Piecewise Two Dimensional Maps and Applications to Cellular Neural Networks

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Abstract. Of concern is a two-dimensional map T of the form. $T(x, y) = (y, F(y) - bx)$. Here F is a three-piece linear map. In this paper, we first prove a theorem which states that a semiconjugate condition for T implies the existence of Smale horseshoe. Second, the theorem is applied to show the spatial chaos of one-dimensional Cellular Neural Networks. We improve a result of Hsu [2000].

I. Introduction

We consider a piecewise two dimensional map of the form

$$T(x, y) = (y, F(y) - bx), \tag{1}$$

where

$$F(y) = \begin{cases} a_1 y + a_0 - a_1 + c_1 & y \geq 1, \\ a_0 y + c_1 & |y| \leq 1, \\ a_{-1} y + a_{-1} - a_0 + c_1 & y \leq -1. \end{cases} \tag{2}$$

Here $a_0 < 0,\ \ a_1, a_{-1} > 1,\ \ b > 0,\ $ and $c_1 \in \mathbb{R}$ is a biased term. The graph of F is given in Figure1.

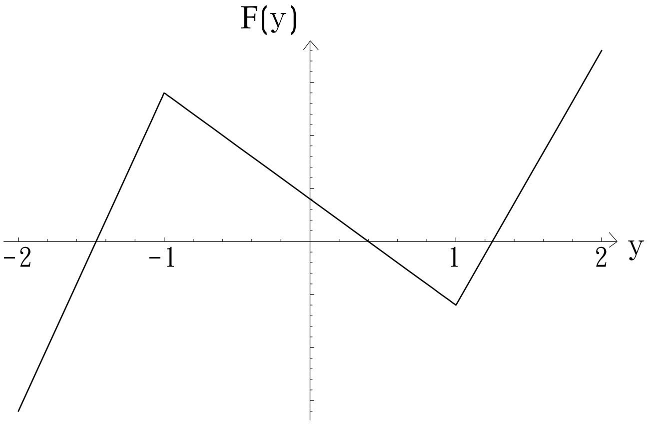

Figure 1.

$a_1 = 1.2,\ \ a_0 = -0.5,\ \ a_{-1} = 1.5,\ \ c_1 = 0.2.$

The motivation for studying such map is, in part, because that the form of the map is a generalized version of Lozi map [Lozi, 1978]. More importantly, the map arises in the study of the complexity of the set of bounded stable stationary solutions of one-dimensional Cellular Neural Networks (CNNs)(See e.g., [Chua, 1998; Chua and Yang, 1998a, 1998b]). In this paper, we first prove a theorem which states that a semiconjugate condition for T implies the existence of Smale horseshoe. Second, we apply the theorem to show the spatial chaos of one-dimensional Cellular Neural Networks. Such CNNs are of the form (e.g., [Ban *et al.*, 2002, 2001; Hsu, 2000]).

$$\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}), \quad i \in \mathbb{Z} \tag{3a}$$

where $f(x)$ is a piecewise-linear output function defined by

$$f(x) = \begin{cases} rx + 1 - r & x \geq 1 \\ x & |x| \leq 1 \\ lx + l - 1 & x \leq -1, \end{cases} \tag{3b}$$

where r and l are positive constants. The quantity z is called threshold or bias term, related to independent voltage sources in electric circuits. The constants α, a, and β are the interaction weights between neighboring cells. The study of the problems for the case that $r = l = 0$ and $\alpha = \beta$ has been established in [Chua, 1998; Chua and Yang, 1998a; Juang and Lin, 2000]. Here we consider $r > 0$ and $l > 0$. The main results on this part are the following. Given α and β, if (z, a) is in a certain parameter region $\Sigma_{\alpha,\beta}$ (see Theorem 3.1), then there exist r and l sufficiently small for which $\Lambda_{l,r}$ (see Theorem 3.1) is a hyperbolic invariant set. Consequently, the spatial entropy of the corresponding set of the bounded, stable stationary solutions is $ln2$.

II. Main Results

We first introduce some notations. Let

$$S = \{(x, y) \in \mathbb{R}^2 : |x| \leq p, \ |y| \leq p\}. \tag{4}$$

Here $p > 1$. Let the four corners of S are labled as

$$K = (p, p), \ L = (p, -p), \ M = (-p, -p), \ N = (-p, p). \tag{5a}$$

Set

$$\overline{K} = (p, 1), \ \overline{L} = (p, -1), \ \overline{M} = (-p, -1), \ \overline{N} = (-p, 1). \tag{5b}$$

The x and y coordinate of K are denoted, respectively, by K^x and K^y.

We next number the following conditions.

$$K_1^y \geq p > 1, \tag{6a}$$

$$\overline{N}_1^y \leq -p, \tag{6b}$$

$$\overline{L}_1^y \geq p, \tag{6c}$$

and

$$M_1^y \leq -p. \tag{6d}$$

Here the subscript denotes the iteration index under the map T. For instance, K_1^y denotes the y coodinate of $T(K) = K_1$. Suppose (6) holds. Then $T(S) \cap S$ has three vertical strips. See Figure 2. Similarly, $T^{-1}(S) \bigcap S$ has three horizontal strips, and $T^{-1}(S) \bigcap S \bigcap T(S)$ has 9 components. By induction $\bigcap\limits_{j=-n}^{n} T^j(S)$ has 9^n components. With this information we can define a semiconjugate

$$h : \Lambda \to \{0,1,2\}^2 \tag{7}$$

which is onto. Here $\Lambda = \bigcap\limits_{j=-\infty}^{\infty} (T^j(S) \bigcap S)$. If the components of Λ are points, then Λ is a Cantor set. This, in turn, implies that the semiconjugacy h is one to one and so is a conjugacy. This motivates the following definition.

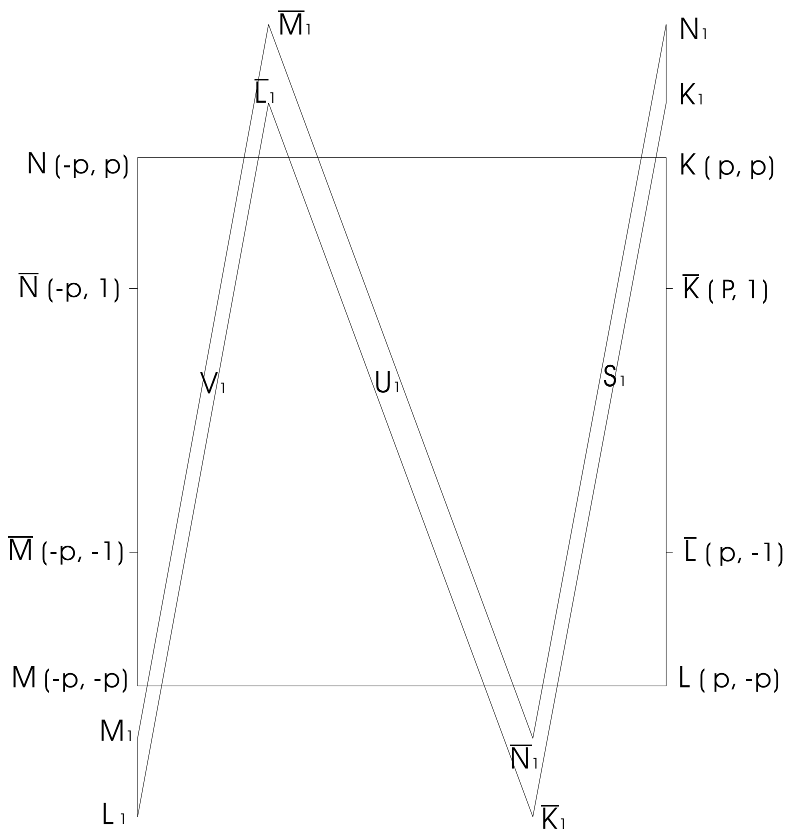

Figure 2.

Definition 1.1 Conditions on b, a_{-1}, a_0, and a_1 so that there exists a $p > 1$ for which (6) holds are called a semiconjugate condition for T.

To prove the main theorem, we need to introduce more notations. Now, $T(S) \bigcap S$, has three vertical strips, say S_1, U_1, and V_1. The one on the right, see Fig.2, is labeled as S_1. Clearly, $T(S_1) \bigcap S$ has also three vertical strips. The strip of $T(S_1) \bigcap S_1$ is to be denoted by S_2. We then define S_n inductinely. Note that S_n, $n \in \mathbb{N}$, are all parallelograms. U_s and V_n are defined similarly.

The parallelogram $N_1K_1\overline{K}_1\overline{N}_1$ see Fig.2, is to be denoted by $\overline{S}_1$. Likewise, $\overline{S}_n$ denotes the parallelogram $N_nK_n\overline{K}_n\overline{N}_n$. The length of the shorter side of the parallelogram S_n(resp., $\overline{S}_n$) is to be denoted by

$$d_n(resp., c_n). \tag{8a}$$

The slope of the longer side of the parallelogram S_n is to be denoted by

$$m_n. \tag{8b}$$

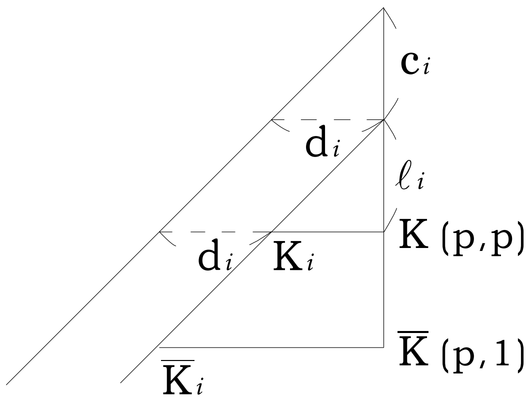

Figure 3.

Lemma 2.1 The following recursive relations hold.
(i) $d_i = \frac{c_i}{m_i}$, $c_{i+1} = bd_i$. (ii) $m_{i+1} = a_1 - \frac{b}{m_i}$, $m_1 = a_1$.

Proof. The first recursive relation is obvious. To see (ii), let l_i be given as in Fig.3. We see then that $K_i = (p - \frac{l_i}{m_i}, p)$ and $\overline{K}_i =$

$(p - \frac{l_i+p-1}{m_i}, 1)$. Now, the slope m_{i+1} = the slope of $\overline{T(K_i)T(\overline{K}_i)}$ = $\overline{K_{i+1}\overline{K_{i+1}}} = \frac{F(p)-F(1)+b(\frac{1-p}{m_i})}{p-1} = a_1 - \frac{b}{m_i}$. □

Lemma 2.2 If $b > 0$ and $a_1 \geq 2(1+b)$, then $\lim_{n\to\infty} c_n = 0$.

Proof. We first prove that $\lim_{n\to\infty} m_n = \frac{a_1+\sqrt{a_1^2-4b}}{2}$. To this end, we see that an induction would yield that $m_i \geq 1$ for all $i \in \mathbb{N}$ and that m_i is decreasing in i. Suppose x is the limit of $\{m_n\}$. Then x must satisfy equation $x = a_1 - \frac{b}{x}$. Upon using the the fact that $m_1 = a_1$, we conclude that $x = \frac{a_1+\sqrt{a_1^2-4b}}{2}$ as asserted. Now, using Lemma 2.1-(i), we get that $d_n = \frac{b^{n-1}d_1}{\prod_{i=2}^{n} m_i}$. Thus,

$$\begin{aligned} d_n &\leq \left(\frac{2b}{a_1+\sqrt{a_1^2-4b}}\right)^{n-1} d_1 \\ &\leq \left(\frac{2b}{a_1}\right)^{n-1} d_1 \\ &\leq \left(\frac{b}{1+b}\right)^{n-1} d_1. \end{aligned}$$

We have just completed the proof of the lemma. □

Similarly, we have the following lemma.

Lemma 2.3 If $b > 0$ and $a_{-1} > 2(1+b)$, then the length of the shorter side of the parallelogram V_n shrinks to zero as $n \to \infty$.

Using Lemmas 2.2 and 2.3, we have the following lemma.

Lemma 2.4 If $b > 0$, $min\{a_1,\ a_{-1}\} > 2(1+b)$, then the length of the shorter side of the parallelogram U_n shrinks to zero as $n \to \infty$.

Remark. The assumptions on Lemmas 2.2-2.4 would also yield that $\bigcap_{j=0}^{-\infty}(T^j(S)\bigcap S)$ are pairwise disjoint horizontal linesegments.

We are now ready to state our main results.

Theorem 2.1 Let F be a piecewise linear map defined as in (2) and the bias term c_1 satisfy the inequality

$$max\{-1-b, a_0+1+b\} < c_1 < min\{1+b, -a_0-1-b\}, \tag{9}$$

then a semiconjugate condition for T implies the conjugate of h.

Proof. Note that $K_1^y \geq p$, (6b) and (6d) are equivalent to the following inequalities.

$$p(a_1-1-b) \geq a_1-a_0-c_1, \tag{10a}$$

$$-a_0+c_1 \geq p(1+b), \tag{10b}$$

$$-a_0-c_1 \geq p(1+b), \tag{10c}$$

and

$$p(a_{-1}-1-b) \geq a_{-1}-a_0+c_1, \tag{10d}$$

respectively. We remark (10b) and (10c) ensure that $-a_0-1-b > 0$, as a result, inequality (9) makes sense. Using (10a) and (10b), we see immediately that

$$\frac{-a_0 + c_1}{b+1} \geq p \geq \frac{a_1 - a_0 - c_1}{a_1 - b - 1}. \tag{11}$$

Note that $a_1 - b - 1$ being positive is guaranteed by the fact that $p > 1$ and the assumptions on c_1. Using (10), we get that

$$a_1 \geq \frac{-2a_0(b+1)}{c_1 - a_0 - 1 - b} = \frac{2(b+1)}{1 + \frac{1+b-c_1}{a_0}} \geq 2(b+1). \tag{12a}$$

The last inequality is justified by the assumptions on c_1. Similarly, we see that

$$a_{-1} \geq \frac{2a_0(b+1)}{c_1 + a_0 + 1 + b} = \frac{2(b+1)}{1 + \frac{1+b+c_1}{a_0}} \geq 2(b+1). \tag{12b}$$

It then follows from Lemmas 2.2-2.4 that $\bigcap\limits_{j=-\infty}^{\infty} (T^j(S) \bigcap S)$ is a Cantor set. We thus complete the proof of the main theorem. $\square$

Remarks.

1. If $F(y)$, as defined in 2, is such that $a_0 > 0$, and a_1, $a_{-1} < -1$, then a similar result can also be obtained.
2. The theorem holds true in general for F being a finitely many piecewise linear map. Specifically, if the bias term c_1 is not " too biased " , then a semiconjugate condition for T implies the existence of Smale horseshoe.

In the following, we give conditions on a_0, a_1, a_{-1}, b and c for which T has a semiconjugate condition.

Theorem 2.2 Let $a_0 < 0$, a_1, $a_{-1} > 1$ and $b > 0$. Suppose $a_0 + 1 + b < 0$, $min\{a_1, a_{-1}\} > 2(1+b)$. Let the bias term c_1 satisfy

(9), and that

$$a_1 \geq \frac{-2a_0(b+1)}{c_1 - a_0 - 1 - b}, \tag{13a}$$

and

$$a_{-1} \geq \frac{2a_0(b+1)}{c_1 + a_0 + 1 + b}. \tag{13b}$$

then there exists a $p > 1$ such that T has a semiconjugate condition.

III. Applications To CNNs

A basic and important class of solutions of (1) is the bounded, stable stationary solutions. In the case that $r = l = 0$ and $\alpha = \beta$, the corresponding stable stationary solutions have been studied in [Chua and Yang, 1998a; Juang and Lin, 2000]. The case that r and l are positive is considered in [Ban *et al.*, 2002, 2001; Hsu, 2000]. The techniques in these two cases are quite different. Specifically, in the latter case, the question of the complexity of the set of stable stationary solutions is converted to asking how chaotic of a map is. If α or $\beta = 0$, then the resulting map is one-dimensional ([Ban *et al.*, 2002, 2001]). If $\alpha, \beta \neq 0$, then the resulting map is a two dimensional of the following form [Hsu, 2000]

$$\begin{aligned} T(x,y) &= (y, \frac{1}{\beta}(\overline{F}(y) - ay - z) - \frac{\alpha}{\beta}x) \\ &=: (y, F(y) - bx). \end{aligned} \tag{14a}$$

Here

$$\overline{F}(y) = \begin{cases} \frac{1}{r}y - \frac{1}{r} + 1 & y \geq 1 \\ y & |y| \leq 1 \\ \frac{1}{l}y - 1 + \frac{1}{l} & y \leq -1. \end{cases} \tag{14b}$$

In [Hsu, 2000], Hsu used a theorem of Afraimovich (see e.g., [Afraimovich, 1993]) as well as a semiconjugate condition to show that in certain parameters' region, the map T has Smale horseshoe structure. However, Afraimovich's Theorem is not needed in this case. Only a semiconjugate condition is required.

To apply Theorem 2.2, we first note that $a_{-1} = \frac{1}{\beta}(\frac{1}{l} - a)$, $a_0 = \frac{1}{\beta}(1 - a)$, $a_1 = \frac{1}{\beta}(\frac{1}{r} - a)$, $c_1 = \frac{-z}{\beta}$, $b = \frac{\alpha}{\beta}$. With the above identifications, we immediately have the following results concerning the complexity of the set of the bounded, stable stationary mosaic solutions of (3). Here the stationary mosaic solutions $(x_i)_{i=-\infty}^{\infty}$ means that $(x_i)_{i=-\infty}^{\infty}$ is a stationary solution of (3) and that $|x_i| > 1$ for all $i \in \mathbb{Z}$. Moreover, the mosaic solutions obtained in the following theorem are bounded and stable (see e.g.,[Chua and Yang, 1998a; Hsu, 2000]).

Define $s = \alpha + a + \beta$. Assume the bias term z satisfy the following inequality.

$$max\{-s + a, s - 2a + 1\} < z < min\{s - a, 2a - 1 - s\}. \qquad (15)$$

Define, respectively, the regions $\Sigma_{\alpha,\beta}$ and $\Sigma_{\alpha,\beta,l,r}$ as follows.

$$\Sigma_{\alpha,\beta} = \{(z, a) \in \mathbb{R}^2 |\ (15)\ holds\}, \qquad (16)$$

and

$$\Sigma_{\alpha,\beta,l,r} = \{(z, a) \in \mathbb{R}^2 | r < r^+,\ and\ \ l < l^+\}. \qquad (17)$$

Here,

$$r^+_{z,\alpha,a,\beta} = \frac{2a - s - 1 - z}{a(1 + s - z) - 2s}, \quad (18a)$$

and

$$l^+_{z,\alpha,a,\beta} = \frac{2a - s - 1 + z}{a(1 + s + z) - 2s}. \quad (18b)$$

We are now in a position to state the following results.

Theorem 3.1 Let α and β be positive numbers and let $a > 1+\alpha+\beta$. Suppose $(z, a) \in \sum_{\alpha,\beta}$. Then there exist r and l sufficiently small, more precisely $0 < r < r^+ = r^+_{z,\alpha,a,\beta}$ and $0 < l < l^+ = l^+_{z,\alpha,a,\beta}$ for which T has a hyperbolic invariant set $\Lambda_{l,r}(z, \alpha, a, \beta) = \Lambda_{l,r}$ in the (x, y) plane such that $T|_{\Lambda_{l,r}}$ is topologically conjugate to a two-side Bernoulli shift of two symbols. Hence, the spatial entropy of the corresponding set of stationary solutions equals to $\ln 2$.

Remarks

1. Note that if $(z, a) \in \Sigma_{\alpha,\beta}$, then $-2s + a(1 + s - z) = a(-z - 1 - s + 2a) + 2(a - 1)(s - a) > 0$ and $-2s + a(1 + s + z) = a(z - 1 - s + 2a) + 2(a - 1)(s - a) > 0$. Consequently, those r^+ and l^+ are positive.

2. Adapting the notations in [Juang and Lin, 2000], we let $\alpha = \beta = a\epsilon$. Then the set $\Sigma_{\alpha,\beta} = \Sigma_\epsilon$ is given in the following figure.

Note that for $0 < \varepsilon < \frac{1}{4}$, $\Sigma_\epsilon \subsetneq [3, 3]_\epsilon$(see Fig.5.1 of [8] for the definition of $[3, 3]_\epsilon$), and for $\frac{1}{4} \leq \epsilon < \frac{1}{2}$, $\Sigma_\epsilon = [3, 3]_\epsilon$ (see Figures 4 and 5). Applying Theorem 3.1, we conclude that let $\alpha = \beta = a\epsilon$,

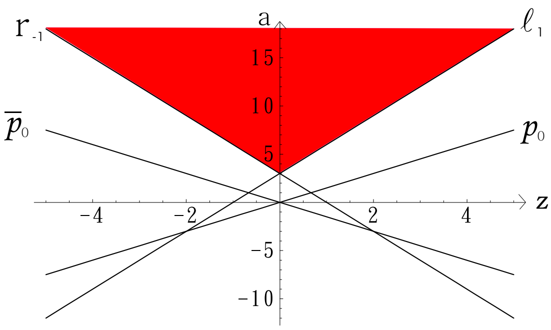

Figure 4.
$\epsilon = \frac{1}{3}$,
$l_1 : -z + a(1-2\epsilon) = 1$, $p_0 : z = 2a\epsilon$,
$r_{-1} : z + a(1-2\epsilon) = 1$, $\bar{p}_0 : z = -2a\epsilon$.

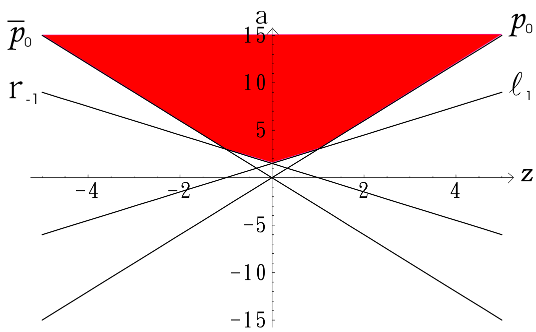

Figure 5.
$\epsilon = \frac{1}{6}$,
$l_1 : -z + a(1-2\epsilon) = 1$, $p_0 : z = 2a\epsilon$,
$r_{-1} : z + a(1-2\epsilon) = 1$, $\bar{p}_0 : z = -2a\epsilon$.

$\frac{1}{4} \leq \varepsilon < \frac{1}{2}$, and if $(z,a) \in \Sigma_\epsilon = [3,3]_\epsilon$, then there exist r and l sufficiently small for which $\Lambda_{l,r}$ is a hyperbolic invariant set. This result generalized those in [Chua, 1998, Chua and Yang, 1998a; Juang and Lin, 2000]. For $0 < \epsilon < \frac{1}{4}$, if $(z,a) \in \Sigma_\epsilon$ and that r, $l > 0$ sufficiently small, then the corresponding set of stable, bounded stationary solutions also has spatial entropy $ln2$.

3. To get a feel how small r and l are required to be. Set $\epsilon = \frac{1}{4}$ and

$z = 0$. We see easily that $r^+ = l^+$ has a maximum $\frac{1}{16}$ for $2 < a < \infty$.

4. Figure6 is a collection of a computer simulation with a set of parameters, satisfying $a > 1 + \alpha + \beta$, $0 < r < r^+ = r^+_{z,\alpha,a,\beta}$ and $0 < l < l^+ = l^+_{z,\alpha,a,\beta}$. Specifically, we choose $\alpha = \beta = 1$, $r = l = 0.005$, $z = 0$, $a = 4$. Each collection in figure6 contains 2 arrays of colors. The first array is the initial outputs. The second array represents the final outputs. If the state x_j of a cell c_j is such $|x_j| < 1$, then we color it with green. If the state x_j of a cell c_j is less than -1 (greater than 1, respectively), then we color it with blue (red, respectively).

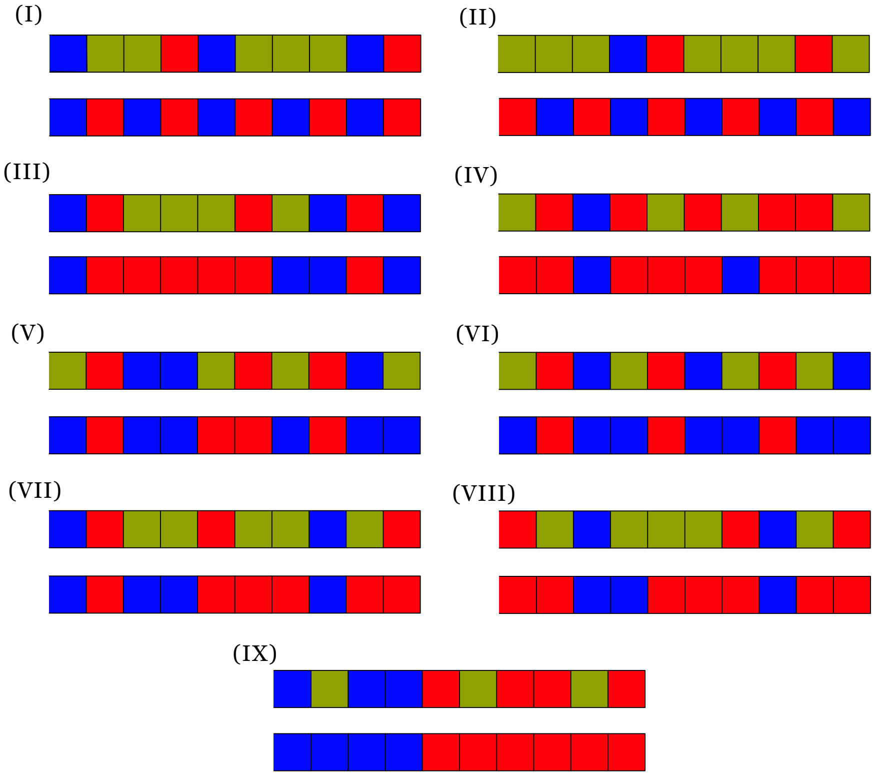

Figure 6.

Acknowledgment

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