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Directed 3-cycle decompositions of complete directed graphs with quadratic leaves

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ABSTRACT

Let G be a vertex-disjoint union of directed cycles in the complete directed graph D_t, let $|E(G)|$ be the number of directed edges of G and suppose $G \neq \vec{C}_2 \cup \vec{C}_3$ or $\vec{C}_5$ if $t = 5$, and $G \neq \vec{C}_3 \cup \vec{C}_3$ if $t = 6$. It is proved in this paper that for each positive integer t, there exist $\vec{C}_3$-decompositions for $D_t - G$ if and only if $t(t-1) - |E(G)| \equiv 0 \pmod 3$.

1. Introduction

A Steiner triple system of order t, denoted $STS(t)$, is a pair $(V, \mathcal{B})$ where V is a t-set and $\mathcal{B}$ is a collection of 3-element subsets (called triples) of V such that each pair of elements occurs in a unique triple. It is well-known that an $STS(t)$ exists if and only if $t \equiv 1$ or $3 \pmod 6$. In terms of graph decompositions, an $STS(t)$ can also be viewed as a partition of the edges of K_t, each element of which induces a triangle C_3; we denote such a decomposition by $C_3|K_t$.

A packing of a graph T with triangles is a partition of the edge set of a subgraph G of T, each element of which induces a triangle; the remainder graph of this packing, also known as the leave, is the subgraph $T - G$ formed from T by removing the edges in G. If the leave is minimum in size (that is, has the least number of edges among all possible leaves of T), then the packing is called a maximum packing. The following result is well-known.

Theorem 1.1 ([5]). *The leaves G for any maximum packing of K_t with triangles are as follows:*

$t \pmod 6$	0	1	2	3	4	5
G	F	$\emptyset$	F	$\emptyset$	F_1	C_4

F is a 1-factor, F_1 is an odd spanning forest with $\frac{t}{2} + 1$ edges (tripole), and C_4 is a cycle of length 4.

It is natural to ask for which subgraphs G of K_t, $C_3|(K_t - G)$. When t is odd and G is a 2-regular graph, the following result has been obtained by Colbourn and Rosa [2].

Theorem 1.2 ([2]). *Let t be an odd positive integer. Let G be a 2-regular subgraph of K_t. If $t = 9$, then suppose that $G \neq C_4 \cup C_5$. Then $C_3|(K_t - G)$ if and only if the number of edges in $K_t - G$ is a multiple of 3.*

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In this paper, we consider the corresponding problem about packings of directed graphs.

A Mendelsohn triple system of order t, denoted $MTS(t)$, is a pair $(V, \mathcal{B})$ where V is a t-set, $\mathcal{B}$ is a collection of cyclically ordered 3-subsets of V(called Mendelsohn triples) such that each ordered pair of V appears in exactly one Mendelsohn triple of $\mathcal{B}$. In terms of graph decomposition, the existence of an $MTS(t)$ is equivalent to partitioning the directed edges (or edges in short) of D_t into a collection of directed 3-cycles. We denote such a decomposition by $\vec{C}_3 \mid D_t$. It is well-known that an $MTS(t)$ exists if and only if $t \equiv 0, 1 \pmod 3$, $t \neq 6$ [1].

In this paper, we shall extend the work of Theorem 1.2 to directed graphs. We consider packings of D_t with Mendelsohn triples and prove the following result.

Theorem 1.3. *Let t be a positive integer. Let G be a vertex-disjoint union of directed cycles in the complete directed graph D_t, and suppose $G \neq \vec{C}_2 \cup \vec{C}_3$ or $\vec{C}_5$ if $t = 5$, and $G \neq \vec{C}_3 \cup \vec{C}_3$ if $t = 6$. Then $\vec{C}_3 \mid (D_t - G)$ if and only if $t(t-1) - |E(G)| \equiv 0 \pmod 3$.*

We can get $2K_t$ from D_t where $2K_t$ is the multigraph in which each pair of vertices is joined by exactly two edges. Thus we have Theorem 1.4.

Theorem 1.4. *Let t be a positive integer. Let G be a vertex-disjoint union of cycles in $2K_t$, and suppose $G \neq C_2 \cup C_3$ if $t = 5$, and $G \neq C_3 \cup C_3$ if $t = 6$. Then $C_3 \mid (2K_t - G)$ if and only if $t(t-1) - |E(G)| \equiv 0 \pmod 3$.*

Note here that the exceptional case $2K_5 - C_5$ can be obtained by direct construction and the other two cases remain impossible.

2. Preliminaries

In this section, we will give some notation, symbols and lemmas which are useful in the proof of our main theorem.

Let D_t be a complete directed graph of order t containing no loops so that for each vertex v in D_t, $\deg^+(v) = \deg^-(v) = t - 1$. Let D_{V_1} denote the directed complete subgraphs of D_t induced by V_1 where $V_1 \subseteq V(D_t)$. Let V_1 be an m-set, V_2 be an n-set and $V_1 \cap V_2 = \emptyset$. A complete bipartite directed graph D_{V_1,V_2} ($D_{m,n}$) contains $2mn$ directed edges; e.g. $D_{3,2} = \{a_ib_j,\ b_ja_i \mid 1 \le i \le 3,\ 1 \le j \le 2\}$ which has 12 edges. (For convenience, we call them edges instead of directed edges.)

The join of two directed graphs G_1 and G_2 is denoted by $G_1 \vee G_2$ where G_1 and G_2 are vertex-disjoint. Then $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup E(D_{|V(G_1)|,|V(G_2)|})$. For convenience, we use $G_1 + G_2$ to denote $E(G_1) \cup E(G_2)$. Obviously, $D_{m+n} = D_m \vee D_n = D_m + D_n + D_{m,n}$. In the following, we denote a directed cycle of length l by $\vec{C}_l$.

Definition 2.1. Let t be a positive integer. If H is a spanning subgraph of G, then H is a t-factor of G if $\deg(v) = t$ for each vertex v in H.

Definition 2.2. Let H be a 2-factor of G. If G is oriented such that for each vertex v in H, $\deg^+(v) = \deg^-(v) = 1$, then H is a directed 2-factor of G.

Lemma 2.1. *Let C be a directed 2-factor. Then $\vec{C}_3 \mid (C \vee K_1)$ where $V(C) \cap V(K_1) = \emptyset$.*

The lemma can be deduced from the following example immediately.

Example 1. Let $V_1 = Z_5$, $V_2 = \{\infty\}$ and $\vec{C}_5 = (0, 1, 2, 3, 4)$. The graph T contains a single point ∞. Then $T \vee \vec{C}_5 = D_{V_1,V_2} + \vec{C}_5 = \{(0, 1, \infty), (1, 2, \infty), (2, 3, \infty), (3, 4, \infty), (4, 0, \infty)\}$.

Theorem 2.1 (*[6]*). *Let G be a t-regular graph with t an even number. Then G can be decomposed into $\frac{t}{2}$ 2-factors.*

An analog of Petersen's 2-factor theorem (Theorem 2.1), in the case of directed graphs, is also needed in our proof. For clarity, we present a proof here.

Theorem 2.2. *Let G be a t-regular digraph, i.e., $\deg^+(v) = \deg^-(v) = t$ for any $v \in V(G)$. Then G can be decomposed into t directed 2-factors (or t 1-regular directed spanning subgraphs).*

Proof. Construct a bipartite graph $H = (V_1, V_2)$ from G by letting $V_1 = V_2 = V(G)$ and a vertex a of V_1 is adjacent to a vertex b of V_2 if and only if (a, b) is an arc in G. By assumption, we conclude that H is t-regular and thus by Hall's condition H has a perfect matching $M = \{a_1b_1, a_2b_2, \dots, a_nb_n\}$ where $n = |V(G)|$.

From M, we obtain a 1-regular directed spanning subgraph $(a_1, b_1, a_{i_1}, b_{i_1}, a_{i_2}, b_{i_2}, \dots)$ of G which is the desired subgraph. Then, the proof follows by considering the perfect matchings one at a time (t of them). □

Since we shall use induction on the order t to prove our main results, the following lemmas which show the direct construction for small orders are essential. We start at $t = 5$ since the cases when $t \le 4$ are easy to be seen.

Lemma 2.2. *If $G = \overrightarrow{C}_2 \cup \overrightarrow{C}_3$ or $\overrightarrow{C}_5$, then $D_5 - G$ has no $\overrightarrow{C}_3$-decomposition.*

Proof. The first case is easy to see. Now, if $G = \overrightarrow{C}_5$, then, without loss of generality, we may let $\overrightarrow{C}_5 = (0, 1, 2, 3, 4)$ where $V(D_5) = Z_5$. Therefore, there are 15 edges in $D_5 - \overrightarrow{C}_5$ which includes all (a, b) with $a, b \in Z_5$ and $a - b \equiv 2$ or $3 \pmod 5$. Since these 10 edges themselves cannot form a directed 3-cycle, we shall need one edge of difference 4 to combine with two of the 10 edges with difference 2 or 3. But, this is impossible by direct checking. □

Lemma 2.3. *If $G = \overrightarrow{C}_3 \cup \overrightarrow{C}_3$, then $D_6 - G$ has no $\overrightarrow{C}_3$-decomposition.*

Proof. Let the two directed 3-cycles be defined on $A = \{a, b, c\}$ and $B = \{d, e, f\}$ respectively. Then there are 18 edges in $D_{A,B}$ which cannot form a directed 3-cycle themselves. Since there are only 6 edges left in A and B, no $\overrightarrow{C}_3$-decompositions can be obtained for $D_6 - G$. □

Lemma 2.4. *Suppose $G \neq \overrightarrow{C}_3 \cup \overrightarrow{C}_3$ if $t = 6$. For $6 \le t \le 12$, if G contains a $\overrightarrow{C}_2$ (or D_2), then $D_t - G$ has a $\overrightarrow{C}_3$-decomposition.*

Proof. The proof follows by adding directed 3-cycles to $G - \overrightarrow{C}_2$ defined on $V(D_t) \setminus V(\overrightarrow{C}_2)$ to obtain a $(t-5)$-regular subgraph H of D_{t-2} ($D_{t-2} - H$ is 2-regular) and then apply Theorem 2.2. □

Note that the number of directed 3-cycles (denoted by N) we add can be seen in the following table. Since they are easy to calculate, we omit the details.

t	6	7	8	9	10	11	12
N	0	2	4	7	10	15	20

Lemma 2.5. *For $7 \le t \le 12$, if G contains a directed 3-cycle, then $D_t - G$ has a $\overrightarrow{C}_3$-decomposition.*

Proof. The idea is similar to the proof of Lemma 2.4 for $8 \le t \le 12$. Instead of making the subgraph H $(t-5)$-regular, we need a $(t-7)$-regular graph H. Therefore, for $8 \le t \le 12$, we have a similar proof. For completeness, we include a $\overrightarrow{C}_3$-decomposition of $D_7 - (\overrightarrow{C}_3 \cup \overrightarrow{C}_3)$ in what follows. Let $V(D_7) = \{a, b, c, d, e, f, g\}$ and let the two directed 3-cycles be defined on $\{a, b, c\}$ and $\{d, e, f\}$ respectively. Then $D_7 - (\{(a, b, c)\} \cup \{(d, e, f)\}) = \{(a, e, b), (b, f, c), (c, d, a), (a, f, e), (b, d, f), (c, e, d), (g, a, d), (g, f, a), (g, c, f), (g, e, c), (b, e, g), (d, b, g)\}$. □

Lemma 2.6. *There exist $\overrightarrow{C}_3$-decompositions for $D_3 - \overrightarrow{C}_3$, $D_6 - \overrightarrow{C}_6$, $D_7 - \overrightarrow{C}_6$, $D_8 - \overrightarrow{C}_8$, $D_9 - \overrightarrow{C}_9$, $D_{11} - \overrightarrow{C}_{11}$, $D_{12} - (\overrightarrow{C}_6 \cup \overrightarrow{C}_6)$, and $D_{12} - \overrightarrow{C}_{12}$.*

Proof. We give the proof by direct construction which can be found in the Appendix. □

Lemma 2.7. *Let G contain two directed cycles with one of them $\overrightarrow{C}_4$ or $\overrightarrow{C}_5$. Then, for $t = 8, 9, 10, 11, 12$, the graph $D_t - G$ has a $\overrightarrow{C}_3$-decomposition if and only if $t(t-1) - |E(G)| \equiv 0 \pmod 3$.*

Proof. The proof can also be found in the Appendix. □

Note that if we need the case $t = 13$ in our main proof, we can decompose $D_{13} - G$ into $\overrightarrow{C}_3$-decomposition by finding four vertex-disjoint 3-cycles in B' which is the set of directed 3-cycles of $D_{13} - G$.

For example, let $G = \overrightarrow{C}_5 \cup \overrightarrow{C}_7$. Let the set of directed 3-cycles be $B'' = \{(e, i, l), (h, c, f), (j, d, a), (b, k, g)\} \subseteq B'$. Let a new vertex be m, then $D_{13} - G = B' - B'' + \{(m, e, i), (m, i, l), (m, l, e), (m, h, c), (m, c, f), (m, f, h), (m, j, d), (m, d, a), (m, a, j), (m, b, k), (m, k, g), (m, g, b)\}$.

On the other hand, if G contains a D_2, then the proof follows by a similar idea as in Lemma 2.4.

With the above preparations, we are now in a position to prove our main results. For readers' convenience, we start with a special case.

Lemma 2.8. *Let t be a positive integer such that $t \equiv 4 \pmod 6$. Let G be a 1-regular directed subgraph of D_t such that $t(t-1) - |E(G)|$ is a multiple of 3 and $\overrightarrow{C}_2$ is not a component of G. Then $D_t - G$ can be decomposed into directed 3-cycles.*

Proof. For $t = 4$ and $t = 10$, the proof follows by direct constructions. Assume that $t \ge 16$. By counting, $|E(G)| \equiv 0 \pmod 3$. For convenience, let $t = 6k + 4$, $V(D_t) = \{v_i \mid i \in Z_{6k+4}\}$ and $V(G) = \{v_i \mid 3l + 1 \le i \le 6k + 3\}$. Now, let $\widetilde{G} = G + B_0$ where $B_0 = \{(v_1, v_2, v_3), (v_4, v_5, v_6), \ldots, (v_{3l-2}, v_{3l-1}, v_{3l})\}$. Let $\widetilde{G}'$ be the graph obtained by reversing the direction of the arcs on $\widetilde{G}$. Then, the following results are easy to see:

(i) By Theorem 1.2, $D_{t-1} - (\widetilde{G} + \widetilde{G}')$ has a $\overrightarrow{C}_3$-decomposition where D_{t-1} is defined on $V(D_t) \setminus \{v_0\}$;

(ii) by Lemma 2.1, $\widetilde{G}' \vee \{v_0\}$ has a $\overrightarrow{C}_3$-decomposition.

Let the collection of directed 3-cycles obtained from (i) and (ii) be denoted by B_1 and B_2 respectively. Then $D_t - G$ can be decomposed into the directed 3-cycle collection $B_0 \cup B_1 \cup B_2$.

To simplify arguments, we also use the following equalities to see the process of obtaining the desired decomposition.

$$\begin{aligned} D_t - G &= D_t - (G + B_0) + B_0 \\ &= D_t - \widetilde{G} + B_0 \\ &= D_{t-1} + D_{V(D_t)\setminus\{v_0\},\{v_0\}} - \widetilde{G} + B_0 \\ &= D_{t-1} - \widetilde{G} - \widetilde{G}' + (D_{V(D_t)\setminus\{v_0\},\{v_0\}} + \widetilde{G}') + B_0 \\ &= B_1 \cup B_2 \cup B_0 \quad \text{where } B_1 = D_{t-1} - \widetilde{G} - \widetilde{G}' \text{ and } B_2 = D_{V(D_t)\setminus\{v_0\},\{v_0\}} + \widetilde{G}'. \quad \square \end{aligned}$$

In order to prove our main theorem, we first consider the case when G is a directed cycle of size very close to t, i.e., either t or $t - 1$.

Theorem 2.3. *For each positive integer t, $\vec{C}_3 \mid (D_t - \vec{C}_t)$ when $t \equiv 0, 2 \pmod 3$ and $\vec{C}_3 \mid (D_t - \vec{C}_{t-1})$ when $t \equiv 1 \pmod 3$.*

Proof. We give the proof by induction. For small cases, the proof can be found in Lemma 2.6. Assuming inductively that the assertion is true for values less than t, we shall prove the assertion is true for t. Note here that when $t \equiv 4 \pmod 6$, we can obtain the result by Lemma 2.8 independently. The proof can be divided into two cases.

Case 1. $t \equiv 0, 2 \pmod 3$.

Case (1.1). $t \equiv 3 \pmod 6$.

For $t \equiv 3 \pmod 6$, let $t = 6k + 3$ and let $V(D_t) = A \cup B$ where $A = \{\infty_i \mid i \in Z_{3k}\}$ and $B = \{i \mid i \in Z_{3k+3}\}$. Furthermore, let $\vec{C}_t = (\infty_0, \infty_1, \dots, \infty_{3k-1}, 0, 1, \dots, 3k + 2)$. Now, by adding $\alpha = \{(k + 2, \infty_{3k-1}, \infty_0), (\infty_0, 3k + 2, k + 2), (\infty_{3k-1}, k + 2, 0), (0, k + 2, 3k + 2)\}$ to $\vec{C}_t$ where α is a set of four Mendelsohn triples, we obtain $\vec{C}_t + \alpha = \vec{C}^{(1)} + \vec{C}^{(2)} + \beta + \{(k + 2, 3k + 2), (0, k + 2)\}$ where $\vec{C}^{(1)} = (\infty_0, \infty_1, \dots, \infty_{3k-1})$, $\vec{C}^{(2)} = (0, 1, \dots, 3k + 2)$ and $\beta = \{(\infty_0, 3k + 2), (\infty_0, k + 2), (\infty_{3k-1}, k + 2), (\infty_{3k-1}, 0)\}$. First, considering $D_t = D_A + D_B + D_{A,B}$, we can easily get $D_A - \vec{C}^{(1)}$ which has a $\vec{C}_3$-decomposition by induction. Second, since $D_{A,B} = D_{A\setminus\{\infty_0,\infty_{3k-1}\},B} + D_{\{\infty_0\},B} + D_{\{\infty_{3k-1}\},B}$, we associate it with $\beta + \{(k+2, 3k+2), (0, k+2)\}$ to obtain $D_{A\setminus\{\infty_0,\infty_{3k-1}\},B} + \{D_{\{\infty_0\},B\setminus\{k+2,3k+2\}} + [f_1 - (k+2, 3k+2)]\}^{\mathrm{I}} + \{D_{\{\infty_{3k-1}\},B\setminus\{k+2,0\}} + [f_2 - (k+2, 0)]\}^{\mathrm{II}} - f_1 - f_2$ where f_1 and f_2 are two different, edge-disjoint directed 2-factors of $D_B - \vec{C}^{(2)}$ such that f_1 contains a 2-cycle $(k+2, 3k+2)$ and f_2 contains a 2-cycle $(k+2, 0)$. In addition, f_1 and f_2 do not contain any edges of difference $k+1$. By Lemma 2.1, there exist $\vec{C}_3$-decompositions for (I) and (II). Finally, $(D_B - \vec{C}^{(2)} - f_1 - f_2)^{\mathrm{III}} + D_{A\setminus\{\infty_0,\infty_{3k-1}\},B}$ is left. Since the edges of difference $k+1$ is not used, let $F_1 = \{(i, i+k+1, i+2k+2) \mid 0 \le i \le k\}$. Then $(D_B - \vec{C}^{(2)} - f_1 - f_2) - F_1$ is a $(3k - 2)$-regular directed graph and can be decomposed into a set of $(3k - 2)$ directed 2-factors, say $H_1, \dots, H_{3k-2}$ by applying Theorem 2.2. For each $i = 1, 2, \dots, 3k - 2$, $H_i + D_{\infty_i,B}$ can be decomposed into a set of directed 3-cycles (denoted by T_2^i) by Lemma 2.1. Thus, $(D_B - \vec{C}^{(2)} - f_1 - f_2) + D_{A\setminus\{\infty_0,\infty_{3k-1}\},B} = F_1 + \bigcup_{i=1}^{3k-2} T_2^i$. We give an example in Fig. 1.

For clarity, the $\vec{C}_3$-decomposition of $D_t - \vec{C}_t$ can be obtained by the following steps.

$$\begin{aligned} D_t - \vec{C}_t &= D_t - (\vec{C}_t + \alpha) + \alpha \\ &= D_t - (\vec{C}^{(1)} + \vec{C}^{(2)} + \beta + \{(k + 2, 3k + 2), (0, k + 2)\}) + \alpha \\ &= D_A + D_B + D_{A\setminus\{\infty_0,\infty_{3k-1}\},B} + D_{\{\infty_0\},B\setminus\{k+2,3k+2\}} + D_{\{\infty_{3k-1}\},B\setminus\{k+2,0\}} + \{(\infty_0, 3k + 2), (\infty_0, k + 2)\} \\ &\quad + \{(\infty_{3k-1}, k + 2), (\infty_{3k-1}, 0)\} - (\vec{C}^{(1)} + \vec{C}^{(2)} + \beta + \{(k + 2, 3k + 2), (0, k + 2)\}) + \alpha \\ &= (D_A - \vec{C}^{(1)}) + [(D_B - \vec{C}^{(2)} - f_1 - f_2) + D_{A\setminus\{\infty_0,\infty_{3k-1}\},B}] \\ &\quad + \{D_{\{\infty_0\},B\setminus\{k+2,3k+2\}} + [f_1 - (k + 2, 3k + 2)]\} + \{D_{\{\infty_{3k-1}\},B\setminus\{k+2,0\}} + [f_2 - (k + 2, 0)]\} + \alpha. \end{aligned}$$

Case (1.2). $t \equiv 0 \pmod 6$.

For $t \equiv 0 \pmod 6$, let $t = 6k$ and $V(D_t) = A \cup B$ where $A = \{\infty_i \mid i \in Z_{3k-1}\}$ and $B = \{i \mid i \in Z_{3k+1}\}$. Furthermore, let $\vec{C}_t = (\infty_0, \infty_1, \dots, \infty_{3k-2}, 0, 1, \dots, 3k)$. Now, by adding $\alpha = \{(k + 2, \infty_{3k-2}, \infty_0)(\infty_0, 3k, k + 2), (\infty_{3k-2}, k + 2, 0), (0, k + 2, 3k)\}$ to $\vec{C}_t$ where α is a set of four Mendelsohn triples, we obtain $\vec{C}_t + \alpha = \vec{C}^{(1)} + \vec{C}^{(2)} + \beta + \{(k + 2, 3k), (0, k + 2)\}$ where $\vec{C}^{(1)} = (\infty_0, \infty_1, \dots, \infty_{3k-2})$, $\vec{C}^{(2)} = (0, 1, \dots, 3k)$ and $\beta = \{(\infty_0, 3k), (\infty_0, k + 2), (\infty_{3k-2}, k + 2), (\infty_{3k-2}, 0)\}$. Moreover, let f_1 and f_2 be two different, edge-disjoint directed 2-factors of $D_B - \vec{C}^{(2)}$ such that

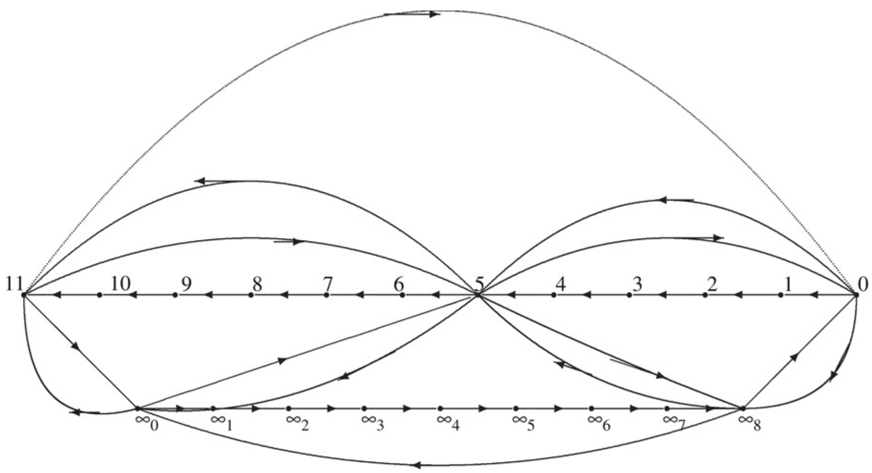

Fig. 1. $t = 21$ for $k = 3$.

f_1 contains a 2-cycle $(k+2, 3k)$ and f_2 contains a 2-cycle $(k+2, 0)$. Then, the main steps of $\vec{C}_3$-decomposition of $D_t - \vec{C}_t$ are as follows.

$$D_t - \vec{C}_t = \alpha + (D_A - \vec{C}^{(1)}) + [(D_B - \vec{C}^{(2)} - f_1 - f_2) + D_{A\setminus\{\infty_0,\infty_{3k-2}\},B}]^{\mathrm{I}}$$
$$+ \{D_{\{\infty_0\},B\setminus\{k+2,3k\}} + [f_1 - (k+2, 3k)]\}^{\mathrm{II}} + \{D_{\{\infty_{3k-2}\},B\setminus\{k+2,0\}} + [f_2 - (k+2, 0)]\}^{\mathrm{III}}.$$

Now, by induction, $D_A - \vec{C}^{(1)}$ has a $\vec{C}_3$-decomposition and the last two parts (II and III) also have $\vec{C}_3$-decompositions (by Lemma 2.1), so it is left to show that (I) has a $\vec{C}_3$-decomposition. $(D_B - \vec{C}^{(2)} - f_1 - f_2)$ is a $(3k-3)$-regular directed graph and by applying Theorem 2.2 and Lemma 2.1, $\vec{C}_3 | (D_B - \vec{C}^{(2)} - f_1 - f_2 + D_{A\setminus\{\infty_0,\infty_{3k-2}\},B})$.

Case (1.3). $t \equiv 2 \pmod 6$.

For $t \equiv 2 \pmod 6$, let $t = 6k+2$ and $V(D_t) = A \cup B$ where $A = \{\infty_i \mid i \in Z_{3k}\}$ and $B = \{i \mid i \in Z_{3k+2}\}$. Furthermore, let $\vec{C}_t = (\infty_0, \infty_1, \ldots, \infty_{3k-1}, 0, 1, \ldots, 3k+1)$. Now, by adding $\alpha = \{(k+2, \infty_{3k-1}, \infty_0), (\infty_0, 3k+1, k+2), (\infty_{3k-1}, k+2, 0), (0, k+2, 3k+1)\}$ to $\vec{C}_t$ where α is a set of four Mendelsohn triples, we obtain $\vec{C}_t + \alpha = \vec{C}^{(1)} + \vec{C}^{(2)} + \beta + \{(k+2, 3k+1), (0, k+2)\}$ where $\vec{C}^{(1)} = (\infty_0, \infty_1, \ldots, \infty_{3k-1})$, $\vec{C}^{(2)} = (0, 1, \ldots, 3k+1)$ and $\beta = \{(\infty_0, 3k+1), (\infty_0, k+2), (\infty_{3k-1}, k+2), (\infty_{3k-1}, 0)\}$. Moreover, let f_1 and f_2 be two different, edge-disjoint directed 2-factors of $D_B - \vec{C}^{(2)}$ where f_1 contains a 2-cycle $(k+2, 3k+1)$ and f_2 contains a 2-cycle $(k+2, 0)$. Then, the $\vec{C}_3$-decomposition of $D_t - \vec{C}_t$ can be obtained by the following steps.

$$D_t - \vec{C}_t = \alpha + (D_A - \vec{C}^{(1)}) + [(D_B - \vec{C}^{(2)} - f_1 - f_2) + D_{A\setminus\{\infty_0,\infty_{3k-1}\},B}]^{\mathrm{I}}$$
$$+ \{D_{\{\infty_0\},B\setminus\{k+2,3k+1\}} + [f_1 - (k+2, 3k+1)]\}^{\mathrm{II}} + \{D_{\{\infty_{3k-1}\},B\setminus\{k+2,0\}} + [f_2 - (k+2, 0)]\}^{\mathrm{III}}.$$

Now, by induction, $D_A - \vec{C}^{(1)}$ has a $\vec{C}_3$-decomposition and the last two parts: (II) and (III) also have $\vec{C}_3$-decompositions (by Lemma 2.1), it is left to show that the part (I) has a $\vec{C}_3$-decomposition. $(D_B - \vec{C}^{(2)} - f_1 - f_2)$ is a $(3k-2)$-regular directed graph and by applying Theorem 2.2 and Lemma 2.1, $\vec{C}_3 | (D_B - \vec{C}^{(2)} - f_1 - f_2 + D_{A\setminus\{\infty_0,\infty_{3k-1}\},B})$.

Case (1.4). $t \equiv 5 \pmod 6$.

For $t \equiv 5 \pmod 6$, let $t = 6k+5$ and let $V(D_t) = A \cup B$ where $A = \{\infty_i \mid i \in Z_{3k}\}$ and $B = \{i \mid i \in Z_{3k+5}\}$. Furthermore, let $\vec{C}_t = (\infty_0, \infty_1, \ldots, \infty_{3k-1}, 0, 1, \ldots, 3k+4)$. Now, by adding $\alpha = \{(k+2, \infty_{3k-1}, \infty_0), (\infty_0, 3k+4, k+2), (\infty_{3k-1}, k+2, 0), (0, k+2, 3k+4)\}$ to $\vec{C}_t$, we obtain $\vec{C}_t + \alpha = \vec{C}^{(1)} + \vec{C}^{(2)} + \beta + \{(k+2, 3k+4), (0, k+2)\}$ where $\vec{C}^{(1)} = (\infty_0, \infty_1, \ldots, \infty_{3k-1})$, $\vec{C}^{(2)} = (0, 1, \ldots, 3k+4)$ and $\beta = \{(\infty_0, 3k+4), (\infty_0, k+2), (\infty_{3k-1}, k+2), (\infty_{3k-1}, 0)\}$. Moreover, let f_1 and f_2 be two different, edge-disjoint directed 2-factors of $D_B - \vec{C}^{(2)}$ where f_1 contains a 2-cycle $(k+2, 3k+4)$ and f_2 contains a 2-cycle $(k+2, 0)$. In addition, f_1 and f_2 do not contain any edges of difference $1, 2, 3k+2$. Then, the $\vec{C}_3$-

decomposition of $D_t - \vec{C}_t$ can be obtained by the following steps.

$$D_t - \vec{C}_t = \alpha + (D_A - \vec{C}^{(1)}) + [(D_B - \vec{C}^{(2)} - f_1 - f_2) + D_{A\setminus\{\infty_0,\infty_{3k-1}\},B}]^{\mathrm{I}} + \{D_{\{\infty_0\},B\setminus\{k+2,3k+4\}} + [f_1 - (k+2, 3k+4)]\}^{\mathrm{II}} + \{D_{\{\infty_{3k-1}\},B\setminus\{k+2,0\}} + [f_2 - (k+2, 0)]\}^{\mathrm{III}}.$$

Now, by induction, $D_A - \vec{C}^{(1)}$ has a $\vec{C}_3$-decomposition and the two parts (II) and (III) also have $\vec{C}_3$-decompositions by Lemma 2.1. It is left to show that the part (I) has a $\vec{C}_3$-decomposition. Since the edges of differences $1, 2, 3k+2$ have not been used, let $F_1 = \{(i, i+1, i+3) \mid i \in Z_{3k+5}\}$. Then $(D_B - \vec{C}^{(2)} - f_1 - f_2) - F_1$ is a $(3k-2)$-regular directed graph and when associated with $D_{A\setminus\{\infty_0,\infty_{3k-1}\},B}$ can be decomposed into a set of directed 3-cycles (denoted by F_2) by applying Theorem 2.2 and Lemma 2.1. Thus, $(D_B - \vec{C}^{(2)} - f_1 - f_2) + D_{A\setminus\{\infty_0,\infty_{3k-1}\},B} = F_1 + F_2$.

Case 2. $t \equiv 1 \pmod 3$.

Since the case $t \equiv 4 \pmod 6$ has been settled by Lemma 2.8, it suffices to consider the case $t \equiv 1 \pmod 6$. Let $t = 6k+1$ and let $V(D_t) = A \cup B$ where $A = \{\infty_i \mid i \in Z_{3k-2}\}$ and $B = \{i \mid i \in Z_{3k+3}\}$. Furthermore, let $\vec{C}_{t-1} = (\infty_0, \infty_1, \ldots, \infty_{3k-4}, 0, 1, \ldots, 3k+2)$. Now, by adding $\alpha = \{(k+2, \infty_{3k-4}, \infty_0), (\infty_0, 3k+2, k+2), (\infty_{3k-4}, k+2, 0), (0, k+2, 3k+2)\}$ to $\vec{C}_{t-1}$, we obtain $\vec{C}_{t-1} + \alpha = \vec{C}^{(1)} + \vec{C}^{(2)} + \beta + \{(k+2, 3k+2), (0, k+2)\}$ where $\vec{C}^{(1)} = (\infty_0, \infty_1, \ldots, \infty_{3k-4})$, $\vec{C}^{(2)} = (0, 1, \ldots, 3k+2)$ and $\beta = \{(\infty_0, 3k+2), (\infty_0, k+2), (\infty_{3k-4}, k+2), (\infty_{3k-4}, 0)\}$. Moreover, let f_1 and f_2 be two different, edge-disjoint directed 2-factors of $D_B - \vec{C}^{(2)}$ where f_1 contains a 2-cycle $(k+2, 3k+2)$ and f_2 contains a 2-cycle $(k+2, 0)$. In addition, f_1 and f_2 do not contain any edges of difference $1, 2, 3k$.

Similar to Case (1.1), we can get $D_A - \vec{C}^{(1)}$. When k is odd, by induction, $D_A - \vec{C}^{(1)}$ has a $\vec{C}_3$-decomposition. When k is even, by Lemma 2.8, $D_A - \vec{C}^{(1)}$ has a $\vec{C}_3$-decomposition. We also associate $D_{A,B}$ with $\beta + \{(k+2, 3k+2), (0, k+2)\}$ and get two parts, one is: $\{D_{\{\infty_0\},B\setminus\{k+2,3k+2\}} + [f_1 - (k+2, 3k+2)]\}^{\mathrm{I}} + \{D_{\{\infty_{3k-4}\},B\setminus\{k+2,0\}} + [f_2 - (k+2, 0)]\}^{\mathrm{II}} - f_1 - f_2$. By Lemma 2.2, there exist $\vec{C}_3$-decompositions for (I) and (II) respectively. Finally, we have $(D_B - \vec{C}^{(2)} - f_1 - f_2) + D_{A\setminus\{\infty_0,\infty_{3k-4}\},B}$ left. Since the edges of differences 1, 2 and $3k$ have not been used, let $F_1 = \{(i, i+1, i+3) \mid i \in Z_{3k+3}\}$. Then $[(D_B - \vec{C}^{(2)} - f_1 - f_2) - F_1]^{\mathrm{III}}$ is a $(3k-4)$-regular directed graph. By Lemma 2.1 and Theorem 2.2, we can get a set of directed 3-cycles, denoted by F_2 by associating it with $D_{A\setminus\{\infty_0,\infty_{3k-4}\},B}$. Thus $(D_B - \vec{C}^{(2)} - f_1 - f_2) + D_{A\setminus\{\infty_0,\infty_{3k-4}\},B} = F_1 + F_2$.

Then, the $\vec{C}_3$-decomposition of $D_t - \vec{C}_t$ can be obtained by the following steps.

$$D_t - \vec{C}_t = (D_A - \vec{C}^{(1)}) + [(D_B - \vec{C}^{(2)} - f_1 - f_2) + D_{A\setminus\{\infty_0,\infty_{3k-4}\},B}] + \{D_{\{\infty_0\},B\setminus\{k+2,3k+2\}} + [f_1 - (k+2, 3k+2)]\} + \{D_{\{\infty_{3k-4}\},B\setminus\{k+2,0\}} + [f_2 - (k+2, 0)]\} + \alpha.$$

This concludes the proof of Theorem 2.3. □

3. Decomposing $D_t - G$

Finally, we will prove Theorem 3.1 by induction.

Theorem 3.1. *Let t be a positive integer and let G be a vertex-disjoint union of directed cycles in D_t while $G \neq \vec{C}_t$ when $t \equiv 0, 2, 3, 5 \pmod 6$ and $G \neq \vec{C}_{t-1}$ when $t \equiv 1 \pmod 3$. Suppose $G \neq \vec{C}_2 \cup \vec{C}_3$ or $\vec{C}_5$ if $t = 5$, and $G \neq \vec{C}_3 \cup \vec{C}_3$ if $t = 6$. Then $\vec{C}_3 | (D_t - G)$ if and only if $t(t-1) - |E(G)| \equiv 0 \pmod 3$.*

Proof. The necessity is obvious. We prove the sufficiency by induction on t. Assuming inductively that the assertion is true for values less than t, we shall prove that it is true for t.

First, we take the following partition of t.

(1) $t \equiv 0 \pmod 6$, $t = 6k = (3k+1) + (3k-1)$;
(2) $t \equiv 3 \pmod 6$, $t = 6k+3 = (3k+3) + 3k$;
(3) $t \equiv 2 \pmod 6$, $t = 6k+2 = (3k+3) + (3k-1)$;
(4) $t \equiv 4 \pmod 6$, $t = 6k+4 = (3k+4) + 3k$;
(5) $t \equiv 1 \pmod 6$, $t = 6k+1 = (3k+3) + (3k-2)$;
(6) $t \equiv 5 \pmod 6$, $t = 6k+5 = (3k+5) + 3k$.

In the following, we will discuss these cases one by one.

Case (1): $t \equiv 0 \pmod 6$.

Let $t = 6k$. By counting, $|E(G)| \equiv 0 \pmod 3$. Let $|E(G)| < 6k$ or $|E(G)| = 6k$. For the former, we can add a directed 3-cycle T to G, where $V(G) \cap V(T) = \emptyset$. This process can be repeated until $|E(G)| = 6k$. Therefore, it is enough to consider the case $|E(G)| = 6k$.

First, if G has a component G_1 such that $|E(G_1)| = 3k - 1$, then let $G_2 = G - G_1$ and $|E(G_2)| = 3k + 1$. We denote $V(G_1)$ by A and $V(G_2)$ by B. Then we have

$$\begin{aligned} D_t - G &= D_A + D_B - (G_1 + G_2) + D_{A,B} \\ &= (D_A - G_1) + (D_B - G_2) + D_{A,B}. \end{aligned}$$

By induction, $D_A - G_1$ has a $\vec{C}_3$-decomposition. Then $D_B - G_2$ is a $(3k - 1)$-regular directed graph and by Theorem 2.2 and Lemma 2.1, $\vec{C}_3 | (D_B - G_2 + D_{B,A})$.

Second, if we cannot get G_1 and G_2 satisfying the condition of the first case, we can rearrange our leave by adding three directed 3-cycles to get G_1^* and G_2^* satisfying $|G_1^*| = 3k-1$ and $|G_2^*| = 3k+1$. Even if all the components of G have cardinality $0 \pmod 3$, we can also rearrange our leave by adding three directed 3-cycles to get G_1^* and G_2^*.

Suppose $\min\{3k - 1 - |E(\bigcup_{\vec{C} \in G} \vec{C})|\} = l > 0$, then let $G_1 = \bigcup_{\vec{C} \in G} \vec{C}$ where $3k - 1 - |E(G_1)| = l$. Furthermore, let $G_2 = G - G_1$. In the following, we will choose a directed cycle $\vec{C}$ from G_2 and divide $\vec{C}$ into two parts: G_1^* and G_2^* such that $|G_1^*| = 3k - 1$ and $|G_2^*| = 3k + 1$. The details are as follows.

For any $\vec{C} \in G_2$, $|E(\vec{C})| > l$ holds. Otherwise, if there exists one cycle denoted by $\vec{C}' \in G_2$ and $|E(\vec{C}')| \le l$, we can get $3k - 1 - |E(G_1 \bigcup \vec{C}')| < l$, a contradiction to the construction of G_1. Choose $\vec{C} \in G_2$ where $\vec{C} = (x_1, x_2, \ldots, x_j)$ and $j \ge l + 1$. If $l = 1$, we can choose $j \ge 4$. Let $x_0 \in V(G_2 - \vec{C})$ and $\alpha = \{(x_0, x_{l+1}, x_l), (x_1, x_0, x_l), (x_j, x_0, x_1)\}$.

Then we have

$$\begin{aligned} \vec{C} + \alpha &= \beta + \vec{C}_l + \vec{C}_{j-l+1} \quad \text{where } \beta = \{(x_1, x_j), (x_1, x_0), (x_l, x_{l+1}), (x_l, x_0)\}, \\ &\quad \vec{C}_l = (x_1, x_2, \ldots, x_l) \text{ and } \vec{C}_{j-l+1} = (x_0, x_{l+1}, \ldots, x_j). \\ \vec{C}_{j-l+1} &= (x_0, x_{l+1}, \ldots, x_j) = (x_{l+1}, x_{l+2}, \ldots, x_j) - x_j x_{l+1} + x_0 x_{l+1} + x_j x_0 \\ &= \vec{C}_{j-l} - x_j x_{l+1} + x_0 x_{l+1} + x_j x_0 \quad \text{where } \vec{C}_{j-l} = (x_{l+1}, x_{l+2}, \ldots, x_j). \end{aligned}$$

Let $G_1^* = G_1 + \vec{C}_l$ and $G_2^* = G_2 - \vec{C} + \vec{C}_{j-l}$. Obviously, $|G_1^*| = 3k - 1$ and $|G_2^*| = 3k + 1$. We denote $V(G_1^*)$ by A and $V(G_2^*)$ by B. Then we have

$$\begin{aligned} D_t - G &= D_B + D_A + D_{A,B} + \alpha - [G_1 + (G_2 - \vec{C}) + (\vec{C} + \alpha)] \\ &= D_B + D_A + D_{A,B} - [G_1 + (G_2 - \vec{C}) + \beta + \vec{C}_l + \vec{C}_{j-l+1}] + \alpha \\ &= \{D_B - [(G_2 - \vec{C} + \vec{C}_{j-l}) - x_j x_{l+1} + x_0 x_{l+1} + x_j x_0]\} + [D_A - (G_1 + \vec{C}_l)] + (D_{A,B} - \beta) + \alpha \\ &= \{D_B - G_2^* + x_j x_{l+1} - x_0 x_{l+1} - x_j x_0 - [f_1 - (x_j, x_0)] - [f_2 - (x_{l+1}, x_0)]\} + (D_A - G_1^*) \\ &\quad + \{D_{\{x_1\}, B \setminus \{x_j, x_0\}} + [f_1 - (x_j, x_0)]\} + \{D_{\{x_l\}, B \setminus \{x_{l+1}, x_0\}} + [f_2 - (x_{l+1}, x_0)]\} + D_{A \setminus \{x_1, x_l\}, B} + \alpha \\ &= \{[D_B - G_2^* - f_1 - f_2 + (x_0, x_j, x_{l+1})] + D_{A \setminus \{x_1, x_l\}, B}\} + (D_A - G_1^*) \\ &\quad + \{D_{\{x_1\}, B \setminus \{x_j, x_0\}} + [f_1 - (x_j, x_0)]\} + \{D_{\{x_l\}, B \setminus \{x_{l+1}, x_0\}} + [f_2 - (x_{l+1}, x_0)]\} + \alpha. \end{aligned}$$

Note that f_1 and f_2 are two different, edge-disjoint directed 2-factors where f_1 contains (x_j, x_0) and f_2 contains (x_{l+1}, x_0). Moreover, f_1 and f_2 are defined on B.

Now, by induction, $D_A - G_1^*$ has a $\vec{C}_3$-decomposition. By Lemma 2.1, $\{D_{\{x_1\}, B \setminus \{x_j, x_0\}} + [f_1 - (x_j, x_0)]\}$ and $\{D_{\{x_l\}, B \setminus \{x_{l+1}, x_0\}} + [f_2 - (x_{l+1}, x_0)]\}$ have $\vec{C}_3$-decompositions. It is left to show that $(D_B - G_2^* - f_1 - f_2) + D_{A \setminus \{x_1, x_l\}, B}$ has a $\vec{C}_3$-decomposition. $D_B - G_2^* - f_1 - f_2$ is a directed $(3k - 3)$-regular graph and by Theorem 2.2 and Lemma 2.1, $\vec{C}_3 | (D_B - G_2^* - f_1 - f_2 + D_{A \setminus \{x_1, x_l\}, B})$.

We give an example in Fig. 2 as follows. The upper ovals represent cycles in G_2, the lower ovals represent the cycles in G_1 and Fig. 2 shows how to get G_1^* and G_2^*.

Case (2). $t \equiv 3 \pmod 6$.

Let $t = 6k + 3$, by counting, $|E(G)| \equiv 0 \pmod 3$. Similar to Case 1, we only consider the case $|E(G)| = 6k + 3$.

First, if G has a component G_1 such that $|E(G_1)| = 3k$, $G_2 = G - G_1$ and $|E(G_2)| = 3k + 3$. We denote $V(G_1)$ by A and $V(G_2)$ by B. Then we have

$$D_t - G = (D_A - G_1) + (D_B - G_2) + D_{B,A}.$$

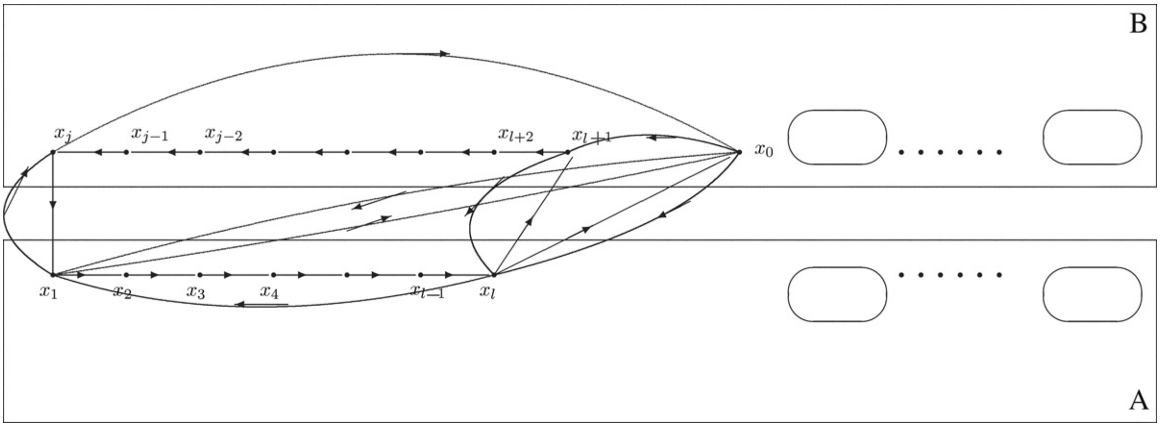

Fig. 2. $t = 6k$.

By induction, $D_A - G_1$ has a $\overrightarrow{C}_3$- decomposition. The difference triple $(k+1, k+1, k+1)$ (short orbit) whose corresponding set of directed 3-cycles is $F_1 = \{(i, i+k+1, i+2k+2) \mid 0 \le i \le k\}$ from $D_B - G_2$. Then $D_B - G_2 - F_1$ is a $3k$-regular directed graph and by Theorem 2.2 and Lemma 2.1, $\overrightarrow{C}_3 \mid (D_B - G_2 - F_1 + D_{B,A})$.

Second, if we cannot get G_1 and G_2 satisfying the condition of the first case, we can rearrange our leave by adding three directed 3-cycles, as follows, to get G_1^* and G_2^* satisfying $|G_1^*| = 3k$ and $|G_2^*| = 3k+3$.

Suppose $\min\{3k - |E(\bigcup_{\overrightarrow{C} \in G} \overrightarrow{C})|\} = l > 0$, then let $G_1 = \bigcup_{\overrightarrow{C} \in G} \overrightarrow{C}$ where $3k - |E(G_1)| = l$. Further, let $G_2 = G - G_1$. In the following, we will choose a directed cycle $\overrightarrow{C}$ from G_2 and divide $\overrightarrow{C}$ into two parts so that we can get G_1^* and G_2^*, $|G_1^*| = 3k$ and $|G_2^*| = 3k+3$. The detail can be found as follows.

For any $\overrightarrow{C} \in G_2$, $|E(\overrightarrow{C})| > l$ holds. Otherwise, if there exists one cycle denoted by $\overrightarrow{C}' \in G_2$ and $|E(\overrightarrow{C}')| \le l$, we can get $3k - |E(G_1 \bigcup \overrightarrow{C}')| < l$, a contradiction to the construction of G_1. Choose $\overrightarrow{C} \in G_2$ where $C = (x_1, x_2, \ldots, x_j)$ and $j \ge l+1$. Let $x_0 \in V(G_2 - \overrightarrow{C})$ and $\alpha = \{(x_0, x_{l+1}, x_l), (x_1, x_0, x_l), (x_j, x_0, x_1)\}$.

Then we have

$$\overrightarrow{C} + \alpha = \beta + \overrightarrow{C}_l + \overrightarrow{C}_{j-l+1} \quad \text{where } \beta = \{(x_1, x_j), (x_1, x_0), (x_l, x_{l+1}), (x_l, x_0)\},$$
$$\overrightarrow{C}_l = (x_1, x_2, \ldots, x_l) \text{ and } \overrightarrow{C}_{j-l+1} = (x_0, x_{l+1}, \ldots, x_j).$$
$$\overrightarrow{C}_{j-l+1} = (x_0, x_{l+1}, \ldots, x_j) = (x_{l+1}, x_{l+2}, \ldots, x_j) - x_j x_{l+1} + x_0 x_{l+1} + x_j x_0$$
$$= \overrightarrow{C}_{j-l} - x_j x_{l+1} + x_0 x_{l+1} + x_j x_0 \quad \text{where } \overrightarrow{C}_{j-l} = (x_{l+1}, x_{l+2}, \ldots, x_j).$$

Let $G_1^* = G_1 + \overrightarrow{C}_l$ and $G_2^* = G_2 - \overrightarrow{C} + \overrightarrow{C}_{j-l}$. Obviously, $|V(G_1^*)| = 3k$ and $|V(G_2^*)| = 3k+3$. We denote $V(G_1^*)$ by A and $V(G_2^*)$ by B.

Similar to Case 1, we have

$$D_t - G = \{[D_B - G_2^* - f_1 - f_2 + (x_0, x_j, x_{l+1})] + D_{A\setminus\{x_1,x_l\},B}\} + (D_A - G_1^*)$$
$$+ \{D_{\{x_1\},B\setminus\{x_j,x_0\}} + [f_1 - (x_j, x_0)]\} + \{D_{\{x_l\},B\setminus\{x_{l+1},x_0\}} + [f_2 - (x_{l+1}, x_0)]\} + \alpha.$$

Note f_1 and f_2 are two different, edge-disjoint directed 2-factors where f_1 contains (x_j, x_0) and f_2 contains (x_{l+1}, x_0). Moreover, f_1 and f_2 are defined on B.

Now, by induction, $D_A - G_1^*$ has a $\overrightarrow{C}_3$-decomposition. By Lemma 2.1, $\{D_{\{x_1\},B\setminus\{x_j,x_0\}} + [f_1 - (x_j, x_0)]\}$ and $\{D_{\{x_l\},B\setminus\{x_{l+1},x_0\}} + [f_2 - (x_{l+1}, x_0)]\}$ have a $\overrightarrow{C}_3$-decomposition.

It is left to show $(D_B - G_2^* - f_1 - f_2) + D_{A\setminus\{x_1,x_l\},B}$ has a $\overrightarrow{C}_3$-decomposition. The difference triple $(k+1, k+1, k+1)$ whose corresponding set of directed 3-cycles is $F_1 = \{(i, i+k+1, i+2k+2) \mid 0 \le i \le k\}$ can be obtained from $D_B - G_2^* - f_1 - f_2$. Then $D_B - G_2^* - f_1 - f_2 - F_1$ is a $(3k-2)$-regular directed graph. We associate it with $D_{A\setminus\{x_1,x_l\},B}$ to get a set of directed 3-cycles (denoted by F_2) by applying Theorem 2.2 and Lemma 2.1. Thus, $D_B - G_2^* - f_1 - f_2 + D_{A\setminus\{x_1,x_l\},B} = F_1 + F_2$.

Case (3). $t \equiv 2 \pmod 6$.

By counting, $|E(G)| \equiv 2 \pmod 3$, let $t = 6k+2$. Similar to Case 1, we only consider the case $|E(G)| = 6k+2$.

First, if G has a component G_1 such that $|E(G_1)| = 3k-1$, $G_2 = G - G_1$ and $|E(G_2)| = 3k+3$.

Then we have

$$D_t - G = (D_A - G_1) + (D_B - G_2) + D_{B,A}.$$

By induction, $D_A - G_1$ has a $\vec{C}_3$- decomposition. The difference triples $(k+1, k+1, k+1)$ and $(2k+2, 2k+2, 2k+2)$ from $D_B - G_2$ can form a set of directed 3-cycles $F_1 = \{(i, i+k+1, i+2k+2), (i, i+2k+2, i+4k+4) \mid 0 \le i \le k\}$. Then $D_B - G_2 - F_1$ is a $(3k-1)$-regular directed graph and when associated with $D_{B,A}$ can be decomposed into a set of directed 3-cycles (denoted by F_2) by applying Theorem 2.2 and Lemma 2.1. Thus, $D_B - G_2 + D_{B,A} = F_1 + F_2$.

Second, suppose $\min\{3k-1-|E(\bigcup_{\vec{C}\in G} \vec{C})|\} = l > 0$, then let $G_1 = \bigcup_{\vec{C}\in G} \vec{C}$ where $3k-1-|E(G_1)| = l$. The remainder of the proof of this case is similar Case 2.

Case (4). $t \equiv 4 \pmod 6$.

By counting $|E(G)| \equiv 0 \pmod 3$, let $t = 6k+4$. Similarly, as that in Case 1, we only consider the case $|E(G)| = 6k+3$.

First, if G has a component G_1 such that $|E(G_1)| = 3k$, $G_2 = G - G_1$, $|E(G_2)| = 3k+3$. We denote $V(G_2)$ by B and $V(D_{6k+4}) \setminus V(G_2)$ by A. Note $|V(D_A)| = 3k+1$.

Then we have

$$D_t - G = (D_A - G_1) + (D_B - G_2) + D_{B,A}.$$

By induction, $D_A - G_1$ has a $\vec{C}_3$- decomposition. The difference triples $(k+1, k+1, k+1)$ and $(2k+2, 2k+2, 2k+2)$ (short orbit) from $D_B - G_2$ can form a collection of directed 3-cycles $F_1 = \{(i, i+k+1, i+2k+2), (i, i+2k+2, i+4k+4) \mid i = 0, 1, \ldots, k\}$. Then $D_B - G_2 - F_1$ is a $3k$-regular directed graph and when associated with $D_{B,A}$ can be decomposed into a set of directed 3-cycles (denoted by F_2) by applying Theorem 2.2 and Lemma 2.1. Thus, $D_B - G_2 + D_{B,A} = F_1 + F_2$.

Second, suppose $\min\{3k-|E(\bigcup_{\vec{C}\in G} \vec{C})|\} = l > 0$, then let $G_1 = \bigcup_{\vec{C}\in G} \vec{C}$ where $3k-|E(G_1)| = l$. The remainder of the proof of this case is similar to Case 2.

Case (5). $t \equiv 1 \pmod 6$.

By counting, $|E(G)| \equiv 0 \pmod 3$, let $t = 6k+1$. Similar to Case 1, we only consider the case $|E(G)| = 6k$.

First, if G has a component G_1 such that $|E(G_1)| = 3k-3$, $G_2 = G - G_1$, $|E(G_2)| = 3k+3$. We denote $V(G_2)$ by B and $V(D_{6k+1}) \setminus V(G_2)$ by A. Note $|V(D_A)| = 3k-2$.

Then we have

$$D_t - G = (D_A - G_1) + (D_B - G_2) + D_{B,A}.$$

By induction, $D_A - G_1$ has a $\vec{C}_3$-decomposition. The difference triple $(1, 2, 3k)$ from $D_B - G_2$ can form a collection of directed 3-cycles $F_1 = \{(i, i+1, i+3) \mid i \in Z_{3k+3}\}$. Then $D_B - G_2 - F_1$ is a $(3k-2)$-regular directed graph. We associate with $D_{B,A}$ to get a set of directed 3-cycles (denoted by F_2) by applying Theorem 2.2 and Lemma 2.1. Thus, $D_B - G_2 + D_{B,A} = F_1 + F_2$.

Second, suppose $\min\{3k-3-|E(\bigcup_{\vec{C}\in G} \vec{C})|\} = l > 0$, then let $G_1 = \bigcup_{\vec{C}\in G} \vec{C}$ where $3k-2-|E(G_1)| = l$. The remainder of the proof of this case is similar to Case 2.

Case (6). $t \equiv 5 \pmod 6$.

Similar to Case 1, we only consider $t = 6k+5$ and $|E(G)| = 6k+5$. Then, the proof follows by a similar argument, we omit the details. □

4. Conclusion

Now, by combining Lemma 2.8, Theorems 2.3 and 3.1, we have proved our main result: Theorem 1.3. We also get Theorem 1.4 as corollary of Theorem 1.3.

Theorem 1.3. *Let t be a positive integer. Let G be a vertex-disjoint union of directed cycles in D_t and suppose $G \neq \vec{C}_2 \cup \vec{C}_3$ or $\vec{C}_5$ if $t = 5$ and $G \neq \vec{C}_3 \cup \vec{C}_3$ if $t = 6$. Then $\vec{C}_3 \mid (D_t - G)$ if and only if $t(t-1) - |E(G)| \equiv 0 \pmod 3$.*

Theorem 1.4. *Let t be a positive integer. Let G be a vertex-disjoint union of cycles in $2K_t$ and suppose $G \neq C_2 \cup C_3$ or $G \neq C_5$ if $t = 5$ and $G \neq C_3 \cup C_3$ if $t = 6$. Then $C_3 \mid (2K_t - G)$ if and only if $t(t-1) - |E(G)| \equiv 0 \pmod 3$.*

The covering of K_t with triangles was first considered by Colbourn and Rosa [3] and then by Fu, Fu and Rodger [4]. Mainly, they prove the following.

Theorem 4.1. *Let G be a 2-regular (not necessarily spanning) subgraph of K_t where t is odd. Then $C_3 \mid (K_t \cup G)$ if and only if the number of edges in $K_t \cup G$ is a multiple of 3.*

Now, by using the results we obtain in this paper, we are able to prove a digraph version. The details are omitted here.

Theorem 4.2. *Let t be a positive integer. Let G be a vertex-disjoint union of directed cycles in D_t and $|E(G)|$ be the number of edges of G. Then $\vec{C}_3 \mid (D_t \cup G)$ if and only if $t(t-1) + |E(G)| \equiv 0 \pmod 3$.*

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Appendix

Lemma 2.6. *There exist* $\overrightarrow{C}_3$*-decompositions for* $D_3 - \overrightarrow{C}_3$, $D_6 - \overrightarrow{C}_6$, $D_7 - \overrightarrow{C}_6$, $D_8 - \overrightarrow{C}_8$, $D_9 - \overrightarrow{C}_9$, $D_{11} - \overrightarrow{C}_{11}$, $D_{12} - \overrightarrow{C}_{12}$ *and* $D_{12} - (\overrightarrow{C}_6 \cup \overrightarrow{C}_6)$.

Proof. We give the proof by direct construction.

Since $\overrightarrow{C}_3|(D_3 - \overrightarrow{C}_3)$ is obvious, we only need to prove other cases.

Let D_6 be defined on Z_6 and take $\overrightarrow{C}_6 = (0, 1, 2, 3, 4, 5)$. Then $D_6 - \overrightarrow{C}_6 = \{(i, i+2, i+4) | i = 0, 1\} \cup \{(i, 3+i, 1+i) | i \in Z_6\}$.

Let D_7 be defined on Z_7 and take $\overrightarrow{C}_6 = (0, 1, 2, 3, 4, 5)$. Then $D_7 - \overrightarrow{C}_6 = \{(4, 0, 2), (1, 3, 2), (3, 5, 4), (3, 0, 6), (6, 4, 2), (5, 2, 0), (5, 6, 2), (6, 1, 4), (0, 4, 1), (6, 0, 3), (3, 1, 5), (1, 6, 5)\}$.

Let D_8 be defined on Z_8 and take $\overrightarrow{C}_8 = (0, 1, 2, 3, 4, 5, 6, 7)$. Then $D_8 - \overrightarrow{C}_8 = \{(5, 4, 3), (4, 7, 2), (5, 0, 2), (6, 0, 3), (1, 0, 5), (3, 0, 7), (3, 7, 1), (1, 5, 3), (6, 3, 2), (7, 6, 5), (7, 4, 1), (5, 2, 7), (6, 1, 4), (4, 0, 6), (2, 0, 4), (1, 6, 2)\}$.

Let D_9 be defined on Z_9 and take $\overrightarrow{C}_9 = (\infty_0, \infty_1, \infty_2, 0, 1, 2, 3, 4, 5)$. Then $D_9 - \overrightarrow{C}_9 = \{(3, \infty_2, \infty_0), (\infty_0, 5, 3), (\infty_2, 3, 0), (0, 3, 5), (\infty_0, 0, 2), (\infty_0, 2, 0), (\infty_0, 1, 4), (\infty_0, 4, 1), (\infty_2, 1, 5), (\infty_2, 5, 1), (\infty_2, 2, 4), (\infty_2, 4, 2), (\infty_1, 1, 0), (\infty_1, 0, 4), (\infty_1, 4, 3), (\infty_1, 3, 1), (\infty_1, 5, 2), (\infty_1, 2, 5), (5, 4, 0), (3, 2, 1), (\infty_0, \infty_2, \infty_1)\}$.

Let D_{11} be defined on Z_{11} and take $\overrightarrow{C}_{11} = (0, 1, 2, \ldots, 10)$. Then $D_{11} - \overrightarrow{C}_{11} = \{(0, 2, 7), (1, 3, 8), (3, 5, 10), (4, 6, 0), (5, 7, 1), (9, 4, 1), (0, 5, 2), (6, 8, 2), (7, 9, 3), (8, 10, 4), (10, 1, 6), (8, 3, 2), (4, 0, 8), (2, 10, 7), (4, 3, 6), (3, 9, 5), (0, 10, 5), (0, 7, 3), (6, 5, 8), (0, 3, 1), (4, 7, 5), (7, 10, 8), (8, 5, 1), (9, 2, 5), (4, 2, 1), (8, 0, 9), (9, 1, 10), (4, 10, 2), (10, 6, 3), (2, 9, 6), (6, 1, 7), (6, 9, 0), (4, 9, 7)\}$.

Let D_{12} be defined on Z_{12} and take $\overrightarrow{C}_{12} = (0, 1, \ldots, 11)$. Then $D_{12} - \overrightarrow{C}_{12} = \{(0, 2, 4), (5, 7, 9), (4, 6, 8), (8, 10, 0), (9, 11, 1), (3, 2, 7), (4, 3, 7), (5, 11, 8), (3, 10, 8), (7, 11, 9), (8, 7, 1), (9, 0, 3), (3, 6, 9), (8, 11, 2), (5, 4, 11), (6, 5, 2), (7, 6, 10), (7, 0, 4), (9, 4, 10), (5, 9, 2), (5, 10, 3), (5, 8, 0), (3, 8, 1), (3, 0, 11), (4, 1, 11), (2, 0, 10), (5, 3, 1), (2, 1, 6), (4, 8, 6), (6, 11, 10), (2, 11, 7), (4, 2, 10), (9, 6, 0), (4, 9, 1), (1, 0, 6), (2, 9, 8), (11, 6, 3), (1, 10, 5), (7, 5, 0), (10, 1, 7)\}$.

Let D_{12} be defined on Z_{12} and $\overrightarrow{C}_6 \cup \overrightarrow{C}_6$ can be obtained by difference 2. Then $D_{12} - (\overrightarrow{C}_6 \cup \overrightarrow{C}_6) = \{(i, i+4, i+8) | i = 0, 1, 2, 3\} \cup \{(i, i+3, i+1), (i, i+7, i+3), (i, i+1, i+6) | i \in Z_{12}\}$. □

Lemma 2.7. *Let G contain two directed cycles with one of them* $\overrightarrow{C}_4$ *or* $\overrightarrow{C}_5$*. Then, for* $t = 8, 9, 10, 11, 12$, $D_t - G$ *has a* $\overrightarrow{C}_3$*-decomposition if and only if* $t(t-1) - |E(G)| \equiv 0 \pmod 3$.

Proof. Case 1. $D_8 - (\overrightarrow{C}_4 \cup \overrightarrow{C}_4)$.

Let D_8 be defined on $\{a, b, c, d, e, f, g, h\}$ and take $\overrightarrow{C}_4 \cup \overrightarrow{C}_4 = (e, a, f, b) \cup (g, c, h, d)$. Then $D_8 - (\overrightarrow{C}_4 \cup \overrightarrow{C}_4) = \{(a, b, c), (b, a, d), (d, a, c), (d, c, b), (a, e, g), (b, f, h), (c, g, f), (d, h, e), (a, h, f), (b, g, e), (c, e, h), (d, f, g), (a, g, h), (d, e, f), (c, f, e), (b, h, g)\}$.

Case 2. $D_9 - (\overrightarrow{C}_4 \cup \overrightarrow{C}_5)$.

Let D_9 be defined on $\{a, b, c, d, e, f, g, h, i\}$ and take $\overrightarrow{C}_4 \cup \overrightarrow{C}_5 = (e, a, f, b) \cup (g, c, h, d, i)$. Then $D_9 - (\overrightarrow{C}_4 \cup \overrightarrow{C}_5) = \{(a, e, i), (b, f, g), (h, c, e), (d, h, f), (i, f, c), (h, i, b), (b, g, h), (d, e, g), (i, d, f), (e, b, i), (i, h, a), (a, g, f), (h, g, a), (f, e, c), (c, g, i), (d, g, e), (e, f, h), (a, b, c), (b, a, d), (d, a, c), (d, c, b)\}$. We denote the set of directed 3-cycles of $D_9 - (\overrightarrow{C}_4 \cup \overrightarrow{C}_5)$ as B which will be used later.

Case 3. $D_{10} - (\overrightarrow{C}_4 \cup \overrightarrow{C}_5)$.

Let D_{10} be defined on $\{a, b, c, d, e, f, g, h, i, j\}$ and take $\overrightarrow{C}_4 \cup \overrightarrow{C}_5 = (e, a, f, b) \cup (g, c, h, d, i)$. Then $D_{10} - (\overrightarrow{C}_4 \cup \overrightarrow{C}_5) = B - \{(e, f, h), (b, a, d), (c, g, i)\} + \{(j, e, f), (j, f, h), (j, h, e), (j, b, a), (j, a, d), (j, d, b), (j, c, g), (j, g, i), (j, i, c)\}$ where B is from Case 2.

Case 4. $D_{11} - (\overrightarrow{C}_4 \cup \overrightarrow{C}_7)$.

Let D_{11} be defined on $\{a, b, c, d, e, f, g, h, i, j, k\}$ and take $\overrightarrow{C}_4 \cup \overrightarrow{C}_7 = (h, j, i, k) \cup (a, b, c, d, e, f, g)$. Then $D_{11} - (\overrightarrow{C}_4 \cup \overrightarrow{C}_7) = \{(j, h, k), (k, g, b), (k, b, d), (k, d, f), (k, e, g), (i, g, c), (i, e, d), (i, d, g), (a, d, b), (a, g, f), (a, e, c), (j, c, f), (j, f, d), (j, d, a), (j, g, e), (j, e, b), (j, b, g), (h, a, f), (h, b, e), (h, e, a), (h, c, g), (h, g, d), (h, d, c), (i, a, k), (i, j, a), (a, c, k), (c, j, k), (h, i, b), (f, i, h), (b, i, f), (k, f, e), (i, c, e), (f, c, b)\}$.

Case 5. $D_{12} - (\overrightarrow{C}_4 \cup \overrightarrow{C}_8)$.

Let D_{12} be defined on $a, b, c, d, e, f, g, h, i, j, k, l$ and take $\overrightarrow{C}_4 \cup \overrightarrow{C}_8 = (k, i, l, j) \cup (a, b, c, d, e, f, g, h)$. Then $D_{12} - (\overrightarrow{C}_4 \cup \overrightarrow{C}_8) = \{(i, k, j), (l, i, j), (k, l, a), (k, a, l), (k, b, h), (k, h, f), (k, f, d), (k, d, g), (k, g, c), (k, c, e), (k, e, b), (l, b, d), (l, d, h),$

(l, h, c), (l, c, f), (l, f, e), (l, e, g), (l, g, b), (a, h, b), (a, e, d), (a, c, g), (a, d, f), (b, g, f), $(h, e, c)\} \cup B'$ where B' is obtained by joining i and j to the directed 2-regular graph defined on $\{a, b, c, d, e, f, g, h\}$,i.e. $B' = D_{\{i,j\},\{a,b,c,d,e,f,g,h\}} \cup (f, c, b, e, a) \cup (g, d, b, f, h) \cup (c, a, g, e, h, d)$. Case $D_{12} - \vec{C}_4 \cup \vec{C}_4 \cup \vec{C}_4$ is similar to this one.

Case 6. $D_{11} - (\vec{C}_5 \cup \vec{C}_6)$.

Let D_{11} be defined on $\{a, b, c, d, e, f, g, h, i, j, k\}$ and take $\vec{C}_5 \cup \vec{C}_6 = (a, b, c, d, e) \cup (f, g, h, i, j, k)$. Then $D_{11} - (\vec{C}_5 \cup \vec{C}_6) = \{(f, c, b), (f, b, a), (f, a, c), (k, e, d), (k, d, c), (k, c, e), (h, a, d), (a, h, d), (a, i, c), (e, i, a), (c, i, e), (j, b, d), (j, d, b), (g, b, e), (g, e, b), (a, k, j), (a, j, g), (a, g, k), (b, i, h), (b, h, k), (b, k, i), (c, h, g), (c, g, j), (c, j, h), (d, g, f), (d, f, i), (d, i, g), (e, f, h), (e, h, j), (e, j, f), (k, g, i), (f, j, i), (k, h, f)\}$.

Case 7. $D_{12} - (\vec{C}_5 \cup \vec{C}_7)$.

Let D_{12} be defined on $\{a, b, c, d, e, f, g, h, i, j, k, l\}$ and $\vec{C}_5 \cup \vec{C}_7 = (h, i, j, k, l) \cup (a, b, c, d, e, f, g)$. Then $D_{12} - (\vec{C}_5 \cup \vec{C}_7) = \{(h, j, i), (j, l, k), (j, h, l), (a, h, k), (a, k, h), (c, i, k), (c, k, i), (e, i, l), (e, l, i), (i, a, d), (i, d, g), (i, g, b), (i, b, f), (i, f, a), (k, b, g), (k, g, d), (k, d, f), (k, f, e), (k, e, b), (l, a, g), (l, g, f), (l, f, d), (l, d, c), (l, c, b), (l, b, a), (h, b, d), (h, d, b), (h, c, f), (h, f, c), (h, g, e), (h, e, g), (a, c, e), (a, e, c), (j, a, f), (j, f, b), (j, b, e), (j, e, d), (j, d, a), (j, c, g), (j, g, c)\}$. □

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