v_1, v_2, \ldots, v_d) | \, v_i \notin \langle W, v_1, v_2, \ldots, v_{i-1} \rangle, i \in [d]\}$ in two ways, we have

$$(q^n - q^k)(q^n - q^{k+1}) \cdots (q^n - q^{k+d-1}) = |D| \cdot (q^d - 1)(q^d - q) \cdots (q^d - q^{d-1}).$$

Hence $|D| = \begin{bmatrix} n-k \\ d \end{bmatrix}_q q^{dk}$ as required. □

Lemma 3.3. (i) *The number of t-cliques of $J(n, t)$ with size l is*

$$u(n, t, l) = \binom{n}{tl} (tl)!/(t!)^l l!. \tag{1}$$

(ii) *The number of strongly t-cliques of $J_q(n, t)$ with size l is*

$$u_q(n, t, l) = \begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} n-t \\ t \end{bmatrix}_q \cdots \begin{bmatrix} n-(l-1)t \\ t \end{bmatrix}_q \cdot \frac{q^{t^2 l(l-1)/2}}{l!}. \tag{2}$$

Proof. (i) Since every tl-subset of $[n]$ forms $\frac{(tl)!}{(t!)^l l!}$ many t-cliques of $J(n, t)$ with size l, (1) holds.

(ii) By Definition 2.3, $u_q(n, t, l)$ is the number of $\{A_1, A_2, \ldots, A_l\} \subseteq \begin{bmatrix} \mathbb{F}_q^n \\ t \end{bmatrix}$ satisfying $\dim(A_1 + A_2 + \cdots + A_l) = tl$. Let $N(n, t, l)$ be the number of ordered tuples $(A_1, A_2, \ldots, A_l)$ of t-subspaces of $\mathbb{F}_q^n$ such that $\dim(A_1 + A_2 + \cdots A_l) = tl$. Then $u_q(n, t, l) = \frac{N(n,t,l)}{l!}$. Hence, if we want to get $u_q(n, t, l)$, it suffices to compute $N(n, t, l)$. There are $\begin{bmatrix} n \\ t \end{bmatrix}_q$ ways to choose A_1, then $\begin{bmatrix} n-t \\ t \end{bmatrix}_q q^{t^2}$ ways to choose A_2 by Lemma 3.2 and so on. It follows that

$$N(n, t, l) = \begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} n-t \\ t \end{bmatrix}_q q^{t \cdot t} \begin{bmatrix} n-2t \\ t \end{bmatrix}_q q^{2t \cdot t} \cdots \begin{bmatrix} n-(l-1)t \\ t \end{bmatrix}_q q^{(l-1)t \cdot t}.$$

Hence (2) holds. □

Theorem 3.4. *Let $1 \leq s \leq d < k$ and $kt \leq n$. Then $J(n, t, d, k)$ is an s^e-disjunct matrix of order $N \times T$ with row weight $u(n - td, t, k - d)$ and column weight $\binom{k}{d}$, where $(N, T) = (u(n, t, d), u(n, t, k))$ and $e = \binom{k-s}{k-d} - 1$.*

Proof. By Lemma 3.3, $J(n,t,d,k)$ is an $N \times T$ matrix with row weight $u(n-td,t,k-d)$ and column weight $\binom{k}{d}$.

Let $C_0, C_1, \ldots, C_s$ be any $s+1$ distinct columns of $J(n,t,d,k)$. For each $i \in [s]$, there exists $P_i \in C_0 \setminus C_i$. Let $E = \{P_i | i \in [s]\}$. Then $|E| \le s$ and $E \subseteq C_0$ but $E \not\subseteq C_i$ for each $i \in [s]$. If $|E| = j$, the number of d-subsets of C_0 containing E is $\binom{k-j}{k-d}$. Since $\binom{k-j}{k-d} \ge \binom{k-s}{k-d}$ whenever $j \le s$, the number of t-cliques with size d contained in C_0 but not contained in C_i for each $i \in [s]$ is at least $\binom{k-s}{k-d}$. □

Corollary 3.5. *Let $1 \le s \le d < k$ and $(k+1)t \le n$. Then $J(n,t,d,k)$ is fully s^e-disjunct with $e = \binom{k-s}{k-d} - 1$ and $e_s = 2\binom{k-s}{k-d}$.*

Proof. In order to prove that $J(n,t,d,k)$ is fully s^e-disjunct, we only need to show that the maximum size of E is obtained in Theorem 3.4. Since $(k+1)t \le n$, there exists a t-clique $T = \{A_1, A_2, \ldots, A_{k+1}\}$ with size $k+1$. Let $C_0 = T - \{A_{k+1}\}$ and $C_i = T - \{A_i\}$ for each $i \in [k]$. Then $|E| = |\{A_i \mid i \in [s]\}| = s$. Similar to Theorem 2.9, it is routine to compute e_s. □

Similar results hold for $J_q(n,t,d,k)$ too, and their proofs will be omitted.

Theorem 3.6. *Let $1 \le s \le d < k$ and $kt \le n$. Then the matrix $J_q(n,t,d,k)$ is an s^e-disjunct matrix of order $N \times T$ with row weight $u_q(n-td,t,k-d)$ and column weight $\binom{k}{d}$, where*

$$(N,T) = (u_q(n,t,d), u_q(n,t,k)) \quad \textit{and} \quad e = \binom{k-s}{k-d} - 1.$$

Corollary 3.7. *Let $1 \le s \le d < k$ and $(k+1)t \le n$. Then $J_q(n,t,d,k)$ is fully s^e-disjunct with $e = \binom{k-s}{k-d} - 1$ and $e_s = 2\binom{k-s}{k-d}$.*

Remarks. (i) By zigzag arguments similar to Theorem 3.1 (ii), $J(tn,t,d,n)$ is d-error-detecting and $\lfloor d/2 \rfloor$ error-correcting.

(ii) The test-to-item ratio of $J(tm,t,d,k)$ (resp. $J_q(tm,t,d,k)$) is much less than that of $M(m,d,k)$ (resp. $J(tm,t,d,k)$).

The following theorem tells us how to choose d and k such that the test-to-item ratio for $J(tm,t,d,k)$ (resp. $J_q(tm,t,d,k)$) is minimized.

Theorem 3.8. *For l goes from 1 to $m \ge 3$,*

(i) *$u(tm,t,l)$ is unimodal and get its peak when $\lfloor l_1 \rfloor \le l \le \lfloor l_2 \rfloor$ where l_1, l_2 satisfying $l_1 + \sqrt[t]{l_1+1} = m$, $l_2 + \frac{t+\sqrt[t]{l_2+1}-1}{t} = m$.*
(ii) *$u_q(tm,t,l)$ is increasing and get its maximum at $l = m$.*

Proof. (i) Suppose $f(l) = \frac{u(tm,t,l+1)}{u(tm,t,l)}$. Then $f(l)$ is a decreasing number series while l goes from 1 to $m-1$. Since $f(1) > 1$ and $f(m-1) = \frac{1}{m} < 1$, $u(tm,t,l)$ is unimodal. Note that

$$f(l) = \frac{u(tm,t,l+1)}{u(tm,t,l)} = \frac{tm-tl}{t\sqrt[t]{l+1}} \cdot \frac{tm-tl-1}{(t-1)\sqrt[t]{l+1}} \cdots \frac{tm-tl-t+1}{\sqrt[t]{l+1}}.$$

Let $u_i = \frac{tm-tl-i}{(t-i)\sqrt[t]{l+1}}$. Then $(u_0)^t \le f(l) \le (u_{t-1})^t$. If $u_0 = 1$, then $l + \sqrt[t]{l+1} = m$; if $u_{t-1} = 1$, then $l + \frac{t+\sqrt[t]{l+1}-1}{t} = m$. The desired results follow.

(ii) Suppose $g(l) = \frac{u_q(tm,t,l+1)}{u_q(tm,t,l)}$. Then

$$g(l) = \frac{(q^{tm} - q^{tl}) \cdots (q^{tm-t+1} - q^{tl})}{(q^t - 1) \cdots (q-1)(l+1)}.$$

It follows that that $g(l)$ is decreasing while l goes from 1 to $m-1$. Since $g(1) > 1$ and $g(m-1) > 1$, $u_q(tm,t,l)$ is increasing while l goes from 1 to m, and achieve the maximum value at $l = m$. □

4. Disjunctness over other distance-regular graphs

In this section, we shall give two constructions of pooling designs associated with antipodal distance-regular graphs and distance-regular graphs of order (r,t), respectively. Since the results are similar to those in Section 3, we shall omit all the proofs in this section.

A distance-regular graph $\Gamma = (X,R)$ of diameter $D \ge 2$ is said to be *antipodal*, if $\partial(x,y) = \partial(x,z) = D$ and $y \ne z$ implies $\partial(y,z) = D$. Let $k_D = p^0_{D,D}$. Then the number of maximal D-clique of Γ is $\frac{|X|}{k_D+1}$. Since any two distinct maximal D-cliques have no common vertices, the number of D-clique with size l of Γ is $\frac{|X|}{k_D+1} \cdot \binom{k_D+1}{l}$.

Table 1
The parameters of s^e-disjunct matrices.

Name	Tests	Items	Ratio (test to item)	s	e	Remarks
$J(n,d,k)$	$\binom{n}{d}$	$\binom{n}{k}$	$\frac{(n-k)!k!}{(n-d)!d!}$	$s \in [d]$	$\binom{k-s}{k-d}-1$	Theorem 2.2
$J_q(n,d,k)$	$\begin{bmatrix} n \\ d \end{bmatrix}_q$	$\begin{bmatrix} n \\ k \end{bmatrix}_q$	$\frac{(q^k-1)\cdots(q^{d+1}-1)}{(q^{n-k}-1)\cdots(q^{n-d+1}-1)}$	$s \in [p_q(1)]$	$e_q(s,1)$	Theorem 2.3
$J(n,d,\mathcal{K})$	$\binom{n}{d}$	$\lvert\mathcal{K}\rvert$	$\frac{\binom{n}{d}}{\lvert\mathcal{K}\rvert}$	$s \in [p_1(r)]$	$e_1(s,r)$	Theorem 2.4
$J_q(n,d,\mathcal{F})$	$\begin{bmatrix} n \\ d \end{bmatrix}_q$	$\lvert\mathcal{F}\rvert$	$\frac{\begin{bmatrix} n \\ d \end{bmatrix}_q}{\lvert\mathcal{F}\rvert}$	$s \in [p_q(r)]$	$e_q(s,r)$	Corollary 2.5
$H(n,q,d,k)$	$\binom{n}{d}q^d$	$\binom{n}{k}q^k$	$\frac{(n-k)!k!q^d}{(n-d)!d!q^k}$	$s \in [d]$	$\binom{k-s}{k-d}-1$	Theorem 2.6
$B_q(n,r,d,k)$	$\begin{bmatrix} n \\ d \end{bmatrix}_q q^{dr}$	$\begin{bmatrix} n \\ k \end{bmatrix}_q q^{kr}$	$\frac{(q^k-1)\cdots(q^{d+1}-1)q^{dr}}{(q^{n-k}-1)\cdots(q^{n-d+1}-1)q^{dk}}$	$s \in [p_q(1)]$	$e_q(s,1)$	Theorem 2.7
$M(2m,d,k)$	$\frac{(2d)!}{2^d d!}\cdot\binom{2m}{2d}$	$\frac{(2k)!}{2^k k!}\cdot\binom{2m}{2k}$	$\frac{(2m-2k)!k!}{(2m-2d)!d!}2^{k-d}$	$s \in [d]$	$\binom{k-s}{k-d}-1$	Theorem 3.1
$J(n,t,d,k)$	$\frac{(td)!}{(t!)^d d!}\cdot\binom{n}{td}$	$\frac{(tk)!}{(t!)^k k!}\cdot\binom{n}{tk}$	$\frac{(n-tk)!k!}{(n-td)!d!}(t!)^{k-d}$	$s \in [d]$	$\binom{k-s}{k-d}-1$	Theorem 3.4
$J_q(n,t,d,k)$	$u_q(n,t,d)$	$u_q(n,t,k)$	$\frac{q^{t(d-k)(k+d-1)/2}k!}{\begin{bmatrix} n-dt \\ t \end{bmatrix}_q\cdots\begin{bmatrix} n-(k-1)t \\ t \end{bmatrix}_q d!}$	$s \in [d]$	$\binom{k-s}{k-d}-1$	Theorem 3.6
$A(n,d,k)$	$\frac{n}{k_D+1}\cdot\binom{k_D+1}{d}$	$\frac{n}{k_D+1}\cdot\binom{k_D+1}{k}$	$\frac{(k_D+1-k)!k!}{(k_D+1-d)!d!}$	$s \in [d]$	$\binom{k-s}{k-d}-1$	Theorem 4.1
$B(r,t;d,k)$	$\frac{n(t+1)}{r+1}\cdot\binom{r+1}{d}$	$\frac{n(t+1)}{r+1}\cdot\binom{r+1}{k}$	$\frac{(r+1-k)!k!}{(r+1-r)!d!}$	$s \in [d]$	$\binom{k-s}{k-d}-1$	Theorem 4.2

A distance-regular graph $\Gamma = (X, R)$ is said to be of *order* (r, t) if, for each vertex $x \in X$, the induced subgraph on $\Gamma(x)$ is a disjoint union of $t+1$ cliques with size r. Then each maximal clique is of size $r+1$, and each vertex is contained in $t+1$ maximal cliques. Denote the set of all maximal cliques by $\mathcal{C}$. By computing the number of the set $\{(x, C) \mid x \in X, C \in \mathcal{C}, x \in C\}$ in two ways, the number of maximal cliques of Γ is $\frac{n(t+1)}{r+1}$; consequently the number of cliques with size l is $\binom{r+1}{l}\cdot\frac{n(t+1)}{r+1}$.

Let Γ be an antipodal distance-regular graph of diameter D with n vertices. For positive integers $1 < d < k < k_D + 1$, let M be the binary matrix whose row (resp. column) indexed by the D-cliques of Γ with size d (resp. k) such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $A(n, d, k)$.

Theorem 4.1. *Let $1 \le s \le d < k < k_D + 1$. Then $A(n, d, k)$ is a fully s^e-disjunct matrix of order $N \times T$ with row weight $\binom{k_D+1-d}{k-d}$ and column weight $\binom{k}{d}$, where*

$$(N, T) = \left(\frac{n}{k_D+1}\cdot\binom{k_D+1}{d}, \frac{n}{k_D+1}\cdot\binom{k_D+1}{k}\right), \quad e = \binom{k-s}{k-d} - 1.$$

Moreover, $e_s = 2\binom{k-s}{k-d}$.

Let Γ be a distance-regular graph of order (r, t). For positive integers $1 < d < k < r + 1$, let M be the binary matrix whose row (resp. column) indexed by the cliques of Γ with size d (resp. k) such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $B(r, t; d, k)$.

Theorem 4.2. *Let $1 \le s \le d < k < r + 1$. Then $B(r, t; d, k)$ is a fully s^e-disjunct matrix of order $N \times T$ with row weight $\binom{r+1-d}{k-d}$ and column weight $\binom{k}{d}$, where*

$$(N, T) = \left(\frac{n(t+1)}{r+1}\cdot\binom{r+1}{d}, \frac{n(t+1)}{r+1}\cdot\binom{r+1}{k}\right), \quad e = \binom{k-s}{k-d} - 1.$$

Moreover, $e_s = 2\binom{k-s}{k-d}$.

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Appendix

See Table 1.

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