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Improved upper and lower bounds on the optimization of mixed chordal ring networks ☆

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ABSTRACT

Recently, Chen, Hwang and Liu [S.K. Chen, F.K. Hwang, Y.C. Liu, Some combinatorial properties of mixed chordal rings, J. Interconnection Networks 1 (2003) 3–16] introduced the mixed chordal ring network as a topology for interconnection networks. In particular, they showed that the amount of hardware and the network structure of the mixed chordal ring network are very comparable to the (directed) double-loop network, yet the mixed chordal ring network can achieve a better diameter than the double-loop network. More precisely, the mixed chordal ring network can achieve diameter about $\sqrt{2N}$ as compared to $\sqrt{3N}$ for the (directed) double-loop network, where N is the number of nodes in the network. One of the most important questions in interconnection networks is, for a given number of nodes, how to find an optimal network (a network with the smallest diameter) and give the construction of such a network. Chen et al. [S.K. Chen, F.K. Hwang, Y.C. Liu, Some combinatorial properties of mixed chordal rings, J. Interconnection Networks 1 (2003) 3–16] gave upper and lower bounds for such an optimization problem on the mixed chordal ring network. In this paper, we improve the upper and lower bounds as $2\lceil\sqrt{N/2}\rceil + 1$ and $\lceil\sqrt{2N} - 3/2\rceil$, respectively. In addition, we correct some deficient contexts in [S.K. Chen, F.K. Hwang, Y.C. Liu, Some combinatorial properties of mixed chordal rings, J. Interconnection Networks 1 (2003) 3–16].

1. Introduction

One of the most important issues in the design of parallel and distributed computing systems is the choice of an interconnection network suitable for a range of different applications. The *diameter* of a network, which is the maximum distance over all node-pairs, represents the maximum transmission delay between two stations. The *ring network* (i.e., the *single-loop network*) is one of the most frequently used topologies for interconnection networks. It has many attractive properties such as simplicity, extendibility, low degree, and ease of implementation. Although it has many attractive properties, it has poor reliability (any failure in an interface or communication link destroys the function of the network) and it has high transmission delay (its diameter equals to $N - 1$ if each link is directed, where N is the number of nodes). As a result, a lot of hybrid topologies utilizing the ring network as a basis for synthesizing richer interconnection schemes have been proposed to improve the reliability and reduce the transmission delay [3,4,6,14,17].

One example of the commonly used extensions for the ring network is the *multi-loop network*, which was first proposed by Wong and Coppersmith in [17] for organizing multi-module memory services. The most studied multi-loop network is possibly the double-loop network. The *double-loop network* $DL(N; a, b)$ is a digraph with N

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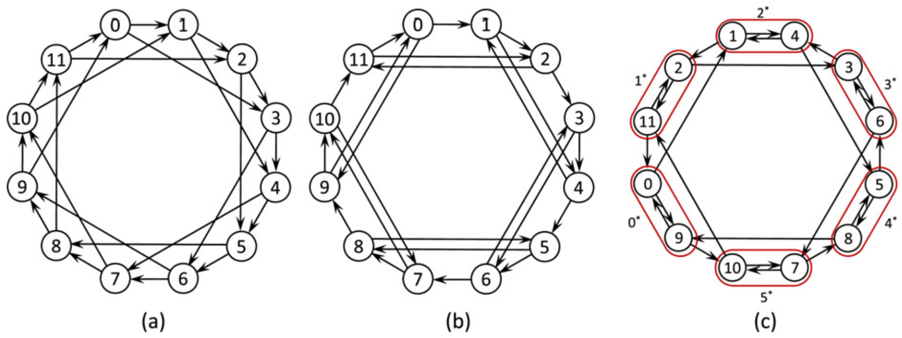

Fig. 1. Examples of the double-loop network and the mixed chordal ring network. (a) $DL(12; 1, 3)$. (b) $MCR(12; 1, 3)$. (c) Embed $MCR(12; 1, 3)$ into $DL(\frac{12}{2}; \frac{1-3}{2}, \frac{1+3}{2})$; i.e., $DL(6; 5, 2)$.

nodes $0, 1, \dots, N-1$ and $2N$ links (also called *steps*) (see Fig. 1(a)):

$$i \to (i+a) \bmod N, \quad i = 0, 1, \dots, N-1,$$

$$i \to (i+b) \bmod N, \quad i = 0, 1, \dots, N-1,$$

where $1 \leqslant a, b < N$, $a \neq b$, and $\gcd(N, a, b) = 1$. Doorn [9] had proven that:

Theorem 1.1. *(See [9].) $DL(N; a, b)$ is strongly 2-connected if and only if $\gcd(N, a, b) = 1$.*

Namely, any node or link failure will not disconnect the network. For a fixed N, let $D_{DL}(N)$ denote the optimal (i.e., smallest) diameter of all double-loop networks with N nodes. Many researchers tried to determine the exact value of $D_{DL}(N)$, but this is a difficult problem even when one of the two steps is 1 [4]; see also [1,7,8,10,12]. Therefore, researchers devoted their attention on finding bounds on $D_{DL}(N)$. A well-known lower bound on $D_{DL}(N)$ is as follows [17]:

$$D_{DL}(N) \geqslant \lceil \sqrt{3N} \rceil - 2. \tag{1}$$

For upper bounds on $D_{DL}(N)$, Hwang and Xu [13] managed to prove, using a heuristic method, that

$$D_{DL}(N) \leqslant \sqrt{3N} + 2(3N)^{1/4} + 5 \quad \text{for } N \geqslant 6348. \tag{2}$$

In [16], Rödseth further improved the above upper bound to be

$$D_{DL}(N) \leqslant \sqrt{3N} + (3N)^{1/4} + \frac{5}{2} \quad \text{for } N \geqslant 1200. \tag{3}$$

Another example of the commonly used extensions for the ring network is the *chordal ring network*; see [3] and [15]. Recently, Chen et al. [6] proposed the mixed chordal ring network as a topology of interconnection networks. The *mixed chordal ring network* $MCR(N; s, w)$, where N is even and both s and w are positive odd, is a digraph with N nodes $0, 1, \dots, N-1$ and $2N$ links of the following types (see Fig. 1(b)):

ring-links:

$$i \to (i+s) \bmod N, \quad i = 0, 1, 2, \dots, N-1,$$

chordal-links:

$$i \to (i+w) \bmod N, \quad i = 1, 3, 5, \dots, N-1,$$

chordal-links:

$$i \to (i-w) \bmod N, \quad i = 0, 2, 4, \dots, N-2.$$

Let $d(N; s, w)$ denote the diameter of $MCR(N; s, w)$. For a fixed positive even integer N, let $D_{MCR}(N)$ denote the optimal (i.e., smallest) diameter of all mixed chordal ring networks with N nodes. It is obvious that each node in the mixed chordal ring network has two in-links and two out-links. Therefore, the mixed chordal ring network is very comparable in hardware to the well-known double-loop network; see [6]. Surprisingly, Theorems 1.2 and 1.3 show that the mixed chordal ring network can achieve a better diameter than the double-loop network (as compared to (1), (2) and (3)).

Theorem 1.2. *(See [6].) $D_{MCR}(N) \geqslant \sqrt{2N} + o(N^{1/2})$.*

Theorem 1.3. *(See [6].) There exists a choice of s and w such that the diameter of $MCR(N; s, w)$ is no larger than $\sqrt{2N} + 3$. In other words, $D_{MCR}(N) \leqslant \sqrt{2N} + 3$.*

Chen et al. [6] also proved:

Theorem 1.4. *(See [6].) $MCR(N; s, w)$ is strongly 2-connected if and only if $\gcd(N, s, w) = 1$.*

The proofs of Theorems 1.3 and 1.4 are based on the idea of embedding a mixed chordal ring network into a double-loop network (see Section 2 for details). Unfortunately, we find that this embedding is not always successful and the proofs of Theorems 1.3 and 1.4 are incomplete (see Section 4 for details). Thus whether $\sqrt{2N} + 3$ is an upper bound on $D_{MCR}(N)$ and whether $\gcd(N, s, w) = 1$ guarantees the strongly 2-connectivity of $MCR(N; s, w)$ remain open. In this paper, we fill these voids by improving the upper and lower bounds on $D_{MCR}(N)$ and correcting the proof of Theorems 1.3 and 1.4. We summarize in Table 1.

This paper is organized as follows. Section 2 gives some preliminaries. Section 3 contains our main results. Section 4 gives a correct proof to Theorem 1.4. Section 5

Table 1

The bounds	In paper [6]	In this paper
Upper bound on $D_{MCR}(N)$	$\sqrt{2N}+3$	$2\lceil\sqrt{N/2}\rceil+1$
Lower bound on $D_{MCR}(N)$	$\sqrt{2N}+o(N^{\frac{1}{2}})$	$\lceil\sqrt{2N}-3/2\rceil$

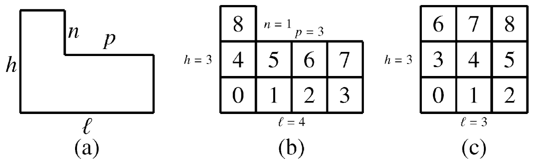

Fig. 2. The minimum distance diagram of double-loop networks. (a) The four parameters. (b) The L-shape of $DL(9;1,4)$. (c) The L-shape of $DL(9;1,3)$.

is for the concluding remarks; some open problems on the double-loop networks and the mixed chordal ring networks are also given here.

2. Preliminaries, assumptions, and embedding

Since the mixed chordal ring network is very related to the double-loop network, we first introduce some terminologies of the double-loop network. Given a $DL(N;a,b)$, a *minimum distance diagram* (MDD) is a diagram with node 0 in cell $(0,0)$ and node v in cell (i,j) if and only if $ia+jb\equiv v \pmod N$ and $i+j$ is the minimum among all (i',j') satisfying the congruence. In other words, a shortest path from 0 to v is through taking i a-links and j b-links (in any order). Note that, in cell (i,j), i (respectively, j) is the column (respectively, row) index. An MDD includes every node exactly once (in case of two shortest paths, the convention is to choose the cell with the smaller row index, i.e., the smaller j).

It had been proven that the MDD of $DL(N;a,b)$ is always an *L-shape* determined by four parameters ℓ, h, p, n [17]; see Fig. 2(a). These four parameters are the lengths of four of the six segments on the boundary of the L-shape. For example, $DL(9;1,4)$ has $\ell=4$, $h=3$, $p=3$, and $n=1$; see Fig. 2(b). An L-shape can degenerate into a rectangle as Fig. 2(c). Fiol et al. [11] observed that an L-shape always tessellates the plane regardless of the L-shape is degenerate or not. Many studies of the double-loop network are based on the L-shape [2,5,11]. One important function of the L-shape is that we can easily compute the diameter of its double-loop network by

$$\max\{\ell+h-p-2,\ \ell+h-n-2\}.$$

Throughout this paper, we will assume that $MCR(N;s,w)$ satisfies the following three conditions:

$$s\neq w,\qquad s+w\neq N,\quad \text{and}\quad \gcd(N,s,w)=1.$$

The reason is as follows. If $s=w$ or $s+w=N$, then $MCR(N;s,w)$ will contain multiple links between two nodes, which means a waste of the hardware. On the other hand, $MCR(N;s,w)$ is strongly connected if and only if $\gcd(N,s,w)=1$. Since we will only talk about strongly connected mixed chordal ring networks, we assume $\gcd(N,s,w)=1$.

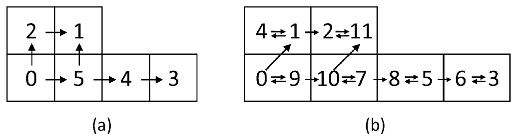

Fig. 3. The pseudo-MDD of a mixed chordal ring network. (a) The MDD of $DL(6;5,2)$. (b) The pseudo-MDD of $MCR(12;1,3)$.

Note that the double-loop network and the mixed chordal ring network are different network topologies: the former is vertex-transitive and the latter may or may not be vertex-transitive. For example, in $MCR(12;3,5)$, node 0 can reach any node within 4 moves, but it takes 5 moves for node 1 to reach node 8.

Chen, Hwang and Liu showed that the mixed chordal ring network $MCR(N;s,w)$ can be embedded into the double-loop network $DL(\frac{N}{2};\frac{s-w}{2},\frac{s+w}{2})$ by combining nodes $2k+1$ and $2k+1+w$ as *supernode* k^* for all $k=0,1,\dots,N/2-1$ [6]. Note that, unless otherwise specified, $\frac{s-w}{2}$ means $(\frac{s-w}{2}) \bmod \frac{N}{2}$, $\frac{s+w}{2}$ means $(\frac{s+w}{2}) \bmod \frac{N}{2}$, all nodes of a mixed chordal ring network are taken modulo N, and all nodes of a double-loop network with $N/2$ nodes are taken modulo $N/2$.

Another way to embed the mixed chordal ring network $MCR(N;s,w)$ into the double-loop network $DL(\frac{N}{2};\frac{s-w}{2},\frac{s+w}{2})$ is to combine nodes $2k$ and $2k-w$ as supernode k^* for all $k=0,1,\dots,N/2-1$. See Fig. 1(c) for an example. Such an embedding results in the same double-loop network $DL(\frac{N}{2};\frac{s-w}{2},\frac{s+w}{2})$ as the one used in [6] but is more natural since node 0 of $MCR(N;s,w)$ is in supernode 0^* of $DL(\frac{N}{2};\frac{s-w}{2},\frac{s+w}{2})$. Thus, throughout this paper, we use $DL(\frac{N}{2};\frac{s-w}{2},\frac{s+w}{2})$ to denote the embedding of combining nodes $2k$ and $2k-w$ as supernode k^*.

Since we can embed a mixed chordal ring network into a double-loop network, we can embed a mixed chordal ring network into the MDD of the corresponding double-loop network. More precisely, given $MCR(N;s,w)$, we replace each node k in the MDD of $DL(\frac{N}{2};\frac{s-w}{2},\frac{s+w}{2})$ with two nodes $2k$ and $2k-w$ in such a way that if k is in cell (i,j), then $2k$ and $2k-w$ are in cells $(2i,j)$ and $(2i+1,j)$, respectively. We call the resultant diagram the *pseudo-MDD* of $MCR(N;s,w)$. See Fig. 3 for an example.

The following lemma had been proven in [6] and it follows from the fact that each move in the MDD of $DL(\frac{N}{2};\frac{s-w}{2},\frac{s+w}{2})$ corresponds to either one or two moves in $MCR(N;s,w)$ (depending on which node in the supernode we start from).

Lemma 2.1. *(See [6].) Suppose $DL(\frac{N}{2};\frac{s-w}{2},\frac{s+w}{2})$ has* L-shape(ℓ,h,p,n)*, then $d(N;s,w)\leqslant 2\max\{\ell,h\}-1$.*

3. Improved bounds on $D_{MCR}(N)$

This section is devoted to improve upper and lower bounds on $D_{MCR}(N)$. Given a $MCR(N;s,w)$, let n_k denote the number of additional nodes that node 0 can reach in k moves. Clearly, $n_0=0$, $n_1=2$ and $n_2=3$. Chen et al. [6] had proven that

$$n_k \leqslant n_{k-1} + 1 \quad \text{for } 2 \leqslant k \leqslant d(N; s, w). \tag{4}$$

In other words, for $k \geqslant 2$, the number of additional nodes that node 0 can reach at the kth move increases by at most 1. We now have the following result.

Theorem 3.1. *$D_{MCR}(N) \geqslant \lceil\sqrt{2N} - 3/2\rceil$ and this bound is tight.*

Proof. By (4),

$$N \leqslant \sum_{k=0}^{d(N;s,w)} (k+1) = \frac{(d(N; s, w) + 2)(d(N; s, w) + 1)}{2}.$$

Therefore, $(d(N; s, w))^2 + 3d(N; s, w) + (2 - 2N) \geqslant 0$. Since $d(N; s, w)$ is positive, $d(N; s, w) \geqslant (\sqrt{8N+1} - 3)/2 > \sqrt{2N} - 3/2$. Since $d(N; s, w)$ is an integer, $d(N; s, w) \geqslant \lceil\sqrt{2N} - 3/2\rceil$. This bound is tight since $d(8; 1, 3) = 3 \geqslant D_{MCR}(8) \geqslant \lceil\sqrt{2 \cdot 8} - 3/2\rceil = 3$. $\square$

We now obtain an upper bound on $D_{MCR}(N)$. The main idea used in obtaining the upper bound is, for each N, to choose s and w suitably so that the corresponding double-loop network $DL(\frac{N}{2}; \frac{s-w}{2}, \frac{s+w}{2})$ has an L-shape(ℓ, h, p, n) with ℓ and h being as small as possible and to apply Lemma 2.1.

Define $\hat{N}$ to be a function of N as follows:

$$\hat{N} = \left\lceil \sqrt{\frac{N}{2}} \right\rceil. \tag{5}$$

According to the parity of $\hat{N}$, define M as follows:

$$M = \begin{cases} \hat{N} & \text{if } \hat{N} \text{ is even,} \\ \hat{N} + 1 & \text{if otherwise.} \end{cases} \tag{6}$$

Lemma 3.2. *Suppose $N \neq 2(4t^2 + 2t - 1)$ for any positive integer t and let M be defined as in* (6). *Then the* L-shape(ℓ, h, p, n) *of $DL(\frac{N}{2}; 1, M)$ satisfies $\ell \leqslant M$ and $h \leqslant M$.*

Proof. Consider $\mathbb{N} = \bigcup_{t=0}^{\infty}[4t^2 + 1, 4(t+1)^2]$. Then $\frac{N}{2} \in [4t^2 + 1, 4(t+1)^2]$ for some non-negative integer t. Thus $M = 2t + 2$. Consider the L-shape(ℓ, h, p, n) of $DL(\frac{N}{2}; 1, M)$. Since

$$M \cdot 1 \equiv 1 \cdot M \left(\text{mod } \frac{N}{2}\right),$$

cell $(M, 0)$ and cell $(0, 1)$ contain the same node. Since $M > 1$, cell $(M, 0)$ is outside the L-shape. Consequently, $\ell \leqslant M$. Now let $N_0(t) = [4t^2 + 1, 4t^2 + 2t - 2]$, $N_1(t) = [4t^2 + 2t - 1, 4t^2 + 4t]$, $N_2(t) = [4t^2 + 4t + 1, 4t^2 + 6t + 2]$, and $N_3(t) = [4t^2 + 6t + 3, 4t^2 + 8t + 4]$. Note that $N_0(0)$, $N_1(0)$, and $N_0(1)$ are empty. Then $\mathbb{N} = \bigcup_{t=0}^{\infty}(N_0(t) \cup N_1(t) \cup N_2(t) \cup N_3(t))$. Suppose $\frac{N}{2} \in N_k(t)$, where $0 \leqslant k \leqslant 3$. Define $N_k^*(t)$ to be the maximum integer in $N_k(t)$. Clearly, $N_k^*(t) = 4t^2 + 2t - 2 + (2t + 2)k$. Suppose $\frac{N}{2} = N_k^*(t) - j$ for some non-negative integer j. Then $0 \leqslant j \leqslant 2t - 3$ if $k = 0$ and $0 \leqslant j \leqslant 2t + 1$ if $1 \leqslant k \leqslant 3$. Again, consider the L-shape(ℓ, h, p, n) of $DL(\frac{N}{2}; 1, M)$. Since

$$\begin{aligned} j \cdot 1 &= N_k^*(t) - \frac{N}{2} \\ &= \left(4t^2 + 2t - 2 + (2t + 2)k\right) - \frac{N}{2} \\ &\equiv (2t - 1 + k)(2t + 2) \left(\text{mod } \frac{N}{2}\right) \\ &= (2t - 1 + k)M \left(\text{mod } \frac{N}{2}\right), \end{aligned}$$

cell $(j, 0)$ and cell $(0, 2t - 1 + k)$ contain the same node. Note that $j \leqslant 2t - 1 + k$ except when $k = 1$ and $j = 2t + 1$, that is, except when $\frac{N}{2} = 4t^2 + 2t - 1$. Hence if $N \neq 2(4t^2 + 2t - 1)$ for any positive integer t, then cell $(0, 2t - 1 + k)$ is outside the L-shape. Consequently, $h \leqslant 2t - 1 + k \leqslant 2t + 2 = M$. $\square$

Lemma 3.3. *Suppose $N = 2(4t^2 + 2t - 1)$ for some positive integer t and let M be defined as in* (6). *Then the* L-shape(ℓ, h, p, n) *of $DL(\frac{N}{2}; 2, M - 1)$ satisfies $\ell \leqslant M - 1$ and $h \leqslant M - 1$.*

Proof. Since $N = 2(4t^2 + 2t - 1)$ for some positive integer t, we have $M = 2t + 2$. Consider the L-shape(ℓ, h, p, n) of $DL(\frac{N}{2}; 2, M - 1)$. Since

$$(2t + 1) \cdot 2 \equiv 2 \cdot (2t + 1) \left(\text{mod } \frac{N}{2}\right),$$

cell $(2t + 1, 0)$ and cell $(0, 2)$ contain the same node. Since t is a positive integer, we have $2t + 1 > 2$. Thus cell $(2t + 1, 0)$ is outside the L-shape. Consequently, $\ell \leqslant 2t + 1 \leqslant M - 1$. Similarly, since

$$(t + 1) \cdot 2 \equiv (2t + 1)(2t + 1) \left(\text{mod } \frac{N}{2}\right),$$

cell $(t + 1, 0)$ and cell $(0, 2t + 1)$ contain the same node. Clearly, $2t + 1 > t + 1$ for $t > 0$; thus cell $(0, 2t + 1)$ is outside the L-shape. Thus $h \leqslant 2t + 1 \leqslant M - 1$. $\square$

Lemma 3.4. *Let M be defined as in* (6). *Then*:

1. *If $N \neq 2(4t^2 + 2t - 1)$ for any positive integer t, then $d(N; M + 1, M - 1) \leqslant 2M - 1$.*
2. *If $N = 2(4t^2 + 2t - 1)$ for some positive integer t, then $d(N; M + 1, M - 3) \leqslant 2M - 3$.*

Proof. Consider the first statement. It is not difficult to verify that both $M + 1$ and $M - 1$ are positive odd integers and $\gcd(N, M + 1, M - 1) = 1$. Thus $MCR(N; M + 1, M - 1)$ is a valid mixed chordal ring network. Since we can embed $MCR(N; M + 1, M - 1)$ into $DL(\frac{N}{2}; 1, M)$, this statement follows directly from Lemmas 2.1 and 3.2. The second statement can be proven similarly except that Lemma 3.2 is replaced with Lemma 3.3. $\square$

Theorem 3.5. *Let $\hat{N}$ be defined as in* (5).

1. *If $\hat{N}$ is even, then $D_{MCR}(N) \leqslant 2\lceil\sqrt{N/2}\rceil - 1$.*
2. *If $\hat{N}$ is odd and $N = 2(4t^2 + 2t - 1)$ for some positive integer t, then $D_{MCR}(N) \leqslant 2\lceil\sqrt{N/2}\rceil - 1$.*

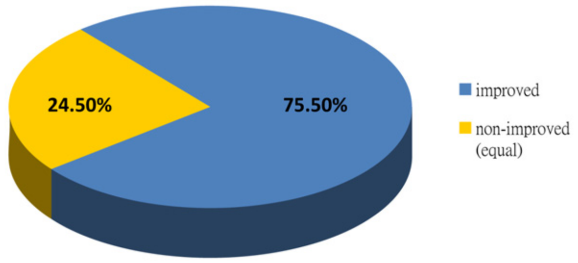

Fig. 4. The improved ratio of our upper bound as compared to the previous upper bound for $N = 6, 8, 10, \ldots, 10004$ (total 5000 N's).

3. *If $\hat{N}$ is odd and $N \neq 2(4t^2 + 2t - 1)$ for any positive integer t, then $D_{MCR}(N) \leqslant 2\lceil\sqrt{N/2}\rceil + 1$.*

Moreover, these bounds are tight.

Proof. Note that if $N = 2(4t^2+2t-1)$ for some positive integer t, then $\hat{N}$ is odd. Thus if $\hat{N}$ is even, then $N \neq 2(4t^2 + 2t - 1)$ for any positive integer t; consequently, $M = \hat{N}$. If $\hat{N}$ is odd and $N = 2(4t^2 + 2t - 1)$ for some positive integer t, then $M = \hat{N} + 1$. If $\hat{N}$ is odd and $N \neq 2(4t^2 + 2t - 1)$ for any positive integer t, then $M = \hat{N} + 1$. Statements 1, 2 and 3 in this theorem now follow from Lemma 3.4. By the aid of a computer program, we obtain $D_{MCR}(20) = 7$, $D_{MCR}(38) = 9$ and $D_{MCR}(12) = 5$. Thus the bound in statement 1 is tight since $D_{MCR}(20) = 7$ and $2\lceil\sqrt{20/2}\rceil - 1 = 7$. The bound in statement 2 is tight since $D_{MCR}(38) = 9$ and $2\lceil\sqrt{38/2}\rceil - 1 = 9$. Similarly, the bound in statement 3 is tight since $D_{MCR}(12) = 5$ and $2\lceil\sqrt{12/2}\rceil - 1 = 5$. □

Remark 1. The previous upper bound on $D_{MCR}(N)$ is $\sqrt{2N} + 3$ [6]. Since $\sqrt{2N} + 3$ is served as an upper bound, we replace it with $\lfloor\sqrt{2N} + 3\rfloor$. The largest upper bound in Theorem 3.5 is $2\lceil\sqrt{N/2}\ \rceil + 1$ and it is always no larger than $\lfloor\sqrt{2N} + 3\rfloor$. To see how good our upper bound $2\lceil\sqrt{N/2}\rceil + 1$ is, we use a computer to obtain results for $N = 6, 8, 10, \ldots, 10004$. Among these 5000 N's, for 3775 (about 75.50%) out of them, our upper bound $2\lceil\sqrt{N/2}\rceil + 1$ improves the previous upper bound $\lfloor\sqrt{2N} + 3\rfloor$; see Fig. 4.

Remark 2. The upper bound $2\lceil\sqrt{N/2}\rceil - 1$ in Theorem 3.5 is no larger than the upper bound $\lceil\sqrt{2N}\rceil + 1$ in Theorem 3.5 and is very close to the lower bound $\lceil\sqrt{2N} - 3/2\rceil$ in Theorem 3.1. In the following, we show that there exist infinite number of N's such that the upper bound $2\lceil\sqrt{N/2}\rceil - 1$ matches the lower bound $\lceil\sqrt{2N} - 3/2\rceil$; in other words, we determine the exact value of $D_{MCR}(N)$ for these N's.

Theorem 3.6. *Suppose $N = 2(4t^2 - t + k)$ for some positive integers t and k, where $1 \leqslant k \leqslant t$. Then*

$$D_{MCR}(N) = 2\lceil\sqrt{N/2}\rceil - 1.$$

Moreover, $d(N; \lceil\sqrt{N/2}\rceil + 1, \lceil\sqrt{N/2}\rceil - 1) = D_{MCR}(N)$.

Proof. Suppose $N = 2(4t^2 - t + k)$ for some positive integer t and k, where $1 \leqslant k \leqslant t$. Then $2(4t^2 - 4t + 1) < N \leqslant 2 \cdot 4t^2$; therefore, $M = \hat{N} = \lceil\sqrt{N/2}\rceil = 2t$. By Lemma 3.4 and Theorem 3.5, $D_{MCR}(N) \leqslant d(N; \lceil\sqrt{N/2}\rceil + 1, \lceil\sqrt{N/2}\rceil - 1) \leqslant 2\lceil\sqrt{N/2}\rceil - 1$. Since $2(4t^2 - t + \frac{1}{4}) < N \leqslant 2(4t^2 + t + \frac{1}{4})$, we have $D_{MCR}(N) \geqslant \lceil\sqrt{2N} - 3/2\rceil = 4t - 1 = 2\lceil\sqrt{N/2}\rceil - 1$. We now have this theorem. □

The N's that satisfy Theorem 3.6 are: 8, 30, 32, 68, 70, 72, 122, ..., and so on. For $N = 6, 8, 10, \ldots, 10004$ (total 5000 N's), about 12.60% out of them satisfy Theorem 3.6 and their optimal diameter can be determined by Theorem 3.6.

4. Strongly connectivity of $MCR(N; s, w)$

We first indicate the problem in the proof of Theorem 1.3 in [6]. To obtain $D_{MCR}(38)$, Chen et al. [6] will use $MCR(38; 7, 5)$ and embed $MCR(38; 7, 5)$ into $DL(19; 1, 6)$. The L-shape of $DL(19; 1, 6)$ has $\ell = 5$ and $h = 7$, which has $h > N' = 6$ and violates

$$\ell \leqslant N' \quad \text{and} \quad h = N' \tag{7}$$

needed in the proof of $D_{MCR}(38) \leqslant \sqrt{2N} + 3$. In fact, we can construct infinite many N's that violates (7); see [6] for more details.

In Section 2, we have shown how to embed the mixed chordal ring network $MCR(N; s, w)$ into the double-loop network $DL(\frac{N}{2}; \frac{s-w}{2}, \frac{s+w}{2})$. However, this embedding sometimes fails. Take $MCR(10; 1, 5)$ as an example; its corresponding double-loop network is $DL(\frac{10}{2}; \frac{1-5}{2}, \frac{1+5}{2})$, i.e., $DL(5; 3, 3)$, which is clearly not a valid double-loop network, yet $MCR(10; 1, 5)$ is a valid mixed chordal ring network. In general, $MCR(2(2k + 1); 1, 2k + 1)$ is embedded into $DL(2k + 1; k + 1, k + 1)$ but $DL(2k + 1; k + 1, k + 1)$ is not a valid double-loop network. The idea used in [6] to prove Theorem 1.4 is to show that $MCR(N; s, w)$ is strongly 2-connected if and only if the corresponding double-loop network $DL(\frac{N}{2}; \frac{s-w}{2}, \frac{s+w}{2})$ is strongly 2-connected. We now correct the proof.

Lemma 4.1. *For $MCR(N; s, w)$,*

1. *if $w \neq \frac{N}{2}$, then $DL(\frac{N}{2}; \frac{s-w}{2}, \frac{s+w}{2})$ is a double-loop network;*
2. *if $w = \frac{N}{2}$, then $DL(\frac{N}{2}; \frac{s-w}{2}, \frac{s+w}{2})$ is not a double-loop network and $MCR(N; s, \frac{N}{2})$ is itself the double-loop network $DL(N; s, \frac{N}{2})$.*

Proof. $DL(\frac{N}{2}; \frac{s-w}{2}, \frac{s+w}{2})$ is not a valid double-loop network whenever $\frac{s-w}{2} \equiv 0 \pmod{\frac{N}{2}}$ or $\frac{s+w}{2} \equiv 0 \pmod{\frac{N}{2}}$ or $\frac{s-w}{2} \equiv \frac{s+w}{2} \pmod{\frac{N}{2}}$ or $\gcd(\frac{N}{2}, \frac{s-w}{2}, \frac{s+w}{2}) \neq 1$. Since we assume $s \neq w$ and $s + w \neq N$, it is impossible that $\frac{s-w}{2} \equiv 0 \pmod{\frac{N}{2}}$ or $\frac{s+w}{2} \equiv 0 \pmod{\frac{N}{2}}$. Also, $\frac{s-w}{2} \equiv \frac{s+w}{2} \pmod{\frac{N}{2}}$ if and only if $w = \frac{N}{2}$. In addition, we have assumed $\gcd(N, s, w) = 1$; therefore $\gcd(\frac{N}{2}, \frac{s-w}{2}, \frac{s+w}{2}) = 1$. Thus we have the first if-statement. When $w = \frac{N}{2}$, $\frac{N}{2} \equiv -\frac{N}{2} \pmod{N}$ occurs and the chordal-links of $MCR(N; s, w)$ become:

$$i \to \left(i + \frac{N}{2}\right) \bmod N, \quad i = 0, 1, \ldots, N - 1.$$

Thus $MCR(N; s, \frac{N}{2})$ is itself the double-loop network $DL(N; s, \frac{N}{2})$ and we have the second if-statement. □

Lemma 4.1 shows that $DL(\frac{N}{2}; \frac{s-w}{2}, \frac{s+w}{2})$ is a valid embedding if and only if $w \neq \frac{N}{2}$. It was proven in [6] that

Lemma 4.2. *$MCR(N; s, w)$ is strongly connected if and only if* $\gcd(N, s, w) = 1$.

Now we give correct proof of Theorem 1.4.

Proof of Theorem 1.4. *Necessity.* This follows directly from Lemma 4.2.

Sufficiency. There are two cases.

Case 1: $w \neq \frac{N}{2}$. Then by Lemma 4.1, $DL(\frac{N}{2}; \frac{s-w}{2}, \frac{s+w}{2})$ is a double-loop network. Since $w \neq \frac{N}{2}$, $\frac{s-w}{2} \neq \frac{s+w}{2}$. Since $\gcd(N, s, w) = 1$, $\gcd(\frac{N}{2}, \frac{s-w}{2}, \frac{s+w}{2}) = 1$. Thus by Theorem 1.1, $DL(\frac{N}{2}; \frac{s-w}{2}, \frac{s+w}{2})$ is strongly 2-connected. Since the two nodes in each super-node can reach each other through the chordal-links between them, $MCR(N; s, w)$ is strongly 2-connected.

Case 2: $w = \frac{N}{2}$. By Lemma 4.1, $MCR(N; s, w)$ is itself the double-loop network $DL(N; s, w)$. Thus by Theorem 1.1 and by the assumption that $\gcd(N, s, w) = 1$, $MCR(N; s, w)$ is strongly 2-connected. □

5. Concluding remarks

In [6], Chen et al. proposed a new network topology called the mixed chordal ring network and discussed its combinatorial properties. They obtained the surprising result that the mixed chordal ring network is comparable in hardware to the well-known double-loop network and yet can achieve a better diameter than the double-loop network. In this paper, we have improved the upper and lower bounds on $D_{MCR}(N)$ (i.e., the optimal diameter of mixed chordal ring networks) as $2\lceil\sqrt{N/2}\rceil + 1$ and $\lceil\sqrt{2N} - 3/2\rceil$, respectively. We have also corrected some deficient contexts in [6].

For the double-loop network, determining the exact value of $D_{DL}(N)$ is a hard problem and even determining $\overline{D}_{DL}(N) = \min_b\{d_{DL}(N; 1, b)\}$, where $d_{DL}(N; 1, b)$ is the diameter of $DL(N; 1, b)$, is a hard problem, too [4]. By (1), (2) and (3), the gap between the upper and the lower bounds on $D_{DL}(N)$ increases by a factor of $(3N)^{1/4}$ and it seems that there is no closed form for $D_{DL}(N)$. For the mixed chordal ring network, we have successfully narrowed the gap between the upper and the lower bounds on $D_{MCR}(N)$ as $2\lceil\sqrt{N/2}\rceil + 1$ and $\lceil\sqrt{2N} - 3/2\rceil$. It has a great probability to determine $D_{MCR}(N)$ and therefore solve this optimization problem in the near future.

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