



Analysis of well residual drawdown after a constant-head test

Hund-Der Yeh *, Chih-Tse Wang

Institute of Environmental Engineering, National Chiao Tung University, Hsinchu 30010, Taiwan

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SUMMARY

A recovery test measures the residual drawdown after an aquifer pumping test has ended and analyzes the recovery data to determine hydrogeological parameters such as transmissivity and storage coefficient. To our knowledge, the solution for the distribution of residual drawdown following a constant-head test has never been presented. In this paper, we first develop a mathematical model that describes the residual drawdown taking into consideration the wellbore-storage effect and the drawdown distribution occurring at the end of a previous constant-head test. Then, the Laplace-domain solution of the model is developed using the Laplace transforms and its time-domain solution is obtained using the Stehfest algorithm. Numerical results show that the distribution of residual drawdown depends on the boundary condition related to the well drawdown and the initial condition related to the aquifer drawdown. The well residual drawdown (i.e., the residual drawdown at wellbore) during the early recovery period will be over-estimated by the approximate residual drawdown solution based on the Theis-type solution and superposition principle due to the neglect of wellbore storage. For a large recovery time, the effect of wellbore storage is negligible and the approximate residual drawdown solution is therefore applicable.

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Introduction

An aquifer test serves mainly to determine the hydrogeological parameters of a site under investigation. The pumping and constant-head tests are the two well-known aquifer tests; the former involves the withdrawal of water at a constant pumping rate while the latter involves the injection or withdrawal of water at the test well with a constant drawdown or buildup at the well throughout the test period. Note that the application of constant-head test is suitable to low-permeability aquifers which are mainly composed of silty or clayey material (Driscoll, 1986; Freeze and Cherry, 1979). The hydrogeological parameters of low-permeability aquifers can be determined by analyzing the measured pumping rate at the test well (Batu, 1998) or drawdown at the observation well (Mishra and Guyonnet, 1992) for the constant-head test. Studies of the drawdown solution for a constant-head test have been presented previously (e.g., van Everdingen and Hurst, 1949; Carslaw and Jaeger, 1959). However, the analytical solution for the drawdown distribution (Carslaw and Jaeger, 1959) developed in terms of an integral form is difficult to accurately resolve. It is because the integrand comprises the product and the square of the Bessel functions with a singularity at the origin. Peng et al. (2002) proposed a unified numerical method to evaluate the analytical solution and gave the dimensionless drawdown in a tabular form with better

accuracy while comparing with the approximate solutions given in Harvard's problem report, (1950) and Jaeger (1956). In addition, a number of studies have also been devoted to developing the approximate drawdown solution for a constant-head test (e.g., Ritchie and Sakakura, 1956; Mishra and Guyonnet, 1992; Hiller and Levy, 1994). Note that Mishra and Guyonnet (1992) proposed an approximate drawdown solution based on the Theis equation (Theis, 1935) and Boltzmann transformation technique. Applying the superposition principle, their Theis-type solution can also be used to analyze the constant-head recovery data, which is similar to the Theis recovery equation.

After the completion of a constant-head test, the water levels in the test well and observation well will start to rise/fall. Such increase/decrease in water levels is referred to as recovery, and the diminishing drawdown that occurs during the same recovery period is termed as the residual drawdown. The analysis of residual drawdown data can provide an independent check on the hydrogeological parameters determined from the previous drawdown data analysis. There are many studies that present data analyses for water-level recovery after constant pumping (e.g., Berg, 1975; Mishra and Chachadi, 1985; Goode, 1997; Batu, 1998; Shapiro et al., 1998; Samani and Pasandi, 2003; Singh, 2003; Todd and Mays, 2005; Willmann et al., 2007). However, the analysis of recovery data after a constant-head test has not been addressed before, and further the related field data are also not available.

The objective of this paper is to develop a mathematical model that describes the residual drawdown by taking into consideration

* Corresponding author. Tel.: +886 3 5731910; fax: +886 3 5725958.
E-mail address: hdych@mail.nctu.edu.tw (H.-D. Yeh).

the effects of the well radius and wellbore storage as well as the drawdown distribution from the previous constant-head test. This model uses the well drawdown at the test well as the boundary condition and the aquifer drawdown at the end of the constant-head test as the initial condition. A new solution to residual drawdown is then developed in the Laplace domain using the Laplace transform technique and its time-domain solution is obtained using the Stehfest algorithm (Stehfest, 1970). This solution can be employed to investigate the effect of the wellbore storage on the residual drawdown and to determine the aquifer parameters if coupled with an optimization algorithm (e.g., Lin and Yeh, 2005; Yeh and Chen, 2007; Yeh et al., 2007), extended Kalman filter method (e.g., Leng and Yeh, 2003; Yeh and Huang, 2005; Huang and Yeh, 2007), or nonlinear least-square method (e.g., Yeh, 1987; Yeh and Han, 1989). In addition, an approximate residual drawdown solution based on the Theis-type solution and superposition principle is also presented and discussed.

Description of the proposed method

A radial groundwater flow equation describing the drawdown distribution in a confined aquifer, which is homogeneous, isotropic, and uniform in thickness, can be written as (Yeh and Yang, 2006):

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} = \frac{S}{T} \frac{\partial s}{\partial t}, \quad r_w \leq r < \infty \text{ and } t > 0 \tag{1}$$

where $s(r, t)$ is the drawdown; t is the time; r is the radial distance from the centerline of the test well; r_w is the radius of the well screen; S and T are the storage coefficient and transmissivity of the aquifer, respectively. Note that the radius of well casing is denoted as r_c .

Assume that a constant-head test has been conducted at a test well by continuously pumping for a period of time. The water level in the test well is maintained at a constant s_0 (i.e., groundwater is withdrawn from the test well that has a fixed drawdown) and the aquifer drawdown is expressed as $s_1(r, t_1)$ at constant-head test time t_1 . Once the constant-head test has terminated, the water level in the test well rises gradually in time and therefore the well residual drawdown, $s_w(t_2)$, reduces with recovery time t_2 . The residual drawdown in the aquifer is expressed as $s_2(r, t_2)$. Consider that the initial drawdown for the constant-head test is zero everywhere (i.e., $s_1(r, t_1 = 0) = 0$) while the initial residual drawdown for the recovery test equals the final drawdown distribution (i.e., $s_2(r, t_2 = 0) = s_1(r, t_1 = t_h)$) at the completion of the constant-head test (at time t_h). Fig. 1 shows the schematic diagram of drawdown distributions at the start of the constant-head test and recovery test.

There is no wellbore-storage effect during the constant-head test because the water level in the test well is kept constant. How-

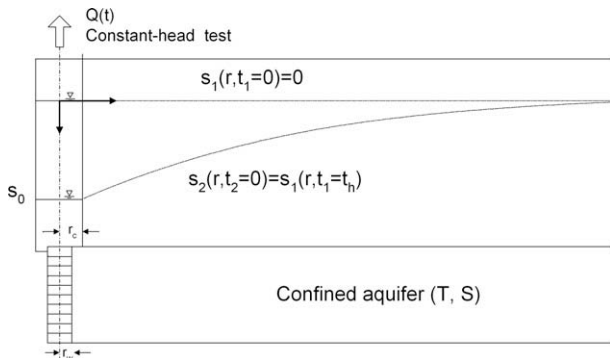


Fig. 1. Schematic diagram of the initial conditions of constant-head test and recovery test.

ever, this effect should be considered in the recovery test because the storage of the well casing will affect the water-level recovery in response to the change in aquifer residual drawdown. Therefore, it is considered in the analysis of residual drawdown in this study.

If a zero drawdown is maintained at an infinite distance from the well, the drawdown solution to Eq. (1), subject to the constant-head boundary condition at the test well, can be obtained from the analogous problem of heat conduction as (Carslaw and Jaeger, 1959, p. 335)

$$s_1(\rho, \tau_1) = s_0 - \frac{2s_0}{\pi} \int_0^\infty \exp\left(-\frac{\tau_1}{\alpha} x^2\right) \frac{Y_0(\rho x)J_0(x) - J_0(\rho x)Y_0(x)}{J_0^2(x) + Y_0^2(x)} \frac{dx}{x}, \quad \rho \geq 1 \tag{2}$$

where $\rho = r/r_w$ is the dimensionless radial distance; $\tau_1 = Tt_1/r_c^2$ is the dimensionless constant-head test time; $\alpha = r_w^2 S/r_c^2$ is the coefficient of wellbore storage; J_0 and Y_0 are the Bessel functions of the first and second kinds of order zero, respectively; and x is a dummy variable. Therefore, after the end of the constant-head test, the residual drawdown will start with $s_1(\rho, \tau_h)$ at the dimensionless time Tt_h/r_c^2 .

The initial conditions for the residual drawdown in the aquifer and the well residual drawdown are, respectively, denoted as

$$s_2(\rho, \tau_2 = 0) = s_1(\rho, \tau_h), \quad 1 \leq \rho < \infty \tag{3}$$

$$s_w(\tau_2 = 0) = s_0 \tag{4}$$

where $\tau_2 = Tt_2/r_c^2$ is the dimensionless recovery time which begins after the constant-head test has ended. Note that the initial condition for the residual drawdown, i.e., Eq. (3), is a function of the radial distance ρ and the constant-head test period, τ_h . In addition, the water starts to flow from the aquifer to the test well once the constant-head test has ended. The continuity requirement for the flow between the confined aquifer and the test well can be expressed as

$$\left. \frac{\partial s_2(\rho, \tau_2)}{\partial \rho} \right|_{\rho=1} = \frac{1}{2} \frac{ds_w(\tau_2)}{d\tau_2}, \quad \tau_2 > 0 \tag{5}$$

The inner and outer boundary conditions for the residual drawdown are, respectively,

$$s_2(\rho = 1, \tau_2) = s_w(\tau_2), \quad \tau_2 > 0 \tag{6}$$

$$s_2(\rho \rightarrow \infty, \tau_2) = 0, \quad \tau_2 > 0 \tag{7}$$

By applying the Laplace transforms, Eq. (1) subject to Eqs. (3)–(7) can be transformed into a non-homogeneous ordinary differential equation in the Laplace domain. The residual drawdown solution is then derived by the method of variation of parameters (Kreyszig, 2006). Appendix “Derivation of Eq. (8)” gives the derivation of the residual drawdown solution and the result is

$$s_2(\rho, \tau_2) = L^{-1}\{s_0\Delta + \Phi_1\Phi_2\Delta + (\Phi_2 - \Phi_3)I_0(\rho\lambda) + \Phi_4K_0(\rho\lambda)\} \tag{8}$$

with

$$\begin{aligned} \Delta &= \frac{K_0(\rho\lambda)}{pK_0(\lambda) + 2\lambda K_1(\lambda)} \\ \Phi_1 &= 2\lambda I_1(\lambda) - pI_0(\lambda) \\ \Phi_2 &= \int_1^\infty \alpha x s_1(x, \tau_h) K_0(\lambda x) dx \\ \Phi_3 &= \int_1^\rho \alpha x s_1(x, \tau_h) K_0(\lambda x) dx \\ \Phi_4 &= \int_1^\rho \alpha x s_1(x, \tau_h) I_0(\lambda x) dx \end{aligned}$$

where L^{-1} indicates the Laplace inverse, $\lambda = \sqrt{ap}$, and p is the Laplace variable.

The first term on the right-hand side (RHS) of Eq. (8) is contributed by the inner boundary condition while the remaining terms are contributed by the initial condition, which is in fact the aquifer drawdown produced by the previous constant-head test. Eq. (8) can be numerically inverted using the Stehfest algorithm (Stehfest, 1970). Note that the first term on the RHS of Eq. (8) is a function of the coefficient of wellbore storage α and the dimensionless recovery time τ_2 , which has no relationship with the initial condition of the recovery test $s_1(\rho, \tau_h)$. Eq. (8) therefore can reduce to the solution of Cooper et al. (Eq. (7), 1967) if $s_1(\rho, \tau_h)$ equals zero for $1 \leq \rho < \infty$.

Moreover, if $\rho = 1$, Eq. (8) can reduce to the well residual drawdown solution as

$$s_w(\tau_2) = L^{-1} \left\{ \frac{s_0 K_0(\sqrt{\alpha p})}{p K_0(\sqrt{\alpha p}) + 2\sqrt{\alpha p} K_1(\sqrt{\alpha p})} + \frac{2\alpha \int_1^\infty x s_1(x, \tau_h) K_0(x\sqrt{\alpha p}) dx}{p K_0(\sqrt{\alpha p}) + 2\sqrt{\alpha p} K_1(\sqrt{\alpha p})} \right\} \quad (9)$$

Note that Eq. (9) can be used to determine the hydrogeological parameters if coupled with an optimization algorithm in analyzing the recovery data.

MG drawdown solution and approximate residual drawdown solution

The constant-head drawdown solution, Eq. (2), is expressed in an integral form that contains an integration ranging from zero to infinity. The integration is difficult to evaluate analytically because the integrand is a function of the Bessel functions and has a singularity at the origin. Although Eq. (2) can be evaluated accurately using a numerical algorithm proposed in Peng et al. (2002) or Yang and Yeh (2002), a simplified formula approximating Eq. (2) should be developed for practical applications.

Mishra and Guyonnet (1992) presented a Theis-type approximate solution for a constant-head test to describe the drawdown distribution as

$$s_1(\rho, \tau_1) = \frac{s_0}{W(\alpha/4\tau_1)} W\left(\frac{\alpha\rho^2}{4\tau_1}\right), \quad \rho \geq 1 \quad (10)$$

where W is the Theis well function and $\alpha\rho^2/\tau_1$ is called the Boltzmann variable. The numerator $W(\alpha\rho^2/4\tau_1)$ is the drawdown distribution due to a constant-rate pumping while the denominator

$W(\alpha/4\tau_1)$ is the drawdown at the well. Therefore, the ratio $W(\alpha\rho^2/4\tau_1)/W(\alpha/4\tau_1)$ represents a normalized well function. Eq. (10), hereinafter called MG drawdown solution, can then be employed to approximately represent the drawdown distribution for a constant-head test.

With regard to the approximation of the residual drawdown distribution, the traditional analysis of a recovery test assumes that a hypothetical recharge well with a recharge rate equaling the pumping rate is used to replace the test well upon completion of the constant-head test (Todd and Mays, 2005). Similar to the Theis recovery equation, the approximate solution for residual drawdown based on the MG drawdown solution and superposition principle can be developed as:

$$s_2(\rho, \tau_2) = \frac{s_0}{W(\alpha/4\tau_h)} \left[W\left(\frac{\alpha\rho^2}{4(\tau_h + \tau_2)}\right) - W\left(\frac{\alpha\rho^2}{4\tau_2}\right) \right], \quad \rho \geq 1 \quad (11)$$

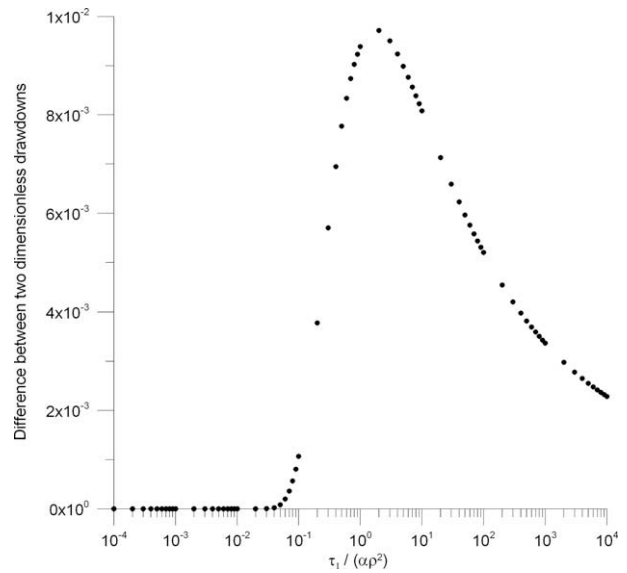


Fig. 3. Difference between Eqs. (2) and (10) as a function of dimensionless time variable $\tau_1/\alpha\rho^2$.

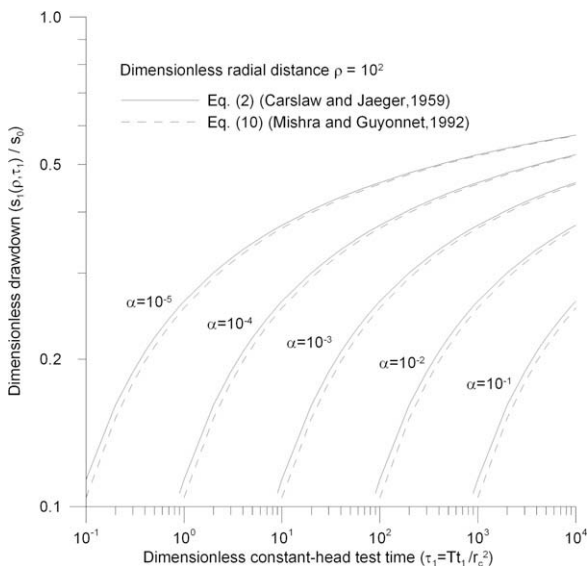


Fig. 2. Dimensionless drawdowns in Eq. (2) (solid lines) and Eq. (10) (dashed lines) versus dimensionless constant-head test time for α ranging from 10^{-5} to 10^{-1} .

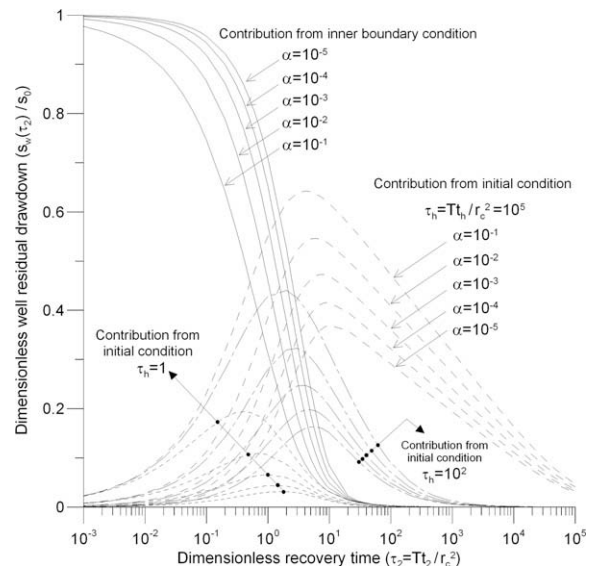


Fig. 4. Dimensionless well residual drawdown of Eq. (9) versus dimensionless recovery time for α ranging from 10^{-5} to 10^{-1} at $\tau_h = 1, 10^2$, and 10^5 , respectively.

This new approximate residual drawdown solution, Eq. (11), is much simpler than Eq. (8). Accordingly, Eq. (11) can approximate Eq. (8) if the effects of the well radius and wellbore storage are very small. Again, the approximate well residual drawdown solution in test well (i.e., $\rho = 1$) can be obtained as:

$$s_w(\tau_2) = \frac{s_0}{W(\alpha/4\tau_h)} \left[W\left(\frac{\alpha}{4(\tau_h + \tau_2)}\right) - W\left(\frac{\alpha}{4\tau_2}\right) \right] \quad (12)$$

In the next section, we examine the effect of wellbore storage on the well residual drawdown and compare Eq. (9) with Eq. (12) at the well, i.e., $\rho = 1$.

Analysis of well residual drawdown

Fig. 2 shows the comparison of the dimensionless drawdowns $s_1(\rho, \tau_1)/s_0$ in Eq. (2) to those in Eq. (10) as a function of the dimensionless test time τ_1 for α ranging from 10^{-5} to 10^{-1} and $\rho = 100$. The solid lines represent the analytical drawdown solution, Eq. (2), and the dashed lines stand for the MG drawdown solution,

Eq. (10). The difference in drawdown solutions decreases with increasing test time. Note that Eq. (2) is evaluated using the numerical approach presented in Peng et al. (2002).

Furthermore, Fig. 3 shows the difference in drawdown solutions versus the inverse of Boltzmann variable, $\tau_1/\alpha\rho^2$, representing a dimensionless time variable ranging from 10^{-4} to 10^4 . The largest difference is 9.72×10^{-3} at $\tau_1/\alpha\rho^2 = 2$, indicating the MG drawdown solution approximates the analytical drawdown solution with errors less than 1%. Mishra and Guyonnet (1992) conclude that their approximation is adequate for practical applications when the dimensionless variable $\tau_1/\alpha\rho^2 \geq 5$, which reveals an estimated error less than 8.99×10^{-3} .

In general, the radius of the well casing generally ranges from 0.05 to 0.25 m and the hydraulic conductivity for silty sand ranges from 8.6×10^{-3} to 8.6 m/day (Batu, 1998). Therefore, the dimensionless constant-head test time τ_1 ranges from $8.6 \cdot t_1$ to $8.6 \times 10^3 \cdot t_1$ (day) if $r_c = 0.10$ m and the thickness of confined aquifer is 10 m. In field applications, for example, τ_1 is usually larger than 1 while the period of constant-head test is more than 3 h.

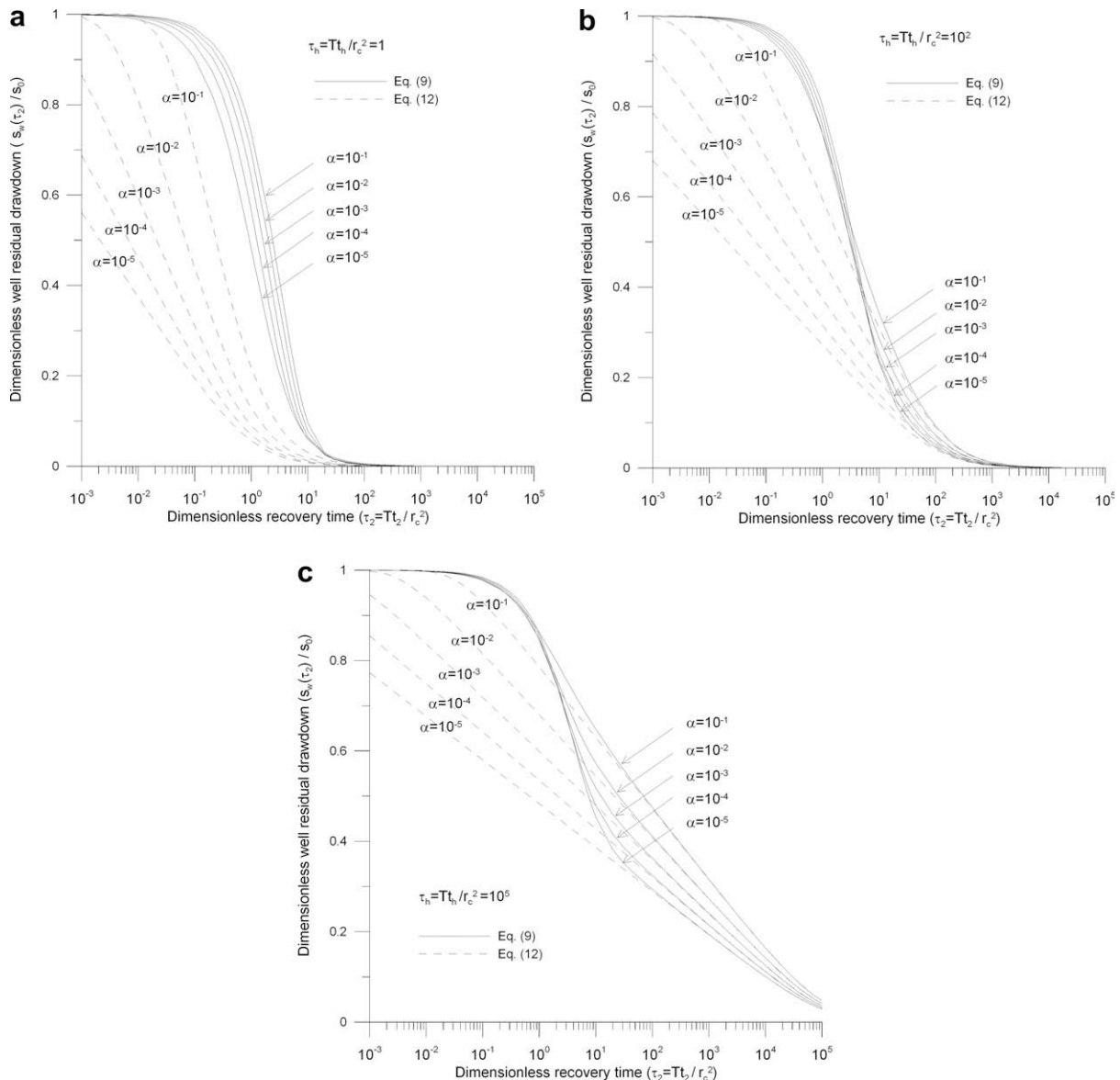


Fig. 5. Dimensionless well residual drawdowns predicted by Eq. (9) (solid lines) and Eq. (12) (dashed lines) versus dimensionless recovery time for α ranging from 10^{-5} to 10^{-1} at (a) $\tau_h = 1$, (b) $\tau_h = 10^2$, and (c) $\tau_h = 10^5$, respectively.

In addition, the value of storage coefficient generally falls within the range of 10^{-5} – 10^{-3} for confined aquifers. Thus, $\tau_1/\alpha\rho^2$ ranges from 10^{-1} to 10 times τ_1 for dimensionless radial distance $\rho = 100$. The difference between Eq. (2) and Eq. (10) is therefore less than 5×10^{-3} m if the constant-head drawdown s_0 is 0.5 m. Under this circumstance, the MG drawdown solution approximates the analytical drawdown solution reasonably well.

The second term on the RHS of Eq. (9) includes an integrand in terms of the product of drawdown distribution $s_1(\rho, \tau_h)$ and modified Bessel function $K_0(\rho\sqrt{\alpha p})$. It is rather difficult to accurately evaluate the integration for that term if the $s_1(\rho, \tau_h)$ is presented by Eq. (2). As mentioned before, the MG drawdown solution, Eq. (10), gives less than 1% error when approximating $s_1(\rho, \tau_h)$. Therefore, Eq. (9) along with Eq. (10) is used to evaluate the well residual drawdown and the results compared with those of the approximation solution, Eq. (12), are given in the following section.

Fig. 4 demonstrates the dimensionless well residual drawdown, i.e., $s_w(\tau_2)/s_0$ of Eq. (9), contributed by the inner boundary condition (solid line) and initial condition (dashed line) versus the dimensionless recovery time τ_2 for α ranging from 10^{-5} to 10^{-1} at $\tau_h = 1, 10^2,$ and 10^5 , respectively. As can be seen, the contribution from the initial condition increases with τ_2 at the beginning and then decreases with time. In addition, the peak value from the initial condition increases from 3.14×10^{-2} to 1.94×10^{-1} , 1.63×10^{-1} to 4.40×10^{-1} , and 3.70×10^{-1} to 6.43×10^{-1} at $\tau_h = 1, 10^2,$ and 10^5 , respectively, when α varies from 10^{-5} to 10^{-1} . This indicates that neglecting contribution from the initial condition may result in an error ranging from 1% to 60% when the period of the constant-head test τ_h varies from 1 to 10^5 . Note that the contribution from the inner boundary condition is independent of τ_h .

Fig. 4 also shows that the effect of α on the contribution from the inner boundary condition is opposite to that from the initial condition. If the diameter of well screen is equal to the well casing, then α reduces to S . The rise of well residual drawdown contributed from the inner boundary condition, shown as the solid line in Fig. 4, is faster for a larger S because a larger S can supply more water to the test well. The well residual drawdown contributed from the initial condition recovers slowly, shown as the dashed line in Fig. 4, for a larger S because a larger S requires more water to raise the residual drawdown of the aquifer.

Fig. 5a–c demonstrates the dimensionless well residual drawdown solution of Eqs. (9) and (12) versus the recovery time τ_2 with α ranging from 10^{-5} to 10^{-1} at $\tau_h = 1, 10^2,$ and 10^5 . As can be seen, Eq. (12) is not suitable for analyzing the recovery data at the beginning, especially when the period of constant-head test, τ_h , is very short. However, Eq. (12) gives a prediction error of less than 0.01 when τ_2 exceeds 50 for τ_h ranging from 1 to 10^5 . In addition, the solid lines in Fig. 5a–c represent the curves of dimensionless well residual drawdown versus dimensionless recovery time for the constant-head test period $\tau_h = 1, 10^2,$ and 10^5 , respectively.

Conclusions

In the past, Theis recovery method was commonly used to check the results obtained from the pumping test data analyzes. However, the residual drawdown solution and the analysis of the recovery data after a constant-head test have so far not been presented. In this paper, we develop a mathematical model that describes the residual drawdown solution for a recovery test in consideration of the wellbore-storage effect and the constant-head drawdown distribution caused by the previous constant-head test. The Laplace-domain and time-domain solutions of this model are obtained by the Laplace transforms and Stehfest algorithm, respectively. The new residual drawdown solution (Eq. (8)) shows that

the water contributing to the residual drawdown comes from two sources; one is from the inner boundary condition related to the well drawdown while the other is from the initial condition related to the aquifer drawdown produced by the previous constant-head test. Eq. (8) reduces to the solution obtained by Cooper et al. (1967) if a zero drawdown is used as the initial condition. The well residual drawdown (Eq. (9)) during the early recovery time will be over-estimated by the approximate well residual drawdown solution (Eq. (12)) due to the neglect of wellbore storage. For a large recovery time, the effect of wellbore storage is negligible and the approximate residual drawdown solution is therefore applicable. However, when the dimensionless recovery time is less than 50, the well residual drawdown solution is suggested to use because the difference between Eq. (9) and Eq. (12) exceeds 0.01 for τ_h ranging from 1 to 10^5 .

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Appendix A. Derivation of Eq. (8)

By taking the Laplace transform with respect to time, the subsidiary equations of Eq. (1) can be obtained as:

$$\frac{d^2 \bar{s}_2(\rho, p)}{d\rho^2} + \frac{1}{\rho} \frac{d\bar{s}_2(\rho, p)}{d\rho} = \alpha [p\bar{s}_2(\rho, p) - s_1(\rho, \tau_h)] \tag{A1}$$

where $\bar{s}_2(\rho, p)$ denotes the Laplace transform of $s_2(\rho, \tau_2)$, $s_1(\rho, \tau_h)$ is the initial condition of $s_2(\rho, \tau_2)$ shown in Eq. (3), and p is a Laplace variable. Similarly, the boundary conditions, i.e., Eqs. (5) and (6), after taking the Laplace transform become

$$\left. \frac{d\bar{s}_2(\rho, p)}{d\rho} \right|_{\rho=1} = \frac{1}{2\alpha} [p\bar{s}_2(\rho = 1, p) - s_0] \tag{A2}$$

$$\bar{s}_2(\infty, p) = 0 \tag{A3}$$

The solution to non-homogeneous linear ordinary differential equation (ODE), Eq. (A1), subjected to the boundary conditions, Eqs. (A2) and (A3), can be found by the method of variation of parameters (Kreyszig, 2006) as

$$\bar{s}_2(\rho, p) = [c_1 I_0(\rho\sqrt{\alpha p}) + c_2 K_0(\rho\sqrt{\alpha p})] + [\phi_1 I_0(\rho\sqrt{\alpha p}) + \phi_2 K_0(\rho\sqrt{\alpha p})] \tag{A4}$$

with

$$c_1 = \alpha \int_1^\infty x s_1(x, \tau_h) K_0(x\sqrt{\alpha p}) dx$$

$$c_2 = \frac{s_0 + [2\sqrt{\alpha p} I_1(\sqrt{\alpha p}) - p I_0(\sqrt{\alpha p})] \cdot [\alpha \int_1^\infty x s_1(x, \tau_h) K_0(x\sqrt{\alpha p}) dx]}{p K_0(\sqrt{\alpha p}) + 2\sqrt{\alpha p} K_1(\sqrt{\alpha p})}$$

$$\phi_1 = -\alpha \int_1^\rho x s_1(x, \tau_h) K_0(x\sqrt{\alpha p}) dx$$

$$\phi_2 = \alpha \int_1^\rho x s_1(x, \tau_h) I_0(x\sqrt{\alpha p}) dx$$

where I_0 and K_0 are the modified Bessel functions of the first and second kinds of order zero, respectively; and I_1 and K_1 are the modified Bessel functions of the first and second kinds of order one, respectively. Note that $I_0(\rho\sqrt{\alpha p})$ and $K_0(\rho\sqrt{\alpha p})$ form a basis for solution of the corresponding homogeneous ODE and the Wronski determinant (Kreyszig, 2006) of those two modified Bessel functions equals $-1/\rho$.

Eq. (A4) is a Laplace-domain solution which can be reduced to that of Cooper et al. (Eq. (7), 1967) if $s_1(\rho, \tau_h) = 0$ for $1 \leq \rho < \infty$. By taking the inverse Laplace transform, the time-domain solution for the residual drawdown can now be expressed as Eq. (8).

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