UNCERTAINTY ANALYSIS BY POINT ESTIMATE METHODS INCORPORATING MARGINAL DISTRIBUTIONS

By Che-Hao Chang,[1] Jinn-Chuang Yang,[2] Member, ASCE, and Yeou-Koung Tung,[3] Member, ASCE

ABSTRACT: The model performance of an engineering system is affected by many variables subject to uncertainty. Point estimate (PE) methods are practical tools to assess the uncertainty features of a model involving multivariate stochastic parameters. Two PE methods have been developed for engineering applications. One is Rosenblueth's PE method, which preserves the first three moments of random variables and the other is Harr's PE method, which reduces the computations of Rosenblueth's method but only as appropriate for application to random variables with normal distributions. In this study, two algorithms are proposed to encompass the advantages of the two PE methods: computational practicality and the handling of mixture distributions. Through a numerical experiment, the proposed methods yielded more accurate estimations than those of Rosenblueth's method with about the same amount of computation as Harr's method. The two proposed methods were also applied to estimate statistical moments of a pier scouring model output to demonstrate their performance in an engineering application.

Downloaded from ascelibrary.org by National Chiao Tung University on 04/30/14. Copyright ASCE. For personal use only; all rights reserved.

INTRODUCTION

In engineering design and analysis, one frequently uses models involving parameters that are subject to uncertainty. Therefore, model outputs on which engineering design and analysis are based are also subject to uncertainty. To perform uncertainty analysis for a model involving many stochastic parameters, point estimate (PE) methods are practical tools. Karmeshu and Lara-Rosano (1987) have shown that the first-order second-moment method is a special case of the PE methods when the uncertainties of random variables are small. The PE methods evaluate uncertainty of a model by computing the model responses at specified points in the parameter space. Proper points for model evaluation should be selected to preserve probabilistic information of the random variables.

Rosenblueth (1975) proposed a PE method for handling random variables with symmetric distributions, which was later extended to handle random variables with nonsymmetric distributions (Rosenblueth 1981). The algorithm, however, is computationally less attractive since the required model evaluations increase rapidly with the number of random variables. An alternative PE method is proposed by Harr (1989) to circumvent the computationally explosive nature of Rosenblueth's algorithm. Li (1992) and Zoppou and Li (1993) have also developed a new algorithm to hold the same order of accuracy as Rosenblueth's method and to reduce the amount of computation. However, the required computation is still far greater than Harr's algorithm unless the random variables are multivariate normal. Chang et al. (1995) showed that the estimated uncertainty feature of model output could be inaccurate if the skewness of a random variable is not accounted for. Nevertheless, the contribution of Harr's PE method to practical uncertainty analysis of engineering problems is valuable.

[1]Res., Energy and Resour. Lab., Industrial Technol. Res. Inst., Hsinchu, Taiwan, ROC.
[2]Prof., Dept. of Civil Engrg., Nat. Chiao-Tung Univ., Hsinchu, Taiwan, ROC.
[3]Prof., Wyoming Water Res. Ctr. and Statistics Dept., Univ. of Wyoming, Laramie, WY 82071.

Note. Discussion open until August 1, 1997. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on February 23, 1994. This paper is part of the ***Journal of Hydraulic Engineering***, Vol. 123, No. 3, March, 1997. ©ASCE, ISSN 0733-9429/97/0003-0244–0250/$4.00 + $.50 per page. Paper No. 7873.

By incorporating a set of semiempirical formulas developed by Der Kiureghian and Liu (1985), this study extended Harr's PE algorithm to allow handling random variables with a mixture of known marginal distributions. Based on the given information about the marginal distributions of random variables, these formulas transform the original nonnormal random variables into equivalent ones in the multivariate standard normal space. Therefore, in the equivalent multivariate standard normal space, the proposed methods, which adopt the fundamental concepts of the Harr algorithm, can operate properly. The selected points in the multivariate standard normal space are transformed back to the original parameter space for evaluating statistical moments of model outputs. Accordingly, the applicability of Harr's PE algorithm for uncertainty analysis is expanded to handle problems involving multivariate nonnormal random variables.

In this paper, two algorithms that consider different expansion points are proposed. The performance of the two proposed algorithms are examined through a numerical experiment and an application is made to a pier scouring model. Specifically, the proposed PE algorithms were compared with Rosenblueth's algorithms on the accuracy of uncertainty analysis under a number of stochastic parameters. Furthermore, the overall performances of the three PE methods were evaluated by fitting Johnson distribution (Johnson and Kotz 1970) curves based on the computed moments.

PROPOSED METHOD 1: MEDIAN-EXPANSION ALGORITHM

Multivariate Normal Space

In the median-expansion algorithm, the vector of stochastic parameters **X** having a multivariate normal distribution are standardized as

$$\mathbf{Y} = \mathbf{D}^{-0.5}(\mathbf{X} - \boldsymbol{\mu}) \quad (1)$$

in which **Y** = vector of the multivariate standard normal random variables; D = a diagonal matrix containing the variances of the stochastic parameters; and μ = vector of the mean values of **X**.

Through an orthogonal transformation, the correlated standard normal variables, **Y**, are decomposed into independent standard normal variables, **Z**, as

$$\mathbf{Z} = \mathbf{L}^{-0.5}\mathbf{V}'\mathbf{Y} = \mathbf{L}^{-0.5}\mathbf{U} \quad (2)$$

in which **U** = vector of uncorrelated random variables in the

J. Hydraul. Eng. 1997.123:244-250.

eigenspace having the mean 0; and covariance matrix **L**, with **L** and **V**, respectively, = eigenvalue and eigenvector matrices associated with the correlation matrix of the stochastic parameters $\mathbf{R}_X$. The eigenvector and eigenvalue matrices satisfy

$$\mathbf{R}_X = \mathbf{V}\mathbf{L}^t \tag{3}$$

where $\mathbf{V} = (v_1, v_2, \ldots, v_n)$, with $v_1, v_2, \ldots, v_n$ being the column vectors of the eigenvectors; and $\mathbf{L} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, with $\lambda_1, \lambda_2, \ldots, \lambda_n$ = corresponding eigenvalues. The transformations provided by (1) and (2) are linear. Therefore, if all the original stochastic parameters were normally distributed, the transformed parameter spaces for **U** are also normal.

In the median-expansion algorithm, a hypersphere with radius $\sqrt{n}$ centered at the origin in the n-dimensional standardized eigenspace is constructed. The points at which model output is to be evaluated are located at the intersections of the hypersphere and the eigenvectors of the correlation matrix of the stochastic parameters. For problems involving n stochastic parameters, model evaluations are performed at the total of $2n$ intersection points whereas $2n$ points are needed by Rosenblueth's method.

Due to the normal distribution and the same scale on each component in the standard space for **Z**, the $2n$ proposed points for model evaluation are located on a hypersurface with an equal probability density function (PDF) value. Fig. 1 schematically shows the point selections by Rosenblueth's algorithm and the proposed method for a bivariate case in the standard normal space. The selected points by the proposed method are located on the ellipse, which is a circle in the standardized eigenspace.

By (1) and (2), the points for model evaluation in the original multivariate normal variable space can be obtained as

$$\mathbf{x}_{k\pm} = \boldsymbol{\mu} \pm \sqrt{n}\mathbf{D}^{0.5}\mathbf{L}^{0.5}\boldsymbol{v}_k, \quad k = 1, \ldots, n \tag{4}$$

where $x_{k\pm} = (x_{k1\pm}, x_{k2\pm}, \ldots, x_{kn\pm})^t$ = a column vector containing the coordinates of the two intersection points on the kth eigenvector in the original normal variable space. At each selected point, the corresponding model output value $w_{k\pm} = g(x_{k\pm})$, for $k = 1$ to n, is computed.

The mth order moment about the origin of the model output is calculated as

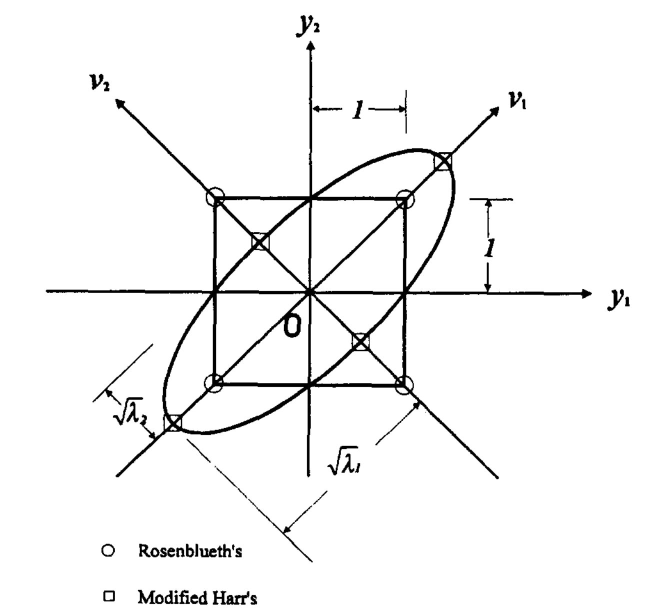

FIG. 1. Selections of Points for Model Evaluation by Different PE Methods

$$E(W^m) = \mu'_{W,m} = \frac{1}{n}\sum_{k=1}^{n}\overline{w_k^m} \tag{5}$$

in which

$$\overline{w_k^m} = \frac{w_{k+}^m + w_{k-}^m}{2} \tag{6}$$

The mth order central moment of the model output W, $\mu_{W,m}$, can be obtained by

$$\mu_m = \sum_{i=0}^{m}(-1)^i C_i^m \mu^i \mu'_{m-i} \tag{7}$$

where $C_i^m = m!/[i!(m-i)!]$, a binomial coefficient.

Incorporating Marginal Distributions of Random Variables

In many practical engineering problems, one often has to deal with random variables having different types of distribution. Such distributional information can have important implications on the results of engineering uncertainty and reliability analyses. The incorporation of marginal distributions information of random variables further enhanced the capability of Harr's PE algorithm, which presently accounts for the first two moments (including correlation) of the involved random variables.

For a mixture of correlated random variables (not necessarily all normal), the proposed methods incorporate the available marginal distribution information by using the set of semiempirical formulas derived by Der Kiureghian and Liu (1985). These formulas transform the correlation coefficient of a pair of nonnormal random variables to the equivalent one in the standard normal space. Through this transformation, the foregoing algorithm for multivariate normal parameters can be performed appropriately.

The distribution types for the correlated random variables pair the formulas are applicable to are shown in Fig. 2. Although the underlying distributions for the applied input data might not be known exactly, one can estimate the uncertainty due to the different selection of distribution types by using the proposed methods. The original multivariate nature of the applied data are not completely preserved, but the marginal distributions are.

The median-expansion algorithm consists of the following steps:

1. Transformation of correlation in nonnormal space to the equivalent normal space—the formulas transform the correlation from the original space to the standard normal space having a probability content equivalent to that of the original space by

$$\rho_{ij}^* = T_{ij} \cdot \rho_{ij} \tag{8}$$

in which ρ_{ij}^* = correlation between two standard normal random variables, Y_i and Y_j, whereas ρ_{ij} is the correlation between the nonnormal stochastic parameters X_i and X_j in the original space; and T_{ij} = a transformation factor that is a function of the marginal distributions and correlation of the two stochastic parameters considered. For each combination of the aforementioned distributions, one corresponding formula exists to compute T_{ij} (see Fig. 2). Given the marginal distributions and correlations for the stochastic parameters, the formulas of Der Kiuregh-ian and Liu (1985) compute the corresponding transformation factor to obtain the equivalent correlation ρ_{ij}^*. After all pairs of stochastic parameters are treated, the correlation matrix in the multivariate standard normal space, $\mathbf{R}_Y$, can be obtained.

Downloaded from ascelibrary.org by National Chiao Tung University on 04/30/14. Copyright ASCE. For personal use only; all rights reserved.

J. Hydraul. Eng. 1997.123:244-250.

Downloaded from ascelibrary.org by National Chiao Tung University on 04/30/14. Copyright ASCE. For personal use only; all rights reserved.

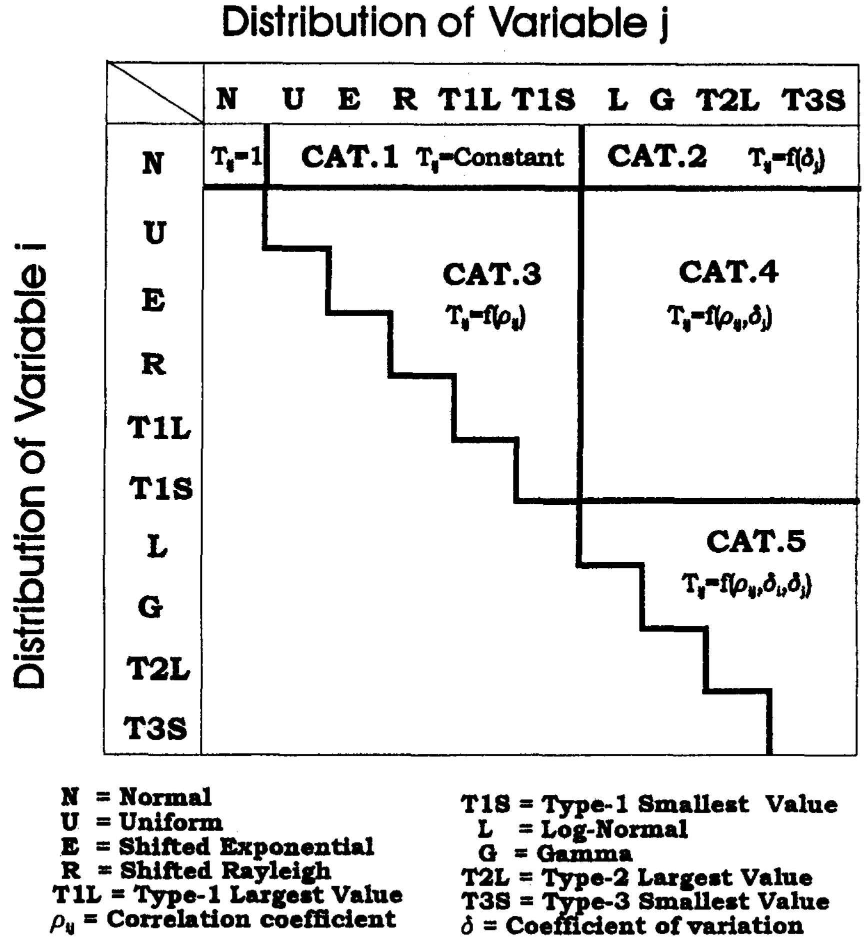

FIG. 2. Schematic Description of Categories of Transformation Factor T_{ij} (Chang et al. 1994)

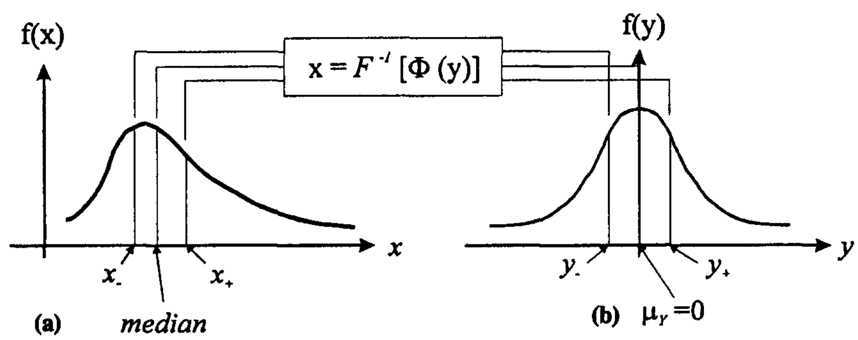

FIG. 3. Schematic Diagram of Transformation by Median-Expansion Algorithm between Nonnormal and Normal Spaces

2. Determine points for model evaluation in the standard normal space—through the transformation by (8), the operation domain is switched to the space in which the transformed random variables are treated as if they were multivariate standard normal random variables with the correlation matrix $\mathbf{R}_Y$. The transformed space is already standard normal. Therefore, the standardization by (1) in the median-expansion algorithm is not needed. Accordingly, (4) can be used to determine the points for model evaluation in the correlated standard normal space as

$$y_{k\pm} = \pm\sqrt{n}\mathbf{L}^{0.5}\mathbf{v}_k, \quad k = 1, \ldots, n \tag{9}$$

3. Inverse transformation—to generate appropriate points for model evaluations in the original space, the points selected in the standard normal space are transformed back to the original space for evaluating the corresponding model output values. The inverse transformation from the standard normal space to the original space can be established by preserving the probability content. The kth pair of selected points in the standard normal space, $y_{k\pm} = (y_{k1\pm}, y_{k2\pm}, \ldots, y_{kn\pm})^t$, can be transformed back to the original space as

$$x_{ki\pm} = F_i^{-1}[\Phi(y_{ki\pm})], \quad i = 1, 2, \ldots, n \tag{10}$$

in which $F_i(\ )$ = cumulative distribution function (CDF) of the ith parameter X_i; and $\Phi(\ )$ = standard normal CDF. Thus, the kth pair of the selected points in the original parameter space $x_{k\pm}$ are obtained for model evaluation. The moments of model output can be estimated by (5)–(7).

The selected points by the median-expansion algorithm for model evaluation are, in essence, the expansion with respect to the mean in the standard normal space that is also the median. Through the inverse transformation by (10), it preserves the median of each stochastic parameter in the original space as shown in Fig. 3. However, when the distribution of the

J. Hydraul. Eng. 1997.123:244-250.

Downloaded from ascelibrary.org by National Chiao Tung University on 04/30/14. Copyright ASCE. For personal use only; all rights reserved.

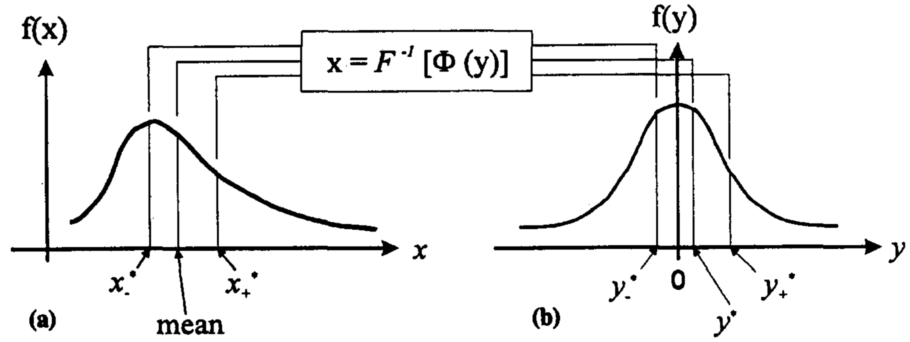

FIG. 4. Schematic Diagram of Transformation by Mean-Expansion Algorithm between Nonnormal and Normal Spaces

stochastic parameter in the original space was not symmetric, the mean and median were different in the original space.

PROPOSED METHOD 2: MEAN-EXPANSION ALGORITHM

The mean-expansion algorithm adjusts the expansion point from the median to the mean of the original stochastic parameters. In doing so, the property of equal PDF for each selected point would no longer be held. Therefore, (5) and (6) must be modified.

Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)'$ represent the means of the stochastic parameters in the original space. In the standard normal space, $\mathbf{y}^* = (y_1^*, y_2^*, \ldots, y_n^*)'$, the equivalent point for the mean of the original distributions can be determined by inverting (10) as

$$y_i^* = \Phi_i^{-1}[F(\mu_i)], \quad i = 1-n \tag{11}$$

Then, the mean-expansion algorithm selects points for model evaluation around the point $\mathbf{y}^*$. Fig. 4 shows the idea of the mean-expansion algorithm for a univariate case. More specifically, the selected points that encompass the origin in the standard normal space is now shifted with respect to $\mathbf{y}^*$. Consequently, the equal PDF is no longer valid at the shifted points. Therefore, for the two kth shifted points in the standard multivariate normal space, $\mathbf{y}_{k\pm}^* = (y_{k1\pm}^*, y_{k2\pm}^*, \ldots, y_{kn\pm}^*)'$, the associated PDF values are used as the weighing factors, $\alpha_{k\pm}$

$$\alpha_{k\pm} = \exp\left(-\frac{1}{2} z_{k\pm}^{*\prime} \mathbf{R}_Z^{-1} z_{k\pm}^*\right) \tag{12}$$

which is the exponential part of the multivariate standard normal PDF since the remaining part is a constant. Using (10), the shifted points in the original space can be obtained. The model output values are computed at these points and are weighed by (12) to estimate the moments. That is, (5) for computing moments of the model output is modified as

$$E(W^m) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\alpha_{k+} + \alpha_{k-}} (\alpha_{k+} W_{k+}^m + \alpha_{k-} W_{k-}^m) \tag{13}$$

NUMERICAL EXPERIMENT AND RESULTS

Experiment

This section describes the experiment for examining the performance of the proposed PE methods as compared with that of Rosenblueth's algorithm. The number of stochastic parameters involved are the main focus to examine its effect on the performance of the three PE methods. Two to 10 stochastic parameters were adopted for this experiment with the model type as $W = \Sigma_i\, iX_i$.

In the numerical experiment, all stochastic parameters were assumed to have lognormal distributions. The mean values and the standard deviations of all stochastic parameters were set to 1.0 and 0.3, respectively. Consequently, the skew coefficient for each stochastic parameter was $\gamma = (0.3)^3 + 3(0.3) = 0.927$. For simplicity, the correlation between the two stochastic parameters, X_i and X_j, was set as

$$\rho_{ij} = 0.9^{|i-j|} \tag{14}$$

Performance Evaluation

Based on the distributional properties of the involved stochastic parameters, 30,000 samples were generated by the multivariate Monte Carlo simulation with known marginal distribution (Chang et al. 1994). In the performance evaluation, the statistical characteristics of the model output from the Monte Carlo simulation were adopted as the true uncertainty features based on which the first four statistical moments of the model output computed by the PE methods were compared. Since the collective behavior of statistical moments can be demonstrated by the corresponding distribution curve, the performance evaluation further examined the goodness of fit of two Johnson distribution curves that were fitted by using the moments from the PE methods and the true values from the Monte Carlo simulation, respectively. The Johnson distribution was adopted for its versatility, covering a great variety of commonly known distributions. The algorithm to determine the Johnson distribution curve based on the first four moments of a random variable was developed by Hill et al. (1976).

Three criteria were used to compare the relative performance as a whole of the three PE methods:

1. Biasness (BIAS)
2. Mean absolute error (MAE)
3. Root mean squared error (RMSE)

$$\text{BIAS} = \int_0^1 (w_{e,p} - w_{t,p})\, dp \tag{15}$$

$$\text{MAE} = \int_0^1 |w_{e,p} - w_{t,p}|\, dp \tag{16}$$

$$\text{RMSE} = \left[\int_0^1 (w_{e,p} - w_{t,p})^2\, dp\right]^{1/2} \tag{17}$$

where $w_{t,p}$ = value of the pth order quantile for the assumed true probability distribution; and $w_{e,p}$ = estimated quantile value. The true quantile values, $w_{t,p}$, were computed using the moments from the Monte Carlo simulation along with the Johnson distribution, whereas the estimated quantiles, $w_{e,p}$, were based on the moments from the PE methods. The integration in the three performance criteria, (15)–(17), was done numerically at several discrete probability values: 0.01, 0.025, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.975, and 0.99. The quantiles at these probability levels were obtained to calculate the values of the three performance criteria.

J. Hydraul. Eng. 1997.123:244-250.

Downloaded from ascelibrary.org by National Chiao Tung University on 04/30/14. Copyright ASCE. For personal use only; all rights reserved.

Results and Comments

Table 1 shows the comparisons of statistical moments estimated by the three PE methods with those of the "true" values from the Monte Carlo simulation (with the superscript "a") for the two model types. The error percentages of estimated moment values by the three PE methods are shown in parentheses below the estimated values. The error percentages were computed by

$$e_{PE} = \frac{\theta_{PE} - \theta_{SIMUL}}{\theta_{SIMUL}} \times 100\% \qquad (18)$$

where θ_{PE} and θ_{SIMUL} = estimated moments by a PE method and the Monte Carlo simulation, respectively.

The relative performance of the three PE methods can be made based on the information provided in Tables 1 and 2. From Table 1, one observes that the three PE methods yield rather close estimations of the mean values for the model output, W. Moreover, the two proposed PE methods provide a better estimation of the standard deviation for W than Rosenblueth's method. Except for the kurtosis obtained by Rosenblueth's method, the accuracy of estimation by the three PE methods deteriorates as the order of moment and the number of stochastic parameters involved increase. Furthermore, the three PE methods tend to overestimate the third and fourth moments, with an increase in the number of stochastic parameters involved. Comparing the two proposed PE methods, the mean-expansion algorithm yields more accurate estimations for the second and third moments. However, the kurtosis is excessively overestimated by the two proposed algorithms.

It is not easy to distinguish the performance just from the moments since no consistently better results were obtained. From the performance criteria MAE and RMSE, given in Table 2, one can observe that the overall performance of the two proposed PE methods is consistently superior to Rosenblueth's method. Between the median and mean-expansion PE algorithms, the latter consistently yields a more precise fitting to the true distribution function for W. However, Table 2 indi-

TABLE 1. Comparison of Estimated Statistical Moments by Different PE Methods for W

Statistical moments (1)	Method (2)	Number of Stochastic Variables						
		Two (3)	Three (4)	Four (5)	Five (6)	Six (7)	Eight (8)	Ten (9)
Mean	SIMUL	2.9993[a]	5.998[a]	10.0096[a]	15.0155[a]	20.9849[a]	35.9992[a]	55.0315[a]
	ROSEN	3.0000	6.0000	10.0000	15.0000	21.0000	36.0000	55.0000
		(0.02%)	(0.03%)	(−0.10%)	(−0.10%)	(0.07%)	(0.00%)	(−0.06%)
	Md-PE	2.9989	5.9991	10.0005	15.0034	21.008	36.0235	55.0482
		(−0.01%)	(0.02%)	(−0.09%)	(−0.08%)	(0.11%)	(0.07%)	(0.03%)
	Mn-PE	3.0001	6.0006	10.0018	15.0044	21.0091	36.0285	55.0687
		(0.03%)	(0.04%)	(−0.08%)	(−0.07%)	(0.12%)	(0.08%)	(0.07%)
Standard deviation	SIMUL	0.8734[a]	1.7329[a]	2.8417[a]	4.2051[a]	5.7919[a]	9.6678[a]	14.5024[a]
	ROSEN	0.8798	1.6831	2.6683	3.7998	5.0461	7.7807	10.7044
		(0.73%)	(−2.87%)	(−6.10%)	(−9.64%)	(−12.88%)	(−19.52%)	(−26.19%)
	Md-PE	0.8555	1.7171	2.8724	4.3222	6.0652	10.4217	15.9195
		(−2.05%)	(−0.91%)	(1.08%)	(2.78%)	(4.72%)	(7.80%)	(9.77%)
	Mn-PE	0.8658	1.7036	2.7952	4.1275	5.6865	9.4308	13.9267
		(−0.87%)	(−1.69%)	(−1.64%)	(−1.85%)	(−1.82%)	(−2.45%)	(−3.97%)
Skewness	SIMUL	0.8545[a]	0.8888[a]	0.8934[a]	0.8559[a]	0.8342[a]	0.822[a]	0.8137[a]
	ROSEN	0.9263	1.0146	1.1022	1.1848	1.2636	1.4136	1.5569
		(8.40%)	(14.15%)	(23.37%)	(38.43%)	(51.47%)	(71.97%)	(91.34%)
	Md-PE	0.3869	0.7402	1.0636	1.3616	1.6367	2.1305	2.5636
		(−54.72%)	(−16.72%)	(19.05%)	(59.08%)	(96.20%)	(159.18%)	(215.05%)
	Mn-PE	0.5588	0.8048	1.0339	1.2479	1.4466	1.8022	2.1066
		(−34.61%)	(−9.45%)	(15.73%)	(45.80%)	(73.41%)	(119.25%)	(158.89%)
Kurtosis	SIMUL	4.2166[a]	4.3697[a]	4.4624[a]	4.2897[a]	4.1956[a]	4.1787[a]	4.1449[a]
	ROSEN	1.9079	2.1781	2.3877	2.5751	2.7625	3.1709	3.6416
		(−54.75%)	(−50.15%)	(−46.49%)	(−39.97%)	(−34.16%)	(−24.12%)	(−12.14%)
	Md-PE	2.0139	3.1305	4.3225	5.5702	6.8552	9.4945	12.1753
		(−52.24%)	(−28.36%)	(−3.14%)	(29.85%)	(63.39%)	(127.21%)	(193.74%)
	Mn-PE	2.3075	3.6482	5.0945	6.6353	8.2548	11.6885	15.3135
		(−45.28%)	(−16.51%)	(14.17%)	(54.68%)	(96.75%)	(179.72%)	(269.45%)

Note: SIMUL = Monte Carlo simulation; ROSEN = Rosenblueth's method; Md-PE = median-expansion algorithm; and Mn-PE = mean-expansion algorithm. The numbers in parentheses indicate the error percentages of estimated moment values by the three PE methods.
[a]True values from the Monte Carlo simulation.

TABLE 2. Comparison of Measures of Goodness of Fit to Distribution of W by Different PE Methods

Performance criteria (1)	Method (2)	Number of Stochastic Variables						
		Two (3)	Three (4)	Four (5)	Five (6)	Six (7)	Eight (8)	Ten (9)
BIAS	ROSEN	NC	0.0008	−0.0076	−0.0095	0.0281	0.002	−0.0649
	Md-PE	0.0039	0.0059	−0.0086	−0.0218	−0.0014	−0.0429	−0.1154
	Mn-PE	0.0047	0.0049	−0.0098	−0.0195	0.006	−0.0146	−0.043
MAE	ROSEN	NC	0.7657	1.2436	1.8547	2.5647	4.0327	6.4676
	Md-PE	0.1135	0.0801	0.1633	0.3953	0.7549	1.9341	3.7122
	Mn-PE	0.0955	0.0304	0.0725	0.2375	0.4572	1.0948	2.0759
RMSE	ROSEN	NC	0.8902	1.4378	2.1525	3.0031	5.0607	7.8158
	Md-PE	0.1515	0.1032	0.2003	0.5017	0.9871	2.5374	4.9122
	Mn-PE	0.1226	0.0467	0.0846	0.2734	0.5307	1.2672	2.3857

Note: ROSEN = Rosenblueth's method; Md-PE = median-expansion algorithm; Mn-PE = mean-expansion algorithm; and NC = not computable.

J. Hydraul. Eng. 1997.123:244-250.

Downloaded from ascelibrary.org by National Chiao Tung University on 04/30/14. Copyright ASCE. For personal use only; all rights reserved.

cates that the measures of goodness of fit for Rosenblueth's method is not computable in the case the two stochastic parameters are involved in. This is because the skew coefficient and kurtosis estimated by Rosenblueth's method are too close to the boundary of impossible region where the skew coefficient, γ, and the kurtosis, κ, held the following relationship (Johnson and Kotz 1970)

$$\kappa < \gamma^2 + 1 \quad (19)$$

Near the boundary of the impossible region, $\kappa = \gamma^2 + 1$, the Johnson distribution curve cannot be determined with stability.

The tendency of the two proposed PE methods to overestimate the moments can be explained by referring to Fig. 1. Using the proposed methods, the two points on the principle component axis with the highest variance contribution would be positioned farther away from the mean than Rosenblueth's PE method in the case of a stronger correlation. The presence of an extraordinary point for model evaluation would significantly influence the value of moment estimation, especially for higher moments. This impact would be amplified as correlation, model nonlinearity, or asymmetry condition of the distributions for stochastic parameters increase.

Regarding the model nonlinearity, other model types like $W = \Sigma_i X_i^2$ and $W = \Pi_i X_i$ have been discussed in the study by Chang (1994). It is indicated that the tendency to overestimate higher moments by the three PE algorithms worsen as the degree of model nonlinearity and the number of stochastic parameters increase. However, the collective behaviors evaluated by fitting the Johnson distribution curve exhibit the consistent improvements by the mean-expansion algorithm throughout the numerical study.

APPLICATION

Bed scouring is a phenomenon in a river caused by the interaction of flow and the river bed. Hydraulic structures such as bridge piers are susceptible to failure under long-term and continuous bed scouring. As required for engineering design as well as for precaution against undesirable consequences, the knowledge of bed scouring around bridge piers is essential. Many models have been developed to predict the potential scour depth around bridge piers. Using such a computer model to aid the design of pier depth is common in modern hydraulic engineering. However, the existence of various uncertainties involved in bed scouring models results in uncertainty in the scour depth prediction required for design.

For the purpose of illustration, a simple pier scour model developed by Johnson (1992) is used here for uncertainty analysis. Focus is placed on the relative performances of the various PE methods in the uncertainty analysis as compared with the Monte Carlo simulation (Chang et al. 1994).

Pier Scouring Model

Johnson (1992) proposed an empirical pier scouring model based on experimental data from various sources

$$D_s = 2.02\lambda y \left(\frac{b}{y}\right)^{0.98} \mathsf{F}^{0.21}\sigma^{-0.24} \quad (20)$$

in which D_s = predicted scour depth; λ = model correction factor; y = flow depth; b = pier width; F = Froude number; and σ = sediment gradation. Because the model is empirical by nature, uncertainties exist in both the model itself and the inputs/parameters involved (Yeh and Tung 1993). Consequently, the scour depth computed from (20) is subject to uncertainty and it is likely that a specified design pier depth could be exceeded, resulting in potential threat to bridge safety.

Uncertainty Analysis of Pier Scouring Model

The stochastic parameters considered in (20) are λ, y, F, and σ. The stochasticity of model correction factor, λ, represents the model uncertainty associated with the pier scouring model whereas the randomness of y, F, and σ are results from model input uncertainties. Their means and coefficients of variation are listed in Table 3. According to Johnson (1992), all stochastic parameters, except the model correction factor λ, are correlated random variables with the correlation matrix given in Table 4. The model correction factor λ is treated here as an independent random variable.

The three PE methods that are used here for the uncertainty analysis of the pier scouring model include: Rosenblueth's, the proposed median-expansion, and the mean-expansion PE methods. In the uncertainty analysis, mixture distributions were adopted to explore the applicability of each method. The distributions used for the stochastic parameters in the pier scouring model were: gamma distribution for λ, lognormal distribution for σ and y, and Weibull distribution for F.

To compare the relative performance in uncertainty analysis among the different methods, results from the Monte Carlo simulation can be used as the true values for comparison. Based on the given marginal distributions and correlations for the stochastic parameters, 100,000 samples were generated from which the statistical moments of scour depth from the pier scouring model were computed.

Table 5 lists the estimated moments of the scour depth from the three PE methods and their error percentages (in parentheses). Under the consideration of the mixture distributions for

TABLE 3. Means and Coefficients of Variation (COV) of Stochastic Parameters Used in Pier Scouring Model [from Johnson (1992)]

Variables (1)	Mean (2)	COV (3)
λ	1.000	0.18
y	4.250	0.20
F	0.537	0.38
σ	4.000	0.20

TABLE 4. Correlation among Stochastic Parameters Used in Pier Scouring Model [from Johnson (1992)]

Variables (1)	λ (2)	y (3)	F (4)	σ (5)
λ	1.00	0.00	0.00	0.00
y	0.00	1.00	−0.33	−0.79
F	0.00	−0.33	1.00	0.29
σ	0.00	−0.79	0.29	1.00

TABLE 5. Comparison of First Four Moments for Random Scour Depth by Various Methods

Moments (1)	Methods ROSEN (2)	Md-PE (3)	Mn-PE (4)	SIMUL (5)
μ	2.5843 (−0.03%)	2.5886 (0.14%)	2.5831 (−0.08%)	2.5851
σ	0.4864 (−0.47%)	0.4915 (0.57%)	0.4853 (−0.70%)	0.4887
γ	0.3899 (−3.08%)	0.5204 (29.36%)	0.3914 (−2.71%)	0.4023
κ	1.4925 (−53.96%)	3.5097 (8.27%)	3.5365 (9.10%)	3.2416

Note: ROSEN = Rosenblueth's method; Md-PE = median-expansion PE method; Mn-PE = mean-expansion PE method; SIMUL = Monte Carlo simulation. Numbers in parentheses indicate error percentages for the estimated moments of the scour depth.

J. Hydraul. Eng. 1997.123:244-250.

the correlated stochastic parameters, all the methods are capable of estimating the first two moments accurately. For higher moments, Table 5 indicates that Rosenblueth's method fails to yield a good estimate for the kurtosis, whereas the median-expansion PE method estimation for the skewness is undesirable. The mean-expansion PE method, however, yields closer estimations for both skew coefficient and kurtosis. From the aspect of computation efficiency, eight model evaluations are needed using the proposed PE methods since four stochastic parameters were involved. However, use of Rosenblueth's PE method requires double the amount of computation.

SUMMARY AND CONCLUSION

In this study, two PE methods were proposed to incorporate the marginal distributions of correlated random variables. The proposed methods integrate Harr's PE procedure along with the formulas that transform the original correlation to the equivalent one in the standard normal space. The performance of the proposed PE methods was evaluated against Rosenblueth's method using asymmetric random variables. Through the numerical experiment, the proposed PE methods consistently reveal superior performance to that of Rosenblueth's method and require much less computation. The input requirements are marginal distributions of involved random variables and their correlations.

The numerical experiment indicated that PE methods, in general, are not necessarily appropriate for the uncertainty analysis of all types of model. The degree of model nonlinearity and number of random variables might have significant impacts on the accuracy of PE methods. In case the model nonlinearity is high, the accuracy of higher-order moment estimations from any PE method should be questioned. Among the two proposed PE methods, the mean-expansion algorithm yields more accurate estimations for the test models.

In the application, the uncertainty analysis of a pier scouring model was performed to demonstrate the relative performance of each PE method for a practical engineering problem. Under the mixture distributions and correlated stochastic parameters considered in this particular application, the three PE methods showed that estimations of the first two moments of the predicted scour depth are as accurate as those obtained from the Monte Carlo simulation with 100,000 model evaluations. However, only the mean-expansion method can yield closer estimates for the two higher moments. This application shows that the uncertainty estimated by the mean-expansion PE method can achieve comparable accuracy with the one from the Monte Carlo simulation with significantly less computations. The latter point is especially important for those uncertainty analyses of models requiring a great amount of computation in themselves.

APPENDIX I. REFERENCES

Chang, C. H. (1994). "Incorporate information of non-normal distributions in uncertainty analysis of hydrosystems," PhD dissertation, Inst. of Civ. Engrg., Nat. Chiao-Tung Univ., Taiwan, ROC, 58–75.

Chang, C. H., Tung, Y. K., and Yang, J. C. (1994). "Monte Carlo simulation for correlated variables with marginal distributions." *J. Hydr. Engrg.*, ASCE, 120(3), 313–331.

Chang, C. H., Tung, Y. K., and Yang, J. C. (1995). "Evaluation of probability point estimate methods." *Appl. Math. Modelling*, 19(2), 95–105.

Der Kiureghian, A., and Liu, P. L. (1985). "Structural reliability under incomplete probability information." *J. Engrg. Mech.*, ASCE, 112(1), 85–104.

Harr, M. E. (1989). "Probabilistic estimates for multivariate analyses." *Appl. Math. Modelling*, 13(5), 313–318.

Hill, I. D., Hill, R., and Holder, R. L. (1976)l. "Algorithm AS 99: fitting Johnson curves by moments." *Appl. Statistics*, 25, 180–189.

Johnson, P. A. (1992). "Reliability-based pier scour engineering." *J. Hydr. Engrg.*, ASCE, 118(10), 1344–1358.

Johnson, N. L., and Kotz, S. (1970). *Continuous univariate distributions—1.* John Wiley & Sons, Inc., New York, N.Y.

Karmeshu, and Lara-Rosano, F. (1987). "Modelling data uncertainty in growth forecasts." *Appl. Math. Modelling*, 11(2), 62–68.

Li, K. S. (1992). "Point-estimate method for calculating statistical moments." *J. Engrg. Mech.*, ASCE, 118(7), 1506–1511.

Rosenblueth, E. (1975). "Point estimates for probability moments." *Proc. Nat. Acad. Sci. USA*, 72(10), 3812.

Rosenblueth, E. (1981). "Two-point estimates in probabilities." *Appl. Math. Modelling*, 5(10), 329–335.

Yeh, K. C., and Tung, Y. K. (1993). "Uncertainty and sensitivity of a pit migration model." *J. Hydr. Engrg.*, ASCE, 119(2), 262–281.

Zoppou, C., and Li, K. S. (1993). "New point estimate method for water resources modeling." *J. Hydr. Engrg.*, ASCE, 119(11), 1300–1307.

APPENDIX II. NOTATION

The following symbols are used in this paper:

- a = correction term in Rosenblueth's PE method;
- $\mathbf{D}$ = diagonal matrix containing variances;
- F = cumulative distribution function;
- $g()$ = model or function;
- $\mathbf{L}$ = diagonal eigenvalue matrix;
- n = number of stochastic parameters;
- p = probability mass;
- $\mathbf{R}_x$ = correlation matrix in the original space;
- $\mathbf{R}_Y$ = correlation matrix in the standard normal space;
- $\mathbf{U}$ = vector of stochastic parameters in the standard eigenspace;
- $\mathbf{V}$ = eigenvectors matrix;
- v = column vector of an eigenvector;
- W = model output;
- w = value of model output;
- $\mathbf{X}$ = vector of stochastic parameters in the original space;
- X = stochastic parameter in the original space;
- x = value of a stochastic parameter in the original space;
- $\mathbf{Y}$ = vector of stochastic parameters in the standard normal space;
- Y = stochastic parameter in the standard normal space;
- y = value of a stochastic parameter in the standard normal space;
- $\mathbf{y}^*$ = vector of mean values in the standard normal space;
- y^* = mean value in the standard normal space;
- $\mathbf{Z}$ = vector of stochastic parameters in the eigenspace;
- α = weighing factor;
- γ = skew coefficient;
- δ = sign indicator;
- θ = estimated moment;
- λ = eigenvalue;
- $\boldsymbol{\mu}$ = vector of mean values;
- μ = mean value;
- ρ = correlation coefficient;
- σ = standard deviation; and
- Φ = standard normal cumulative distribution function.

J. Hydraul. Eng. 1997.123:244-250.

Downloaded from ascelibrary.org by National Chiao Tung University on 04/30/14. Copyright ASCE. For personal use only; all rights reserved.