Global consensus for discrete-time competitive systems

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Abstract

Grossberg established a remarkable convergence theorem for a class of competitive systems without knowing and using Lyapunov function for the systems. We present the parallel investigations for the discrete-time version of the Grossberg's model. Through developing an extended component-competing analysis for the coupled system, without knowing a Lyapunov function and applying the LaSalle's invariance principle, the global pattern formation or the so-called global consensus for the system can be achieved. A numerical simulation is performed to illustrate the present theory.

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1. Introduction

"Convergence of dynamics" means that every solution tends to a single stationary solution, as time tends to infinity. One of the commonest ways to establish convergence of dynamics is to find a Lyapunov function for the system, that is, a continuous real-valued function V on state space, which is nonincreasing along trajectories of the system. One then applies the LaSalle's invariance principle to conclude the convergence. For example, Cohen and Grossberg [9] derived a convergence theorem for neural network systems of the form:

$$\dot{x}_i = a_i(\mathbf{x})\left[b_i(x_i) - \sum_{j=1}^{n} \omega_{ij} g_j(x_j)\right], \quad i = 1, 2, \ldots, n, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $a_i \geqslant 0$, $g_i' \geqslant 0$ for all j. There exists a Lyapunov function for (1):

$$V(\mathbf{x}) = -\sum_{i=1}^{n} \int_0^{x_i} b_i(\xi) g_i'(\xi) \mathrm{d}\xi + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij} g_i(x_i) g_i(x_j).$$

They showed that if $a_i > 0$, the matrix $[\omega_{ij}]$ of coupling weights is symmetric, and $g_i' > 0$ for every i, then V is a strict Lyapunov function and therefore the system is quasi-convergent, see also [19]. Forti and Tesi [10] proved the global stability for the Hopfield-type neural network of the form:

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$$\dot{x}_i = -\mu_i x_i + \sum_{j=1}^{n} T_{ij} g_j(x_j) + I_i, \tag{2}$$

where $\mu_i > 0$, g_i is a nondecreasing function, $i = 1, 2, \ldots, n$. Again, the results obtained therein employed a Lyapunov function of the so-called generalized Lur'e–Postnikov type. However, it is not always possible to find a suitable Lyapunov function when considering convergent dynamics.

Grossberg [11] considered a class of "competitive systems" of the form

$$\dot{x}_i = a_i(\mathbf{x})[b_i(x_i) - C(x_1, x_2, \ldots, x_n)], \quad i = 1, 2, \ldots, n, \tag{3}$$

where $a_i > 0, \partial C / \partial x_i \geqslant 0, \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. System (3) was proposed as a mathematical model for the resolution to a dilemma in science for hundred of years: How do arbitrarily many individuals, populations, or states, each obey unique and personal laws, succeed in harmoniously interacting with each other to form some sort of stable society, or collective mode of behavior. Such a system can have any number of competing populations, any interpopulation signal functions $b_i(x_i)$, any mean competition function, or adaptation level $C(\mathbf{x})$, and any state-dependent amplifications $a_i(\mathbf{x})$ of the competitive balance. A suitable Lyapunov function for system (3) is not known. The work in [11] employed a skillful component-competing analysis to prove that any initial value. $\mathbf{x}(0) \geqslant 0$ (i.e. $x_i(0) \geqslant 0$, for any i) evolves to a limiting pattern $\mathbf{x}(\infty) = (x_1(\infty), x_2(\infty), \ldots, x_n(\infty))$ with $0 \leqslant x_i(\infty) := \lim_{t \to \infty} x_i(t) < \infty$, under some conditions on a_i, b_i, C. Systems of the form (3) include the generalized Volterra–Lotka systems and an inhibitory network [14].

Recently, Cohen–Grossberg's model has attracted much scientific interests, cf. [1,8,15,21]. As we observe, Grossberg's model (3) has more general form than the Cohen–Grossberg's model (1). Indeed, while the coupling terms are in the form of linear summation in system (1), no special form is demanded for the adaptation level $C(x_1, x_2, \ldots, x_n)$ in (3). On the other hand, there have been growing interests in discrete-time systems, cf. [2–7,9,10,12,13,16–18]. These investigations include the discrete-time counterparts of the Hopfield's models and the cellular neural networks.

System (3) can be approximated, via Euler's difference scheme or delta-operator circuit implementation [13], by

$$x_i((k+1)\delta) = x_i(k\delta) + \delta a_i(\mathbf{x}(k\delta))[b_i(x_i(k\delta)) - C(\mathbf{x}(k\delta))], \tag{4}$$

where one takes $x_i(k\delta)$ as the kth iteration of x_i. In this study, we consider the following discrete-time version of the Grossberg's model, namely,

$$x_i(k+1) = x_i(k) + \beta a_i(\mathbf{x}(k))[b_i(x_i(k)) - C(\mathbf{x}(k))], \tag{5}$$

where $i = 1, 2, \ldots, n, k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. We first establish the theory for (5) with $\beta = 1$, i.e.

$$x_i(k+1) = x_i(k) + a_i(\mathbf{x}(k))[b_i(x_i(k)) - c(\mathbf{x}(k))]. \tag{6}$$

The results are then extended to (5). The main goal of this investigation is to explore the conditions on functions a_i, b_i, and C, under which system (5) or (6) possesses global limiting patterns $\mathbf{x}(\infty) := (x_1(\infty), x_2(\infty), \ldots, x_n(\infty))$ with $-\infty < x_i(\infty) := \lim_{t \to \infty} x_i(t) < \infty$ for every i, given any initial value $\mathbf{x}(0)$. Extension for the global consensus or global pattern formation from the continuous-time systems to the one for discrete-time models is by no means trivial. In fact, the orbits for the discrete-time dynamical systems are even more unpredictable. Additional elaborations need to be made in deriving the conditions to conclude the discrete-time global pattern formations.

Below, in Section 2, we state the main results of this presentation. In Section 3, we justify three key lemmas for Theorem 1 and give an example with numerical illustration. A conclusion is drawn in Section 4.

2. Main results

Definition 1 (*Global consensus*). A discrete-time competitive system $\mathbf{x}(k+1) = F(\mathbf{x}(k))$ is said to achieve *global consensus* (or *global pattern formation*) if, given any initial value $\mathbf{x}(0) \in \mathbb{R}^n$, the limit $x_i(\infty) := \lim_{k \to \infty} x_i(k)$ exists, for all $i = 1, 2, \ldots, n$.

We describe the following conditions for our main results.
Condition (A1): Each $a_i(\mathbf{x})$ is continuous, and

$$0 < a_i(\mathbf{x}) \leqslant 1, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n, \quad i = 1, 2, \ldots, n. \tag{7}$$

Condition (A2): $C(\mathbf{x})$ is bounded and continuously differentiable with bounded derivatives; namely, there exist constants M_1, M_2, r_j such that for all $\mathbf{x} \in \mathbb{R}^n$,

$$M_1 \leqslant C(\mathbf{x}) \leqslant M_2, \tag{8}$$

$$0 \leqslant \frac{\partial C}{\partial x_j}(\mathbf{x}) \leqslant r_j, \quad j = 1, 2, \ldots, n. \tag{9}$$

Condition (A3): $b_i(\xi)$ is continuously differentiable, strictly decreasing and there exist $d_i > 0$, $l_i, u_i \in \mathbb{R}$ such that for all $i = 1, 2, \ldots, n$.

$$-d_i \leqslant b_i'(\xi) < 0 \quad \text{for all } \xi \in \mathbb{R}, \tag{10}$$

$$b_i(\xi) > M_2, \quad \text{for } \xi \leqslant l_i, \quad \text{and} \quad b_i(\xi) < M_1, \quad \text{for } \xi \geqslant u_i. \tag{11}$$

Condition (A4): For $i = 1, 2, \ldots, n$,

$$0 < d_i \leqslant 1 - \sum_{j=1}^{n} r_j < 1. \tag{12}$$

For later uses, we set

$$d := \min\{d_i : i = 1, 2, \ldots, n\}, \tag{13}$$

$$M := \max\{|M_1|, |M_2|\}. \tag{14}$$

Theorem 1. *System* (6) *with a_i, b_i, and C satisfying conditions* (A1)–(A4) *achieves global consensus.*

The proof of Theorem 1 consists of three lemmas which will be stated below. We consider the following conditions for an extension of Theorem 1 to system (5).

Condition (A1)′: Each $a_i(\mathbf{x})$ is continuous, and there exists $B > 0$ such that

$$0 < a_i(\mathbf{x}) \leqslant B, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n, \quad i = 1, 2, \ldots, n. \tag{15}$$

Condition (A4)′: There exists $\beta > 0$ such that for $i = 1, 2, \ldots, n$,

$$0 < d_i \leqslant \frac{1}{\beta} - \sum_{j=1}^{n} r_j < \frac{1}{\beta}. \tag{16}$$

Condition (A4)″: There exist B, $\beta > 0$ such that for $i = 1, 2, \ldots, n$,

$$0 < d_i \leqslant \frac{1}{B\beta} - \sum_{j=1}^{n} r_j < \frac{1}{B\beta}. \quad \square \tag{17}$$

Corollary 1. *System* (5) *with a_i, b_i, and C satisfying conditions* (A1)–(A3), *and* (A4)′ *achieves global consensus.*

In fact, we only need that function a_i is continuous, positive and bounded above by some real number $B > 0$, for all i, instead of condition (A1). *This is due to that system* (6) *can be rewritten as*

$$x_i(k + 1) = x_i(k) + \frac{a_i(\mathbf{x}(k))}{B}[Bb_i(x_i(k)) - BC(\mathbf{x}(k))].$$

We thus obtain the following corollary.

Corollary 2. *System* (5) *with a_i, b_i, and C satisfying condition* (A1)′,(A2), (A3), *and* (A4)″ *achieves global consensus.*

Remark 1. From Corollary 2, we find that the smaller β in (5) (δ in (4)) is, the weaker restrictions on functions a_1, b_i, C are.

We define $\Delta x_i(k) := x_i(k + 1) - x_i(k)$, so that system (6) becomes

$$\Delta x_i(k) = a_i(\mathbf{x}(k))[b_i(x_i(k)) - C(\mathbf{x}(k))]. \tag{18}$$

In order to state the key lemmas for our main result, Theorem 1, we introduce some notations and definition:

$$g_i(k) = b_i(x_i(k)) - C(\mathbf{x}(k)), \quad \Delta g_i(k) = g_i(k + 1) - g_i(k),$$
$$\hat{g}(k) = \max\{g_i(k) : i = 1, 2, \ldots n\}, \quad \check{g}(k) = \min\{g_i(k) : i = 1, 2, \ldots n\},$$
$$I(k) = \min\{i : g_i(k) = \hat{g}(k)\}, \quad J(k) = \min\{i : g_i(k) = \check{g}(k)\},$$
$$\hat{x}(k) = x_{I(k)}(k), \quad \check{x}(k) = x_{J(k)}(k),$$
$$\hat{b}(k) = b_{I(k)}(\hat{x}(k)), \quad \check{b}(k) = b_{J(k)}(x(k)),$$
$$\Delta \hat{b}(k) = \hat{b}(k + 1) - \hat{b}(k), \quad \Delta \check{b}(k) = \check{b}(k + 1) - \check{b}(k),$$
$$\Delta b_i(x_i(k)) = b_i(x_i(k + 1)) - b_i(x_i(k)).$$

Definition 2. (i) A jump of type-1 is said to occur from i to j at the kth iteration if $I(k) = i$, $I(k + 1) = j$. (ii) A jump of type-2 is said to occur from i to j at the kth iteration if $J(k) = i$, $J(k + 1) = j$.

Lemma 1. *Consider system* (6) *with* a_i, b_i, *and* C *satisfying* (7), (8), (10) *and* (11). *Given any initial value* $\mathbf{x}(0) \in \mathbb{R}^n$, $\{\mathbf{x}(k)\}$ *will be attracted to some compact set in* $\mathbb{R}^n$. *Hence sequence* $\{x_i(k) | k \in \mathbb{N}_0\}$ *are bounded above and below for all* $i = 1, 2, \ldots, n$.

We consider an arbitrary orbit $\{\mathbf{x}(k) | k \in \mathbb{N}_0\}$. *Then, by* Lemma 1, $\{|a_i(\mathbf{x}(k))||k \in \mathbb{N}_0\}$ *is bounded below by some positive number, say* $0 < \rho_i \leqslant |a_i(\mathbf{x}(k))|$ *for all* $k \in \mathbb{N}_0$ *and* $\{b'(x_i(k)) | k \in \mathbb{N}_0\}$ *are bounded above by some negative number, say* $b'_i(x_i(k)) < -\varepsilon_i < 0$ *for all* $k \in \mathbb{N}_0$. *We define*

$$\rho := \min\{\rho_i : i = 1, 2, \ldots, n, \}, \quad \varepsilon := \min\{\varepsilon_i : i = 1, 2, \ldots, n\}. \tag{19}$$

Lemma 2. *Consider system* (6) *with* a_i, b_i, *and* C *satisfying* (7), (9), (10) *and* (12). *Then*

(I) *for function* $\hat{g}$, *either case* $(\hat{g}\text{-}(i))$ *or case* $(\hat{g}\text{-}(ii))$ *holds, where*

$(\hat{g}\text{-}(i)) : \hat{g}(k) < 0, \quad$ *for all* $k \in \mathbb{N}_0$,
$(\hat{g}\text{-}(ii)) : \hat{g}(k) \geqslant 0, \quad$ *for all* $k \geqslant K_1, \quad$ *for some* $K_1 \in \mathbb{N}_0$;

(II) *for function* $\breve{g}$, *either case* $(\breve{g} - (i))$ *or case* $(\breve{g}\text{-}(ii))$ *holds, where*

$(\breve{g}\text{-}(i)) : \breve{g}(k) < 0, \quad$ *for all* $k \in \mathbb{N}_0$,
$(\breve{g}\text{-}(ii)) : \breve{g}(k) \leqslant 0, \quad$ *for all* $k \geqslant K_2, \quad$ *for some* $K_2 \in \mathbb{N}_0$;

If Lemma 2 is valid, there are only four possibilities to consider.

Case (i): Both $(\hat{g}\text{-}(i))$ and $(\breve{g}\text{-}(i))$ hold. This case is impossible from our definition of $\hat{g}$ and $\breve{g}$.
Case (ii): Both $(\hat{g}\text{-}(i))$ and $(\breve{g}\text{-}(ii))$ hold. Then sequence $\{x_i(k)\}$ will always be nonincreasing as k is increasing, for all $i = 1, 2, \ldots, n$. By Lemma 1, $\{x_i(k)\}$ are bounded below for every i, hence the limit $x_i(\infty)$ exists, for every $i = 1, 2, \ldots, n$.
Case (iii): Both $(\hat{g}\text{-}(ii))$ and $(\breve{g}\text{-}(i))$ hold. Then sequence $\{x_i(k)\}$ will always increase as k increases, for all $i = 1, 2, \ldots, n$. By Lemma 1, $\{x_i(k)\}$ are bounded above for every i, hence the limit $x_i(\infty)$ exists, for every $i = 1, 2, \ldots, n$.
Case (iv): Both $(\breve{g}\text{-}(ii))$ and $(\hat{g}\text{-}(ii))$ hold.

Accordingly, we are left with the case (iv) only, for the conclusion of global consensus for system (6). We thus assume that $\hat{g}(0) \geqslant 0, \breve{g}(0) \leqslant 0$, without loss of generality.

Lemma 3. *Consider system* (6) *with* a_i, b_i, *and* C *satisfying conditions* $(A1)$–$(A4)$. *Then*
(i) $\lim_{k\to\infty}\hat{b}(k) = \lim_{k\to\infty} C(\mathbf{x}(k))$, *(ii)* $\lim_{k\to\infty}\breve{b}(k) = \lim_{k\to\infty} C(\mathbf{x}(k))$.

If Lemma 3 holds, then

$$\lim_{k\to\infty}\hat{b}(k) = \lim_{k\to\infty} C(\mathbf{x}(k)) = \lim_{k\to\infty}\breve{b}(k) =: \overline{B}. \tag{20}$$

On the other hand, $\breve{g}(k) \leqslant g_i(k) \leqslant \hat{g}(k)$, *for each* $i = 1, 2, \ldots, n$, *for all* $k \in \mathbb{N}_0$. *Equivalently,*

$$\breve{b}(k) - C(\mathbf{x}(k)) \leqslant b_i(x_i(k)) - C(\mathbf{x}(k)) \leqslant \hat{b}(k) - c(\mathbf{x}(k)),$$

for all $k \in \mathbb{N}_0$. *Thus,* $\breve{b}(k) \leqslant b_i(x_i(k)) \leqslant \hat{b}(k)$ *for all* $k \in \mathbb{N}_0$

Therefore

$$\lim_{k\to\infty}\breve{b}(k) \leqslant \lim_{k\to\infty} b_i(x_i(k)) \leqslant \lim_{k\to\infty}\hat{b}(k).$$

We obtain

$$\lim_{k\to\infty}\hat{b}(k) = \lim_{k\to\infty} b_i(x_i(k)) = \lim_{k\to\infty}\breve{b}(k) = \overline{B},$$

by (20). *Therefore we conclude that*

$$\lim_{k\to\infty} b_i(x_i(k)) = \overline{B}, \quad \text{for all } i = 1, 2, \ldots n. \tag{21}$$

Subsequently, $\lim_{k\to\infty} x_i(k)$ *exists, for every* $i = 1, 2, \ldots, n$, *by* (10) *and* (21). *Hence, global consensus of system* (6) *is achieved, if functions* a_i, b_i, *and* C *satisfy conditions* $(A1)$–$(A4)$.

3. Proofs of lemmas

Proof of Lemma 1. For a given initial vale $\mathbf{x}(0)$, we consider the iteration sequence $\{\mathbf{x}(k)\}$ and their components $x_i(k)$. We divide the proof into seven steps.

(i) By (8) and (11), $b_i(x_i) - C(\mathbf{x}) < 0$, for all $x_i \geqslant u_i$. Therefore

$$\Delta x_i(k) = a_i(\mathbf{x}(k))[b_i(x_i(k)) - C(\mathbf{x}(k))] < 0, \tag{22}$$

if $x_i(k) > u_i$. Similarly, $b_i(x_i) - C(\mathbf{x}) > 0$, for all $x_i \leqslant l_i$. By (8) and (11). Therefore

$$\Delta x_i(k) = a_i(\mathbf{x}(k))[b_i(x_i(k)) - C(\mathbf{x}(k))] > 0, \tag{23}$$

if $x_i(k) \leqslant l_i$. In addition, we have

$$|b_i(x_i(k))| \leqslant d_i|x_i(k)| + |b_i(0)|, \quad \text{for all } k \in \mathbb{N}_0, \tag{24}$$

since that $b_i(x_i(k)) - b_i(0) = b_i'(\cdot)x_i(k)$, and $|b_i(x_i(k))| \leqslant |b_i(0)| + d_i|x_i(k)|$, where " $\cdot$ " means some real number between $x_i(k)$ and 0.

(ii) Let L_i be fixed constants. Since $(d_i|x_i| + L_i)/|x_i| = d_i + L_i/|x_i| \to d_i < 1$, as $|x_i| \to \infty$ we have

$$d_i|x_i| + L_i < \tilde{d}_i|x_i|, \quad \text{if } |x_i| \geqslant \tilde{u}_i, \tag{25}$$

for some constants $\tilde{u}_i > 0$ and $\tilde{d}_i$, with $0 < d_i < \tilde{d}_i < 1$.

(iii) It follows from (7), (14), and (24) that

$$|\Delta x_i(k)| = |a_i(\mathbf{x}(k))[b_i(x_i(k)) - C(\mathbf{x}(k))]| \leqslant |b_i(x_i(k)) - C(x(k))| \leqslant d_i|x_i(k)| + |b_i(0)| + |C(\mathbf{x}(k))|$$
$$\leqslant d_i|x_i(k)| + |b_i(0)| + M.$$

Hence, by choosing $L_i = |b_i(0)| + M$ in (25), there exist constants $\tilde{u}_i$ and $\tilde{d}_i$ with $\tilde{u}_i > 0, 0 < d_i < \tilde{d}_i < 1$ such that

$$|\Delta x_i(k)| < \tilde{d}_i|x_i(k)| < |x_i(k)|, \quad \text{if } |x_i(k)| > \tilde{u}_i. \tag{26}$$

(iv) Set, for each $i, \tilde{q}_i := \max\{|u_i|, |l_i|, \tilde{u}_i\}$ Let $\widetilde{Q} := [-\tilde{q}_1, \tilde{q}_1] \times \ldots \times [-\tilde{q}_n, \tilde{q}_n]$. Then $\widetilde{Q}$ is a compact set, hence $|a_i(\mathbf{x})[b_i(x_i) - C(\mathbf{x})]|$ is bounded on $\widetilde{Q}$, say

$$|a_i(\mathbf{x})[b_i(x_i) - C(\mathbf{x})]| \leqslant K, \tag{27}$$

for all $\mathbf{x} \in \widetilde{Q}$, for all i. Set $q_i := \tilde{q}_i + K$, and $Q := [-q_1, q_2] \times \cdots \times [-q_n, q_n]$.

(v) We show that if $-q_i \leqslant x_i(0) \leqslant q_i$, then $-q_i < x_i(k) < q_i$, for all $k \in \mathbb{N}_0$.

Case (a): If $x_i(0) \in [-q_i, -\tilde{q}_i]$, then $\Delta x_i(0) > 0$, due to $x_i(0) \leqslant -\tilde{q}_i \leqslant l_i$; in addition, $|\Delta x_i(0)| < |x_i(0)|$, due to $x_i(0) \leqslant -\tilde{u}_i$. Hence $x_i(1)$ still stays in $(-q_i, \tilde{q}_i)$ or moves into $(-q_i, \tilde{q}_i)$ If the former case occurs, we consider $x_i(1)$ as case (a) again. If the latter case occurs, we consider $x_i(1)$ as in the following case (b).

Case (b): If $x_i(0) \in (-\tilde{q}_i, \tilde{q}_i)$, then $|\Delta x_i(0)| < K$, by (27). Hence $x_i(1)$ will stay in $[-q_i, -\tilde{q}_i]$ or $(-\tilde{q}_i, \tilde{q}_i)$ or $[\tilde{q}_i, q_i]$. Then we can still consider $x_i(1)$ as in case (a), case (b), and case (c), respectively.

Case (c): If $x_i(0) \in [\tilde{q}_i, q_i]$, then $\Delta x_i(0) < 0$, by $x_i(0) \geqslant \tilde{q}_i \geqslant \tilde{u}_i$ and $|\Delta x_i(0)| < |x_i(0)|$, by $x_i(0) \geqslant \tilde{u}_i$. Hence $x_i(1)$ still stays in $[\tilde{q}_i, q_i]$, or moves into $(-\tilde{q}_i, \tilde{q}_i)$. If the former case occurs, we consider $x_i(1)$ as in case (c) again. If the latter case occurs, we consider $x_i(1)$ as in case (b). From the above arguments, we find that if $-q_i \leqslant x_i(0) \leqslant q_i$, then $-q_i < x_i(1) < q_i$, and we can prove that $-q_i < x_i(k) < q_i$, for all $k \geqslant 2$, by induction.

(vi): If $x_i(0) < -q_i$, then

Case (d): $\{x_i(k)\}$ either increases as k increases and remains bounded above by $-q_i$;

Case (e): $\{x_i(k)\}$ enters $[-q_i, q_i]$ at some iteration, and never leaves $[-q_i, q_i]$ again.

(vii) if $x_i(0) > q_i$, then

case (f): $\{x_i(k)\}$ either decreases as k increases and remains bounded below by q_i;

case (g): $\{x_i(k)\}$ enters $[-q_i, q_i]$ at some iteration, and never leaves $[-q_i, q_i]$ again.

We find that no matter which of cases (d)–(g) occurs, $\{x_i(k) : k \in \text{open } \mathbb{N}_0\}$ are bounded above and below for all i. Therefore, $\{|a_i(\mathbf{x}(k))|\}$ are bounded below by some positive number, say $0 < \rho_i \leqslant |a_i(\mathbf{x}(k))|$, and $\{b_i'(x_i(k))\}$ are bounded above by some negative number, say $b_i'(x_i(k)) \leqslant -\varepsilon_i < 0$. In fact, it is impossible for the above case (d) and case (f) to occur. This is due to that if case (d) occurs, then

$$b_i(x_i(k)) - C(\mathbf{x}(k)) = b_i(x_i(k)) - b_i(l_i) + b_i(l_i) - C(\mathbf{x}(k)) > b_i(x_i(k)) - b_i(l_i) = b_i'(\cdot)[x_i(k) - l_i] \geqslant \varepsilon_i K$$

for all $x_i(k) \leqslant -q_i \leqslant l_i - K$, where "$\cdot$" means some real number between $x_i(k)$ and l_i. Therefore, $\Delta x_i(k) = a_i(\mathbf{x}(k))[b_i(x_i(k)) - C(\mathbf{x}(k))] > \varepsilon_i K \rho_i$. Hence $\{x_i(k)\}$ increases unboundedly and yields a contradiction. Therefore case (d) never occurs. Similarly, case (f) never occurs. By the arguments above, we find that given any initial value $\mathbf{x}(0), \{x(k) : k \in \mathbb{N}_0\}$ is attracted to set Q. $\square$

Proof of Lemma 2. For function $\hat{g}$, if $\hat{g}(k) \geqslant 0$ for some k, say $I(k) = i$, then $g_j(k) \leqslant g_i(k)$, for all $j \neq i$. Consider two possibilities: $|\Delta g_i(k)| \leqslant g_i(k)$, and $|\Delta g_i(k)| > g_i(k)$.

Case (i): $|\Delta g_i(k)| \leqslant g_i(k)$. It follows that
$$\hat{g}(k+1) \geqslant g_i(k+1) = g_i(k) + \Delta g_i(k) \geqslant 0.$$
Case (ii): $|\Delta g_i(k)| > g_i(k)$. Let us elaborate:

$$\Delta g_i(k) = g_i(k+1) - g_i(k) = b_i(x_i(k+1)) - C(\mathbf{x}(k+1)) - [b_i(x_i(k)) - C(\mathbf{x}(k))]$$
$$= b_i(x_i(k+1)) - b_i(x_i(k)) - [C(\mathbf{x}(k+1)) - C(x(k))]$$
$$= b_i'(\cdot)[x_i(k+1) - x_i(k)] - \sum_{j=1}^{n} \frac{\partial C}{\partial x_j}(\bullet)[x_j(k+1) - x_j(k)],$$

where "$\cdot$" means some real number between $x_i(k+1)$ and $x_i(k)$, "$\bullet$" means some point on the segment connecting $\mathbf{x}(k+1)$ and $\mathbf{x}(k)$. Thus,

$$\Delta g_i(k) = b_i'(\cdot)a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^{n} \frac{\partial C}{\partial x_j}(\bullet)a_j(\mathbf{x}(k))g_i(k) \geqslant -d_i a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^{n} r_j a_j(\mathbf{x}(k))g_i(k)$$

$$\geqslant -d_i g_i(k) - \sum_{j=1}^{n} r_j g_i(k) = \left[-d_i - \sum_{j=1}^{n} r_j\right] g_j(k) \geqslant -g_i(k),$$

by (7), (8), (10) and (12) and $g_i(k) \geqslant \max\{g_j(k), 0\}$, for all j. Hence $\Delta g_i(k) > 0$, since $|\Delta g_i(k)| > g_i(k)$ and $\Delta g_i(k) \geqslant -g_i(k)$. Therefore, $\hat{g}(k+1) \geqslant g_i(k+1) = g_i(k) + \Delta g_i(k) > 0$.

For function $\breve{g}$, if $\breve{g}(k) \leqslant 0$ for some k, say $J(k) = i$, then $g_j(k) \geqslant g_i(k)$, for all $j \neq i$. Hence either $|\Delta g_i(k)| \leqslant -g_i(k)$ or $|\Delta g_i(k)| > -g_i(k)$ holds.

Case (i): $|\Delta g_i(k)| \leqslant -g_i(k)$. It follows that $\breve{g}(k+1) \leqslant g_i(k+1) = g_i(k) + \Delta g_i(k) \leqslant 0$.
Case (ii): $|\Delta g_i(k)| > -g_i(k)$. We compute

$$\Delta g_i(k) = g_i(k+1) - g_i(k) = b_i(x_i(k+1)) - C(\mathbf{x}(k+1)) - [b_i(x_i(k)) - C(x(k))]$$
$$= b_i(x_i(k+1)) - b_i(x_i(k)) - [C(\mathbf{x}(k+1)) - C(\mathbf{x}(k))]$$
$$= b_i'(\cdot)[x_i(k+1) - x_i(k)] - \sum_{j=1}^{n} \frac{\partial C}{\partial x_j}(\bullet)[x_j(k+1) - x_j(k)],$$

where "$\cdot$" means some real number between $x_i(k+1)$ and $x_i(k)$, "$\bullet$" means some point on the segment connecting $\mathbf{x}(k+1)$ and $\mathbf{x}(k)$. Thus

$$|\Delta g_i(k)| = b_i'(\cdot)a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^{n} \frac{\partial C}{\partial x_j}(\bullet)a_j(\mathbf{x}(k))g_i(k) \leqslant -d_i a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^{n} r_j a_j(\mathbf{x}(k))g_i(k)$$

$$\leqslant -d_i g_i(k) - \sum_{j=1}^{n} r_j g_i(k) = \left[d_i + \sum_{j=1}^{n} r_j\right](-g_i(k)) \leqslant -g_i(k),$$

by (7), (8), (10) and (12) and $g_i(k) \leqslant \min\{g_j(k), 0\}$, for all j. Hence $\Delta g_i(k) < 0$, since $|\Delta g_i(k)| > -g_i(k)$ and $\Delta g_i(k) \leqslant -g_i(k)$. Therefore, $\breve{g}(k+1) \leqslant g_i(k+1) = g_i(k) + \Delta g_i(k) < 0$.

From the above arguments, we find that function $\hat{g}$ may keep negative at all iterations. But once it becomes nonnegative at some iteration, it will always remain nonnegative after that iteration. Similarly, $\breve{g}$ may keep positive at all iterations. But once it becomes non-positive at some iteration, it will always be nonpositive after that iteration. This completes the Proof of Lemma 2. With Lemma 2 and previous arguments, we assume that $\hat{g}(0) \geqslant 0, \breve{g}(0) \leqslant 0$, without loss of generality. $\square$

Proof of Lemma 3. We assert that $\lim_{k \to \infty} \hat{b}(k)$ exists, and denote it by $\widehat{B}$; moreover, we shall justify that $\lim_{k \to \infty} C(\mathbf{x}(k)) = \widehat{B}$.

Case (i): There exist finitely many jumps of type-1. In this case, there exist some $K_3 \in \mathbb{N}$, and some i, say $i = 1$, such that $\hat{g}(k) = g_1(k) \geqslant 0$, for all $k \geqslant K_3$. Hence $\{x_1(k)\}$ will be non-decreasing as k is increasing. By Lemma 1, $\{x_1(k)\}$ are bounded above. Therefore, $\lim_{k \to \infty} x_1(k)$ exists, hence $\lim_{k \to \infty} b_1(x_1(k))$ exists, denoted by $\breve{B}$. Equivalently, $\lim_{k \to \infty} \hat{b}(k) = \widehat{B}$. Next, we justify that $\lim_{k \to \infty} C(\mathbf{x}(k)) = \widehat{B}$. Assume otherwise, $\lim_{k \to \infty} C(\mathbf{x}(k)) \neq \widehat{B}$. It follows from $\hat{g}(k) = g_1(k) \geqslant 0$, for all $k \geqslant K_3$, that $b_1(x_1(k)) \geqslant C(\mathbf{x}(k))$, for all $k \geqslant K_3$. There exist some $\varepsilon > 0$, and a sequence $\{k_\ell\}_{\ell=1}^{\infty}$ of positive integers with $k_1 > K_3$ such that $|C(\mathbf{x}(k_\ell)) - \hat{B}| > \varepsilon$, for all $\ell \in \mathbb{N}$. Because $\lim_{k \to \infty} b_1(x_1(k)) = \widehat{B}$, for such ε, there exists $K_4 \in \mathbb{N}$, such that $|b_1(x_1(k)) - \widehat{B}| \leqslant \frac{\varepsilon}{2}$, for all $k \geqslant K_4$. Therefore $g_1(k_\ell) = b_1(x_1(k_\ell)) - C(\mathbf{x}(k_\ell)) > \frac{\varepsilon}{2}$, for all $k_\ell \geqslant K_4$. We find that $\{x_1(k)\}$ is always increasing after the K_4th iteration. In fact,

$$\Delta x_1(k_\ell) = a_1(\mathbf{x}(k_\ell))[b_i(x_i(k_\ell)) - C(\mathbf{x}(k_\ell))] > \rho \frac{\varepsilon}{2},$$

if $k_l \geqslant K_4$. Hence $\{x_1(k)\}$ will increase unboundedly, and yield a contradiction to Lemma 1.

Case (ii): There exist infinitely many jumps of type-1. We shall justify that $\{\hat{b}(k)\}$ decreases as $\{k\} \uparrow \infty$. Consider a fixed $k \in \mathbb{N}_0$.

Subcase (a): There is no jump of type-1 occurring at the kth iteration. Suppose $I(k) = I(k+1) = i$, then $g_i(k) \geqslant 0$, $g_i(k+1) \geqslant 0$. In addition, $\hat{b}(k+1) = b_i(x_i(k+1)) \leqslant b_i(x_i(k)) = \hat{b}(k)$, thanks to (10), and $\Delta x_i(k) = a_i(\mathbf{x}(k))g_i(k) \geqslant 0$. Thus $\{\hat{b}(k)\}$ decreases as k increases.

Subcase (b): There is a jump of type-1 occurring at the kth iteration and $g_i(k) \geqslant 0$, $g_j(k) \geqslant 0$, where $I(k) = i \neq I(k+1) = j$. We derive that $\hat{b}(k+1) = b_j(x_j(k+1)) \leqslant b_j(x_j(k)) \leqslant b_i(x_i(k)) = \hat{b}(k)$, due to (10), $\Delta x_j(k) = a_j(\mathbf{x}(k))g_j(k) \geqslant 0$, and by $I(k) = i \neq j$.

Subcase (c): There is a jump of type-1 occurring at the kth iteration and $g_i(k) \geqslant 0$, $g_j(k) < 0$, where $I(k) = i \neq I(k+1) = j$. Notably, we still have $g_j(k+1) \geqslant 0$. We claim that

$$b_j(x_j(k+1)) - b_j(x_j(k)) \leqslant b_i(x_i(k)) - b_j(x_j(k)). \tag{28}$$

Indeed, LHS $= b_j'(\cdot)\Delta x_j(k) = b_j'(\cdot)a_j(\mathbf{x}(k))g_j(k) \leqslant b_j'(\cdot)g_j(k) \leqslant -d_jg_j(k) \leqslant g_i(k) - g_j(k) = b_i(x_i(k)) - b_j(x_j(k)) =$ RHS, by (7), (10), $(1 - d_i)g_j(k) < 0 \leqslant g_i(k)$ and $g_j(k) < 0$. Herein, "$\cdot$" is as previous use. Hence, $\hat{b}(k+1) = b_j(x_j(k+1)) \leqslant b_i(x_i(k)) = \hat{b}(k)$.

All the above cases indicate that $\{\hat{b}(k)\}$ decreases as $\{k\}$ increases. By Lemma 1, $\{\mathbf{x}(k)\}$ are attracted into some compact set Q in $\mathbb{R}^n$. Therefore, $\{b_i(x_i(k))\}$ are bounded below, and so are $\{\hat{b}(k)\}$. Hence $\{\hat{b}(k)\}$ decreases and converges to some number $\widehat{B}$ as k tends to infinity (denoted by $\{\hat{b}(k)\} \downarrow \widehat{B}$).

Next, we verify that $\lim_{k \to \infty} C(\mathbf{x}(k)) = \widehat{B}$. Assume otherwise: $\lim_{k \to \infty} C(\mathbf{x}(k)) \neq \widehat{B}$, then there exist some $\mu > 0$, and a sequence $\{k_l\}_{l=1}^{\infty}$ of positive integers, such that

$$|C(\mathbf{x}(k_l)) - \widehat{B}| > \frac{\mu}{\varepsilon \rho}, \tag{29}$$

where ε, ρ are defined in (19). Because $\{\hat{b}(k)\} \downarrow \widehat{B}$, for $\lambda := \min\left\{\frac{\mu}{\varepsilon\rho}, \mu\right\} > 0$ there exists $L \in \mathbb{N}$ such that

$$\widehat{B} \leqslant b_{I(k)}(x_{I(k)}(k)) \leqslant \widehat{B} + \lambda, \tag{30}$$

for all $k \geqslant L$. Moreover

$$\hat{g}(\ell) = b_{I(\ell)}(x_{I(\ell)}(\ell)) - C(\mathbf{x}(\ell)) \geqslant 0, \tag{31}$$

for all $\ell \in \mathbb{R}$. Consider the k_Lth iteration and note that $k_L > L$. By (29)–(31), we have

$$\hat{g}(k_L) = b_1(x_1(k_L)) - C(\mathbf{x}(k_L)) > \frac{\mu}{\varepsilon\rho},$$

where, for convenience, we set $I(k_L) = 1$. The jump of type-1 may or may not occur at the k_Lth iteration. If it does not occur, then

$$|\Delta\hat{b}(k_L)| = |\hat{b}(k_L + 1) - \hat{b}(k_L)| = |b_1(x_1(k_L + 1)) - b_1(x_1(k_L))| = b_1'(\cdot)|x_1(k_L + 1) - x_1(k_L)|$$

$$= |b_1'(\cdot)||a_1(\mathbf{x}(k_L))||\hat{g}(k_L)| = b_1'(\cdot)||a_1(\mathbf{x}(k_L))||\hat{g}(k_L)| > \varepsilon\rho\frac{\mu}{\varepsilon\rho} = \mu.$$

But this is impossible, because of (30). Suppose a jump of type-1 takes place at the k_Lth iteration. Assume that $I(k_L + 1) = 2$. Below we consider three different cases for $b_2(x_2(k_L))$.

Subcase (a): $\hat{B} \leqslant b_2(x_2(k_L)) < b_1(x_1(k))$. Then $g_2(k_L) > \frac{\mu}{\varepsilon\rho}$, and $|\Delta b_2(x_2(k_L))| = |b_2'(\cdot)||a_2(\mathbf{x}(k_L))||g_2(k_L)| > \varepsilon\rho\frac{\mu}{\varepsilon\rho} = \mu$, which is impossible, due to (30).

Subcase (b): $\hat{B} > b_2(x_2(k_L)) \geqslant C(\mathbf{x}(k_L))$. Then $g_2(k_L \geqslant 0)$, and $x_2(k_L + 1) \geqslant x_2(k_L)$. Thus, $\hat{b}(k_L + 1) = b_2(x_2(k_L + 1)) \leqslant b_2(x_2(k_L)) < \hat{B}$, which is impossible, since $\{\hat{b}(k)\} \downarrow \hat{B}$.

Subcase (c): $b_2(x_2(k_L)) < C(\mathbf{x}(k_L))$. Then $g_2(k_L) < 0$, and

$$\Delta b_2(x_2(k_L)) = b_2(x_2(k_L + 1)) - b_2(x_2(k_L)) = b_2'(\cdot)a_2(\mathbf{x}(k_L))g_2(k_L) \leqslant -d_2 g_2(k_L) \leqslant -g_2(k_L).$$

Thus, $b_2(x_2(k_L + 1)) = b_2(x_2(k_L)) + \Delta b_2(x_2(k_L)) < b_2(x_2(k_L)) - g_2(k_L) = C(\mathbf{x}(k_L))$. Hence $\hat{b}(k_L + 1) = b_2(x_2(k_L + 1)) < C(\mathbf{x}(k_L)) < \hat{B}$, which is impossible, since $\{\hat{b}(k)\} \downarrow \hat{B}$.

From the above discussions, we conclude that $\lim_{k\to\infty} C(\mathbf{x}(k)) = \hat{B}$.

The second part of the lemma asserts that $\lim_{k\to\infty}\check{b}(k)$ exists, denoted by $\check{B}$, and $\lim_{k\to\infty} C(\mathbf{x}(fc)) = \check{B}$. The scheme of the proof parallels the first part and is thus omitted. □

We provide an example to illustrate our theory.

Example 1. Consider the following system in the form (6) with $n = 2$:

$$x_1(k + 1) = x_1(k) + [0.1\sin(x_1(k)) + 0.3]\{-0.2x_1(k) - [0.1\tanh(x_1(k)) + 0.2\tanh(x_2(k))]\}$$
$$x_2(k + 1) = x_2(k) + [0.1\sin(x_2(k)) + 0.3]\{-0.3x_2(k) - [0.1\tanh(x_1(k)) + 0.2\tanh(x_2(k))]\}.$$

We check that condition (A1) holds:

$$0 < a_1(\mathbf{x}) = 0.1\sin(x_1) + 0.3 \leqslant 1, 0 < a_2(\mathbf{x}) = 0.1\sin(x_2) + 0.3 \leqslant 1,$$

where $\mathbf{x} = (x_1, x_2)$. Condition (A2) also holds, since that with $M_1 = -0.3$, $M_2 = 0.3$, $r_i = 0.1$ and $r_2 = 0.2$, it follows that for all $\mathbf{x} = (x_1, x_2)$,

$$M_1 \leqslant C(\mathbf{x}) = 0.1\tanh(x_1) + 0.2\tanh(x_2) \leqslant M_2;$$
$$0 \leqslant \frac{\partial C}{\partial x_1}\mathbf{x} \leqslant r_1, \quad 0 \leqslant \frac{\partial C}{\partial x_2}\mathbf{x} \leqslant r_2.$$

In addition, condition (A3) holds, since that $b_1(\xi) = -0.2\xi$ and $b_2(\xi) = -0.3\xi$ are continuously differentiable, strictly decreasing and there exist $d_1 = 0.2$ and $d_2 = 0.3$ such that

$$-d_1 \leqslant b_1'(\xi) < 0, \quad -d_2 \leqslant b_1'(\xi) < 0 \quad \text{for all } \xi \in \mathbb{R}.$$

Moreover, (11) holds trivially. Finally, condition (A4) also holds obviously. Hence the system achieves global consensus, due to Theorem 1. The numerical simulation for the system with 15 initial points is illustrated in Fig. 1.
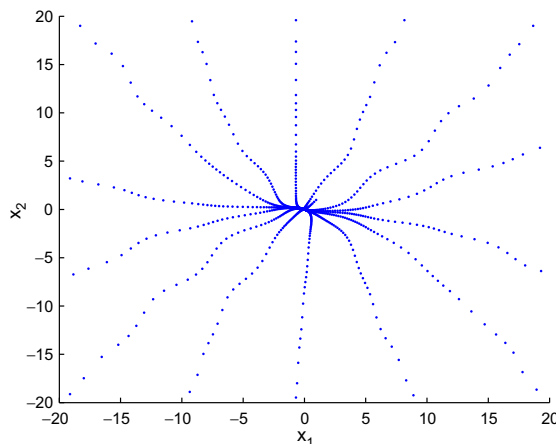

Fig. 1. Illustration for the orbits of Example 1.

4. Conclusions

For the investigations of stability and convergence for neural networks and other systems, in continuous-time or discrete-time, knowing a Lyapunov function and applying the associated arguments are very effective. However, finding a Lyapunov function is not always a feasible task, especially for those systems in general form. Indeed, it is rather difficult or even impossible to exploit a suitable Lyapunov function for system (3) and its discrete-time version (5).

In comparing the analysis for the proofs of Theorem 1 and the continuous model, we find that the behaviors of discrete-time orbits $\{\mathbf{x}(k)\}$ are more unpredictable than the ones for continuous-time orbits $\{\mathbf{x}(t)\}$. Hence, instead of that only upper growth $\hat{g}(t)$ for $g_i(t)$ is considered for the continuous-time case, both upper growth $\hat{g}(k)$ and lower growth $\check{g}(k)$ of $g_i(k)$ need to be controlled for the discrete-time situation. Accordingly, additional conditions on functions a_i, b_i, C are required for estimations of $\Delta x_i(k)$ and monotonicity of $\{\hat{b}(k)\}$ and $\{\check{b}(k)\}$, to achieve the global consensus for the discrete-time system (4).

In this presentation, we have successfully developed an analytical component-competing approach to conclude global pattern formation for the discrete-time model (5). The technique itself is an interesting and efficient methodology for studying stability and global dynamics for coupled systems. The component-competing analysis developed herein has also been applied to delayed recurrent neural networks successfully [20].

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