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Mutually Independent Hamiltonian Connectivity of (n,k)-Star Graphs

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Abstract. A graph G is hamiltonian connected if there exists a hamiltonian path joining any two distinct nodes of G. Two hamiltonian paths $P_1 = \langle u_1, u_2, \ldots, u_{\nu(G)} \rangle$ and $P_2 = \langle v_1, v_2, \ldots, v_{\nu(G)} \rangle$ of G from u to v are independent if $u = u_1 = v_1$, $v = u_{\nu(G)} = v_{\nu(G)}$, and $u_i \neq v_i$ for every $1 < i < \nu(G)$. A set of hamiltonian paths, $\{P_1, P_2, \ldots, P_k\}$, of G from u to v are mutually independent if any two different hamiltonian paths are independent from u to v. A graph is k mutually independent hamiltonian connected if for any two distinct nodes u and v, there are k mutually independent hamiltonian paths from u to v. The mutually independent hamiltonian connectivity of a graph G, $IHP(G)$, is the maximum integer k such that G is k mutually independent hamiltonian connected. Let n and k be any two distinct positive integers with $n-k \geq 2$. We use $S_{n,k}$ to denote the (n,k)-star graph. In this paper, we prove that $IHP(S_{n,k}) = n-2$ except for $S_{4,2}$ such that $IHP(S_{4,2}) = 1$.

Keywords: hamiltonian, hamiltonian connected, (n,k)-star graphs

1. Introduction

For notations of graph theory, we refer to [2]. A (v_1, v_k)-*path* P is a sequence of adjacent nodes from v_1 to v_k, written as $\langle v_1, v_2, \ldots, v_k \rangle$, in which the nodes $v_1, v_2, \ldots, v_k$ are distinct. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_1 \rangle$ and we use $P(i)$ to denote the i-th node v_i of P. We also write the path $\langle v_1, v_2, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k \rangle$ where Q is a (v_i, v_j)-path. A path of a graph G is *hamiltonian* if it contains all nodes of G. A graph G is *hamiltonian connected* if, for any two distinct nodes of G, there

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is a hamiltonian path of G between them. A *hamiltonian cycle* of G is a cycle that contains all nodes of G. A graph is *hamiltonian* if it has a hamiltonian cycle.

We say two hamiltonian paths $P_1 = \langle u = u_1, u_2, \ldots, u_{\nu(G)} = v\rangle$ and $P_2 = \langle u = v_1, v_2, \ldots, v_{\nu(G)} = v\rangle$ of G from u to v are *independent* if $u_i \neq v_i$ for every $1 < i < \nu(G)$. A set of hamiltonian paths, $\{P_1, P_2, \ldots, P_k\}$, of G from u to v is *mutually independent* if any two different hamiltonian paths are independent from u to v, and we say there are k *mutually independent hamiltonian paths* from u to v at the same time.

A graph is k *mutually independent hamiltonian connected* if, for any two distinct nodes u and v, there are k mutually independent hamiltonian paths from u to v. Moreover, the *mutually independent hamiltonian connectivity* of a graph G, $IHP(G)$, is the maximum integer k such that G is k mutually independent hamiltonian connected if G is hamiltonian connected, and 0 otherwise. Some results on mutually independent hamiltonian paths have been introduced by Lin et al. in [10], Sun et al. in [12], and Teng et al. in [13].

The concept of mutually independent hamiltonian paths arises from the following application. If there are k pieces of data needed to be sent from u to v, and the data need to be processed at every node (and the process takes time), then we want mutually independent hamiltonian paths from u to v so that there will be no waiting time at a processor.

Assume that G is a graph with at least three nodes. Suppose that u and v are adjacent in G. Let $P = \langle u = v_1, v_2, \ldots, v_{\nu(G)} = v\rangle$ be any hamiltonian path of G from u to v. Since $\nu(G) \geq 3$, $v_2 \neq v$, then there are at most $d_G(u) - 1$ mutually independent hamiltonian paths in G from u to v. Hence, $IHP(G) \leq \delta(G) - 1$ where $\delta(G) = \min\{d_G(x) \mid x \in V\}$.

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph, in which the nodes correspond to processors and the edges correspond to connections. Hence, we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for interconnection networks. Thus, many families of interconnection networks are proposed [1,4,8,11]. In particular, the (n,k)-star graph, $S_{n,k}$, which proposed by Chiang and Chen [4] has been recognized as an attractive interconnection network. For this reason, many properties of the (n,k)-star graphs are studied [3,5–7,9,10].

In this paper, we prove that $IHP(S_{n,k}) = n-2$ for any $n-k \geq 2$ except for $S_{4,2}$, and $IHP(S_{4,2}) = 1$. In the following section, we give the definition and some properties of $S_{n,k}$. In Section 3, we prove our main result. The conclusion is given in Section 4.

2. Preliminary

Assume that n and k are two positive integers with $n > k$. We use $\langle n\rangle$ to denote the set $\{1, 2, \ldots, n\}$. The (n,k)-star graph, $S_{n,k}$, is a graph with node set $V(S_{n,k}) = \{u_1u_2\cdots u_k \mid u_i \in \langle n\rangle$ and $u_i \neq u_j$ for $i \neq j\}$. Adjacency is defined as follows: A node $u_1u_2\cdots u_i\cdots u_k$ is adjacent to (1) the node $u_iu_2u_3\cdots u_{i-1}u_1u_{i+1}\cdots u_k$, where $2 \leq i \leq k$ (i.e., swap u_i with u_1), and (2) the node $xu_2u_3\cdots u_k$ where $x \in \langle n\rangle - \{u_i \mid 1 \leq i \leq k\}$.

The edges of type (1) are referred to *i-edge* and the edges of type (2) are referred to as 1-*edges*.

By definition, $S_{n,k}$ is an $(n-1)$-regular graph with $n!/(n-k)!$ nodes. Moreover, it is node transitive [4]. We use boldface to denote nodes in $S_{n,k}$. Let $\mathbf{u} = u_1u_2\cdots u_k$ be any node of $S_{n,k}$. We say that u_i is the i-th coordinate of $\mathbf{u}$, denoted by $(\mathbf{u})_i$, for $1 \le i \le k$. By the definition of $S_{n,k}$, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-edge with $2 \le i \le k$. For this reason, we use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$ for $2 \le i \le k$. Obviously, $\left((\mathbf{u})^i\right)^i = \mathbf{u}$. For $1 \le i \le n$, let $S_{n,k}^{\{i\}}$ denote the subgraph of $S_{n,k}$ induced by those nodes $\mathbf{u}$ with $(\mathbf{u})_k = i$. In [4], it was shown that $S_{n,k}$ can be decomposed into n subgraphs $S_{n,k}^{\{i\}}$, $1 \le i \le n$, such that each subgraph $S_{n,k}^{\{i\}}$ is isomorphic to $S_{n-1,k-1}$. Thus, the (n,k)-star graph can be constructed recursively. Obviously, $\mathbf{u} \in S_{n,k}^{\{(\mathbf{u})_k\}}$. Let $I \subseteq \langle n \rangle$. We use $S_{n,k}^I$ to denote the subgraphs of $S_{n,k}$ induced by those nodes $\mathbf{u}$ with $(\mathbf{u})_k \in I$. For $1 \le i \le n$ and $1 \le j \le n$ with $i \ne j$, we use $E_{n,k}^{i,j}$ to denote the set of edges between $S_{n,k}^{\{i\}}$ and $S_{n,k}^{\{j\}}$. The (n,k)-star graphs $S_{3,1}$, $S_{4,1}$, $S_{4,2}$, and $S_{5,2}$ are shown in Figure 1.

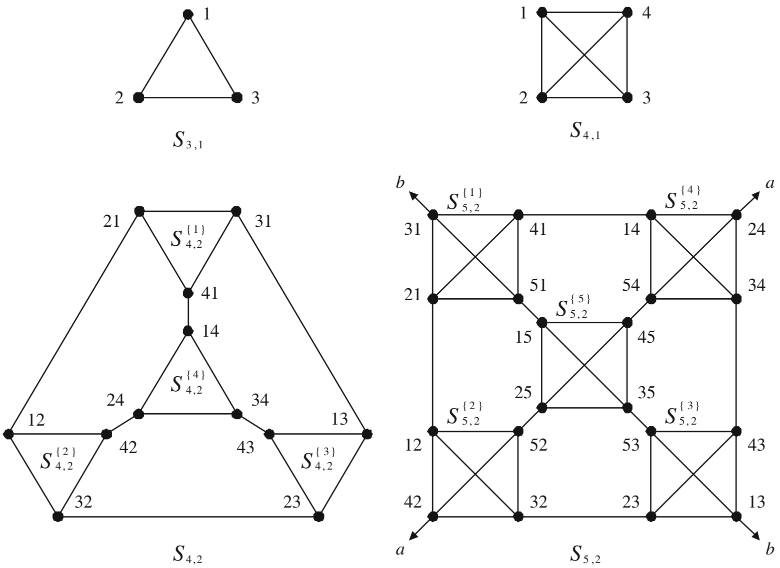

Figure 1: Some examples of the (n,k)-star graphs: $S_{3,1}$, $S_{4,1}$, $S_{4,2}$, and $S_{5,2}$.

Lemma 2.1. ([4]) *For* $k \ge 2$, $\left|E_{n,k}^{i,j}\right| = (n-2)!/(n-k)!$.

Since $S_{n,1}$ is isomorphic to the complete graph K_n with n nodes, we have the following result.

Lemma 2.2. *For* $n \ge 3$, $S_{n,1} - F$ *is hamiltonian connected if* $F \subseteq V(S_{n,1})$ *with* $|F| \le n-2$.

Theorem 2.3. ([6]) *Let n and k be any two positive integers with $n-k \geq 2$, and let $F \subseteq V(S_{n,k}) \cup E(S_{n,k})$. Then $S_{n,k} - F$ is hamiltonian connected if $|F| \leq n-4$ and $S_{n,k} - F$ is hamiltonian if $|F| \leq n-3$.*

Lemma 2.4. *Let n and k be any two integers with $k \geq 2$ and $n-k \geq 2$. Let $\mathbf{u}$ be any node of $S_{n,k}$, and let i be any positive integer with $1 \leq i \leq n$. Then there is a hamiltonian path P of $S_{n,k} - \{\mathbf{u}\}$ joining a node $\mathbf{x}$ to a node $\mathbf{y}$ where $(\mathbf{x})_1 = (\mathbf{u})_k$ and $(\mathbf{y})_1 = i$.*

Proof. By Theorem 2.3 and Lemma 2.2, this statement holds for $n \geq 5$ and $k = 1$. Therefore, we only need to check the case of $n = 4$ and $k = 2$. Since $S_{4,2}$ is node transitive, we assume that $\mathbf{u} = 14$. Note that $(\mathbf{u})_2 = 4$. We list all of the desired $\mathbf{x}$, $\mathbf{y}$, and hamiltonian paths of $S_{4,2} - \{\mathbf{u} = 14\}$ for each $1 \leq i \leq 4$ in Table 1. Hence, this statement is proved. ∎

Table 1: The desired hamiltonian paths for $S_{4,2} - \{14\}$ of Lemma 2.4.

$\{i=1\}$	$\langle \mathbf{x} = 41, 21, 31, 13, 23, 43, 34, 24, 42, 32, 12 = \mathbf{y} \rangle$
$\{i=2\}$	$\langle \mathbf{x} = 41, 21, 31, 13, 43, 34, 24, 42, 12, 32, 23 = \mathbf{y} \rangle$
$\{i=3\}$	$\langle \mathbf{x} = 41, 21, 31, 13, 23, 43, 34, 24, 42, 12, 32 = \mathbf{y} \rangle$
$\{i=4\}$	$\langle \mathbf{x} = 41, 21, 31, 13, 23, 32, 12, 42, 24, 34, 43 = \mathbf{y} \rangle$

Lemma 2.5. *Let n and k be any two integers with $n \geq 5$, $k \geq 2$, and $n-k \geq 2$. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct nodes of $S_{n,k}$ with $(\mathbf{u})_t = (\mathbf{v})_t$ for every $2 \leq t \leq k$, and let i and j be two integers in $\langle n \rangle$ with $i < j$. Then there is a hamiltonian path P of $S_{n,k} - \{\mathbf{u}, \mathbf{v}\}$ joining a node $\mathbf{x}$ to a node $\mathbf{y}$ with $(\mathbf{x})_1 = i$ and $(\mathbf{y})_1 = j$.*

Proof. Without loss of generality, we assume that $(\mathbf{u})_1 = 1$, $(\mathbf{v})_1 = 2$, and $(\mathbf{u})_t = (\mathbf{v})_t = n-k+t$ for every $2 \leq t \leq k$. Since $i < j$, $i \neq n$, and $j \neq 1$. We have the following cases:

Case 1. $n = 5$ and $k = 2$. Since $(\mathbf{u})_1 = 1$, $(\mathbf{v})_1 = 2$, and $(\mathbf{u})_2 = (\mathbf{v})_2 = 5$, we have $\mathbf{u} = 15$ and $\mathbf{v} = 25$. We list all of the desired hamiltonian paths of $S_{5,2} - \{\mathbf{u}, \mathbf{v}\}$ for all possible $1 \leq i < j \leq 5$ in Table 2.

Case 2. $n = 5$ and $k = 3$. Since $(\mathbf{u})_1 = 1$, $(\mathbf{v})_1 = 2$, and $(\mathbf{u})_t = (\mathbf{v})_t = t+2$ for every $2 \leq t \leq 3$, we have $\mathbf{u} = 145$ and $\mathbf{v} = 245$. Let $\mathbf{x}_1 = 135$, $\mathbf{x}_2 = 235$, $\mathbf{x}_3 = 325$, $\mathbf{x}_4 = 425$, and $\mathbf{p}_0 = 345$. Depending on the value i, we set $\mathbf{x}$ and R as

$$\mathbf{x} = \mathbf{x}_1 \text{ and } R = \langle 135, 315, 415, 215, 125, 425, 325, 235, 435, 345 = \mathbf{p}_0 \rangle \text{ if } i = 1;$$

$$\mathbf{x} = \mathbf{x}_2 \text{ and } R = \langle 235, 325, 425, 125, 215, 415, 315, 135, 435, 345 = \mathbf{p}_0 \rangle \text{ if } i = 2;$$

$$\mathbf{x} = \mathbf{x}_3 \text{ and } R = \langle 325, 425, 125, 215, 415, 315, 135, 235, 435, 345 = \mathbf{p}_0 \rangle \text{ if } i = 3;$$

$$\mathbf{x} = \mathbf{x}_4 \text{ and } R = \langle 425, 325, 125, 215, 415, 315, 135, 235, 435, 345 = \mathbf{p}_0 \rangle \text{ if } i = 4.$$

Table 2: The desired hamiltonian paths for Case 1 of Lemma 2.5.

$\{i=1, j=2\}$	$\langle 12, 42, 52, 32, 23, 13, 43, 53, 35, 45, 54, 34, 24, 14, 41, 31, 51, 21\rangle$
$\{i=1, j=3\}$	$\langle 12, 42, 52, 32, 23, 13, 43, 53, 35, 45, 54, 34, 24, 14, 41, 21, 51, 31\rangle$
$\{i=1, j=4\}$	$\langle 12, 52, 32, 42, 24, 34, 54, 14, 41, 21, 51, 31, 13, 23, 43, 53, 35, 45\rangle$
$\{i=1, j=5\}$	$\langle 12, 42, 52, 32, 23, 13, 43, 53, 35, 45, 54, 34, 24, 14, 41, 31, 21, 51\rangle$
$\{i=2, j=3\}$	$\langle 23, 13, 43, 53, 35, 45, 54, 34, 24, 14, 41, 31, 51, 21, 12, 42, 52, 32\rangle$
$\{i=2, j=4\}$	$\langle 23, 13, 43, 53, 35, 45, 54, 34, 24, 14, 41, 31, 51, 21, 12, 32, 52, 42\rangle$
$\{i=2, j=5\}$	$\langle 23, 13, 43, 53, 35, 45, 54, 34, 24, 14, 41, 31, 51, 21, 12, 42, 32, 52\rangle$
$\{i=3, j=4\}$	$\langle 35, 45, 54, 14, 24, 34, 43, 53, 23, 13, 31, 41, 51, 21, 12, 32, 52, 42\rangle$
$\{i=3, j=5\}$	$\langle 35, 45, 54, 14, 24, 34, 43, 53, 23, 13, 31, 41, 51, 21, 12, 32, 42, 52\rangle$
$\{i=4, j=5\}$	$\langle 45, 35, 53, 23, 13, 43, 34, 24, 54, 14, 41, 31, 51, 21, 12, 32, 42, 52\rangle$

Note that R is a hamiltonian path of $S_{5,3}^{\{5\}} - \{\mathbf{u}, \mathbf{v}\}$ joining $\mathbf{x}$ to $\mathbf{p}_0$. We set $\mathbf{p}_1 = 453$, $\mathbf{p}_2 = 254$, and $\mathbf{p}_3 = 152$. By Lemma 2.1, $\left|\left\{\mathbf{z} \mid \mathbf{z} \in S_{5,3}^{\{1\}} \text{ and } (\mathbf{z})_1 = j\right\}\right| = \left|E_{5,3}^{1,j}\right| = 3$ for $2 \le j \le 5$. We can choose a node $\mathbf{p}_4$ in $S_{5,3}^{\{1\}} - \left\{(\mathbf{p}_3)^3\right\}$ with $(\mathbf{p}_4)_1 = j$. By Theorem 2.3, there is a hamiltonian path Q_t of $S_{5,3}^{\{(\mathbf{p}_t)_3\}}$ joining $(\mathbf{p}_{t-1})^3$ to $\mathbf{p}_t$ for every $1 \le t \le 4$. We set $\mathbf{y} = \mathbf{p}_4$ and $P = \left\langle \mathbf{x}, R, \mathbf{p}_0, (\mathbf{p}_0)^3, Q_1, \mathbf{p}_1, (\mathbf{p}_1)^3, Q_2, \mathbf{p}_2, \ldots, (\mathbf{p}_3)^3, Q_4, \mathbf{p}_4 = \mathbf{y}\right\rangle$. Then P forms the desired path.

Case 3. $n \ge 6$, $k \ge 2$, and $n-k \ge 2$. Let $\mathbf{x}$ and $\mathbf{y}$ be any two nodes of $S_{n,k} - \{\mathbf{u}, \mathbf{v}\}$ with $(\mathbf{x})_1 = i$ and $(\mathbf{y})_1 = j$. By Theorem 2.3, there is a hamiltonian path P of $S_{n,k} - \{\mathbf{u}, \mathbf{v}\}$ joining $\mathbf{x}$ to $\mathbf{y}$.

Thus, the statement is proved. ∎

Theorem 2.6. *Let n and k be any two integers with $n-k \ge 2$ and $k \ge 3$, and let $I = \{i_1, i_2, \ldots, i_r\}$ be any subset of $\langle n\rangle$ for $1 \le r \le n$. Assume that $\mathbf{u}$ and $\mathbf{v}$ are two distinct nodes of $S_{n,k}$ with $\mathbf{u} \in S_{n,k}^{\{i_1\}}$ and $\mathbf{v} \in S_{n,k}^{\{i_r\}}$. Then there is a hamiltonian path $H = \langle \mathbf{u} = \mathbf{x}_1, H_1, \mathbf{y}_1, \mathbf{x}_2, H_2, \mathbf{y}_2, \ldots, \mathbf{x}_r, H_r, \mathbf{y}_r = \mathbf{v}\rangle$ of $S_{n,k}^I$ joining $\mathbf{u}$ to $\mathbf{v}$ such that H_j is a hamiltonian path of $S_{n,k}^{\{i_j\}}$ joining $\mathbf{x}_j$ to $\mathbf{y}_j$ for every $1 \le j \le r$.*

Proof. We set $\mathbf{x}_1 = \mathbf{u}$ and $\mathbf{y}_r = \mathbf{v}$. Since $S_{n,k}^{\{i_j\}}$ is isomorphic to $S_{n-1,k-1}$ for every $j \in \langle r\rangle$, this statement holds for $r = 1$ by Theorem 2.3. Hence, we assume that $r \ge 2$.

Since $k \ge 3$ and $n-k \ge 2$, by Lemma 2.1, $\left|E_{n,k}^{i_j, i_{j+1}}\right| = (n-2)!/(n-k)! \ge 3$ for every $j \in \langle r-1\rangle$. We choose $(\mathbf{y}_j, \mathbf{x}_{j+1}) \in E_{n,k}^{i_j, i_{j+1}}$ for every $j \in \langle r-1\rangle$ with $\mathbf{y}_j \in S_{n,k}^{\{i_j\}}$, $\mathbf{x}_{j+1} \in S_{n,k}^{\{i_{j+1}\}}$, $\mathbf{y}_1 \ne \mathbf{u}$, and $\mathbf{x}_r \ne \mathbf{v}$. By Theorem 2.3, there is a hamiltonian path H_j of $S_{n,k}^{\{i_j\}}$ joining $\mathbf{x}_j$ to $\mathbf{y}_j$ for every $j \in \langle r\rangle$. Then $H = \langle \mathbf{u} = \mathbf{x}_1, H_1, \mathbf{y}_1, \mathbf{x}_2, H_2, \mathbf{y}_2, \ldots, \mathbf{x}_r, H_r, \mathbf{y}_r = \mathbf{v}\rangle$ forms the desired path. See Figure 2 as an illustration. ∎

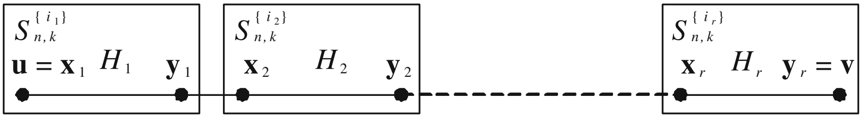

Figure 2: An illustration for Theorem 2.6.

3. Mutually Independent Hamiltonian Paths of $S_{n,k}$

We deal with the cases $k = 1$, $k = 2$, and $k \geq 3$ into the following three subsections.

3.1. $IHP(S_{n,1})$

Theorem 3.1. *$IHP(S_{n,1}) = n-2$ if $n \geq 3$.*

Proof. Since $\delta(S_{n,1}) = n-1$ if $n \geq 3$ and $IHP(G) \leq \delta(G)-1$ for every graph G with at least three nodes, $IHP(S_{n,1}) \leq n-2$ for every $n \geq 3$. To prove the statement, we need to construct $(n-2)$ mutually independent hamiltonian paths of $S_{n,1}$ between any two distinct nodes. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct nodes of $S_{n,1}$. Since $S_{n,1}$ is isomorphic to the complete graph K_n with node set $\{v_1, v_2, \ldots, v_n\}$, we assume that $\mathbf{u} = v_n$, $\mathbf{v} = v_{n-1}$. For every $1 \leq i \leq n-2$ and $1 \leq j \leq n-2$, we set $\mathbf{z}_j^i$ as

$$\mathbf{z}_j^i = \begin{cases} v_{i+j-1}, & \text{if } i+j \leq n-1, \text{ and} \\ v_{i+j-n+1}, & \text{if } n \leq i+j. \end{cases}$$

We set $P_i = \langle \mathbf{u}, \mathbf{z}_1^i, \mathbf{z}_2^i, \ldots, \mathbf{z}_{n-2}^i, \mathbf{v} \rangle$ for every $1 \leq i \leq n-2$. Then $\{P_1, P_2, \ldots, P_{n-2}\}$ forms a set of $(n-2)$ mutually independent hamiltonian paths of $S_{n,1}$ from $\mathbf{u}$ to $\mathbf{v}$. ∎

3.2. $IHP(S_{n,2})$

Theorem 3.2. *$IHP(S_{4,2}) = 1$.*

Proof. By Theorem 2.3, $S_{4,2}$ is hamiltonian connected. Therefore, $IHP(S_{4,2}) \geq 1$. Using depth-first search, we list all hamiltonian paths of $S_{4,2}$ from 12 to 14 in Table 3. Obviously, $P_1(6) = P_2(6) = 43$. Thus, $IHP(S_{4,2}) < 2$. Hence, $IHP(S_{4,2}) = 1$. ∎

Table 3: All hamiltonian paths of $S_{4,2}$ between nodes 12 and 14.

$P_1 = \langle 12, 21, 41, 31, 13, 43, 23, 32, 42, 24, 34, 14 \rangle$
$P_2 = \langle 12, 32, 42, 24, 34, 43, 23, 13, 31, 21, 41, 14 \rangle$

Lemma 3.3. *Let* $\mathbf{u}$ *be any node of* $S_{n,1}$*, and let* $\{i_1, i_2, \ldots, i_m\}$ *be any non-empty subset of* $\langle n \rangle - \{(\mathbf{u})_1\}$ *with* $n \geq 2$*. Then, there are m hamiltonian paths* $P_1, P_2, \ldots, P_m$ *of* $S_{n,1}$ *such that*

(1) P_j *joins* $\mathbf{u}$ *to node* $\mathbf{x}_j$ *with* $\mathbf{x}_j = (\mathbf{x}_j)_1 = i_j$ *for every* $1 \leq j \leq m$*, and*
(2) $|\{P_1(j), P_2(j), \ldots, P_m(j)\}| = m$ *for every* $2 \leq j \leq n$*.*

Proof. Without loss of generality, we assume that $\mathbf{u} = (\mathbf{u})_1 = n$ and $\mathbf{x}_j = (\mathbf{x}_j)_1 = i_j = j$ for every $1 \leq j \leq n-1$. For every $1 \leq j \leq n-1$ and for every $1 \leq l \leq n-1$, we set $\mathbf{z}_l^j$ as

$$\mathbf{z}_l^j = \begin{cases} j+l-1, & \text{if } j+l \leq n, \text{ and} \\ j+l-n, & \text{if } n+1 \leq j+l. \end{cases}$$

We set $\mathbf{x}_j = j = \mathbf{z}_{n-1}^j$ and $P_j = \left\langle \mathbf{u}, \mathbf{z}_1^j, \mathbf{z}_2^j, \ldots, \mathbf{z}_{n-1}^j = \mathbf{x}_j \right\rangle$ for every $1 \leq j \leq n-1$. Then $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ form the desired paths. ∎

Theorem 3.4. *For* $n \geq 5$*,* $IHP(S_{n,2}) = n-2$*.*

Proof. Since $IHP(G) \leq \delta(G) - 1$ for every graph G with at least three nodes, $IHP(S_{n,2}) \leq n-2$ if $n \geq 5$. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct nodes of $S_{n,2}$. We need to construct $(n-2)$ mutually independent hamiltonian paths of $S_{n,2}$ from $\mathbf{u}$ to $\mathbf{v}$. Since $S_{n,k}$ is node transitive, we assume that $\mathbf{u} = 12$. According to the position of $\mathbf{v}$, we have the following cases:

Case 1. $\mathbf{v} = 21$. By Lemma 3.3, there are $(n-2)$ hamiltonian paths $H_1, H_2, \ldots, H_{n-2}$ of $S_{n,2}^{\{2\}}$ such that

(1) H_i joins $\mathbf{u}$ to node $\mathbf{x}_i$ with $(\mathbf{x}_i)_1 = i+2$ for every $1 \leq i \leq n-2$, and
(2) $|\{H_1(j), H_2(j), \ldots, H_{n-2}(j)\}| = n-2$ for every $2 \leq j \leq n-1$.

Again, there are $(n-2)$ hamiltonian paths $Q_1, Q_2, \ldots, Q_{n-2}$ of $S_{n,2}^{\{1\}}$ such that

(1) Q_1 joins $\mathbf{v}$ to $\mathbf{y}_1 = n1$ and Q_i joins $\mathbf{v}$ to $\mathbf{y}_i = (i+1)1$ for every $2 \leq i \leq n-2$, and
(2) $|\{Q_1(j), Q_2(j), \ldots, Q_{n-2}(j)\}| = n-2$ for every $2 \leq j \leq n-1$.

Let A be a matrix of order $(n-2) \times (n-2)$ defined by

$$a_{i,j} = \begin{cases} i+j+1, & \text{if } i+j \leq n-1, \text{ and} \\ i+j-n+3, & \text{if } n \leq i+j. \end{cases}$$

More precisely,

$$A = \begin{bmatrix} 3 & 4 & \cdots & n-2 & n-1 & n \\ 4 & 5 & \cdots & n-1 & n & 3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & 3 & \cdots & n-3 & n-2 & n-1 \end{bmatrix}.$$

Obviously, $a_{i,1}a_{i,2}\cdots a_{i,n-1}$ forms a permutation of the set $\{3,4,\dots,n\}$ for every i with $1 \le i \le n-2$. Moreover, $a_{i,j} \neq a_{i',j}$ for any $1 \le i < i' \le n-2$ and $1 \le j \le n-2$. In other words, A forms a square Latin square with entries in $\{3,4,\dots,n\}$.

We set $\mathbf{p}_0^i = \mathbf{x}_i$ and $\mathbf{p}_{n-2}^i = (\mathbf{y}_i)^2$ for every $1 \le i \le n-2$. For every $1 \le i \le n-2$ and $1 \le j \le n-3$, we set $\mathbf{p}_j^i = a_{i,j+1}a_{i,j}$. By Theorem 2.3, there is a hamiltonian path T_j^i of $S_{n,2}^{\{a_{i,j}\}}$ joining $\left(\mathbf{p}_{j-1}^i\right)^2$ to $\mathbf{p}_j^i$ for every $1 \le i \le n-2$ and $1 \le j \le n-2$. We set $P_i = \left\langle \mathbf{u}, H_i, \mathbf{x}_i = \mathbf{p}_0^i, \left(\mathbf{p}_0^i\right)^2, T_1^i, \mathbf{p}_1^i, \left(\mathbf{p}_1^i\right)^2, T_2^i, \mathbf{p}_2^i, \dots, \left(\mathbf{p}_{n-3}^i\right)^2, T_{n-2}^i, \mathbf{p}_{n-2}^i = (\mathbf{y}_i)^3, \mathbf{y}_i, Q_i^{-1}, \mathbf{v} \right\rangle$ for every $1 \le i \le n-2$. Then $\{P_1, P_2, \dots, P_{n-2}\}$ forms a set of $(n-2)$ mutually independent hamiltonian paths of $S_{n,2}$ from $\mathbf{u}$ to $\mathbf{v}$. See Figure 3 as an illustration of this case for $S_{6,2}$.

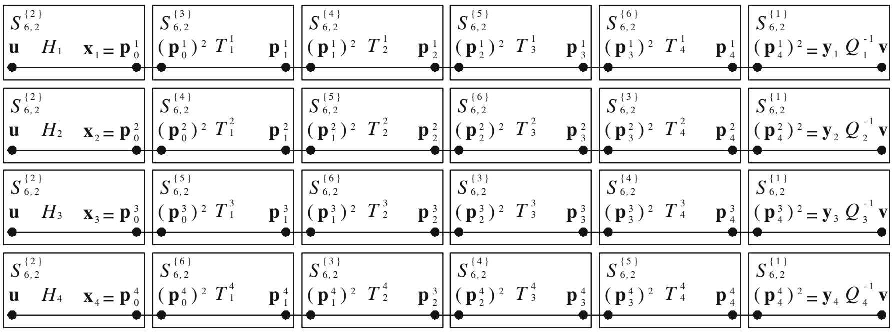

Figure 3: An illustration for Case 1 of Theorem 3.4 on $S_{6,2}$.

Case 2. Either $(\mathbf{v})_1 = 2$ and $(\mathbf{v})_2 \neq 1$ or $(\mathbf{v})_2 = 1$ and $(\mathbf{v})_1 \neq 2$. Without loss of generality, we assume that $(\mathbf{u})_1 \neq (\mathbf{v})_2$. Moreover, we assume that $\mathbf{u} = 12$ and $\mathbf{v} = 23$. By Lemma 3.3, there are $(n-3)$ hamiltonian paths $H_1, H_2, \dots, H_{n-3}$ of $S_{n,2}^{\{2\}}$ such that

(1) H_i joins $\mathbf{u}$ to node $\mathbf{x}_i = (i+3)2$ for every $1 \le i \le n-3$, and
(2) $|\{H_1(j), H_2(j), \dots, H_{n-3}(j)\}| = n-3$ for every $2 \le j \le n-1$.

Again, there are $(n-3)$ hamiltonian paths $Q_1, Q_2, \dots, Q_{n-3}$ of $S_{n,2}^{\{3\}}$ such that

(1) Q_1 joins $\mathbf{v}$ to $\mathbf{y}_1 = 13$ and Q_i joins $\mathbf{v}$ to $\mathbf{y}_i = (i+2)3$ for every $2 \le i \le n-3$, and
(2) $|\{H_1(j), H_2(j), \dots, H_{n-3}(j)\}| = n-3$ for every $2 \le j \le n-1$.

Let A be a matrix of order $(n-3) \times (n-2)$ defined by

$$a_{i,j} = \begin{cases} i+j+2, & \text{if } i+j \le n-2, \\ 1, & \text{if } i+j = n-1, \text{ and} \\ i+j-n+4, & \text{if } n \le i+j. \end{cases}$$

More precisely,

$$A = \begin{bmatrix} 4 & 5 & 6 & \cdots & n-1 & n & 1 \\ 5 & 6 & 7 & \cdots & n & 1 & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & 1 & 4 & \cdots & n-3 & n-2 & n-1 \end{bmatrix}.$$

We set $\mathbf{p}_0^i = \mathbf{x}_i$ and $\mathbf{p}_{n-2}^i = (\mathbf{y}_i)^2$ for every $1 \le i \le n-2$. For every $1 \le i \le n-3$ and $1 \le j \le n-3$, we set $\mathbf{p}_j^i = a_{i,j+1}a_{i,j}$. By Theorem 2.3, there is a hamiltonian path T_j^i of $S_{n,2}^{\{a_{i,j}\}}$ joining $\left(\mathbf{p}_{j-1}^i\right)^2$ to $\mathbf{p}_j^i$ for every $1 \le i \le n-3$ and $1 \le j \le n-2$. We set P_i $= \left\langle \mathbf{u}, H_i, \mathbf{x}_i = \mathbf{p}_0^i, \left(\mathbf{p}_0^i\right)^2, T_1^i, \mathbf{p}_1^i, \left(\mathbf{p}_1^i\right)^2, T_2^i, \mathbf{p}_2^i, \ldots, \left(\mathbf{p}_{n-3}^i\right)^2, T_{n-2}^i, \mathbf{p}_{n-2}^i = (\mathbf{y}_i)^2, \mathbf{y}_i, Q_i^{-1}, \mathbf{v} \right\rangle$ the $(n-3)$ paths for every $1 \le i \le n-3$.

Let $b_1 = 1$, $b_2 = 3$, $b_i = i+1$ for every $3 \le i \le n-1$, and $b_n = 2$. We set $\mathbf{w}_0 = \mathbf{u}$, $\mathbf{w}_i = b_i b_{i+1}$ for every $1 \le i \le n-1$, and $\mathbf{w}_n = (\mathbf{v})^2$. Note that $(\mathbf{w}_1)^2 = 13$, $\mathbf{w}_2 = 43$, $(\mathbf{w}_{n-1})^2 = n2$, and $\mathbf{w}_n = 32$. By Theorem 2.3, there is a hamiltonian path W_i of $S_{n,2}^{\{b_i\}}$ joining $(\mathbf{w}_{i-1})^2$ to $\mathbf{w}_i$ for every $i \in \langle n-1 \rangle - \{2\}$. By Lemma 2.2, there is a hamiltonian path W_2 of $S_{n,2}^{\{3\}} - \{\mathbf{v}\}$ joining $(\mathbf{w}_1)^2$ to $\mathbf{w}_2$, and a hamiltonian path W_n of $S_{n,2}^{\{2\}} - \{\mathbf{u}\}$ joining $(\mathbf{w}_{n-1})^2$ to $\mathbf{w}_n$. We set $P_{n-2} = \left\langle \mathbf{u} = \mathbf{w}_0, (\mathbf{w}_0)^2, W_1, \mathbf{w}_1, (\mathbf{w}_1)^2, W_2, \mathbf{w}_2, \ldots, (\mathbf{w}_{n-1})^2, W_n, \mathbf{w}_n = (\mathbf{v})^2, \mathbf{v} \right\rangle$.

Then $\{P_1, P_2, \ldots, P_{n-2}\}$ forms a set of $(n-2)$ mutually independent hamiltonian paths of $S_{n,2}$ from $\mathbf{u}$ to $\mathbf{v}$. See Figure 4 as an illustration of this case for $S_{6,2}$.

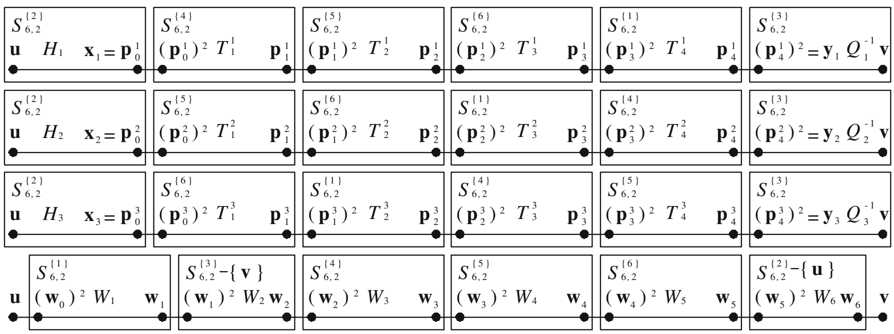

Figure 4: An illustration for Case 2 of Theorem 3.4 on $S_{6,2}$.

Case 3. $(\mathbf{v})_2 = 2$. Without loss of generality, we assume $\mathbf{v} = 32$. Suppose that $n = 5$. We show that there are 3 mutually independent hamiltonian paths on $S_{5,2}$ joining $\mathbf{u}$ to $\mathbf{v}$ by exhibiting the three required hamiltonian paths in Table 4.

Table 4: The 3 mutually independent hamiltonian paths of $S_{5,2}$ between 12 and 32.

$P_1 = \langle 12, 42, 24, 34, 54, 14, 41, 51, 21, 31, 13, 43, 23, 53, 35, 45, 15, 25, 52, 32\rangle$
$P_2 = \langle 12, 52, 25, 15, 45, 35, 53, 43, 23, 13, 31, 21, 51, 41, 14, 34, 54, 24, 42, 32\rangle$
$P_3 = \langle 12, 21, 31, 41, 51, 15, 45, 35, 25, 52, 42, 24, 14, 54, 34, 43, 53, 13, 23, 32\rangle$

Therefore, we assume that $n \geq 6$. For every $1 \leq i \leq n-3$ and for every $1 \leq j \leq n-3$, we set $\mathbf{z}^i_j$ as

$$\mathbf{z}^i_j = \begin{cases} (i+j+4)2, & \text{if } i+j \leq n-4, \\ (i+j-n+7)2, & \text{if } n-3 \leq i+j \leq 2n-7, \text{ and} \\ 42, & \text{if } i+j = 2n-6. \end{cases}$$

We set $H_i = \left\langle \mathbf{u}, \mathbf{z}^i_1, \mathbf{z}^i_2, \ldots, \mathbf{z}^i_{n-4}\right\rangle$ the $(n-3)$ paths for every $1 \leq i \leq n-3$. Obviously, H_i is a hamiltonian path of $S^{\{2\}}_{n,2} - \left\{\mathbf{v}, \mathbf{z}^i_{n-3}\right\}$ joining $\mathbf{u}$ to $\mathbf{z}^i_{n-4}$ for every $1 \leq i \leq n-3$, and $|\{H_1(j), H_2(j), \ldots, H_{n-3}(j)\}| = n-3$ for every $2 \leq j \leq n-3$. Let A be a matrix of order $(n-3)\times(n-1)$ defined by

$$a_{i,j} = \begin{cases} i-j+4, & \text{if } i \neq n-3 \text{ and } j \leq i, \\ 1, & \text{if } j-1 = i, \\ 3, & \text{if } i \neq n-3 \text{ and } j-2 = i, \text{or } i = n-3 \text{ and } j = 2, \\ n+i-j+3, & \text{if } i \neq n-3 \text{ and } j-3 \geq i, \\ n, & \text{if } i = n-3 \text{ and } j = 1, \\ n-j+2, & \text{if } i = n-3 \text{ and } 3 \leq j \leq n-3, \text{ and} \\ 4, & \text{if } i = n-3 \text{ and } j = n-1. \end{cases}$$

More precisely,

$$A = \begin{bmatrix} 4 & 1 & 3 & n & n-1 & \cdots & 8 & 7 & 6 & 5 \\ 5 & 4 & 1 & 3 & n & \cdots & 9 & 8 & 7 & 6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n-1 & n-2 & n-3 & n-4 & n-5 & \cdots & 4 & 1 & 3 & n \\ n & 3 & n-1 & n-2 & n-3 & \cdots & 6 & 5 & 1 & 4 \end{bmatrix}.$$

Let $\mathbf{q}_1 = 24$, $\mathbf{q}_2 = 54$, $\mathbf{q}_3 = 34$, $\mathbf{q}_i = (i+2)4$ for every $4 \leq i \leq n-2$, and $\mathbf{q}_{n-1} = 14$. Then we set $Q_1 = \langle \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{n-1}\rangle$, $Q_2 = \langle \mathbf{q}_2, \mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_4, \ldots, \mathbf{q}_{n-1}\rangle$, and $Q_3 = \langle \mathbf{q}_{n-1}, \mathbf{q}_2, \mathbf{q}_3, \ldots, \mathbf{q}_{n-2}, \mathbf{q}_1\rangle$. Obviously, $\left(\mathbf{z}^1_{n-4}\right)^2 = \mathbf{q}_1$, and Q_i is a hamiltonian path of $S^{\{4\}}_{n,2}$ for every $1 \leq i \leq 3$.

We construct P_1 as follows. Let $\mathbf{p}^1_1 = \mathbf{q}_{n-1}$, $\mathbf{p}^1_{n-1} = \left(\mathbf{z}^1_{n-3}\right)^2$, and $\mathbf{p}^1_j = a_{1,j+1}a_{1,j}$ for every j, $2 \leq j \leq n-2$. By Theorem 2.3, there is a hamiltonian path T^1_j of $S^{\{a_{1,j}\}}_{n,2}$

joining $\left(\mathbf{p}_{j-1}^1\right)^2$ to $\mathbf{p}_j^1$ for every $2 \le j \le n-1$. Then we set $P_1 = \left\langle \mathbf{u}, H_1, \mathbf{z}_{n-4}^1, \mathbf{q}_1, Q_1, \mathbf{q}_{n-1} = \mathbf{p}_1^1, \left(\mathbf{p}_1^1\right)^2, T_2^1, \mathbf{p}_2^1, \ldots, \left(\mathbf{p}_{n-2}^1\right)^2, T_{n-1}^1, \mathbf{p}_{n-1}^1 = \left(\mathbf{z}_{n-3}^1\right)^2, \mathbf{z}_{n-3}^1, \mathbf{v} \right\rangle$.

We construct P_2 as follows. Let $\mathbf{p}_2^2 = \mathbf{q}_{n-1}$, $\mathbf{p}_{n-1}^2 = \left(\mathbf{z}_{n-3}^2\right)^2$, and $\mathbf{p}_j^2 = a_{2,j+1}a_{2,j}$ for every $3 \le j \le n-2$. By Theorem 2.3, there is a hamiltonian path T_j^2 of $S_{n,2}^{\{a_{2,j}\}}$ joining $\left(\mathbf{p}_{j-1}^2\right)^2$ to $\mathbf{p}_j^2$ for every $3 \le j \le n-1$, and a hamiltonian path H of $S_{n,2}^{\{5\}}$ joining $\left(\mathbf{z}_{n-4}^2\right)^2$ to $(\mathbf{q}_2)^2$. Then we set $P_2 = \left\langle \mathbf{u}, H_2, \mathbf{z}_{n-4}^2, \left(\mathbf{z}_{n-4}^2\right)^2, H, (\mathbf{q}_2)^2, \mathbf{q}_2, Q_2, \mathbf{q}_{n-1} = \mathbf{p}_2^2, \left(\mathbf{p}_2^2\right)^2, T_3^2, \mathbf{p}_3^2, \ldots, \left(\mathbf{p}_{n-2}^2\right)^2, T_{n-1}^2, \mathbf{p}_{n-1}^2 = \left(\mathbf{z}_{n-3}^2\right)^2, \mathbf{z}_{n-3}^2, \mathbf{v} \right\rangle$.

We construct the path P_i for every $3 \le i \le n-3$ as follows. Let $\mathbf{p}_0^i = \mathbf{z}_{n-4}^i$ and $\mathbf{p}_{n-1}^i = \left(\mathbf{z}_{n-3}^i\right)^2$ for every $3 \le i \le n-3$. We set $\mathbf{p}_j^i = a_{i,j+1}a_{i,j}$ for every $3 \le i \le n-3$ and $1 \le j \le n-2$. By Theorem 2.3, there is a hamiltonian path T_j^i of $S_{n,2}^{\{a_{i,j}\}}$ joining $\left(\mathbf{p}_{j-1}^i\right)^2$ to $\mathbf{p}_j^i$ for every $3 \le i \le n-3$ and $1 \le j \le n-1$. We set $P_i = \left\langle \mathbf{u}, H_i, \mathbf{z}_{n-4}^i = \mathbf{p}_0^i, \left(\mathbf{p}_0^i\right)^2, T_1^i, \mathbf{p}_1^i, \left(\mathbf{p}_1^i\right)^2, T_2^i, \mathbf{p}_2^i, \ldots, \left(\mathbf{p}_{n-2}^i\right)^2, T_{n-1}^i, \mathbf{p}_{n-1}^i = \left(\mathbf{z}_{n-3}^i\right)^2, \mathbf{z}_{n-3}^i, \mathbf{v} \right\rangle$ the $(n-5)$ paths for every $3 \le i \le n-3$.

We construct P_{n-2} as follows. Let $\mathbf{x}_1 = n2$, $\mathbf{x}_i = (n-i+1)(n-i+2)$ for every $2 \le i \le n-4$, and $\mathbf{x}_{n-3} = 35$. By Lemma 2.2, there is a hamiltonian path W_1 of $S_{n,2}^{\{2\}} - \{\mathbf{u}, \mathbf{v}\}$ joining $(\mathbf{q}_1)^2$ to $\mathbf{x}_1$. By Theorem 2.3, there is a hamiltonian path W_0 of $S_{n,2}^{\{1\}}$ joining $(\mathbf{u})^2$ to $(\mathbf{q}_{n-1})^2$. Again, there is a hamiltonian path W_i of $S_{n,2}^{\{n-i+2\}}$ joining $(\mathbf{x}_{i-1})^2$ to $\mathbf{x}_i$ for every $2 \le i \le n-3$. Moreover, there is a hamiltonian path W_{n-2} of $S_{n,2}^{\{3\}}$ joining $(\mathbf{x}_{n-3})^2$ to $(\mathbf{v})^2$. We set $P_{n-2} = \langle \mathbf{u}, (\mathbf{u})^2, W_0, (\mathbf{q}_{n-1})^2, \mathbf{q}_{n-1}, Q_3, \mathbf{q}_1, (\mathbf{q}_1)^2, W_1, \mathbf{x}_1, (\mathbf{x}_1)^2, W_2, \mathbf{x}_2, \ldots, (\mathbf{x}_{n-4})^2, W_{n-3}, \mathbf{x}_{n-3}, (\mathbf{x}_{n-3})^2, W_{n-2}, (\mathbf{v})^2, \mathbf{v} \rangle$.

Then $\{P_1, P_2, \ldots, P_{n-2}\}$ forms a set of $(n-2)$ mutually independent hamiltonian paths of $S_{n,2}$ from $\mathbf{u}$ to $\mathbf{v}$. See Figure 5 as an illustration of this case for $S_{6,2}$.

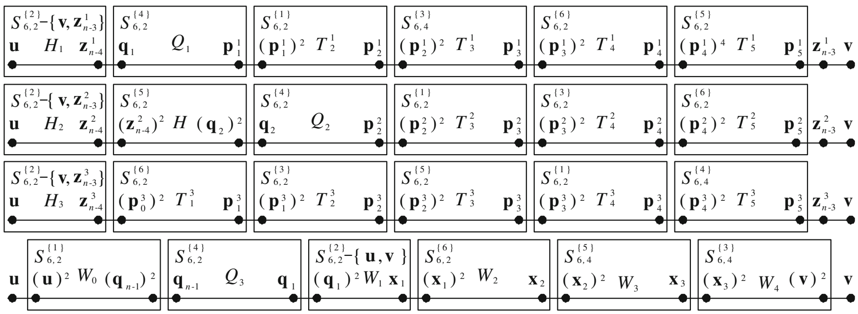

Figure 5: An illustration for Case 3 of Theorem 3.4 on $S_{6,2}$.

Case 4. $(\mathbf{v})_1 = 1$. Without loss of generality, we assume that $\mathbf{v} = 13$. By Lemma 3.3, there are $n-2$ hamiltonian paths $H_1, H_2, \ldots, H_{n-2}$ of $S_{n,2}^{\{2\}}$ such that

(1) H_i joins $\mathbf{u}$ to $\mathbf{x}_i = (i+4)2$ for every $1 \le i \le n-4$, $\mathbf{x}_{n-3} = 32$, and $\mathbf{x}_{n-2} = 42$, and
(2) $|\{H_1(j), H_2(j), \dots, H_{n-2}(j)\}| = n-2$ for every $2 \le j \le n-1$.

Let A be a matrix of order $(n-2) \times (n-1)$ defined by

$$a_{i,j} = \begin{cases} i+j+3, & \text{if } i+j \le n-3, \\ i+j-n+5, & \text{if } n-2 \le i+j \le n-1, \\ 1, & \text{if } i+j = n, \\ i+j-n+4, & \text{if } n+1 \le i+j \le 2n-4, \text{ and} \\ 3, & \text{if } i+j = 2n-3. \end{cases}$$

More precisely,

$$A = \begin{bmatrix} 5 & 6 & 7 & 8 & \cdots & n & 3 & 4 & 1 \\ 6 & 7 & 8 & 9 & \cdots & 3 & 4 & 1 & 5 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n & 3 & 4 & 1 & \cdots & n-4 & n-3 & n-2 & n-1 \\ 3 & 4 & 1 & 5 & \cdots & n-3 & n-2 & n-1 & n \\ 4 & 1 & 5 & 6 & \cdots & n-2 & n-1 & n & 3 \end{bmatrix}.$$

We set $\mathbf{p}_0^1 = \mathbf{x}_1$, $\mathbf{p}_j^1 = a_{1,j+1}a_{1,j}$ for every $1 \le j \le n-2$, and $\mathbf{p}_{n-1}^1 = (\mathbf{v})^2$. Note that $\left\{ \left(\mathbf{p}_{n-4}^1\right)^2 = n3, \mathbf{p}_{n-3}^1 = 43 \right\} \subset V\left(S_{n,2}^{\{3\}}\right) - \{\mathbf{v}\}$. By Lemma 2.2, there is a hamiltonian path T_j^1 of $S_{n,2}^{\{a_{1,j}\}}$ joining $\left(\mathbf{p}_{j-1}^1\right)^2$ to $\mathbf{p}_j^1$ for every $j \in \langle n-1 \rangle - \{n-3\}$, and there is a hamiltonian path T_{n-3}^1 of $S_{n,2}^{\{3\}} - \{\mathbf{v}\}$ joining $\left(\mathbf{p}_{n-4}^1\right)^2$ to $\mathbf{p}_{n-3}^1$. We set $P_1 = \left\langle \mathbf{u}, H_1, \mathbf{x}_1 = \mathbf{p}_0^1, \left(\mathbf{p}_0^1\right)^2, T_1^1, \mathbf{p}_1^1, \left(\mathbf{p}_1^1\right)^2, T_2^1, \mathbf{p}_2^1, \dots, \left(\mathbf{p}_{n-2}^1\right)^2, T_{n-1}^1, \mathbf{p}_{n-1}^1 = (\mathbf{v})^2, \mathbf{v} \right\rangle$.

Let $\mathbf{y}_i = a_{i+1,n-1}3$ for every $1 \le i \le n-4$. For every $2 \le i \le n-3$, we set $\mathbf{p}_0^i = \mathbf{x}_i$, $\mathbf{p}_j^i = a_{i,j+1}a_{i,j}$ for every $1 \le j \le n-2$, and $\mathbf{p}_{n-1}^i = (\mathbf{y}_{i-1})^2$. Note that $\left\{ \left(\mathbf{p}_0^{n-3}\right)^2 = 23, \mathbf{p}_1^{n-3} = 43 \right\} \subset V\left(S_{n,2}^{\{3\}}\right) - \{\mathbf{v}, \mathbf{y}_{n-4}\}$ and $\left\{ \left(\mathbf{p}_{n-i-4}^{i+1}\right)^2 = n3, \mathbf{p}_{n-i-3}^{i+1} = 43 \right\} \subset V\left(S_{n,2}^{\{3\}}\right) - \{\mathbf{v}, \mathbf{y}_i\}$ for every $1 \le i \le n-5$. For every $2 \le i \le n-3$, by Lemma 2.2, there is a hamiltonian path T_j^i of $S_{n,2}^{\{a_{i,j}\}}$ joining the node $\left(\mathbf{p}_{j-1}^i\right)^2$ to $\mathbf{p}_j^i$ for every $j \in \langle n-1 \rangle - \{n-i-2\}$, and there is a hamiltonian path T_{n-i-2}^i of $S_{n,2}^{\{3\}} - \{\mathbf{v}, \mathbf{y}_{i-1}\}$ joining $\left(\mathbf{p}_{n-i-3}^i\right)^2$ to $\mathbf{p}_{n-i-2}^i$. We set $P_i = \left\langle \mathbf{u}, H_i, \mathbf{x}_i = \mathbf{p}_0^i, \left(\mathbf{p}_0^i\right)^2, T_1^i, \mathbf{p}_1^i, \left(\mathbf{p}_1^i\right)^2, T_2^i, \mathbf{p}_2^i, \dots, \left(\mathbf{p}_{n-2}^i\right)^2, T_{n-1}^i, \mathbf{p}_{n-1}^i = (\mathbf{y}_{i-1})^2, \mathbf{y}_{i-1}, \mathbf{v} \right\rangle$ the $(n-4)$ paths for every $2 \le i \le n-3$.

We set $\mathbf{p}_0^{n-2} = \mathbf{x}_{n-2}$, $\mathbf{p}_j^{n-2} = a_{n-2,j+1}a_{n-2,j}$ for every $1 \le j \le n-3$, and $\mathbf{p}_{n-2}^{n-2} = 4n$. By Theorem 2.3, there is a hamiltonian path T_j^{n-2} of $S_{n,2}^{\{a_{n-2,j}\}}$ joining $\left(\mathbf{p}_{j-1}^{n-2}\right)^2$ to $\mathbf{p}_j^{n-2}$ for every j in $\langle n-2 \rangle - \{1\}$. We set $\mathbf{w} = 34$ and $\mathbf{z} = 23$. By Lemma 2.2, there is a hamiltonian path T_1^{n-2} of $S_{n,2}^{\{4\}} - \left\{ \mathbf{w}, \left(\mathbf{p}_{n-2}^{n-2}\right)^2 \right\}$ joining $\left(\mathbf{p}_0^{n-2}\right)^2$ to $\mathbf{p}_1^{n-2}$. Again, there is a hamiltonian path Q of $S_{n,2}^{\{3\}} - \{\mathbf{v}\}$ joining $(\mathbf{w})^2$ to $\mathbf{z}$. Then we set the path P_{n-2} $= \left\langle \mathbf{u}, H_{n-2}, \mathbf{x}_{n-2} = \mathbf{p}_0^{n-2}, \left(\mathbf{p}_0^{n-2}\right)^2, T_1^{n-2}, \mathbf{p}_1^{n-2}, \left(\mathbf{p}_1^{n-2}\right)^2, T_2^{n-2}, \mathbf{p}_2^{n-2}, \ldots, \left(\mathbf{p}_{n-3}^{n-2}\right)^2, T_{n-2}^{n-2}, \mathbf{p}_{n-2}^{n-2}, \left(\mathbf{p}_{n-2}^{n-2}\right)^2, \mathbf{w}, (\mathbf{w})^2, Q, \mathbf{z}, \mathbf{v} \right\rangle$.

Then $\{P_1, P_2, \ldots, P_{n-2}\}$ forms a set of $(n-2)$ mutually independent hamiltonian paths of $S_{n,2}$ from $\mathbf{u}$ to $\mathbf{v}$. See Figure 6 as an illustration of this case for $S_{6,2}$.

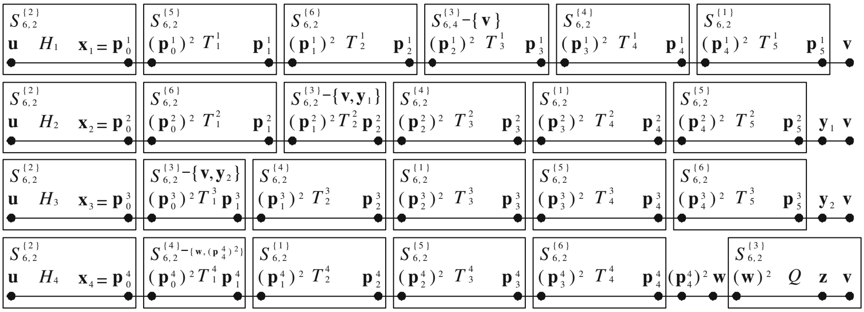

Figure 6: An illustration for Case 4 of Theorem 3.4 on $S_{6,2}$.

Case 5. $(\mathbf{v})_1, (\mathbf{v})_2 \notin \{1, 2\}$. Without loss of generality, we assume that $\mathbf{v}$ is the node 34. By Lemma 3.3, there are $(n-3)$ hamiltonian paths $H_1, H_2, \ldots, H_{n-3}$ of $S_{n,2}^{\{2\}}$ such that

(1) H_i joins $\mathbf{u}$ to $\mathbf{x}_i = (i+4)2$ for every $1 \le i \le n-4$ and H_{n-3} joins $\mathbf{u}$ to $\mathbf{x}_{n-3} = 32$, and
(2) $|\{H_1(j), H_2(j), \ldots, H_{n-3}(j)\}| = n-3$ for every $2 \le j \le n-1$.

Again, there are $(n-3)$ hamiltonian paths $Q_1, Q_2, \ldots, Q_{n-3}$ of $S_{n,2}^{\{4\}}$ such that

(1) Q_1 joins $\mathbf{v}$ to $\mathbf{y}_1 = 14$ and Q_i joins $\mathbf{v}$ to $\mathbf{y}_i = (i+3)4$ for every $2 \le i \le n-3$, and
(2) $|\{Q_1(j), Q_2(j), \ldots, Q_{n-3}(j)\}| = n-3$ for every $2 \le j \le n-1$.

Let A be a matrix of order $(n-3) \times (n-2)$ as

$$a_{i,j} = \begin{cases} i+j+3, & \text{if } i+j \le n-3, \\ 3, & \text{if } i+j = n-2, \\ 1, & \text{if } i+j = n-1, \text{ and} \\ i+j-n+5, & \text{if } n \le i+j. \end{cases}$$

More precisely,

$$A = \begin{bmatrix} 5 & 6 & 7 & 8 & \cdots & n-1 & n & 3 & 1 \\ 6 & 7 & 8 & 9 & \cdots & n & 3 & 1 & 5 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n & 3 & 1 & 5 & \cdots & n-4 & n-3 & n-2 & n-1 \\ 3 & 1 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n \end{bmatrix}.$$

We set $\mathbf{p}_0^i = \mathbf{x}_i$ and $\mathbf{p}_{n-2}^i = (\mathbf{y}_i)^2$ for every $1 \le i \le n-3$. For every $1 \le i \le n-3$ and every $1 \le j \le n-3$, we set $\mathbf{p}_j^i = a_{i,j+1}a_{i,j}$. By Theorem 2.3, there is a hamiltonian path T_j^i of $S_{n,2}^{\{a_{i,j}\}}$ joining the node $\left(\mathbf{p}_{j-1}^i\right)^2$ to $\mathbf{p}_j^i$ for every $1 \le i \le n-3$ and for every $1 \le j \le n-2$. We set $P_i = \left\langle \mathbf{u}, H_i, \mathbf{x}_i = \mathbf{p}_0^i, \left(\mathbf{p}_0^i\right)^2, T_1^i, \mathbf{p}_1^i, \left(\mathbf{p}_1^i\right)^2, T_2^i, \mathbf{p}_2^i, \dots, \left(\mathbf{p}_{n-3}^i\right)^2, T_{n-2}^i, \mathbf{p}_{n-2}^i, \left(\mathbf{p}_{n-2}^i\right)^2 = \mathbf{y}_i, Q_i^{-1}, \mathbf{v} \right\rangle$ the $(n-3)$ paths for every $1 \le i \le n-3$.

Let $b_1 = 1$, $b_2 = 4$, $b_i = i+2$ for every $3 \le i \le n-2$, $b_{n-1} = 2$, and $b_n = 3$. We set $\mathbf{w}_0 = \mathbf{u}$, $\mathbf{w}_i = b_{i+1}b_i$ for every $1 \le i \le n-1$, and $\mathbf{w}_n = (\mathbf{v})^2$. Note that $(\mathbf{w}_1)^2 = 14$, $\mathbf{w}_2 = 54$, $(\mathbf{w}_{n-2})^2 = n2$, and $\mathbf{w}_{n-1} = 32$. By Lemma 2.2, there is a hamiltonian path W_i of $S_{n,2}^{\{b_i\}}$ joining the node $(\mathbf{w}_{i-1})^2$ to $\mathbf{w}_i$ for every $i \in \langle n \rangle - \{2, n-1\}$, and there is a hamiltonian path W_2 of $S_{n,2}^{\{4\}} - \{\mathbf{v}\}$ joining the node $(\mathbf{w}_1)^2$ to $\mathbf{w}_2$. Again, there is a hamiltonian path W_{n-1} of $S_{n,2}^{\{2\}} - \{\mathbf{u}\}$ joining the node $(\mathbf{w}_{n-2})^2$ to $\mathbf{w}_{n-1}$. We set the path $P_{n-2} = \langle \mathbf{u} = \mathbf{w}_0, (\mathbf{w}_0)^2, W_1, \mathbf{w}_1, (\mathbf{w}_1)^2, W_2, \mathbf{w}_2, \dots, (\mathbf{w}_{n-1})^2, W_n, \mathbf{w}_n = (\mathbf{v})^2, \mathbf{v} \rangle$.

Then $\{P_1, P_2, \dots, P_{n-2}\}$ forms a set of $(n-2)$ mutually independent hamiltonian paths of $S_{n,2}$ from $\mathbf{u}$ to $\mathbf{v}$. See Figure 7 as an illustration of this case for $S_{6,2}$.

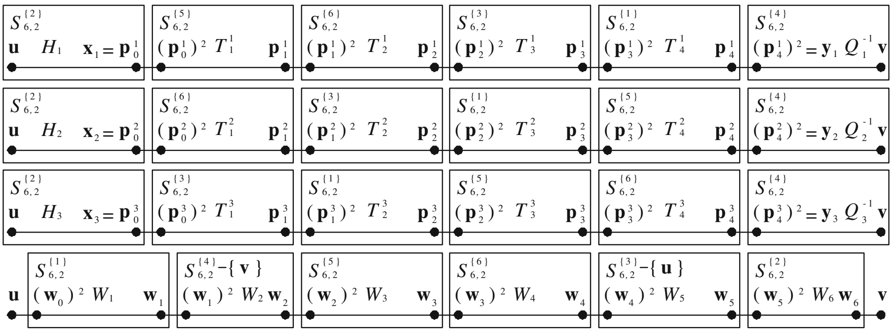

Figure 7: An illustration for Case 5 of Theorem 3.4 on $S_{6,2}$.

Hence, the statement is proved. ∎

Note that the matrices A used in Theorem 3.4 are rectangular Latin squares. They will be useful technique in the sequel of the paper, e.g., in Lemmas 3.5, 3.6, 3.7, and Theorem 3.8.

3.3. *IHP*$(S_{n,k})$ for $3 \le k \le n-2$

Lemma 3.5. *Let $\{i_1, i_2, \dots, i_{n-1}\}$ be any subset of $\langle n \rangle$, and let* $\mathbf{u}$ *be any node of $S_{n,k}$ with $2 \le k \le n-2$. There are $(n-1)$ hamiltonian paths $P_1, P_2, \dots, P_{n-1}$ of $S_{n,k}$ such that*

(1) *P_j joins* $\mathbf{u}$ *to some node $\mathbf{z}_j$ with $(\mathbf{z}_j)_1 = i_j$ for every $1 \le j \le n-1$, and*
(2) *$|\{P_1(j), P_2(j), \dots, P_{n-1}(j)\}| = n-1$ for every $2 \le j \le \nu(S_{n,k})$.*

Proof. We prove the statement by induction on k. Suppose that $n=4$ and $k=2$. Since $S_{4,2}$ is node transitive, we assume that $\mathbf{u}=14$. We prove the statement holds on $S_{4,2}$ by exhibiting the three required hamiltonian paths in Table 5.

Table 5: All cases of the required hamiltonian paths for $S_{4,2}$ of Lemma 3.5 .

$\{i_1,i_2,i_3\}=\{1,2,3\}$	$P_1=\langle 14,34,24,42,32,23,43,13,31,41,21,12\rangle$ $P_2=\langle 14,41,21,31,13,43,34,24,42,12,32,23\rangle$ $P_3=\langle 14,24,42,32,12,21,41,31,13,23,43,34\rangle$
$\{i_1,i_2,i_3\}=\{1,2,4\}$	$P_1=\langle 14,34,24,42,32,23,43,13,31,41,21,12\rangle$ $P_2=\langle 14,41,31,21,12,32,42,24,34,43,13,23\rangle$ $P_3=\langle 14,24,34,43,23,13,31,41,21,12,32,42\rangle$
$\{i_1,i_2,i_3\}=\{1,3,4\}$	$P_1=\langle 14,24,34,43,23,32,42,12,21,41,31,13\rangle$ $P_2=\langle 14,41,21,31,13,23,43,34,24,42,12,32\rangle$ $P_3=\langle 14,34,24,42,32,12,21,41,31,13,23,43\rangle$
$\{i_1,i_2,i_3\}=\{2,3,4\}$	$P_1=\langle 14,24,34,43,13,31,41,21,12,42,32,23\rangle$ $P_2=\langle 14,41,21,12,32,42,24,34,43,23,13,31\rangle$ $P_3=\langle 14,34,24,42,12,32,23,43,13,31,21,41\rangle$

Suppose that $n\geq 5$ and $k=2$. Since $S_{n,2}$ is node transitive, we assume that $\mathbf{u}=1n$. Without loss of generality, we suppose that $i_1<i_2<\cdots<i_{n-1}$. Obviously, $i_j=j$ or $j+1$ for $1\leq j\leq n-1$. Hence, $i_j\notin\{j+2,j+3\}$ for every $1\leq j\leq n-4$, $i_{n-3}\notin\{1,n-1\}$, $i_{n-2}\notin\{1,2\}$, and $i_{n-1}\notin\{2,3\}$. By Lemma 3.3, there are $(n-2)$ hamiltonian paths $H_1,H_2,\ldots,H_{n-2}$ of $S_{n,2}^{\{n\}}$ such that

(1) H_j joins $\mathbf{u}$ to node $\mathbf{x}_j=(j+1)n$ for every $1\leq j\leq n-2$, and
(2) $|\{H_1(j),H_2(j),\ldots,H_{n-2}(j)\}|=n-2$ for every $2\leq j\leq n-1$.

We choose the node $\mathbf{z}_j=i_j(j+2)$ for $1\leq j\leq n-3$, the node $\mathbf{z}_{n-2}=i_{n-2}1$. Let A be a matrix of order $(n-2)\times(n-1)$ as

$$a_{j,l}=\begin{cases} j-l+2, & \text{if } l\leq j+1, \text{ and}\\ j-l+n+1, & \text{if } j+2\leq l.\end{cases}$$

More precisely,

$$A=\begin{bmatrix} 2 & 1 & n-1 & n-2 & \cdots & 4 & 3\\ 3 & 2 & 1 & n-1 & \cdots & 5 & 4\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ n-2 & n-3 & n-4 & n-5 & \cdots & 1 & n-1\\ n-1 & n-2 & n-3 & n-4 & \cdots & 2 & 1\end{bmatrix}.$$

We construct the path P_2 as follows. Let $\mathbf{p}_0^2=\mathbf{x}_2$ and $\mathbf{p}_l^2=a_{2,l+1}a_{2,l}$ for every $1\leq l\leq n-2$. Note that $\left(\mathbf{p}_l^2\right)^2\in S_{n,2}^{\{a_{2,l+1}\}}-\left\{\mathbf{p}_{l+1}^2\right\}$ for every $0\leq l\leq n-2$. Since $i_2\notin\{4,5\}$, we can choose a node $\mathbf{p}_{n-1}^2=i_2a_{2,n-1}=\mathbf{z}_2$ in $S_{n,2}^{\{a_{2,n-2}\}}$. By Lemma 2.2, there is a hamiltonian path T in $S_{n,2}^{\{3\}} - \left\{ \left(\mathbf{p}_0^2\right)^2 \right\}$ joining $\mathbf{y} = 43$ to $\mathbf{p}_1^2$. Again, there is a hamiltonian path T_l^2 in $S_{n,2}^{\{a_{2,l}\}}$ joining $\left(\mathbf{p}_{l-1}^2\right)^2$ to $\mathbf{p}_l^2$ for every $2 \le l \le n-1$. We set the path $P_2 = \left\langle \mathbf{u}, H_2, \mathbf{x}_2 = \mathbf{p}_0^2, \left(\mathbf{p}_0^2\right)^2, \mathbf{y}, T, \mathbf{p}_1^2, \left(\mathbf{p}_1^2\right)^2, T_2^2, \mathbf{p}_2^2, \left(\mathbf{p}_2^2\right)^2, T_3^2, \mathbf{p}_3^2, \ldots, \left(\mathbf{p}_{n-2}^2\right)^2, T_{n-1}^2, \mathbf{p}_{n-1}^2 = \mathbf{z}_2 \right\rangle$.

For every $j \in \langle n-2 \rangle - \{2\}$, we construct P_j as follows. Let $\mathbf{p}_0^j = \mathbf{x}_j$ for every $j \in \langle n-2 \rangle - \{2\}$. We set $\mathbf{p}_l^j = a_{j,l+1}a_{j,l}$ for every $1 \le j \le n-2$ and $1 \le l \le n-2$. Note that $\left(\mathbf{p}_l^j\right)^2 \in S_{n,2}^{\{a_{j,l+1}\}} - \left\{\mathbf{p}_{l+1}^j\right\}$ for every $j \in \langle n-2 \rangle - \{2\}$ and $0 \le l \le n-2$. Since $i_j \notin \{j+2, j+3\}$ for every $j \in \langle n-2 \rangle - \{2\}$, $i_{n-3} \notin \{1, n-1\}$, and $i_{n-2} \notin \{1, 2\}$, we can choose a node $\mathbf{p}_{n-1}^j = i_j a_{j,n-1} = \mathbf{z}_j$ in $S_{n,2}^{\{a_{j,n-2}\}}$. By Lemma 2.2, there is a hamiltonian path T_l^j in $S_{n,2}^{\{a_{j,l}\}}$ joining $\left(\mathbf{p}_{l-1}^j\right)^2$ to $\mathbf{p}_l^j$ for every $j \in \langle n-2 \rangle - \{2\}$ and $1 \le l \le n-1$. We set $P_j = \left\langle \mathbf{u}, H_j, \mathbf{x}_j = \mathbf{p}_0^j, \left(\mathbf{p}_0^j\right)^2, T_1^j, \mathbf{p}_1^j, \left(\mathbf{p}_1^j\right)^2, T_2^j, \mathbf{p}_2^j, \ldots, \left(\mathbf{p}_{n-2}^j\right)^2, T_{n-1}^j, \mathbf{p}_{n-1}^j = \mathbf{z}_j \right\rangle$ the $(n-3)$ paths for every $j \in \langle n-2 \rangle - \{2\}$.

Let P_{n-1} be constructed as follows. Let $\mathbf{x} = 13$, $\mathbf{q}_0 = n3$, and $\mathbf{q}_i = (n-i)(n-i+1)$ for every $1 \le i \le n-2$. Since $i_{n-1} \notin \{2,3\}$, we choose a node $\mathbf{z} = i_{n-1}2$ in $S_{n,2}^{\{2\}}$. By Theorem 2.3, there is a hamiltonian path R of $S_{n,2}^{\{1\}}$ joining $(\mathbf{u})^2 = n1$ to $(\mathbf{x})^2 = 31$. Again, there is a hamiltonian path Q_{n-1} of $S_{n,2}^{\{2\}}$ joining $(\mathbf{q}_{n-2})^2$ to $\mathbf{z}$. By Lemma 2.2, there is a hamiltonian path Q_1 of $S_{n,2}^{\{n\}} - \{\mathbf{u}\}$ joining $(\mathbf{q}_0)^2$ to $\mathbf{q}_1$. And, there is a hamiltonian path Q_i of $S_{n,2}^{\{n-i+1\}} - \{\mathbf{q}_i\}$ joining $(\mathbf{q}_{i-1})^2$ to $\mathbf{w}_i = (n-i-1)(n-i+1)$ for $2 \le i \le n-3$. Moreover, there is a hamiltonian path Q_{n-2} of $S_{n,2}^{\{3\}} - \{\mathbf{x}, \mathbf{q}_0\}$ joining $(\mathbf{q}_{n-3})^2$ to $\mathbf{q}_{n-2}$. We set $\mathbf{z}_{n-1} = \mathbf{z}$ and $P_{n-1} = \langle \mathbf{u}, (\mathbf{u})^2, R, (\mathbf{x})^2, \mathbf{x}, \mathbf{q}_0, (\mathbf{q}_0)^2, Q_1, \mathbf{q}_1, (\mathbf{q}_1)^2, Q_2, \mathbf{w}_2, \mathbf{q}_2, (\mathbf{q}_2)^2, Q_3, \mathbf{w}_3, \mathbf{q}_3, (\mathbf{q}_3)^2, Q_4, \mathbf{w}_4, \mathbf{q}_4, \ldots, (\mathbf{q}_{n-3})^2, Q_{n-2}, \mathbf{q}_{n-2}, (\mathbf{q}_{n-2})^2, Q_{n-1}, \mathbf{z} = \mathbf{z}_{n-1} \rangle$. Then $P_1, P_2, \ldots, P_{n-1}$ form the desired paths of $S_{n,2}$. Note that $\mathbf{q}_i = (n-i)(n-i+1) = \mathbf{p}_{i+1}^1$, hence $\mathbf{q}_i$ does not equal $T_{i+1}^1(2)$ for all $2 \le i \le n-3$. Hence $\{P_1, P_2, \ldots, P_{n-1}\}$ forms a set of desired hamiltonian paths of $S_{n,2}$ for $n \ge 5$. See Figure 8 as an illustration of this case for $S_{6,2}$.

Suppose that this statement holds for $S_{m,l}$ for every $4 \le m \le n-1$, $2 \le l \le k-1$, and $l \le m-2$. Let $\mathbf{u}$ be any node of $S_{n,k}$. Since $S_{n,k}$ is node transitive, we assume that $\mathbf{u}$ is a node in $S_{n,k}$ with $(\mathbf{u})_1 = 1$ and $(\mathbf{u})_i = n-k+i$ for every $2 \le i \le k$. Note that $\mathbf{u} \in S_{n,k}^{\{n\}}$. Without loss of generality, we suppose that $i_1 < i_2 < \cdots < i_{n-3} < i_{n-1} < i_{n-2}$. Obviously, $i_j \ne j+2$ for every $1 \le j \le n-3$, $i_{n-2} \ne 2$, and $i_{n-1} \ne n$. By the induction hypothesis, there are $(n-2)$ hamiltonian paths $H_1, H_2, \ldots, H_{n-2}$ of $S_{n,k}^{\{n\}}$ such that H_i is joining $\mathbf{u}$ to a node $\mathbf{v}_j$ such that

(1) $(\mathbf{v}_j)_1 = (H_j((n-1)!/(n-k-1)!))_1 = j+3$ for every $1 \le j \le n-4$, $(\mathbf{v}_{n-3})_1 = (H_{n-3}((n-1)!/(n-k-1)!))_1 = 1$, $(\mathbf{v}_{n-2})_1 = (H_{n-2}((n-1)!/(n-k-1)!))_1 = 3$, and

(2) $|\{H_1(j), H_2(j), \ldots, H_{n-2}(j)\}| = n-2$ for $2 \le j \le (n-1)!/(n-k-1)!$.

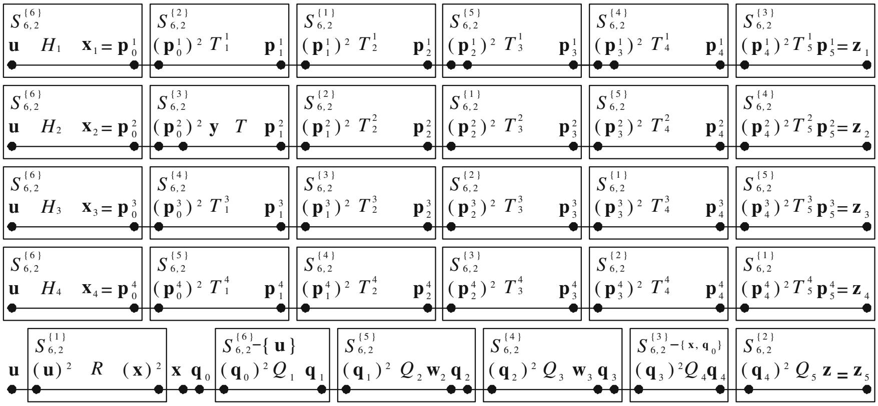

Figure 8: An illustration for Lemma 3.5 on $S_{6,2}$.

Since $n \geq 5$ and $k \geq 3$, by Lemma 2.1, $|E_{n,k}^{j,l}| = (n-2)!/(n-k)! \geq 3$. We choose a node $\mathbf{z}_j \in S_{n,k}^{\{j+2\}}$ with $(\mathbf{z}_j)_1 = i_j$ for $1 \leq j \leq n-3$, a node $\mathbf{z}_{n-2} \in S_{n,k}^{\{2\}}$ with $(\mathbf{z}_{n-2})_1 = i_{n-2}$, and a node $\mathbf{z}_{n-1} \in S_{n,k}^{\{n\}} - \{\mathbf{u}\}$ with $(\mathbf{z}_{n-1})_1 = i_{n-1}$. Let B be a matrix of order $(n-2) \times (n-1)$ as

$$b_{j,l} = \begin{cases} j+l+2, & \text{if } j \leq n-3 \text{ and } j+l \leq n-3, \\ j+l-n+3, & \text{if } j \leq n-3 \text{ and } n-2 \leq j+l, \\ l+2, & \text{if } j = n-2 \text{ and } l \leq n-3, \text{ and} \\ l-n+3, & \text{if } j = n-2 \text{ and } n-2 \leq l \leq n-1. \end{cases}$$

More precisely,

$$B = \begin{bmatrix} 4 & 5 & \cdots & n-2 & n-1 & 1 & 2 & 3 \\ 5 & 6 & \cdots & n-1 & 1 & 2 & 3 & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n-1 & 1 & \cdots & n-6 & n-5 & n-4 & n-3 & n-2 \\ 1 & 2 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 \\ 3 & 4 & \cdots & n-3 & n-2 & n-1 & 1 & 2 \end{bmatrix}.$$

Let j be any index with $1 \leq j \leq n-2$. By Theorem 2.6, there is a hamiltonian path $W_j = \left\langle \mathbf{x}_1^j, T_1^j, \mathbf{y}_1^j, \mathbf{x}_2^j, T_2^j, \mathbf{y}_2^j, \ldots, \mathbf{x}_{n-1}^j, T_{n-1}^j, \mathbf{y}_{n-1}^j \right\rangle$ of $S_{n,k}^{\langle n-1 \rangle}$ joining the node $(\mathbf{v}_j)^k$ to $\mathbf{z}_j$ such that $\mathbf{x}_1^j = (\mathbf{v}_j)^k$, $\mathbf{y}_{n-1}^j = \mathbf{z}_j$, and T_l^j is a hamiltonian path of $S_{n,k}^{\{b_{j,l}\}}$ joining $\mathbf{x}_l^j$ to $\mathbf{y}_l^j$ for every $1 \leq l \leq n-1$. We set $P_i = \langle \mathbf{u}, H_i, \mathbf{v}_i, (\mathbf{v}_i)^k = \mathbf{x}_1^i, W_i, \mathbf{y}_{n-1}^i = \mathbf{z}_i \rangle$ the $(n-2)$ paths for every $1 \leq i \leq n-2$.

Since $S_{n,k}^{\{n\}}$ is isomorphic to $S_{n-1,k-1}$, by Lemma 2.4, there is a hamiltonian path R of $S_{n,k}^{\{n\}} - \{\mathbf{u}\}$ joining a node $\mathbf{w}$ to a node $\mathbf{z}_{n-1}$ with $(\mathbf{w})_1 = (\mathbf{u})_{k-1} = n-1$ and

$(\mathbf{z}_{n-1})_1 = i_{n-1}$. By Theorem 2.6, there is a hamiltonian path $W = \left\langle \mathbf{x}_1^{n-1}, T_1^{n-1}, \mathbf{y}_1^{n-1}, \mathbf{x}_2^{n-1}, T_2^{n-1}, \mathbf{y}_2^{n-1}, \ldots, \mathbf{x}_{n-1}^{n-1}, T_{n-1}^{n-1}, \mathbf{y}_{n-1}^{n-1} \right\rangle$ of $S_{n,k}^{\langle n-1\rangle}$ joining the node $(\mathbf{u})^k$ to the node $(\mathbf{w})^k$ such that $\mathbf{x}_1^{n-1} = (\mathbf{u})^k$, $\mathbf{y}_{n-1}^{n-1} = (\mathbf{w})^k$, and T_i^{n-1} is a hamiltonian path of $S_{n,k}^{\{i\}}$ joining $\mathbf{x}_i^{n-1}$ to $\mathbf{y}_i^{n-1}$ for $1 \le i \le n-1$. We set $P_{n-1} = \langle \mathbf{u}, (\mathbf{u})^k, W, (\mathbf{w})^k, \mathbf{w}, R, \mathbf{z}_{n-1} \rangle$.

Then $\{P_1, P_2, \ldots, P_{n-1}\}$ forms a set of desired paths of $S_{n,k}$. Note that $\left(\mathbf{y}_1^{n-1}\right)_1 = 2$, $\left(\mathbf{x}_1^{n-3}\right)_1 = n$, $\left(\mathbf{y}_i^{n-1}\right)_1 = i+1$, and $\left(\mathbf{x}_i^{n-3}\right)_1 = i-1$ for every $2 \le i \le n-1$. See Figure 9 for an illustration of this case for $S_{6,3}$.

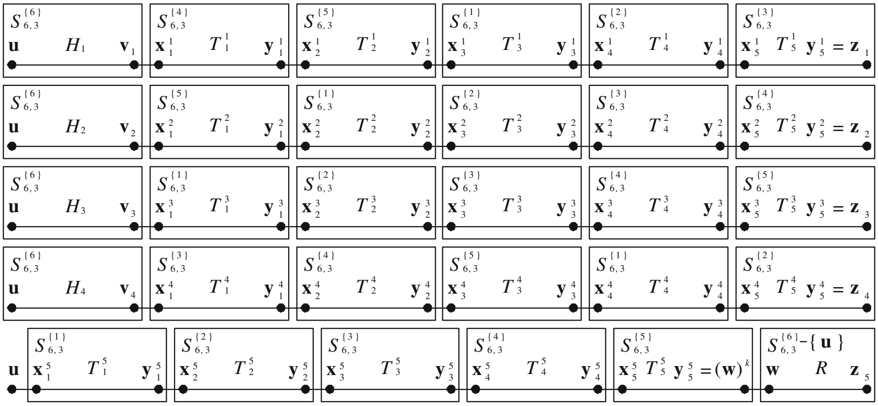

Figure 9: An illustration for Lemma 3.5 on $S_{6,3}$.

Hence, the statement is proved. ∎

Lemma 3.6. *Let* $\mathbf{u} = 1n$, $\mathbf{v} = 2n$, $\mathbf{y}_i = (i+2)n$ *for every* $1 \le i \le n-3$, *and* $\mathbf{y}_{n-2} = n2$ *be nodes in* $S_{n,2}$ *with* $n \ge 5$. *Then there are* $(n-2)$ *paths* $P_1, P_2, \ldots, P_{n-2}$ *of* $S_{n,2}$ *such that*

(1) P_i *is a hamiltonian path of* $S_{n,2} - \{\mathbf{v}, \mathbf{y}_i\}$ *joining* $\mathbf{u}$ *to a node* $\mathbf{x}_i$ *with* $(\mathbf{x}_i)_1 = i+1$ *for every* $1 \le i \le n-2$, *and*
(2) $|\{P_1(j), P_2(j), \ldots, P_{n-2}(j)\}| = n-2$ *for every* $2 \le j \le n(n-1)-2$.

Proof. We have $\{\mathbf{u}, \mathbf{v}, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n-3}\} = V\left(S_{n,2}^{\{n\}}\right)$ and $\mathbf{y}_{n-2} \in S_{n,2}^{\{2\}}$. For every $1 \le i \le n-3$ and $1 \le j \le n-4$, let $\mathbf{z}_j^i$ be as

$$\mathbf{z}_j^i = \begin{cases} (i+j+2)n, & \text{if } i+j \le n-3, \text{ and} \\ (i+j-n+5)n, & \text{if } n-2 \le i+j. \end{cases}$$

We set $R_i = \langle \mathbf{u}, \mathbf{z}_1^i, \mathbf{z}_2^i, \ldots, \mathbf{z}_{n-4}^i \rangle$ the $(n-3)$ paths for every $1 \le i \le n-3$. Obviously, R_i is a hamiltonian path of $S_{n,2}^{\{n\}} - \{\mathbf{v}, \mathbf{y}_i\}$ joining $\mathbf{u}$ to $\mathbf{z}_{n-4}^i$ for every $1 \le i \le n-3$, and $|\{R_1(j), R_2(j), \ldots, R_{n-3}(j)\}| = n-3$ for every $2 \le j \le n-3$. Moreover,

$\left(\mathbf{z}^1_{n-4}\right)_1 = n-1$ and $\left(\mathbf{z}^i_{n-4}\right)_1 = i+1$ for every $2 \le i \le n-3$. Let C be a matrix of order $(n-3)\times(n-1)$ defined by

$$c_{i,j} = \begin{cases} n-1, & \text{if } i=1 \text{ and } j=1,\\ 5-j, & \text{if } i=1 \text{ and } 2\le j\le 4,\\ j-1, & \text{if } i=1 \text{ and } 5\le j,\\ 4-j, & \text{if } i=2 \text{ and } 1\le j\le 3,\\ j, & \text{if } i=2 \text{ and } 4\le j,\\ i+j, & \text{if } 3\le i \text{ and } i+j\le n-1,\\ n-i-j+3, & \text{if } 3\le i \text{ and } n\le i+j\le n+2, \text{ and}\\ i+j-n+1, & \text{if } 3\le i \text{ and } n+3\le i+j. \end{cases}$$

More precisely,

$$C = \begin{bmatrix} n-1 & 3 & 2 & 1 & 4 & \dots & n-5 & n-4 & n-3 & n-2\\ 3 & 2 & 1 & 4 & 5 & \dots & n-4 & n-3 & n-2 & n-1\\ 4 & 5 & 6 & 7 & 8 & \dots & n-1 & 3 & 2 & 1\\ 5 & 6 & 7 & 8 & 9 & \dots & 3 & 2 & 1 & 4\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ n-2 & n-1 & 3 & 2 & 1 & \dots & n-6 & n-5 & n-4 & n-3 \end{bmatrix}.$$

For every $1\le i\le n-3$, we set $\mathbf{p}^i_0 = \mathbf{z}^i_{n-4}$, $\mathbf{p}^i_j = c_{i,j+1}c_{i,j}$ for every $1\le j\le n-2$, and $\mathbf{p}^i_{n-1} = (i+1)c_{i,n-1}$. Note that $\mathbf{p}^i_j \in S^{\{c_{i,j}\}}_{n,2}$ and $\left(\mathbf{p}^i_j\right)^2 \in S^{\{c_{i,j+1}\}}_{n,2} - \left\{\mathbf{p}^i_{j+1}\right\}$ for every $1\le i\le n-3$ and $0\le j\le n-2$.

By Theorem 2.3, there is a hamiltonian path T^2_j of $S^{\{c_{2,j}\}}_{n,2}$ joining $\left(\mathbf{p}^2_{j-1}\right)^2$ to $\mathbf{p}^2_j$ for every $j\in\langle n-1\rangle - \{2\}$, and there is a hamiltonian path T of $S^{\{c_{2,2}=2\}}_{n,2} - \left\{\left(\mathbf{p}^2_1\right)^2\right\}$ joining the node $\mathbf{w}=52$ to $\mathbf{p}^2_2$. We set $T^2_2 = \left\langle \left(\mathbf{p}^2_1\right)^2, \mathbf{w}, T, \mathbf{p}^2_2, \right\rangle$. By Theorem 2.3, there is a hamiltonian path T^i_j of $S^{\{c_{i,j}\}}_{n,2}$ joining $\left(\mathbf{p}^i_{j-1}\right)^2$ to $\mathbf{p}^i_j$ for every $i\in\langle n-3\rangle$ and $1\le j\le n-1$ except $(i,j)=(2,2)$. We set $P_i = \left\langle \mathbf{u}, R_i, \mathbf{p}^i_0, \left(\mathbf{p}^i_0\right)^2, T^i_1, \mathbf{p}^i_1, \left(\mathbf{p}^i_1\right)^2, T^i_2, \mathbf{p}^i_2, \dots, \left(\mathbf{p}^i_{n-2}\right)^2, T^i_{n-1}, \mathbf{p}^i_{n-1}\right\rangle$ the $(n-3)$ paths for every $i\in\langle n-3\rangle$.

We set $d_1=1$, $d_2=2$, $d_i=i+1$ for $3\le i\le n-2$, $d_{n-1}=3$, and $d_n=n$. We set $\mathbf{q}_1=(\mathbf{u})^2$ and $\mathbf{q}_i = d_{i-1}d_i$ for every $2\le i\le n$. Obviously, $\mathbf{q}_i$ is in $S^{\{d_i\}}_{n,2}$ and $(\mathbf{q}_i)^2$ is in $S^{\{d_{i-1}\}}_{n,2} - \{\mathbf{q}_{i-1}\}$ for every $2\le i\le n$. Moreover, $\mathbf{q}_2, (\mathbf{q}_3)^2 \in S^{\{2\}}_{n,2} - \left\{\mathbf{y}_{n-2} = (\mathbf{v})^2\right\}$ and $\mathbf{q}_n \in S^{\{n\}}_{n,2} - \{\mathbf{u},\mathbf{v}\}$. By Lemma 2.2, there is a hamiltonian path Q_i of $S^{\{d_i\}}_{n,2}$ joining node $\mathbf{q}_i$ to node $(\mathbf{q}_{i+1})^2$ for every $i\in\langle n-1\rangle - \{2\}$. Again, there is a hamiltonian path Q_2 of $S^{\{d_2\}}_{n,2} - \left\{(\mathbf{q}_3)^2, \mathbf{y}_{n-2}\right\}$ joining $\mathbf{q}_2$ to $\mathbf{w}=52$, and there is a hamiltonian path Q_n of $S^{\{d_n\}}_{n,2} - \{\mathbf{u},\mathbf{v}\}$ joining the node $\mathbf{q}_n = 3n$ to the node $\mathbf{z} = (n-1)n$. Then we set the path

$P_{n-2} = \left\langle \mathbf{u}, (\mathbf{u})^2 = \mathbf{q}_1, Q_1, (\mathbf{q}_2)^2, \mathbf{q}_2, Q_2, \mathbf{w}, (\mathbf{q}_3)^2, \ldots, \mathbf{q}_{n-1}, Q_{n-1}, (\mathbf{q}_n)^2, \mathbf{q}_n, Q_n, \mathbf{z} \right\rangle$.

Then $P_1, P_2, \ldots, P_{n-2}$ form the desired paths. See Figure 10 for an illustration of this case for $S_{6,2}$. ∎

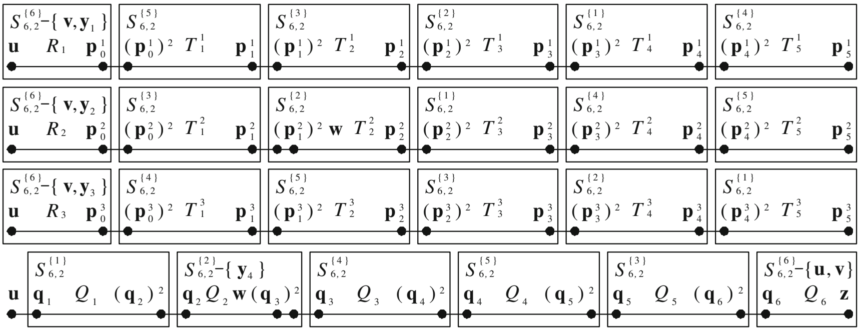

Figure 10: An illustration for Lemma 3.6 on $S_{6,2}$.

Lemma 3.7. *Let n and k be any two positive integers with $n \geq 5$, $k \geq 2$, and $n-k \geq 2$. Let $\mathbf{u}$ and $\mathbf{v}$ be two nodes of $S_{n,k}$ with $(\mathbf{u})_1 = 1$, $(\mathbf{v})_1 = 2$, and $(\mathbf{u})_i = (\mathbf{v})_i = n-k+i$ for every $2 \leq i \leq k$, and let $\mathbf{y}_i \in N_{S_{n,k}}(\mathbf{v})$ with $(\mathbf{y}_i)_1 = i+2$ for every $1 \leq i \leq n-2$. Then there are $(n-2)$ paths $P_1, P_2, \ldots, P_{n-2}$ of $S_{n,k}$ such that*

(1) *P_i is a hamiltonian path of $S_{n,k} - \{\mathbf{v}, \mathbf{y}_i\}$ joining $\mathbf{u}$ to a node $\mathbf{x}_i$ with $(\mathbf{x}_i)_1 = i+1$ for every $1 \leq i \leq n-2$, and*
(2) *$|\{P_1(j), P_2(j), \ldots, P_{n-2}(j)\}| = n-2$ for every $2 \leq j \leq n!/(n-k)!-2$.*

Proof. Note that Lemma 3.6 proves the case $k = 2$. Note that $\{\mathbf{u}, \mathbf{v}, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n-3}\} \subset S_{n,k}^{\{n\}}$ and $\mathbf{y}_{n-2} \in S_{n,k}^{\{2\}}$. We have the following cases:

Case 1. $n = 5$ and $k = 3$. We have $\mathbf{u} = 145$, $\mathbf{v} = 245$, $\mathbf{y}_1 = 345$, $\mathbf{y}_2 = 425$, and $\mathbf{y}_3 = 542$. We prove the statement holds on $S_{5,3}$ by listing every path in Table 6.

Table 6: Desired hamiltonian paths of $S_{5,3} - \{245, \mathbf{y}_i\}$ for Lemma 3.7.

$\mathbf{y}_1 = 345$	⟨145, 415, 315, 215, 125, 425, 325, 235, 135, 435, 534, 354, 154, 254, 524, 324, 124, 214, 514, 314, 134, 234, 432, 532, 352, 152, 452, 542, 342, 142, 412, 512, 312, 132, 231, 531, 431, 341, 541, 241, 421, 321, 521, 251, 451, 351, 153, 453, 543, 243, 143, 413, 513, 213, 123, 423, 523, 253⟩
$\mathbf{y}_2 = 425$	⟨145, 345, 435, 235, 135, 315, 415, 215, 125, 325, 523, 423, 243, 143, 543, 453, 253, 153, 513, 413, 213, 123, 321, 421, 521, 251, 351, 451, 541, 241, 341, 431, 531, 231, 132, 532, 352, 452, 152, 512, 312, 412, 142, 542, 342, 432, 234, 324, 124, 524, 254, 154, 514, 214, 314, 134, 534, 354⟩
$\mathbf{y}_3 = 524$	⟨145, 541, 241, 341, 431, 231, 321, 421, 521, 251, 451, 351, 531, 135, 315, 415, 215, 125, 425, 325, 235, 435, 345, 543, 143, 243, 423, 523, 123, 213, 413, 513, 153, 253, 453, 354, 154, 254, 524, 124, 324, 234, 534, 134, 314, 514, 214, 412, 142, 342, 432, 532, 132, 312, 512, 152, 352, 452⟩

Case 2. $n \geq 6$, $k \geq 3$, and $n-k \geq 2$. We prove the case by induction on n and k. By Lemma 3.6, this statement holds for every $n \geq 5$ and $k = 2$. By Case 1, this statement

holds for $n = 5$ and $k = 3$. Therefore, we assume that this statement holds on $S_{m,l}$ for every $5 \le m < n$ and for every $2 \le l < k$ with $m - l \ge 2$. By induction, there are $(n-3)$ paths $W_1, W_2, \ldots, W_{n-3}$ of $S_{n,k}^{\{n\}}$ such that

(1) W_i is a hamiltonian path of $S_{n,k}^{\{n\}} - \{\mathbf{v}, \mathbf{y}_i\}$ joining $\mathbf{u}$ to a node $\mathbf{w}_i$ with $(\mathbf{w}_i)_1 = i+1$ for every $1 \le i \le n-3$, and
(2) $|\{W_1(j), W_2(j), \ldots, W_{n-3}(j)\}| = n-3$ for every $2 \le j \le (n-1)!/(n-k)! - 2$.

Let C be a matrix of order $(n-3) \times (n-1)$ defined by

$$c_{i,j} = \begin{cases} i+j, & \text{if } i+j \le n-2, \\ 1, & \text{if } i+j = n-1, \\ n-1, & \text{if } i+j = n, \text{ and} \\ i+j-n+1, & \text{if } i+j \ge n+1. \end{cases}$$

More precisely,

$$C = \begin{bmatrix} 2 & 3 & 4 & 5 & \ldots & n-3 & n-2 & 1 & n-1 \\ 3 & 4 & 5 & 6 & \ldots & n-2 & 1 & n-1 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n-2 & 1 & n-1 & 2 & \ldots & n-6 & n-5 & n-4 & n-3 \end{bmatrix}.$$

We set $\mathbf{p}_0^i = \mathbf{w}_i$ for every $1 \le i \le n-3$. We choose a node $\mathbf{p}_j^i$ in $S_{n,k}^{\{c_{i,j}\}}$ with $\left(\mathbf{p}_j^i\right)_1 = c_{i,j+1}$ for every $1 \le i \le n-3$ and $1 \le j \le n-2$, and we choose a node $\mathbf{p}_{n-1}^i$ in $S_{n,k}^{\{c_{i,n-1}\}}$ with $\left(\mathbf{p}_{n-1}^i\right)_1 = i+1$ for every $1 \le i \le n-3$. Note that $\left(\mathbf{p}_j^i\right)^k$ is a node in $S_{n,k}^{\{c_{i,j+1}\}} - \left\{\mathbf{p}_{j+1}^i\right\}$ for every $1 \le i \le n-3$ and for every $0 \le j \le n-2$. By Theorem 2.3, there is a hamiltonian path T_j^i of $S_{n,k}^{\{c_{i,j}\}}$ joining the node $\left(\mathbf{p}_{j-1}^i\right)^k$ to $\mathbf{p}_j^i$ for every $1 \le i \le n-3$ and for every $1 \le j \le n-1$. Let $\mathbf{x}_i = \mathbf{p}_{n-1}^i$ for $1 \le i \le n-3$, then we set $P_i = \left\langle \mathbf{u}, W_i, \mathbf{w}_i = \mathbf{p}_0^i, \left(\mathbf{p}_0^i\right)^k, T_1^i, \mathbf{p}_1^i, \left(\mathbf{p}_1^i\right)^k, \mathbf{p}_2^i, \ldots, \left(\mathbf{p}_{n-2}^i\right)^k, T_{n-1}^i, \mathbf{p}_{n-1}^i = \mathbf{x}_i \right\rangle$ the $(n-3)$ paths for every $1 \le i \le n-3$.

We set $d_1 = 1, d_2 = n, d_3 = n-1$, and $d_i = i-2$ for every $4 \le i \le n$. Since $n \ge 6$, $k \ge 3$, and $n-k \ge 2$, by Lemma 2.1, $\left|E_{n,k}^{d_i,d_{i+1}}\right| = (n-2)!/(n-k)! \ge 3$. By Lemma 2.5, there is a hamiltonian path Q_2 of $S_{n,k}^{\{d_2=n\}} - \{\mathbf{u}, \mathbf{v}\}$ joining a node $\mathbf{x}$ to a node $\mathbf{y}$ with $(\mathbf{x})_1 = 1$ and $(\mathbf{y})_1 = n-1$. We set $\mathbf{q}_1 = (\mathbf{u})^k$, $\mathbf{q}_2 = \mathbf{x}$, and $\mathbf{q}_3 = (\mathbf{y})^k$. We choose a node $\mathbf{q}_i$ in $S_{n,k}^{\{d_i\}}$ with $(\mathbf{q}_i)_1 = d_{i-1}$ for every $4 \le i \le n$. Obviously, $(\mathbf{q}_2)^k \in S_{n,k}^{\{d_1\}} - \{\mathbf{q}_1\}$ and $(\mathbf{q}_i)^k \in S_{n,k}^{\{d_{i-1}\}} - \{\mathbf{q}_{i-1}\}$ for every $3 \le i \le n$. Moreover, $\left\{\mathbf{q}_4, (\mathbf{q}_5)^k\right\} \subset V\left(S_{n,k}^{\{d_4\}}\right) - \{\mathbf{y}_{n-2}\}$. By Theorem 2.3, there is a hamiltonian path Q_i of $S_{n,k}^{\{d_i\}}$ joining $\mathbf{q}_i$ to $(\mathbf{q}_{i+1})^k$ for every $i \in \langle n-1 \rangle - \{2, 4\}$. Again, there is a hamiltonian path Q_4 of $S_{n,k}^{\{d_4\}} - \{\mathbf{y}_{n-2}\}$ joining $\mathbf{q}_4$ to the node $(\mathbf{q}_5)^k$. Moreover, there is a hamiltonian path Q_n of $S_{n,k}^{\{d_n\}}$ joining $\mathbf{q}_n$ to a node $\mathbf{z}$ with $(\mathbf{z})_1 = n-1$. Then we set $P_{n-2} = \left\langle \mathbf{u}, (\mathbf{u})^k = \mathbf{q}_1, Q_1, (\mathbf{q}_2)^k, \mathbf{q}_2, Q_2, (\mathbf{q}_3)^k, \ldots, \mathbf{q}_{n-1}, Q_{n-1}, (\mathbf{q}_n)^k, \mathbf{q}_n, Q_n, \mathbf{z} \right\rangle$.

Then $P_1, P_2, \ldots, P_{n-2}$ form the desired paths of $S_{n,k}$. Note that $\left((\mathbf{q}_4)^k\right)_1 = 2$ and $\left(\left(\mathbf{p}_2^{n-3}\right)^k\right)_1 = 1$. See Figure 11 for an illustration of this case for $S_{6,3}$. ∎

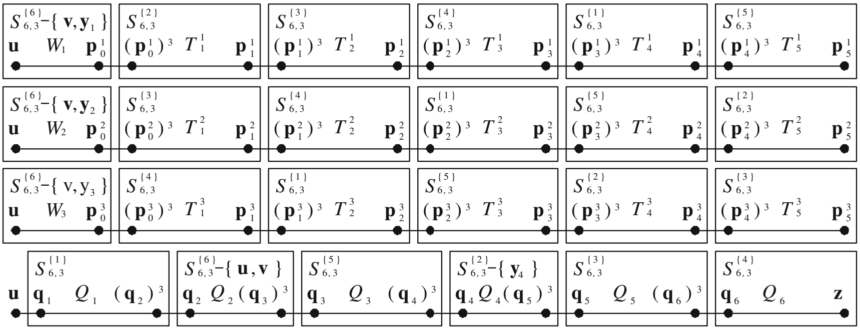

Figure 11: An illustration for Lemma 3.7 on $S_{6,3}$.

Theorem 3.8. *For* $3 \leq k \leq n-2$, $IHP(S_{n,k}) = n-2$.

Proof. Since $\delta(S_{n,k}) = n-1$ and $IHP(S_{n,k}) \leq \delta(S_{n,k}) - 1$, $IHP(S_{n,k}) \leq n-2$. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct nodes of $S_{n,k}$. To prove the statement is true, we need to construct $(n-2)$ mutually independent hamiltonian paths of $S_{n,k}$ between $\mathbf{u}$ and $\mathbf{v}$. We have the following cases:

Case 1. $(\mathbf{u})_i \neq (\mathbf{v})_i$ for some $2 \leq i \leq k$. Without loss of generality, we assume that $(\mathbf{u})_k \neq (\mathbf{v})_k$. Moreover, we assume that $(\mathbf{u})_k = n$ and $(\mathbf{v})_k = n-1$. Let C be a matrix of order $(n-2) \times (n-2)$ defined by

$$c_{i,j} = \begin{cases} i+j-1, & \text{if } i+j \leq n-1, \text{ and} \\ i+j-n+1, & \text{if } n \leq i+j. \end{cases}$$

More precisely,

$$C = \begin{bmatrix} 1 & 2 & 3 & \ldots & n-2 \\ 2 & 3 & 4 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2 & 1 & 2 & \ldots & n-3 \end{bmatrix}.$$

By Lemma 3.5, there are hamiltonian paths $Q_1, Q_2, \ldots, Q_{n-2}$ of $S_{n,k}^{\{n\}}$ such that Q_i is a hamiltonian path of $S_{n,k}^{\{n\}}$ joining $\mathbf{u}$ to a node $\mathbf{s}_i$ with

(1) $(\mathbf{s}_i)_1 = c_{i,1}$ for every $1 \leq i \leq n-2$, and

(2) $|\{Q_1(j), Q_2(j), \ldots, Q_{n-2}(j)\}| = n-2$ for every $2 \le j \le (n-1)!/(n-k)!$.

Again, there are hamiltonian paths $R_1, R_2, \ldots, R_{n-2}$ of $S_{n,k}^{\{n-1\}}$ such that R_i is a hamiltonian path of $S_{n,k}^{\{n-1\}}$ joining $\mathbf{v}$ to a node $\mathbf{t}_i$ with

(1) $(\mathbf{t}_i)_1 = c_{i,(n-2)}$ for every $1 \le i \le n-2$, and
(2) $|\{R_1(j), R_2(j), \ldots, R_{n-2}(j)\}| = n-2$ for every $2 \le j \le (n-1)!/(n-k)!$.

For $1 \le i \le n-2$, by Theorem 2.6, there exists a hamiltonian path $W_i = \langle \mathbf{x}_1^i, T_1^i, \mathbf{y}_1^i, \mathbf{x}_2^i, T_2^i, \mathbf{y}_2^i, \ldots, \mathbf{x}_{n-2}^i, T_{n-2}^i, \mathbf{y}_{n-2}^i \rangle$ of $S_{n,k}^{\langle n-2 \rangle}$ joining the node $\mathbf{x}_1^i$ to the node $\mathbf{y}_{n-2}^i$ such that $\mathbf{x}_1^i = (\mathbf{s}_i)^k$, $\mathbf{y}_{n-2}^i = (\mathbf{t}_i)^k$, and T_j^i is a hamiltonian path of $S_{n,k}^{\{c_{i,j}\}}$ joining $\mathbf{x}_j^i$ to $\mathbf{y}_j^i$ for every $1 \le j \le n-2$. Let P_i be the path $\langle \mathbf{u}, Q_i, \mathbf{s}_i, (\mathbf{s}_i)^k = \mathbf{x}_1^i, W_i, \mathbf{y}_{n-2}^i = (\mathbf{t}_i)^k, \mathbf{t}_i, R_i^{-1}, \mathbf{v} \rangle$ for every $1 \le i \le n-2$. Hence, $\{P_1, P_2, \ldots, P_{n-2}\}$ form a set of $(n-2)$ mutually independent hamiltonian paths of $S_{n,k}$ from $\mathbf{u}$ to $\mathbf{v}$. See Figure 12 for an illustration of this case for $n = 6$ and $k = 4$.

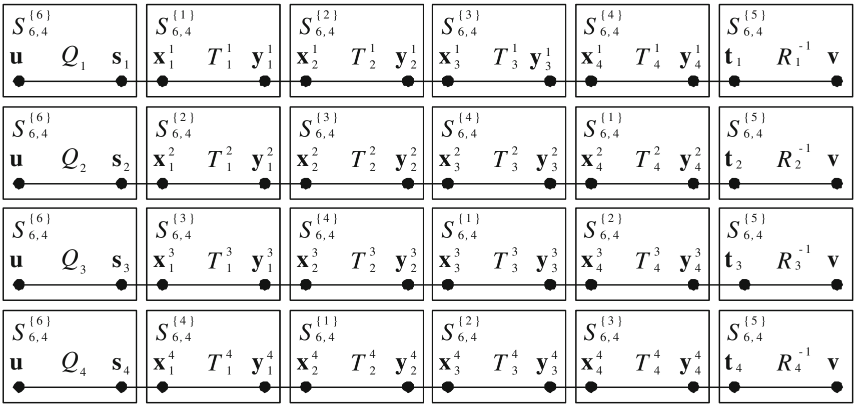

Figure 12: An illustration for Case 1 of Theorem 3.8 for $n = 6$ and $k = 4$.

Case 2. $(\mathbf{u})_i = (\mathbf{v})_i$ for every $2 \le i \le k$. Without loss of generality, we assume that $(\mathbf{u})_1 = 1$, $(\mathbf{v})_1 = 2$, and $(\mathbf{u})_i = (\mathbf{v})_i = n-k+i$ for every $2 \le i \le k$. Suppose that $n = 5$ and $k = 3$. We prove there are 3 mutually independent hamiltonian paths on $S_{5,3}$ joining $\mathbf{u} = 145$ to $\mathbf{v} = 245$ by exhibiting the three required hamiltonian paths in Table 7.

Hence, we assume that $n \ge 6$, $k \ge 3$, and $n-k \ge 2$. Let $\mathbf{y}_i$ be a node of $N_{S_{n,k}}(\mathbf{v})$ with $(\mathbf{y}_i)_1 = i+2$ for every $1 \le i \le n-2$. Obviously, $\mathbf{u} \in S_{n,k}^{\{n\}}$, $\mathbf{v} \in S_{n,k}^{\{n\}}$, $\mathbf{y}_i \in S_{n,k}^{\{n\}}$ for every $1 \le i \le n-3$, and $\mathbf{y}_{n-2} \in S_{n,k}^{\{2\}}$. By Lemma 3.7, there are $(n-3)$ paths $R_1, R_2, \ldots, R_{n-3}$ of $S_{n,k}^{\{n\}}$ such that

(1) R_i is a hamiltonian path of $S_{n,k}^{\{n\}} - \{\mathbf{v}, \mathbf{y}_i\}$ joining $\mathbf{u}$ to a node $\mathbf{x}_i$ with $(\mathbf{x}_i)_1 = i+1$ for every $1 \le i \le n-3$, and

Table 7: The 3 mutually independent hamiltonian paths of $S_{5,3}$ between 145 and 245.

$P_1 = \langle$145, 345, 435, 135, 235, 325, 523, 423, 243, 143, 543, 453, 253, 153, 513, 413, 213, 123, 321, 521, 421, 241, 341, 541, 451, 251, 351, 531, 431, 231, 132, 532, 352, 452, 152, 512, 312, 412, 142, 542, 342, 432, 234, 534, 134, 314, 214, 124, 324, 524, 254, 354, 154, 514, 415, 315, 215, 125, 425, 245$\rangle$
$P_2 = \langle$145, 415, 315, 215, 125, 425, 524, 324, 124, 214, 514, 314, 134, 234, 534, 354, 154, 254, 452, 352, 532, 132, 432, 342, 542, 142, 412, 312, 512, 152, 251, 451, 541, 341, 241, 421, 521, 321, 231, 431, 531, 351, 153, 513, 413, 213, 123, 423, 243, 143, 543, 453, 253, 523, 325, 235, 135, 435, 345, 245$\rangle$
$P_3 = \langle$145, 541, 341, 241, 421, 521, 321, 231, 431, 531, 351, 251, 451, 154, 354, 254, 524, 124, 324, 234, 534, 134, 314, 214, 514, 415, 315, 215, 125, 425, 325, 235, 135, 435, 345, 543, 143, 413, 213, 513, 153, 453, 253, 523, 123, 423, 243, 342, 142, 412, 512, 312, 132, 432, 532, 352, 152, 452, 542, 245$\rangle$

(2) $|\{R_1(j), R_2(j), \ldots, R_{n-3}(j)\}| = n-3$ for every $2 \le j \le (n-1)!/(n-k)! - 2$.

Let C be a matrix of order $(n-3)\times(n-1)$ defined by

$$c_{i,j} = \begin{cases} i-j+2, & \text{if } j \le i+1, \text{ and} \\ i+n-j+1, & \text{if } i+2 \le j. \end{cases}$$

More precisely,

$$C = \begin{bmatrix} 2 & 1 & n-1 & n-2 & \ldots & 4 & 3 \\ 3 & 2 & 1 & n-1 & \ldots & 5 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-2 & n-3 & n-4 & n-5 & \ldots & 1 & n-1 \end{bmatrix}.$$

We set $\mathbf{p}_0^i = \mathbf{x}_i$ and $\mathbf{p}_{n-1}^i = (\mathbf{y}_i)^k$ for every $1 \le i \le n-3$. We choose a node $\mathbf{p}_j^i$ in $S_{n,k}^{\{c_{i,j}\}}$ with $\left(\mathbf{p}_j^i\right)_1 = c_{i,j+1}$ for every $1 \le i \le n-3$ and for every $1 \le j \le n-2$. Note that the node $\left(\mathbf{p}_j^i\right)^k$ is in $S_{n,k}^{\{c_{i,j+1}\}} - \left\{\mathbf{p}_{j+1}^i\right\}$ for every $1 \le i \le n-3$ and $0 \le j \le n-2$. By Theorem 2.3, there is a hamiltonian path T_j^i of $S_{n,k}^{\{c_{i,j}\}}$ joining the node $\left(\mathbf{p}_{j-1}^i\right)^k$ to the node $\mathbf{p}_j^i$ for every $1 \le i \le n-3$ and $1 \le j \le n-1$. We set $P_i = \left\langle \mathbf{u}, R_i, \mathbf{x}_i = \mathbf{p}_0^i, (\mathbf{p}_0^i)^k, T_1^i, \mathbf{p}_1^i, (\mathbf{p}_1^i)^k, T_2^i, \mathbf{p}_2^i, \ldots, (\mathbf{p}_{n-2}^i)^k, T_{n-1}^i, \mathbf{p}_{n-1}^i = (\mathbf{y}_i)^k, \mathbf{y}_i, \mathbf{v} \right\rangle$ the $(n-3)$ paths for every $1 \le i \le n-3$.

By Lemma 2.5, there is a hamiltonian path Q_2 of $S_{n,k}^{\{n\}} - \{\mathbf{u}, \mathbf{v}\}$ joining a node $\mathbf{x}$ to a node $\mathbf{y}$ with $(\mathbf{x})_1 = 1$ and $(\mathbf{y})_1 = n-1$. We set $\mathbf{q}_0 = \mathbf{u}$, $\mathbf{q}_1 = (\mathbf{x})^k$, $\mathbf{q}_2 = y$, and $\mathbf{q}_n = \mathbf{y}_{n-2}$. We choose a node $\mathbf{q}_i \in S_{n,k}^{\{n-i+2\}}$ for every $3 \le i \le n-1$ with $(\mathbf{q}_i)_1 = n-i+1$. Note that node $(\mathbf{q}_0)^k$ is in $S_{n,k}^{\{1\}} - \{\mathbf{q}_1\}$ and node $(\mathbf{q}_i)^k$ is in $S_{n,k}^{\{n-i+1\}} - \{\mathbf{q}_{i+1}\}$ for every $2 \le i \le n-1$. By Theorem 2.3, there is a hamiltonian path Q_1 of $S_{n,k}^{\{1\}}$ joining node $(\mathbf{q}_0)^k$ to node $\mathbf{q}_1$. Again, there is a hamiltonian path Q_i of $S_{n,k}^{\{n-i+2\}}$ joining $(\mathbf{q}_{i-1})^k$ to $\mathbf{q}_i$ for every $3 \le i \le n$. Then we set the path $P_{n-2} = \langle \mathbf{u} = \mathbf{q}_0, (\mathbf{q}_0)^k, Q_1, \mathbf{q}_1, (\mathbf{q}_1)^k = \mathbf{x}, Q_2, \mathbf{y} = \mathbf{q}_2, \ldots, (\mathbf{q}_{n-2})^k, Q_{n-1}, \mathbf{q}_{n-1}, (\mathbf{q}_{n-1})^k, Q_n, \mathbf{q}_n = \mathbf{y}_{n-2}, \mathbf{v} \rangle$. Then $\{P_1, P_2, \ldots, P_{n-2}\}$ forms a set of $(n-2)$ mutually independent hamiltonian paths of $S_{n,k}$ joining $\mathbf{u}$ to $\mathbf{v}$. Note that $(\mathbf{q}_3)_1 = n-2 \neq 1 = \left((\mathbf{p}_2^1)^k\right)_1$ and

$(\mathbf{q}_i)_1 = n-i+1 \neq n-i+3 = \left(\left(\mathbf{p}^1_{i-1}\right)^k\right)_1$ for every $4 \leq i \leq n-1$. See Figure 13 for an illustration of this case for $n=6$ and $k=4$.

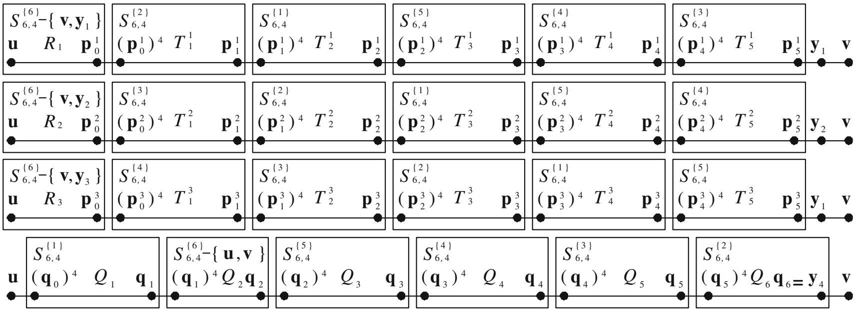

Figure 13: An illustration for Case 2 of Theorem 3.8 for $n=6$ and $k=4$.

Thus, the statement is proved. ∎

4. Conclusion

Lin et al. proposed the concept of mutually independent hamiltonian paths in [10]. The mutually independent hamiltonian paths of $S_{n,n-1}$ is studied in [10]. In this paper, we denote by $IHP(G)$. Using Theorems 3.1, 3.2, 3.4, and 3.8, we have the following result.

Theorem 4.1. *For* $1 \leq k \leq n-2$, $IHP(S_{n,k}) = n-2$ *except for* $S_{4,2}$ *such that* $IHP(S_{4,2}) = 1$.

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