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Symplectic synchronization of different chaotic systems

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Abstract

In this paper, a new symplectic synchronization of chaotic systems is studied. Traditional generalized synchronizations are special cases of the symplectic synchronization. A sufficient condition is given for the asymptotical stability of the null solution of an error dynamics. The symplectic synchronization may be applied to the design of secure communication. Finally, numerical results are studied for a Quantum-CNN oscillators synchronized with a Rössler system in three different cases.

1. Introduction

Many approaches have been presented for the synchronization of chaotic systems [2–6]. There are a chaotic master system and either an identical or a different slave system. Our goal is the synchronization of the chaotic master and the chaotic slave by coupling or by other methods.

Among many kinds of synchronizations [7], generalized synchronization is investigated [8–12]. There exists a functional relationship between the states of the master and that of the slave. In this paper, a new synchronization

$$y = H(x, y, t) + F(t) \tag{1}$$

is studied, where x, y are the state vectors of the "master" and of the "slave", respectively, $F(t)$ is a given function of time in different form, such as a regular or a chaotic function. When $H(x, y, t) = x$, Eq. (1) reduces to the generalized synchronization given in [1]. Therefore this paper is an extension of [1].

In Eq. (1), the final desired state y of the "slave" system not only depends upon the "master" system state x but also depends upon the "slave" system state y itself. Therefore the "slave" system is not a traditional pure slave obeying the "master" system completely but plays a role to determine the final desired state of the "slave" system. In other words, it plays an "interwined" role, so we call this kind of synchronization "symplectic synchronization"[1], and call the "master" system partner A, the "slave" system partner B.

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[1] The term "symplectic" comes from the Greek for "interwined". H. Weyl first introduced the term in 1939 in his book "The Classical Groups" (p. 165 in both the first edition, 1939, and second edition, 1946, Princeton University Press).

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When $H(x,y,t) = H(x,t)$, Eq. (1) becomes

$$y = H(x,t) + F(t) \tag{2}$$

which reduces to generalized synchronization. Therefore generalized synchronization is a special case of the symplectic synchronization. There exists great potential of the application of the symplectic synchronization. For instance, when the symplectically synchronized chaotic signal is used as a signal carrier, the secure communication is more difficult to be deciphered.

As numerical examples, recently developed Quantum Cellular Neural Network (Quantum-CNN) chaotic oscillator is used to synchronize with different systems, respectively. Quantum-CNN oscillator equations are derived from a Schrödinger equation taking into account quantum dots cellular automata structures to which in the last decade a wide interest has been devoted, with particular attention towards quantum computing [13].

This paper is organized as follows. In Section 2, by the Lyapunov asymptotical stability theorem, a symplectic synchronization scheme is given. In Section 3, various feedback controllers are designed for the symplectic synchronization of the Quantum-CNN oscillator and a Rössler system. Numerical simulations are also given in Section 3. Finally, some concluding remarks are given in Section 4.

2. Symplectic synchronization scheme

There are two different nonlinear chaotic systems. The partner A controls the partner B partially. The partner A is given by

$$\dot{x} = f(x) \tag{3}$$

where $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in R^n$ is a state vector and f is a vector function.

The partner B is given by

$$\dot{y} = g(y) \tag{4a}$$

where $y = [y_1, y_2, \ldots, y_n]^{\mathrm{T}} \in R^n$ is a state vector, and g is a vector function different from f.

After a controller $u(t)$ is added, partner B becomes

$$\dot{y} = g(y) + u(t) \tag{4b}$$

where $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}} \in R^n$ is the control vector.

Our goal is to design the controller $u(t)$ so that the state vector y of the partner B asymptotically approaches $H(x,y,t) + F(t)$, a given function $H(x,y,t)$ plus a given vector function $F(t) = [F_1(t), F_2(t), \ldots, F_n(t)]^{\mathrm{T}}$ which is a regular or a chaotic function of time. Define error vector $e(t) = [e_1, e_2, \ldots, e_n]^{\mathrm{T}}$:

$$e = H(x,y,t) - y + F(t) \tag{5}$$

$$\lim_{t\to\infty} e = 0 \tag{6}$$

is demanded.

From Eq. (5), it is obtained that

$$\dot{e} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial t} - \dot{y} + \dot{F}(t) \tag{7}$$

By Eqs. (3), (4a) and (4b), (7) becomes

$$\dot{e} = \frac{\partial H}{\partial x}f(x) + \frac{\partial H}{\partial y}g(y) + \frac{\partial H}{\partial t} - g(y) - u(t) + \dot{F}(t) \tag{8}$$

A positive definite Lyapnuov function $V(e)$ is chosen:

$$V(e) = \frac{1}{2}e^{\mathrm{T}}e \tag{9}$$

Its derivative along any solution of Eq. (8) is

$$\dot{V}(e) = e^{\mathrm{T}}\left\{\frac{\partial H}{\partial x}f(x) + \frac{\partial H}{\partial y}g(y) + \frac{\partial H}{\partial t} - g(y) + \dot{F}(t) - u(t)\right\}. \tag{10}$$

In Eq. (10), $u(t)$ is designed so that $\dot{V} = e^{\mathrm{T}}C_{n\times n}e$ where $C_{n\times n}$ is a diagonal negative definite matrix. $\dot{V}$ is a negative definite function of e. By Lyapunov theorem of asymptotical stability

$$\lim_{t\to\infty} e = 0$$

The symplectic synchronization is obtained [14–16].

3. Numerical results for the symplectic chaos synchronization of Quantum-CNN oscillator and Rössler System

Case I: A cubic symplectic synchronization

For a two-cell Quantum-CNN, following differential equations are obtained [13]

$$\begin{cases} \dot{x}_1 = -2a_1\sqrt{1-x_1^2}\sin x_2 \\ \dot{x}_2 = -\omega_1(x_1-x_3) + 2a_1\dfrac{x_1}{\sqrt{1-x_1^2}}\cos x_2 \\ \dot{x}_3 = -2a_2\sqrt{1-x_3^2}\sin x_4 \\ \dot{x}_4 = -\omega_2(x_3-x_1) + 2a_2\dfrac{x_3}{\sqrt{1-x_3^2}}\cos x_4 \end{cases} \tag{11}$$

where x_1, x_3 are polarizations, x_2, x_4 are quantum phase displacements, a_1 and a_2 are proportional to the inter-dot energy inside each cell and ω_1 and ω_2 are the parameters that weigh the effects on the cell of the difference of polarization of the neighboring cells, like the cloning templates in traditional CNNs. When $a_1 = 19.4$, $a_2 = 13.1$, $\omega_1 = 9.529$ and $\omega_2 = 7.94$, the system is chaotic.

A chaotic Rössler system is described by

$$\begin{cases} \dot{y}_1 = -y_2 - y_3 \\ \dot{y}_2 = y_1 - \alpha y_2 + y_4 \\ \dot{y}_3 = y_1 y_3 + \beta \\ \dot{y}_4 = \gamma y_3 + \sigma y_4 \end{cases} \tag{12}$$

where $\alpha = 0.5$, $\beta = 0.52$, $\gamma = 0.5$, $\sigma = 0.05$.

For symplectic synchronization of these two systems, u_1, u_2, u_3 and u_4 are added to the four equations of Eq. (12), respectively:

$$\begin{cases} \dot{y}_1 = -y_2 - y_3 + u_1 \\ \dot{y}_2 = y_1 - \alpha y_2 + y_4 + u_2 \\ \dot{y}_3 = y_1 y_3 + \beta + u_3 \\ \dot{y}_4 = \gamma y_3 + \sigma y_4 + u_4 \end{cases} \tag{13}$$

The initial values of the states of the Quantum-CNN system and of the Rössler system are taken as $x_1(0) = 0.8$, $x_2(0) = -0.77$, $x_3(0) = -0.72$, $x_4(0) = 0.57$, $y_1(0) = 0.3$, $y_2(0) = -0.4$, $y_3(0) = -0.7$ and $y_4(0) = 0.15$.

We take $F_1(t) = x_4^3(t)$, $F_2(t) = x_1^3(t)$, $F_3(t) = x_2^3(t)$, and $F_4(t) = x_3^3(t)$. They are chaotic functions of time. $H_i(x,y,t) = -x_i^2 y_i$ $(i = 1,2,3,4)$ are given. By Eq. (6) we have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(-x_i^2 y_i - y_i + x_j^3) = 0, \quad i = 1,2,3,4 \quad j = \begin{cases} 4, & i = 1 \\ i-1, & i \neq 1 \end{cases} \tag{14}$$

From Eq. (7) we have

$$\dot{e}_i = -2\dot{x}_i x_i y_i - x_i^2 \dot{y}_i - \dot{y}_i + 3\dot{x}_j x_j^2, \quad i = 1,2,3,4 \quad j = \begin{cases} 4, & i = 1 \\ i-1, & i \neq 1 \end{cases} \tag{15}$$

Eq. (8) can be expressed as

$$\begin{aligned} \dot{e}_1 = {} & 2y_1 x_1\left(2a_1\sqrt{1-x_1^2}\sin x_2\right) + (y_2+y_3)x_1^2 + y_2 + y_3 - u_1 \\ & + 3x_4^2\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) \end{aligned}$$

$$\dot{e}_2 = -2y_2x_2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)-(y_1-\alpha y_2+y_4)x_2^2 \\ -y_1+\alpha y_2-y_4-u_2+3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right)$$

$$\dot{e}_3 = 2y_3x_3\left(2a_2\sqrt{1-x_3^2}\sin x_4\right)-(y_1y_3+\beta)x_2^3-y_1y_3-\beta-u_3 \\ +3x_2^2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)$$

$$\dot{e}_4 = -2y_4x_4\left(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)-(\gamma y_3+\sigma y_4)x_4^2-\gamma y_3 \\ -\sigma y_4-u_4+3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right)$$

where $e_1=-x_1^2y_1-y_1+x_4^3$, $e_2=-x_2^2y_2-y_2+x_1^3$, $e_3=-x_3^2y_3-y_3+x_2^3$ and $e_4=-x_4^2y_4-y_4+x_3^3$.

Choose a positive definite Lyapunov function:

$$V(e_1,e_2,e_3,e_4)=\frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2) \tag{17}$$

Its time derivative along any solution of Eq. (16) is

$$\begin{aligned}\dot{V} = {} & e_1\left\{2y_1x_1\left(2a_1\sqrt{1-x_1^2}\sin x_2\right)+(y_2+y_3)x_1^2+y_2+y_3\right. \\ & \left.+3x_4^2\left(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)-u_1\right\} \\ & +e_2\left\{-2y_2x_2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)-(y_1-\alpha y_2+y_4)x_2^2\right. \\ & \left.-y_1+\alpha y_2-y_4+3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right)-u_2\right\} \\ & +e_3\left\{2y_3x_3\left(2a_2\sqrt{1-x_3^2}\sin x_4\right)-(y_1y_3+\beta)x_2^3-y_1y_3-\beta\right. \\ & \left.+3x_2^2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)-u_3\right\} \\ & +e_4\left\{-2y_4x_4\left(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)-(\gamma y_3+\sigma y_4)x_4^2-\gamma y_3\right. \\ & \left.-\sigma y_4+3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right)-u_4\right\}\end{aligned} \tag{18}$$

Choose

$$u_1 = 2y_1x_1\left(2a_1\sqrt{1-x_1^2}\sin x_2\right)+(y_2+y_3)x_1^2+y_2+y_3 \\ +3x_4^2\left(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)-y_1x_1^2-y_1+x_4^3$$

$$u_2 = -2y_2x_2\left(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)-(y_1-\alpha y_2+y_4)x_2^2 \\ -y_1-y_4+3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right)-\alpha(y_2x_2^2-x_1^3)$$

$$u_3 = 2y_3x_3\left(2a_2\sqrt{1-x_3^2}\sin x_4\right) - (y_1y_3+\beta)x_2^3 - y_1y_3 - \beta + 3x_2^2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right) - y_3x_3^2 - y_3 + x_2^3$$
$$u_4 = -2y_4x_4\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) - (\gamma y_3+\sigma y_4)x_4^2 - \gamma y_3 + 3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right) - \sigma(y_4x_4^2 + 2y_4 - x_3^3)$$

Eq. (18) becomes

$$\dot{V} = -(e_1^2 + \alpha e_2^2 + e_3^2 + \sigma e_4^2) < 0 \tag{19}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. Cubic symplectic synchronization of the Quantum-CNN system and the Rössler system is achieved. The numerical results are shown in Fig. 1. After 5 s, the motion trajectories enter a chaotic attractor.

Case II: A time delay symplectic synchronization

We take $F_1(t) = x_1(t-T)$, $F_2(t) = x_2(t-T)$, $F_3(t) = x_3(t-T)$ and $F_4(t) = x_4(t-T)$. They are chaotic functions of time, where time delay $T = 1$ s is a positive constant. $H_i(x,y,t) = (x_i^2 + y_i)(e^{-t}+2)$ $(i = 1,2,3,4)$ are given. By Eq. (6) we have

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}((x_i^2+y_i)(e^{-t}+2) - y_i + x_i(t-T)) = 0, \quad i = 1,2,3,4 \tag{20}$$

From Eq. (7) we have

$$\dot{e}_i = (2x_i\dot{x}_i + \dot{y}_i)(e^{-t}+2) - e^{-t}(x_i^2+y_i) - \dot{y}_i + \dot{x}_i(t-T), \quad i = 1,2,3,4 \tag{21}$$

Eq. (8) is expressed as

$$\begin{aligned}
\dot{e}_1 &= 2x_1\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right)(e^{-t}+2) + (-y_2-y_3)(e^{-t}+2) - (x_1^2+y_1)e^{-t} \\
&\quad + y_2 + y_3 - u_1 - 2a_1\sqrt{1-x_1^2(t-T)}\sin x_2(t-T) \\
\dot{e}_2 &= 2x_2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)(e^{-t}+2) + (y_1 - \alpha y_2 + y_4)(e^{-t}+2) \\
&\quad - (x_2^2+y_2)e^{-t} - y_1 + \alpha y_2 - y_4 - u_2 - \omega_1(x_1(t-T) - x_3(t-T)) \\
&\quad + 2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T) \\
\dot{e}_3 &= 2x_3\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right)(e^{-t}+2) + (y_1y_3+\beta)(e^{-t}+2) - (x_3^2+y_3)e^{-t} \\
&\quad - y_1y_3 - \beta - u_3 - 2a_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) \\
\dot{e}_4 &= 2x_4\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)(e^{-t}+2) + (\gamma y_3 + \sigma y_4)(e^{-t}+2) \\
&\quad - (x_4^2+y_4)e^{-t} - \gamma y_3 - \sigma y_4 - u_4 - \omega_2(x_3(t-T) - x_1(t-T)) \\
&\quad + 2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)
\end{aligned} \tag{22}$$

where $e_1 = (x_1^2+y_1)(e^{-t}+2) - y_1 + x_1(t-T)$, $e_2 = (x_2^2+y_2)(e^{-t}+2) - y_2 + x_2(t-T)$, $e_3 = (x_3^2+y_3)(e^{-t}+2) - y_3 + x_3(t-T)$, $e_4 = (x_4^2+y_4)(e^{-t}+2) - y_4 + x_4(t-T)$.

Choose a positive definite Lyapunov function:

$$V(e_1,e_2,e_3,e_4) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{23}$$

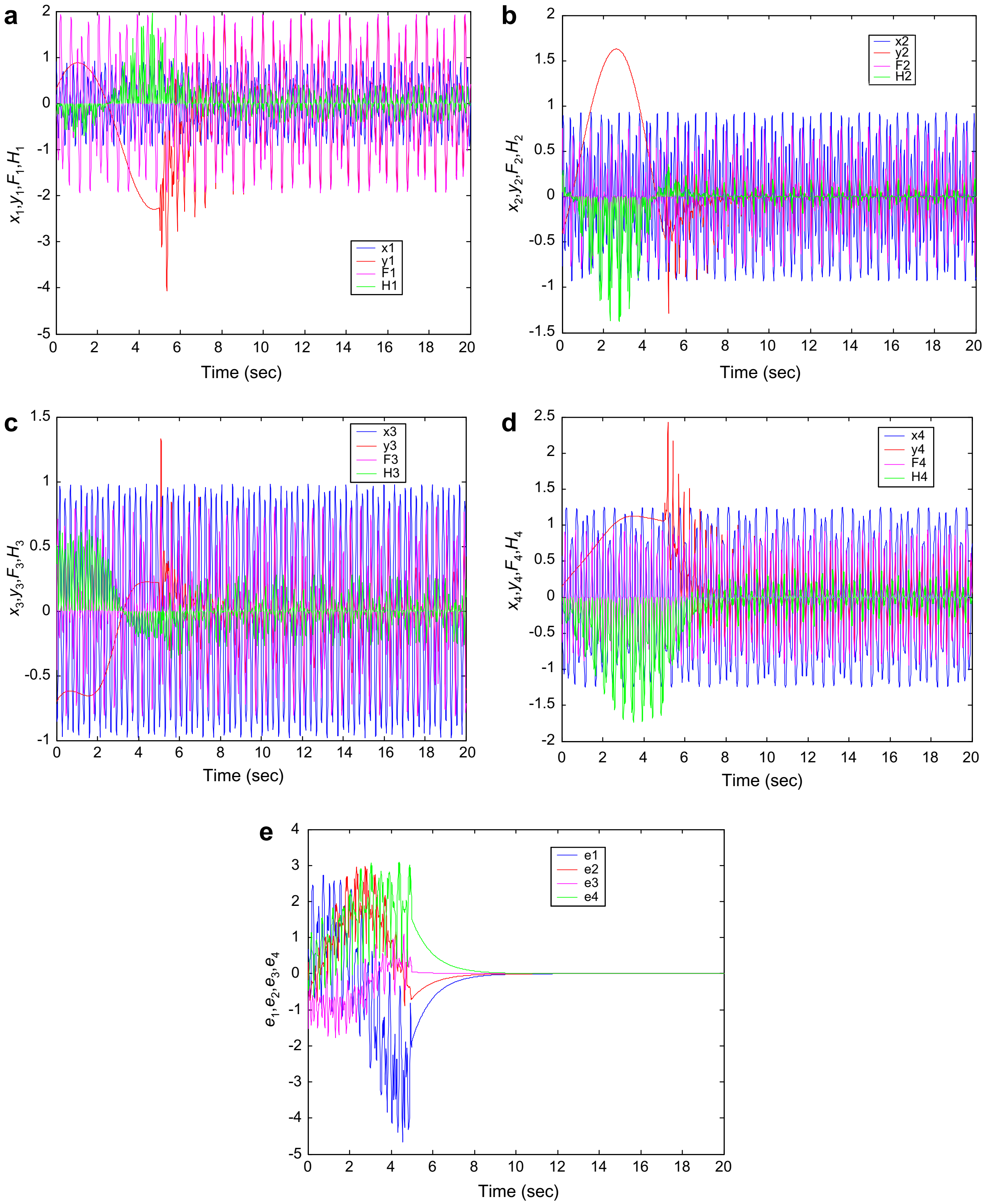

Fig. 1. Time histories of states, state errors, F_1, F_2, F_3, F_4, H_1, H_2, H_3 and H_4 for Case I.

Its time derivative along any solution of Eq. (22) is

$$\begin{aligned}
\dot{V} &= e_1\Big\{2x_1\Big(-2a_1\sqrt{1-x_1^2}\sin x_2\Big)(e^{-t}+2)+(-y_2-y_3)(e^{-t}+2)-(x_1^2+y_1)e^{-t}\\
&\quad +y_2+y_3-2a_1\sqrt{1-x_1^2(t-T)}\sin x_2(t-T)-u_1\Big\}\\
&\quad +e_2\Big\{2x_2\Big(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\Big)(e^{-t}+2)+(y_1-\alpha y_2+y_4)(e^{-t}+2)-(x_2^2+y_2)e^{-t}\\
&\quad -y_1+\alpha y_2-y_4-\omega_1(x_1(t-T)-x_3(t-T))+2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T)-u_2\Big\}\\
&\quad +e_3\Big\{2x_3\Big(-2a_2\sqrt{1-x_3^2}\sin x_4\Big)(e^{-t}+2)+(y_1y_3+\beta)(e^{-t}+2)-(x_3^2+y_3)e^{-t}\\
&\quad -y_1y_3-\beta-2a_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T)-u_3\Big\}\\
&\quad +e_4\Big\{2x_4\Big(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\Big)(e^{-t}+2)+(\gamma y_3+\sigma y_4)(e^{-t}+2)-(x_4^2+y_4)e^{-t}\\
&\quad -\gamma y_3-\sigma y_4-\omega_2(x_3(t-T)-x_1(t-T))+2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)-u_4\Big\}
\end{aligned} \tag{24}$$

Choose

$$\begin{aligned}
u_1 &= 2x_1\Big(-2a_1\sqrt{1-x_1^2}\sin x_2\Big)(e^{-t}+2)+(-y_2-y_3)(e^{-t}+2)-(x_1^2+y_1)e^{-t}+y_2+y_3\\
&\quad -2a_1\sqrt{1-x_1^2(t-T)}\sin x_2(t-T)+(x_1^2+y_1)(e^{-t}+2)-y_1+x_1(t-T)\\
u_2 &= 2x_2\Big(-\omega_1(x_1-x_3)+2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\Big)(e^{-t}+2)+(y_1-\alpha y_2+y_4)(e^{-t}+2)-(x_2^2+y_2)e^{-t}\\
&\quad -y_1-y_4-\omega_1(x_1(t-T)-x_3(t-T))+2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T)\\
&\quad +\alpha((x_2^2+y_2)(e^{-t}+2)+x_2(t-T))\\
u_3 &= 2x_3\Big(-2a_2\sqrt{1-x_3^2}\sin x_4\Big)(e^{-t}+2)+(y_1y_3+\beta)(e^{-t}+2)-(x_3^2+y_3)e^{-t}\\
&\quad -y_1y_3-\beta-2a_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T)+(x_3^2+y_3)(e^{-t}+2)-y_3+x_3(t-T)\\
u_4 &= 2x_4\Big(-\omega_2(x_3-x_1)+2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\Big)(e^{-t}+2)+(\gamma y_3+\sigma y_4)(e^{-t}+2)-(x_4^2+y_4)e^{-t}\\
&\quad -\gamma y_3-\omega_2(x_3(t-T)-x_1(t-T))+2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)\\
&\quad +\sigma((x_4^2+y_4)(e^{-t}+2)-2y_4+x_4(t-T))
\end{aligned}$$

Eq. (24) becomes

$$\dot{V} = -(e_1^2+\alpha e_2^2+e_3^2+\sigma e_4^2) < 0 \tag{25}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. Time delay symplectic synchronization of the Quantum-CNN system and the Rössler system is achieved. The numerical results are shown in Fig. 2. After 5 s, the motion trajectories enter a chaotic attractor.

Case III: A cubic time delay symplectic synchronization

We take $F_1(t)=x_4(t)x_1(t-T)$, $F_2(t)=x_1(t)x_2(t-T)$, $F_3(t)=x_2(t)x_3(t-T)$ and $F_4(t)=x_3(t)x_4(t-T)$, where $T=1$ sec is a positive constant time delay. They are chaotic functions of time. $H_i(x,y,t)=x_i^3-(y_i^3\sin\varpi_i t-1)\sin\varpi_i t$ $(i=1,2,3,4)$ are given. By Eq. (5) we have

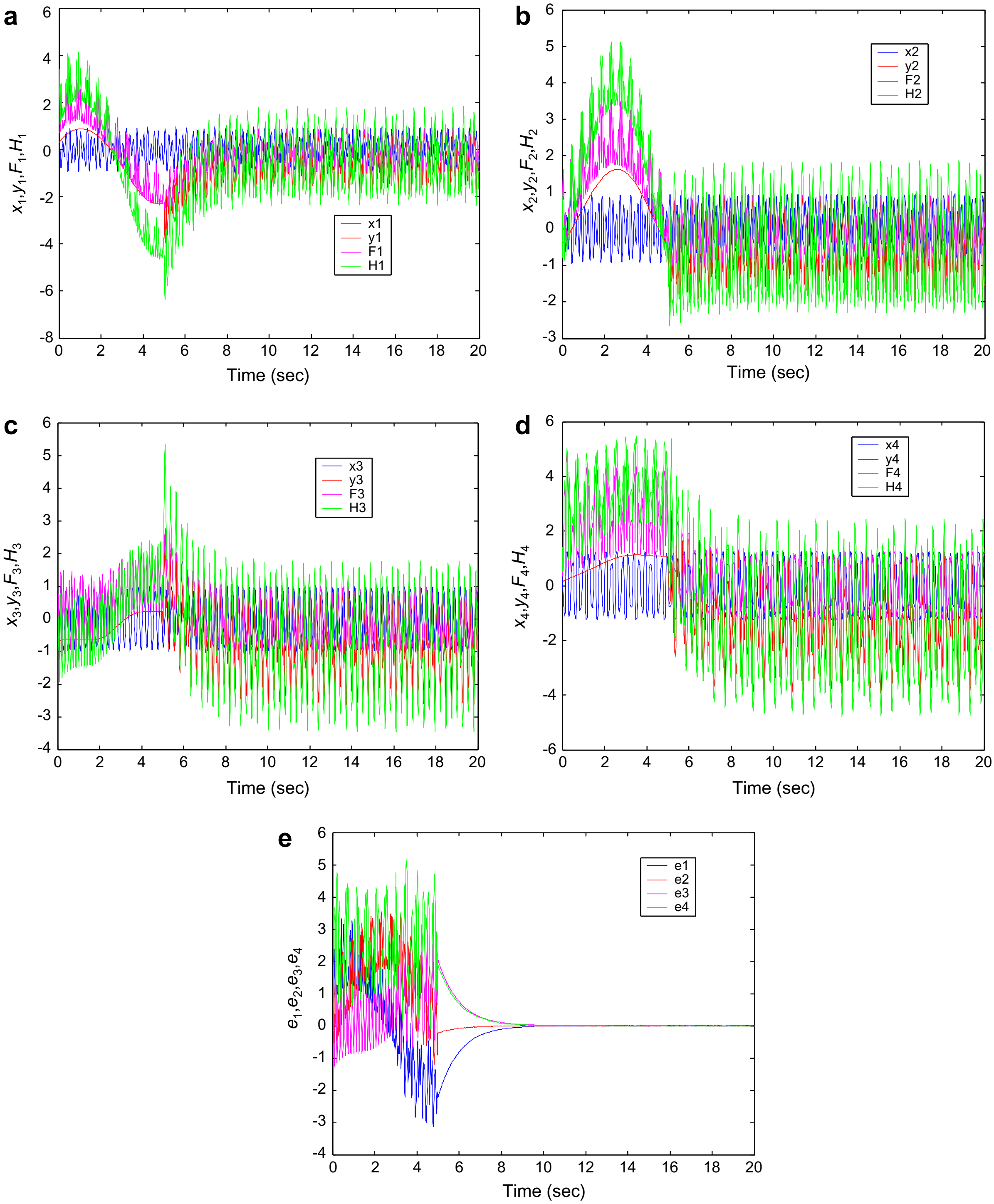

Fig. 2. Time histories of states, state errors, F_1, F_2, F_3, F_4, H_1, H_2, H_3 and H_4 for Case II.

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i^3 - (y_i^3 \sin \varpi_i t - 1)\sin \varpi_i t - y_i + x_j x_i(t - T)) = 0, \quad i = 1,2,3,4, \quad j = \begin{cases} 4, & i = 1 \\ i-1, & i \neq 1 \end{cases} \tag{26}$$

From Eq. (7) we have

$$\dot{e}_i = (3\dot{x}_i x_i^2 - (3\dot{y}_i y_i^2 \sin\varpi_i t + y_i^3 \varpi_i \cos\varpi_i t)\sin\varpi_i t - (y_i^3 \sin\varpi_i t - 1)\varpi_i \cos\varpi_i t - \dot{y}_i + \dot{x}_j x_i(t-T) + x_j \dot{x}_i(t-T),$$
$$i = 1,2,3,4, \quad j = \begin{cases} 4, & i = 1 \\ i-1, & i \neq 1 \end{cases} \tag{27}$$

Eq. (8) is expressed as

$$\begin{aligned}
\dot{e}_1 &= 3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right) - 3y_1^2(-y_2-y_3)\sin^2\varpi_1 t - y_1^3\varpi_1\sin 2\varpi_1 t \\
&\quad + \varpi_1\cos\varpi_1 t + y_2 + y_3 - u_1 + \left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)x_1(t-T) \\
&\quad - 2a_1x_4\sqrt{1-x_1^2(t-T)}\sin x_2(t-T) \\
\dot{e}_2 &= 3x_2^2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right) - 3y_2^2(y_1-\alpha y_2+y_4)\sin^2\varpi_2 t \\
&\quad - y_2^3\varpi_2\sin 2\varpi_2 t + \varpi_2\cos\varpi_2 t - y_1 + \alpha y_2 - y_4 - u_2 - 2a_1x_2(t-T)\sqrt{1-x_1^2}\sin x_2 \\
&\quad + x_1(-\omega_1(x_1(t-T)-x_3(t-T))) + 2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T) \\
\dot{e}_3 &= 3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right) - 3y_3^2(y_1y_3+\beta)\sin^2\varpi_3 t + y_3^3\varpi_3\sin 2\varpi_3 t \\
&\quad + \varpi_3\cos\varpi_3 t - y_1y_3 - \beta - u_3 + \left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)x_3(t-T) \\
&\quad - 2a_2x_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) \\
\dot{e}_4 &= 3x_4^2\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) - 3y_4^2(\gamma y_3+\sigma y_4)\sin^2\varpi_4 t \\
&\quad + y_4^3\varpi_4\sin 2\varpi_4 t + \varpi_4\cos\varpi_4 t - \gamma y_3 - \sigma y_4 - u_4 - 2a_2x_4(t-T)\sqrt{1-x_3^2}\sin x_4 \\
&\quad + x_3\left(-\omega_2(x_3(t-T)-x_1(t-T)) + 2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)\right)
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
e_1 &= x_1^3 - (y_1^3\sin\varpi_1 t - 1)\sin\varpi_1 t - y_1 + x_4(t)x_1(t-T) \\
e_2 &= x_2^3 - (y_2^3\sin\varpi_2 t - 1)\sin\varpi_2 t - y_2 + x_1(t)x_2(t-T) \\
e_3 &= x_3^3 - (y_3^3\sin\varpi_3 t - 1)\sin\varpi_3 t - y_3 + x_2(t)x_3(t-T) \\
e_4 &= x_4^3 - (y_4^3\sin\varpi_4 t - 1)\sin\varpi_4 t - y_4 + x_3(t)x_4(t-T)
\end{aligned}$$

Choose a positive definite Lyapunov function:

$$V(e_1,e_2,e_3,e_4) = \frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2) \tag{29}$$

Its time derivative along any solution of Eq. (28) is

$$\begin{aligned}
\dot{V} &= e_1\Bigg\{3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right) - 3y_1^2(-y_2-y_3)\sin^2\varpi_1 t - y_1^3\varpi_1\sin 2\varpi_1 t + \varpi_1\cos\varpi_1 t + y_2 + y_3 \\
&\quad + \left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)x_1(t-T) - 2a_1x_4\sqrt{1-x_1^2(t-T)}\sin x_2(t-T) - u_1\Bigg\} \\
&\quad + e_2\Bigg\{3x_2^2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right) - 3y_2^2(y_1-\alpha y_2+y_4)\sin^2\varpi_2 t - y_2^3\varpi_2\sin 2\varpi_2 t
\end{aligned}$$

$$+\varpi_2\cos\varpi_2 t - y_1 + \alpha y_2 - y_4 - 2a_1x_2(t-T)\sqrt{1-x_1^2}\sin x_2 + x_1(-\omega_1(x_1(t-T)-x_3(t-T))$$
$$\left. +2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T) - u_2\right\}$$
$$+e_3\left\{3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right) - 3y_3^2(y_1y_3+\beta)\sin^2\varpi_3 t + y_3^3\varpi_3\sin 2\varpi_3 t + \varpi_3\cos\varpi_3 t - y_1y_3\right.$$
$$\left. -\beta + \left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)x_3(t-T) - 2a_2x_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) - u_3\right\}$$
$$+e_4\left\{3x_4^2\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) - 3y_4^2(\gamma y_3+\sigma y_4)\sin^2\varpi_4 t + y_4^3\varpi_4\sin 2\varpi_4 t\right.$$
$$+\varpi_4\cos\varpi_4 t - \gamma y_3 - \sigma y_4 - 2a_2x_4(t-T)\sqrt{1-x_3^2}\sin x_4 + x_3(-\omega_2(x_3(t-T)-x_1(t-T))$$
$$\left. +2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)) - u_4\right\}$$

Choose

$$u_1 = 3x_1^2\left(-2a_1\sqrt{1-x_1^2}\sin x_2\right) - 3y_1^2(-y_2-y_3)\sin^2\varpi_1 t - y_1^3\varpi_1\sin 2\varpi_1 t$$
$$+\varpi_1\cos\varpi_1 t + y_2 + y_3 + \left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right)x_1(t-T)$$
$$-2a_1x_4\sqrt{1-x_1^2(t-T)}\sin x_2(t-T) + x_1^3 - (y_1^3\sin\varpi_1 t - 1)\sin\varpi_1 t - y_1 + x_4(t)x_1(t-T)$$
$$u_2 = 3x_2^2\left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right) - 3y_2^2(y_1 - \alpha y_2 + y_4)\sin^2\varpi_2 t$$
$$-y_2^3\varpi_2\sin 2\varpi_2 t + \varpi_2\cos\varpi_2 t - y_1 - y_4 - 2a_1x_2(t-T)\sqrt{1-x_1^2}\sin x_2$$
$$+x_1(-\omega_1(x_1(t-T)-x_3(t-T))) + 2a_1\frac{x_1(t-T)}{\sqrt{1-x_1^2(t-T)}}\cos x_2(t-T)$$
$$+\alpha(x_2^3 - (y_2^3\sin\varpi_2 t - 1)\sin\varpi_2 t + x_1(t)x_2(t-T))$$
$$u_3 = 3x_3^2\left(-2a_2\sqrt{1-x_3^2}\sin x_4\right) - 3y_3^2(y_1y_3+\beta)\sin^2\varpi_3 t$$
$$+y_3^3\varpi_3\sin 2\varpi_3 t + \varpi_3\cos\varpi_3 t - y_1y_3 - \beta + \left(-\omega_1(x_1-x_3) + 2a_1\frac{x_1}{\sqrt{1-x_1^2}}\cos x_2\right)x_3(t-T)$$
$$-2a_2x_2\sqrt{1-x_3^2(t-T)}\sin x_4(t-T) + x_3^3 - (y_3^3\sin\varpi_3 t - 1)\sin\varpi_3 t - y_3 + x_2(t)x_3(t-T)$$
$$u_4 = 3x_4^2\left(-\omega_2(x_3-x_1) + 2a_2\frac{x_3}{\sqrt{1-x_3^2}}\cos x_4\right) - 3y_4^2(\gamma y_3 - \sigma y_4)\sin^2\varpi_4 t$$
$$+y_4^3\varpi_4\sin 2\varpi_4 t + \varpi_4\cos\varpi_4 t - \gamma y_3 - 2a_2x_4(t-T)\sqrt{1-x_3^2}\sin x_4$$
$$+x_3\left(-\omega_2(x_3(t-T)-x_1(t-T)) + 2a_2\frac{x_3(t-T)}{\sqrt{1-x_3^2(t-T)}}\cos x_4(t-T)\right)$$
$$+\sigma(x_4^3 - (y_4^3\sin\varpi_4 t - 1)\sin\varpi_4 t - 2y_4 + x_3(t)x_4(t-T))$$

Eq. (30) becomes

$$\dot{V} = -(e_1^2 + \alpha e_2^2 + e_3^2 + \sigma e_4^2) < 0 \tag{31}$$

which is negative definite. The Lyapunov asymptotical stability theorem is satisfied. Cubic time delay symplectic synchronization of the Quantum-CNN system and the Rössler system is achieved. The numerical results are shown in Fig. 3. After 5 s, the motion trajectories enter a chaotic attractor.

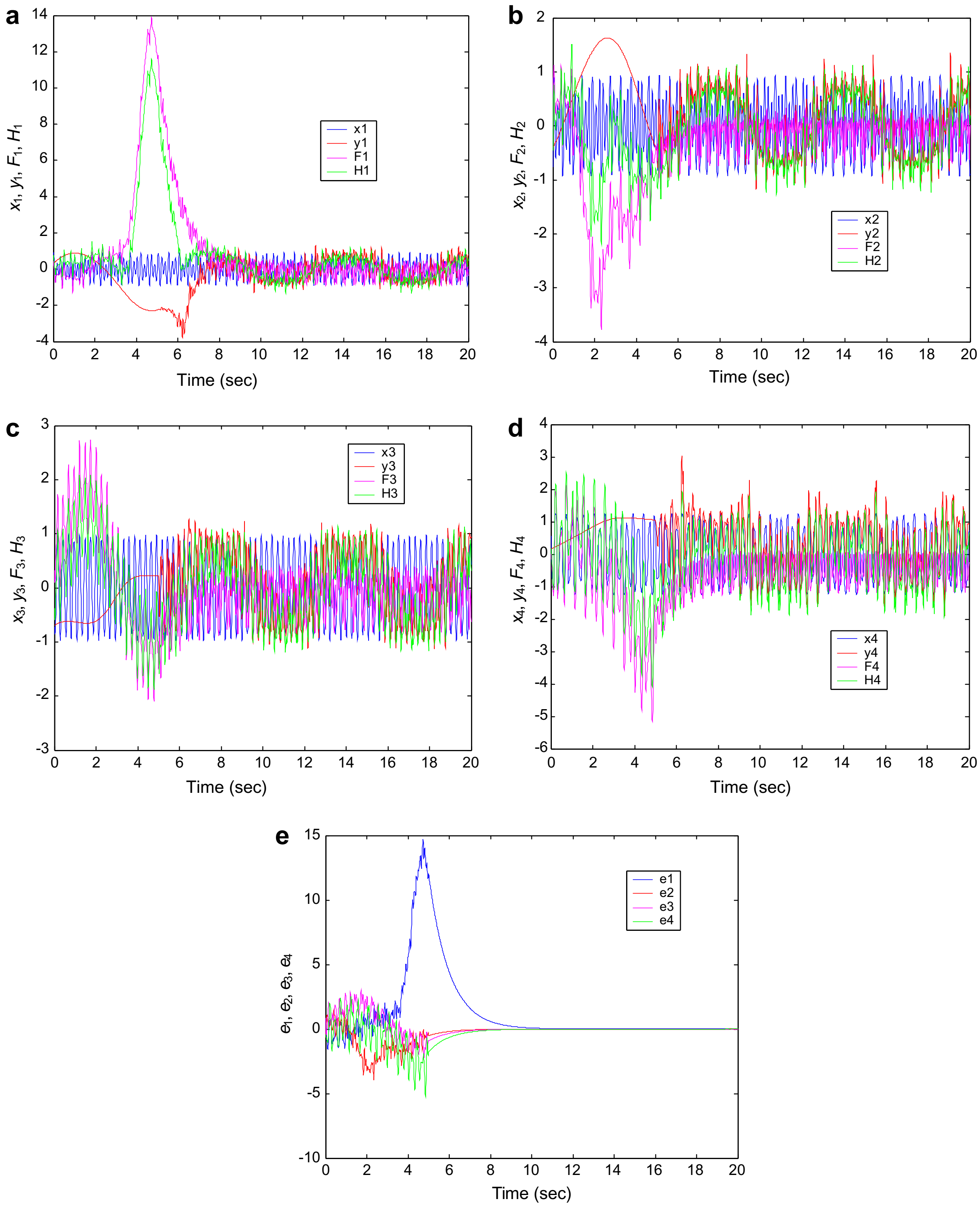

Fig. 3. Time histories of states, state errors, F_1, F_2, F_3, F_4, H_1, H_2, H_3 and H_4 for Case III.

4. Conclusions

A new symplectic synchronization of a Quantum-CNN chaotic oscillator and a Rössler system is obtained by the Lyapunov asymptotical stability theorem. Two different chaotic dynamical systems, the Quantum-CNN system and the Rössler system, are in symplectic synchronization for three cases: the cubic symplectic synchronization, the time delay symplectic synchronization and the cubic time delay symplectic synchronization. Symplectic synchronization of chaotic systems can be used to increase the security of secret communication.

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