國 立 交 通 大 學

應用數學系碩士班

碩 士 論 文

複數網格型耦合系統在 Julia 集上的同步化

The Synchronization on the Julia Set for Complex Valued Coupled Map Lattices

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中 華 民 國 一 零 二 年 六 月

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應 用 數 學 系
碩 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of

Master

in

Applied Mathematics

June 2013

Hsinchu, Taiwan, Republic of China

中 華 民 國 一百零二 年 六 月

誌　謝

猶記兩年前的新生座談會，才剛認識校園沒多久，如今即將畢業，兩年來的點滴歷歷在目，這些曾經，竟已在不覺中成為我人生的一部分，非常感謝生命道路上遇到的大家，沒有你們扶持與關愛，就沒有現在的我。

碩班兩年，首先要感謝我的指導教授，莊重老師，謝謝莊老師不畏辛勞的指導我，包容我的不足，更要感謝莊老師總是耐心與細心地教導資質平庸的我，在學術研究外，莊老師的做人處事以及把學生視如己親的照顧，著實是我學習的典範，希望我也能把從莊老師身上學到的寶貴經驗，實際應用在我的人生中，成為如莊老師般的模範老師！

再者，要感謝所有曾教導我的老師們，碩班的課業不同於大學，真的非常謝謝老師們用心的教導；也很感謝曾給予我幫助的學長們，謝謝您們不嫌棄我是個愛發問問題的學妹，總是熱心的給予協助；還要感謝我的好同學們，謝謝你們，讓碩班的生活多采多姿，我們相聚的日子，相信會是我這輩子最美好的回憶；最後，我要感謝我的家人們，謝謝您們在我身邊給我支持鼓勵與照顧，家，是一輩子的避風港，很開心能當爸媽的女兒，能當兩位弟弟的姊姊，有您們的陪伴，我很幸福，謝謝您們！謹以此篇論文獻上誠摯的感謝！謝謝大家！

複數網格型耦合系統在 Julia 集上的同步化

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摘　　　要

本篇論文主要目的是為了解決 Julia 集上複數耦合網格系統（Complex Valued Coupled Map Lattices（CCMLs））的同步現象。首先，我們介紹一個研究全域同步與局部同步的統一形式。其次，我們解決一個 inf min max 的問題，這個問題，是在給定的一群耦合矩陣中，找到一個耦合矩陣和其相對應的耦合係數使得系統的同步收斂速度最快。最後，我們給出對應系統在 Julia 集上的全域同步與局部同步之結果。

The Synchronization on the Julia Set for Complex Valued Coupled Map Lattices

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June 2013

Abstract

The purpose of this thesis is to address synchronous chaos on a Julia set in complex-valued coupled map lattices (CCMLs). Our main results contain the following. First, a unified formulation for the study of global and local synchronization CCMLs is presented. Second, we solve an inf min max problem for which its solution gives the fastest synchronized rate among a class of coupling matrices. Third, various results for global and local synchronization on the Julia set are presented.

Keywords: Complex-Valued Coupled Map Lattices, Julia Set, Fastest Synchronized Network.

Contents

1 Introduction

In recent years, models of complex-valued neural networks have widened the scope of application in optoelectronics, image, remote sensing, quantum neural devices and systems, spatiotemporal analysis of physiological neural systems, and artificial neural information processing [1, 2]. One typical such model is a fully connected complex-valued neural network as the mathematical extension of an ordinary real-valued Hopfield network. Their basic mathematical theory and applications have been extensively studied. Coupled map lattices (CMLs) are comparable to neural networks in that the value of each oscillator in both models depends on the neighbors. The major differences lie on the facts that the dynamics on the (uncoupled) oscillator for CMLs is usually assumed to be chaotic while that of the neural networks is simple. One of the major mathematical questions for neural networks, among others, is the global attractivity of the model. On the other hand, CMLs are predominantly used to qualitatively study the chaotic dynamics of spatially extended systems. Another interesting form of dynamical behavior occurs in CMLs when all of the individual systems or oscillators acquire identical chaotic behavior. Such synchronized behavior of a network can be constructed as models in many systems of interest in physics, biology, and engineering. Some progress in the theory of synchronization has been made in CMLs. Indeed, for general (real) CMLs, the study of local synchronization can be found in [3, 4, 5, 6, 7]. Not much progress has been made for the investigation of its global synchronization. There are, however, globally synchronous results for some special cases (see e.g., [8, 9]).

The purpose of this thesis is to investigate the theory of synchronization of CCMLs. There are some notable differences between the (real) CMLs and CCMLs as far as the synchrony is concerned. The same function when considered in the complex plane generates much more complex and interesting dynamics. For instance, the real valued function $f(x) = x^2 + c$ has simple dynamics whenever $|c|$ is small. However, its counterpart, defined in the complex plane, could generate chaotic dynamics on its Julia set. Moreover, if the coupling coefficient between two nodes is assigned to be a complex number, equipped with both the amplitude and phase, then finding the optimal coupling coefficient yielding the fastest synchronization speed is a taunting task. It should be remarked that finding the optimal coupling, leading to an min max problem, is a key step toward establishing both local and global synchronization theory of CMLs. In this thesis, we first present a unified formulation for the study of both local and global synchronization of CCMLs. Second, the problem of constructing a network for which the fastest synchronized speed can be made by choosing a suitable coupling coefficient is investigated. The question then becomes an inf min max problem. In particular, we prove that given a class of coupling matrices of size 4, the equality between the nonzero and non-diagonal elements gives the fastest synchronized speed. Third, synchronized theory on its Julia set is presented.

We conclude this introductory section by mentioning the organization of the thesis. The unified formulation for investigating both local and global synchronization theory of CCMLs is presented in Section 2. The results concerning the min max problem is placed in Section 3. The needed results for Julia set are contained in Section 4. The main results, the inf min max problem and the synchronization on its Julia set, are recorded in Section 5. Some concluding remarks about future research are addressed in Section 6.

2 Unified Framework

Consider a network of (CCMLs) consisting of m oscillators. The equations of the motion then read as follow.

$$z_i(n+1) = f(z_i(n)) + d\left(\sum_{k=1}^{m} g_{ik} h(z_k(n))\right), \; i = 1, \ldots, m. \tag{2.1}$$

Here $f : \mathbb{C} \to \mathbb{C}$, represents the individual complex-valued function, and $h : \mathbb{C} \to \mathbb{C}$ is an arbitrary nonlinear function to give how each oscillator's variables are used in the coupling. The quantities $g_{ik} \in \mathbb{C}$ are the coupling coefficients between the oscillators i and k. To consider the notion of synchronization, we assume that

$$\sum_{k=1}^{m} g_{ik} = 0 \quad \text{for each } i\ , \tag{2.2a}$$

and

$$0 \text{ is the simple eigenvalue of the coupling matrix } \mathbf{G} = (g_{ik}). \tag{2.2b}$$

The quantity d represents the coupling strength, which is also allowed to be a complex-valued number. In vector-matrix form with $h = f$, (2.1) becomes

$$\mathbf{z}(n+1) = \mathbf{F}(\mathbf{z}(n)) + d\mathbf{G}\mathbf{F}(\mathbf{z}(n)). \tag{2.3}$$

Here, $\mathbf{z}(n) = (z_1(n), \ldots, z_m(n))^T$ and $\mathbf{F}(\mathbf{z}(n)) = (f(z_1(n)), \ldots, f(z_m(n)))^T$. In the following, we shall derive a unified formulation for the study of both local and global synchronization. To this end, we first make a coordinate change to decompose the synchronous manifold. Let $\mathbf{A}$ be an $m \times m$ matrix of the form

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} =: \begin{pmatrix} \mathbf{C} \\ \mathbf{e}^T \end{pmatrix},$$

where $\mathbf{e}^T = (1, 1, \ldots, 1)$.
It is then easy to see that $\mathbf{CC^T}$ is invertible and that

$$\mathbf{A}^{-1} = \left(\mathbf{C}^T(\mathbf{CC^T})^{-1} \,\middle|\, \frac{\mathbf{e}}{m} \right).$$

Multiplying $\mathbf{A}$ to both sides of Equation (2.3), we get

$$\begin{aligned} \mathbf{Az}(n+1) &= \mathbf{AF}(\mathbf{z}(n)) + d\mathbf{AGA}^{-1}\mathbf{AF}(\mathbf{z}(n)) \\ &= \mathbf{AF}(\mathbf{z}(n)) + d\begin{pmatrix} \mathbf{CGC}^T(\mathbf{CC}^T)^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}\mathbf{AF}(\mathbf{z}(n)). \end{aligned} \tag{2.4a}$$

Let

$$\mathbf{G}^+ := \mathbf{CGC}^T(\mathbf{CC}^T)^{-1}, \tag{2.4b}$$

$$\begin{pmatrix} z_1(n) - z_2(n) \\ z_2(n) - z_3(n) \\ \vdots \\ z_{m-1}(n) - z_m(n) \\ \sum_{i=1}^{m} z_i(n) \end{pmatrix} := \begin{pmatrix} e_1(n+1) \\ \vdots \\ e_{m-1}(n+1) \\ e_s(n+1) \end{pmatrix}, \; \mathbf{e}(n+1) := \begin{pmatrix} e_1(n+1) \\ \vdots \\ e_{m-1}(n+1) \end{pmatrix}$$

and

$$\mathbf{AF}(\mathbf{z}(n)) = \begin{pmatrix} \mathbf{D}(n)\mathbf{e}(n) \\ * \end{pmatrix}.$$

Here $\mathbf{D}(n)$ is an $(m-1)\times(m-1)$ diagonal matrix of the form

$$\mathbf{D}(n) = diag\left(\frac{f(z_1(n)) - f(z_2(n))}{z_1(n) - z_2(n)}, \frac{f(z_2(n)) - f(z_3(n))}{z_2(n) - z_3(n)}, \ldots, \frac{f(z_{m-1}(n)) - f(z_m(n))}{z_{m-1}(n) - z_m(n)}\right).$$

Then we have that the dynamics of $\mathbf{e}(n)$ is satisfied by the following equation

$$\mathbf{e}(n+1) = (\mathbf{I} + d\mathbf{G}^{+})\mathbf{D}(n)\mathbf{e}(n). \tag{2.5}$$

The task of obtaining global synchronization of system (2.1) is now reduced to showing that the origin is globally asymptotically stable with respect to system (2.5). It should be remarked that for the study of local synchronization, $\mathbf{D}(n)$ reduces to the form $f'(z(n))\mathbf{I}$. Here $\{z(n)\}_{n=0}^{\infty}$ is the orbit defined by (2.1) on the synchronous manifold. Consequently, we say that (2.1) is locally synchronized provided that the origin of the linear system

$$\mathbf{e}(n+1) = f'(z(n))(\mathbf{I} + d\mathbf{G}^{+})\mathbf{e}(n) \tag{2.6}$$

is asymptotically stable. Note that (2.6) is equivalent to the well-known master stability equation [7, 10].

3 Min Max Problem

In view of (2.5) and (2.6), to study the synchronous dynamics of (2.1), we need find an optimal coupling coefficient d so that the spectral radius of $1 + d\mathbf{G}^{+}$ is the smallest. We are then led to consider the following min max problem. Assume that (2.2a) and (2.2b) are satisfied. We also assume from here on that

$$\text{real parts of the eigenvalues of } \mathbf{G} \text{ are non-positive.} \tag{3.1}$$

Let the spectrum $\sigma(\mathbf{G})$ of $\mathbf{G}$ be denoted by

$$\sigma(\mathbf{G}) = \{\lambda_1 = 0, \lambda_2, \lambda_3, \ldots, \lambda_m\}.$$

Here $\mathrm{Re}(\lambda_j) \leq 0,\ 2 \leq j \leq m$, and $0 < |\lambda_2| \leq |\lambda_3| \leq \cdots \leq |\lambda_m|$. It then follows from (2.4a) and (2.4b) that

$$\sigma(\mathbf{G}) = \sigma(\mathbf{A}\mathbf{G}\mathbf{A}^{-1}) = \sigma(\mathbf{G}^{+}) \cup \{0\}.$$

Consequently, $\sigma(\mathbf{G}^+) = \{\lambda_2, \lambda_3, \ldots, \lambda_m\}$. To minimize the spectral radius of $1 + d\mathbf{G}^+$, we consider the following min max problem:

$$
\begin{aligned}
\min_{d\in\mathbb{C}} \max_{2\le j\le m} |1 + d\lambda_j| &= \min_{-\pi\le\theta\le\pi} \min_{r\in\mathbb{R}^+} \max_{2\le j\le m} |1 + re^{i\theta}\lambda_j| \\
&= \min_{-\theta_{p_1}\le\theta\le\theta_{p_2}} \min_{r\in\mathbb{R}^+} \max_{2\le j\le m} |1 + re^{i\theta}\lambda_j| \\
&=: \min_{-\theta_{p_1}\le\theta\le\theta_{p_2}} \min_{r\in\mathbb{R}^+} \max_{2\le j\le m} \gamma_j(r,\theta,\theta_j),
\end{aligned}
\tag{3.2}
$$

where $d = re^{i\theta}$, $\lambda_j = r_j e^{i\theta_j}$, $r_j \in \mathbb{R}^+$, $\frac{\pi}{2} \le \theta_j \le \frac{3\pi}{2}$, $\theta_{p_1} = \min\left\{\theta_j - \frac{\pi}{2} : \frac{\pi}{2} \le \theta_j \le \frac{3\pi}{2}\right\}$ and $\theta_{p_2} = \min\left\{-\theta_j - \frac{\pi}{2} : -\frac{3\pi}{2} \le \theta_j \le -\frac{\pi}{2}\right\}$.

The second equality holds due to the fact that if $\alpha \in [-\pi, -\theta_{p_1}) \cup (\theta_{p_2}, \pi]$ and $\theta \in [-\theta_{p_1}, \theta_{p_2}]$, then

$$\gamma_j^2(r,\alpha,\theta_j) = 1 + 2rr_j\cos(\alpha+\theta_j) + r^2|\lambda_j|^2 \ge \gamma_j^2(r,\theta,\theta_j).$$

For each fixed $\theta \in [-\theta_{p_1}, \theta_{p_2}]$, we first need to solve the following min max problem

$$\min_{r\in\mathbb{R}^+} \max_{2\le j\le m} \gamma_j(r,\theta,\theta_j) = \min_{r\in\mathbb{R}^+} \max_{2\le j\le m} \gamma_j(r,0,\theta+\theta_j) =: \min_{r\in\mathbb{R}^+} \gamma(r,\theta). \tag{3.3}$$

The solution to (3.3) with $\theta = 0$ was given in [7]. By treating eigenvalues having the form $\lambda_j e^{i(\theta+\theta_j)}$, the case $\theta \ne 0$ becomes similarly to the case $\theta = 0$. Hence, for each θ, we may apply the efficient procedure proposed in [7] for solving (3.3). For ease of reference, we also recall a result from [7].

Theorem 3.1. (Theorem 2.4 of [7]) *Suppose the $m \times m$ coupling matrix $\mathbf{G}$ has non-positive real eigenvalues. Denote by $\{\lambda_i\}_{i=2}^{\overline{m}}$, $\overline{m} \le m$, where λ_i are eigenvalues of $\mathbf{G}$ and $0 < |\lambda_2| < |\lambda_3| < \cdots < |\lambda_{\overline{m}}|$. Then*

$$d = \frac{-2}{\lambda_2 + \lambda_{\overline{m}}} =: d_f \tag{3.4a}$$

solves min max problem (3.3). Moreover,

$$\min_{d\in\mathbb{R}} \max_{2\le j\le m} |1 + d\lambda_j| = \frac{\lambda_{\overline{m}} - \lambda_2}{\lambda_{\overline{m}} + \lambda_2} =: \delta. \tag{3.4b}$$

It should be noted that d_f gives the fastest convergence rate δ of the initial values toward the synchronous state. We next show that given a real-valued matrix $\mathbf{G}$, there exists a unique positive real d solves the min max (3.2).

Proposition 3.1. *Let* $\mathbf{G} \in \mathbb{R}^{m\times m}$*. Then*

$$\min_{d\in\mathbb{C}} \max_{2\le j\le m} |1 + d\lambda_j| = \min_{d\in\mathbb{R}} \max_{2\le j\le m} |1 + d\lambda_j| = \min_{d>0} \max_{2\le j\le m} |1 + d\lambda_j|. \tag{3.5}$$

Proof: For $\mathbf{G} \in \mathbb{R}^{m\times m}$, $\theta_{p_1} = \theta_{p_2}$. Let λ_j be an eigenvalue of $\mathbf{G}$. If λ_j is real and negative, then

$$\gamma_j^2(r, \theta, \theta_j = \pi) \ge \gamma_j^2(r, 0, \pi) \text{ for any } \theta \in [-\theta_{p_1}, \theta_{p_2}].$$

Suppose $\lambda_j = r_j e^{i\theta_j}$ is complex. Then $\bar{\lambda}_j$ is also an eigenvalue of $\mathbf{G}$. Without lose of generality, we may assume that

$$\bar{\lambda}_j = \lambda_{j+1} = r_{j+1} e^{i\theta_{j+1}}, \text{ where } r_{j+1} = r_j \text{ and } \theta_{j+1} = -\theta_j.$$

For any $\theta \in [-\theta_{p_1}, \theta_{p_2}]$, we have

$$\begin{aligned} \max\{\gamma_j^2(r, \theta, \theta_j), \gamma_{j+1}^2(r, \theta, \theta_{j+1})\} &\ge \max\{\gamma_j^2(r, 0, \theta_j), \gamma_{j+1}^2(r, 0, \theta_{j+1})\} \\ &= \gamma_j^2(r, 0, \theta_j) \\ &= \gamma_{j+1}^2(r, 0, \theta_{j+1}). \end{aligned}$$

Thus,

$$\max_{2\le j\le m} \gamma_j(r, \theta, \theta_j) \ge \max_{2\le j\le m} \gamma_j(r, 0, \theta_j).$$

We have just completed the proof of the proposition.

Remark 3.1. *For a complex-valued* $\mathbf{G}$*, finding the solution for* (3.2) *is more challenging. For each* θ*, the synchronization curve* S_θ*, composed a certain of pieces of* $|1 + d\lambda_j|$*, termed transverse Lyapunov exponent curves (LECs), can be found efficiently as described in* [7]*. This, in turn, gives the fastest convergence rate* $\delta = \delta(\theta)$ *occurring at* $d = r(\theta)e^{i\theta}$*. As one gradually various* θ *from* $-\theta_{p_1}$ *to* θ_{p_2}*, the pieces of LECs for the corresponding synchronization curve are most likely to change. To illustrate our point, let* $\mathbf{G}$ *be a circulant matrix* [11] *of the form*

$$circ(-10 + 4i, 9 - 5i, 0, 0, 1 + i).$$

The eigenvalues of $\mathbf{G}$ *is* $\lambda_2 \approx -1.2035 + 10.3724i$, $\lambda_3 \approx -12.6162 - 4.8445i$, $\lambda_4 \approx -14.5635 + 11.9383i$, *and* $\lambda_5 = -21.6169 + 2.5338i$. *Using the procedure provided in* [7], *we have that for* $-0.1155 \leq \theta \leq 0.0570$, *the synchronization curve is composed of the LECs corresponding to* $|1+d\lambda_2|$, $|1+d\lambda_4|$ *and* $|1+d\lambda_5|$. *As* θ *keeps varying, we list the exact pieces of the LECs for its associated synchronization curve in Table 3.1. The graph of* $\delta(\theta)$ *is shown in Fig3.1. Pictorially, we have that*

$$\theta \approx 0.5440 \text{ and } r \approx 0.051344,$$

solves min max (3.2). *Moreover,*

$$\min_{d \in \mathbb{C}} \max_{2 \leq j \leq 5} |1 + d\lambda_j| \approx 0.793995.$$

$-\theta_{p_1} \leq \theta \leq \theta_{p_2}$	LECs
$-\theta_{p_1} = -0.1155 \leq \theta \leq 0.0570$	$(2, 4, 5)$
$0.0570 \leq \theta \leq 0.4920$	$(2, 5)$
$0.4920 \leq \theta \leq 0.6440$	$(2, 3, 5)$
$0.6440 \leq \theta \leq 1.2041 = \theta_{p_2}$	$(3, 5)$

Table 3.1: The synchronization curve S_θ is decided by three LECs, $|1+d\lambda_2|$, $|1+d\lambda_4|$ and $|1+d\lambda_5|$, whenever $-0.1155 \leq \theta \leq 0.0570$. The numbers in other columns of the table is similarly explained.

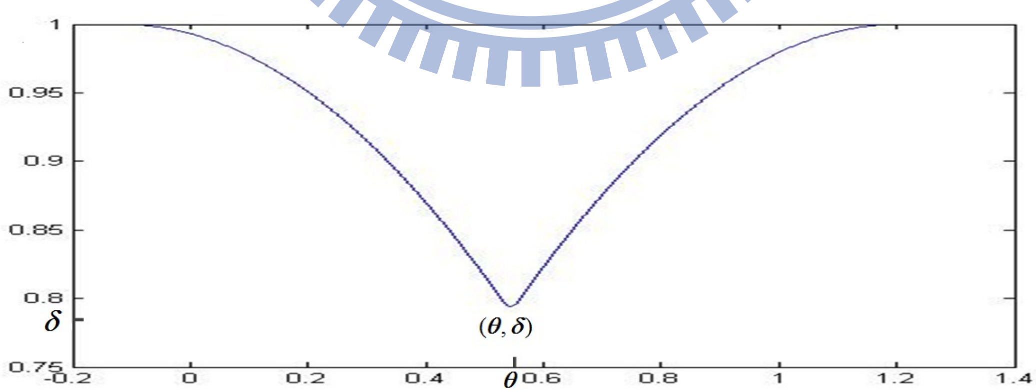

Figure 3.1: The graph of $\delta(\theta)$, $-\theta_{p_1} \leq \theta \leq \theta_{p_2}$. Its minimum occurs at $(r, \delta) \approx (0.051344, 0.793995)$

.

4 Julia Set

Since synchronization of CCMLs is considered on its Julia set, we shall recall some well-known definitions and results (see e.g., [12]). Some needed estimates for the size of Julia sets are also derived in this section. We shall concentrate on considering polynomial maps of the form,

$$f_c(z) = z^2 + c, \text{where } z, c \in \mathbb{C} \text{ and } g_c(z) = z^3 + cz. \tag{4.1}$$

Definition 4.1. *Let $P_c : \mathbb{C} \to \mathbb{C}$ be a polynomial with a parameter $c \in \mathbb{C}$. Then Julia set of P_c, denote by $J(P_c)$, is the closure of the set of repelling periodic points of P_c.*

Proposition 4.1. (See e.g., [12]) *Suppose $|c| < \frac{1}{4}$* (resp., $|c| < 1$). *Then $J(f_c)$* (resp., $J(g_c)$) *is a simple closed curve. If, in addition, c is complex, then $J(f_c)$* (resp., $J(g_c)$) *contains no smooth arcs.*

The proof for $J(g_c)$ being a simple curve with $|c| < 1$ is similar to that of $J(g_c)$ with $|c| < \frac{1}{4}$, and is, thus, omitted.

Theorem 4.1. (See e.g., [12]) *$J(P_c)$ is a perfect set and is completely invariant. Moreover, P_c is chaotic on $J(P_c)$ in the sense of Devaney.*

Remark 4.1.

(i) *Clearly, the Julia set $P(f_0)$ is the unit circle, and so $f_0 : P(f_0) \to P(f_0)$ reduces to the chaotic map of the form $\theta \to 2\theta$.*

(ii) *Let $f_{-2}(z) = z^2 - 2$. Then $J(f_{-2})$ is the closed interval* $[-2, 2]$ (see e.g., Example 5.11 of [12]). *Moreover, the map $f_{-2}(z) : J(f_{-2}) \to J(f_{-2})$ is topological conjugate to the map $x \to 4x(1-x)$.*

In the following, the size of the Julia sets of f_c and g_c are to be estimated.

Proposition 4.2.

(i) *Let $r = \frac{1+\sqrt{2}}{2}$. Then $J(f_c) \subset B_r(0)$, where $|c| < \frac{1}{4}$.*

(ii) *Let* $r = \sqrt{2}$. *Then* $J(g_c) \subset B_r(0)$, *where* $|c| < 1$.

Proof: To prove (i), we have that

$$|f_c(z)| \geq |z|^2 - |c| \geq |z|\left(|z| - \frac{1}{4|z|}\right).$$

Now, if $z \notin B_r(0)$, then $|z| > \frac{1+\sqrt{2}}{2}$. Consequently, $|z| - \frac{1}{4|z|} > 1$. Therefore, if $z \notin B_r(0)$, then $|f_c(z)| > |z|$. Consequently, $|f_c^n(z)| \to \infty$ as $n \to \infty$. It then follows from Theorem 4.1 and Proposition 4.1 that $J(f_c) \subset B_r(0)$ as claimed.

To see (ii), we have that for $z \notin B_r(0)$,

$$|g_c(z)| > |z|^3 - |z| = |z|\left(|z|^2 - 1\right) > |z|.$$

Similarly, we conclude that $J(g_c) \subset B_r(0)$.

5 Main Results

In this section, the questions of fastest synchronized network and the theory of synchronization on its Julia set are to be addressed. To this end, we begin with considering how the network of the system should be constructed to have its system synchronized fastest among a class of coupling matrices. Let $\mathscr{A}$ be a class of coupling matrices satisfying (2.2a), (2.2b) and (3.1). The above mentioned problem is then amount to solving

$$\inf_{\mathbf{G}\in\mathscr{A}} \min_{d\in\mathbb{C}} \max_{2\leq j\leq m} |1 + d\lambda_j| := \inf_{\mathbf{G}\in\mathscr{A}} \delta(\mathbf{G}). \tag{5.1}$$

Here, δ, which depends on $\mathbf{G}$, is defined in Theorem 3.1 such inf, if exists, is called the fastest synchronized rate among the class $\mathscr{A}$. To simplified problem, we shall consider the set of circulant matrices of the form

$$circ(c_1, c_m, c_{m-1}, \ldots, c_3, c_2). \tag{5.2a}$$

Note that the spectrum (use e.g., [11]) of the circulant matrix of the form in (5.2a) is

$$\left\{c_1 + c_m\omega_j + \cdots + c_2\omega_j^{m-1} : \omega_j = exp\left(\frac{2(j-1)\pi i}{m}\right),\ j = 1, \ldots, m\right\}.$$

To get some theoretical results, we further simplify our case.
Specifically, let

$$\mathscr{A} = \{\mathbf{G} = circ(-a_1 - a_2 - a_3, a_1, a_2, a_3) : \text{exactly one of } a_1, a_2 \text{ and } a_3 \text{ is zero, the others are positive real}\}. \tag{5.2b}$$

To emphasize the dependence of $\mathbf{G}$, d_f and δ on a_i, we shall write, for instance, $\mathbf{G} = \mathbf{G}(a_1, a_2)$, $d_f = d_f(a_1, a_2)$ and $\delta = \delta(a_1, a_2)$. The dependency on a_i will be dropped, should no confusion arise.

Theorem 5.1. *Let $\mathscr{A}$ be given as in (5.2b). Then*

$$\inf_{\mathbf{G}\in\mathscr{A}} \delta(\mathbf{G}) = \min_{\mathbf{G}\in\mathscr{A}} \delta(\mathbf{G}) = \frac{1}{3}. \tag{5.3}$$

Moreover, the min min max is achieved whenever two nonzero a_i are equal.

Proof: Let $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$, where $\mathscr{A}_i \subset \mathscr{A}$ and $\mathbf{G} \in \mathscr{A}_i$ if and only if $a_i = 0$, $i = 1, 2, 3$. We shall start out with finding $\inf_{\mathbf{G}\in\mathscr{A}_2} \delta(\mathbf{G})$. Let

$$\mathbf{G} = circ(-a_1 - a_3, a_1, 0, a_3).$$

Then for $\mathbf{G} \in \mathscr{A}_2$,

$$\begin{aligned}
\sigma(\mathbf{G}^+) &= \{-2(a_1 + a_3), -(a_1 + a_3) + i(a_1 - a_3), -(a_1 + a_3) - i(a_1 - a_3)\} \\
&= \{-2r(\cos\theta + \sin\theta), -r[(\cos\theta + \sin\theta) - i(\cos\theta - \sin\theta)], \\
&\quad -r[(\cos\theta + \sin\theta) + i(\cos\theta - \sin\theta)]\}.
\end{aligned}$$

Here $r = \sqrt{a_1^2 + a_3^2}$ and $0 \le \theta \le \frac{\pi}{2}$. In view of (3.5), we may assume without lose of generality that

$$\begin{aligned}
\sigma(\mathbf{G}^+) &= \{-2(\cos\theta + \sin\theta), -(\cos\theta + \sin\theta) + i(\cos\theta - \sin\theta), \\
&\quad -(\cos\theta + \sin\theta) - i(\cos\theta - \sin\theta)\} \\
&:= \{\gamma_1, \gamma_2, \gamma_3\}.
\end{aligned}$$

Note that $|1 + d\gamma_2| = |1 + d\gamma_3|$. Moreover, we have, via Proposition 3.1, that

$$\min_{d\in\mathbb{C}} \max_{1\le j\le 2} |1 + d\gamma_j| = \min_{d>0} \max_{1\le j\le 2} |1 + d\gamma_j| := \min_{d>0} \max_{1\le j\le 2} \gamma_j(d).$$

Let $\Gamma_j(d) = \gamma_j^2(d),\ j = 1, 2$ and $d > 0$. Then

$$\Gamma_1(d) = (1 - 2d(\cos\theta + \sin\theta))^2$$

and

$$\Gamma_2(d) = 2d^2 - 2d(\cos\theta + \sin\theta) + 1.$$

Clearly, two parabolas Γ_1 and Γ_2 intersect at $d = 0$ and $d = \dfrac{\cos\theta + \sin\theta}{1 + 2\sin 2\theta}$.
Let A and C be, respectively, the $d-$coordinate of the vertices of the parabolas Γ_1 and Γ_2. Let $B \neq 0$ be the $d-$coordinate of the intersection of Γ_1 and Γ_2. In particular,

$$A = \frac{1}{2(\cos\theta + \sin\theta)}, B = \frac{\cos\theta + \sin\theta}{1 + 2\sin 2\theta} \text{ and } C = \frac{\cos\theta + \sin\theta}{2}.$$

For $0 \le \theta \le \dfrac{\pi}{12}$ *or* $\dfrac{5\pi}{12} \le \theta \le \dfrac{\pi}{2}$, $A \le C \le B$. For $\dfrac{\pi}{12} \le \theta \le \dfrac{5\pi}{12}$, $A \le B \le C$.

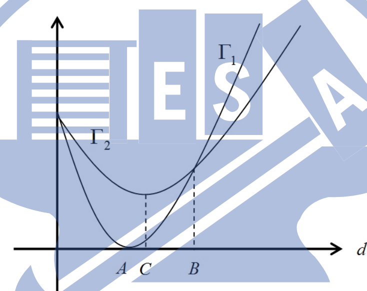

Figure 5.1: The case that $0 \le \theta \le \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12} \le \theta \le \dfrac{\pi}{2}$. Then $A \le C \le B$.

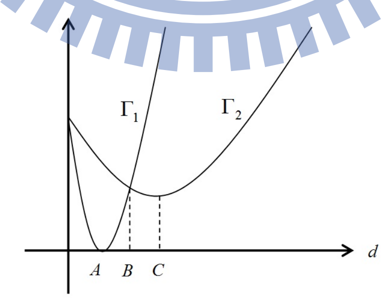

Figure 5.2: The case that $\dfrac{\pi}{12} \le \theta \le \dfrac{5\pi}{12}$. Then $A \le B \le C$.

Typical graphs of Γ_1 and Γ_2 are shown in Figs 5.1 and 5.2. From Fig 5.1, we have that for $0 \le \theta \le \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12} \le \theta \le \dfrac{\pi}{2}$,

$$\delta(\mathbf{G}) = (\Gamma_2(C))^{\frac{1}{2}} = \left(\frac{1-\sin 2\theta}{2}\right)^{\frac{1}{2}} \geq \frac{1}{2}.$$

For $\frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12}$,

$$\delta(\mathbf{G}) = |1 + B\gamma_1| = \frac{1}{1 + 2\sin 2\theta} \geq \frac{1}{3}.$$

Consequently,

$$\inf_{\mathbf{G}\in\mathscr{A}_2} \delta(\mathbf{G}) = \min_{\mathbf{G}\in\mathscr{A}_2} \delta(\mathbf{G}) = \frac{1}{3}.$$

Furthermore, such minimum can be achieved provided that $\theta = \frac{\pi}{4}$, or, equivalently, $a_3 = a_1$, which corresponds to a symmetric $\mathbf{G}$.

Let $\mathbf{G} \in \mathscr{A}_3$ and $\mathbf{H} \in \mathscr{A}_1$. Then

$$\sigma(\mathbf{G}^+) = \{-2a_1, -(a_1 + 2a_2) + a_1 i, -(a_1 + 2a_2) - a_1 i\}$$

and

$$\sigma(\mathbf{H}) = \{-2a_3, -(a_3 + 2a_2) + a_3 i, -(a_3 + 2a_2) - a_3 i\}.$$

Hence,

$$\inf_{\mathbf{G}\in\mathscr{A}_3} \delta(\mathbf{G}) = \inf_{\mathbf{G}\in\mathscr{A}_1} \delta(\mathbf{G}).$$

To complete the proof of the theorem, it then suffices to show that

$$\inf_{\mathbf{G}\in\mathscr{A}_3} \delta(\mathbf{G}) \geq \frac{1}{3}.$$

As in the case of $\inf_{\mathbf{G}\in\mathscr{A}_2} \delta(\mathbf{G})$, the corresponding Γ_1 and Γ_2 are, respectively, given as in the following.

$$\Gamma_1(d) = (1 - 2da_1)^2 \text{ and } \Gamma_2(d) = (1 - d(a_1 + 2a_2))^2 + (da_1)^2.$$

Here d is defined on $(0, \infty)$. A direct calculation would yield that Γ_1 and Γ_2 intersects at $d = 0$ and $d = \frac{2a_2 - a_1}{2a_2^2 + 2a_1 a_2 - a_1^2} =: \bar{a}$. Moreover,

$$\Gamma_2(d) \geq \Gamma_1(d) \text{ (resp., } \Gamma_2(d) \leq \Gamma_1(d)) \text{ provided that } d \geq \bar{a} \text{ (resp., } 0 < d \leq \bar{a}). \quad (5.4)$$

To further pursue our goal, we need to know the relative position of $\bar{a}$ and the minimum points of Γ_1 and Γ_2 on the real line. To this end, let $a_2 = ra_1$. Then,

$$\text{for } 0 < r < \frac{\sqrt{3}-1}{2} \text{ or } r > \frac{1}{2} \text{ (resp., } \frac{\sqrt{3}-1}{2} < r < \frac{1}{2}\text{),}$$
$$\text{we have that } \bar{a} > 0 \text{ (resp., } \bar{a} < 0\text{).} \tag{5.5}$$

Moreover, $\bar{a} = 0$ provided that $r = \dfrac{1}{2}$. The proof then breaks into three cases.

(**Case i**) $\dfrac{\sqrt{3}-1}{2} < r \leq \dfrac{1}{2}$.

Combining (5.4) and (5.5), we get that, for

$$\frac{\sqrt{3}-1}{2} < r \leq \frac{1}{2},\ \Gamma_2(d) \geq \Gamma_1(d) \text{ for } d \in (0, \infty).$$

Therefore, if

$$a_2 = ra_1,\ \frac{\sqrt{3}-1}{2} < r \leq \frac{1}{2},$$

then the corresponding $\delta(a_1, a_2)$ has the property that

$$\delta(a_1, a_2) = (\Gamma_2(d_c))^{\frac{1}{2}} = \left(\frac{1}{1 + (1+2r)^2}\right)^{\frac{1}{2}} \geq \frac{1}{\sqrt{5}},$$

where d_c is the minimum point of the parabola $\Gamma_2(d)$. Note that

$$d_c = \frac{1+2r}{a_1(1 + (1+2r))^2} =: \frac{1}{a_1} h_1(r).$$

(**Case ii**) $r \geq \dfrac{1}{2}$.

In this case, $\bar{a} > 0$ and so

$$\min_{d>0} \max_{1 \leq j \leq 2} \Gamma_j(d) = \left\{ \min_{0<d\leq\bar{a}} \Gamma_1(d),\ \min_{d \geq \bar{a}} \Gamma_2(d) \right\}.$$

Writing $\bar{a}$ in terms of a_1 and r, we get that

$$\bar{a} = \frac{2r-1}{a_1(2r^2 + 2r - 1)} =: \frac{1}{a_1} h_2(r).$$

Let r_c be the unique real solution of

$$4r^3 - 2r^2 - 1 = 0.$$

Here $\frac{1}{2} < r_c < 1$.

Some direct calculations would yield that

$$h_2(r) \geq h_1(r), \text{or, equivalently, } \bar{a} \geq d_c$$
$$(\text{resp., } h_2(r) \leq h_1(r), \text{or, equivalently, } \bar{a} \leq d_c),$$

provided that $r \geq r_c$ (resp., $r \leq r_c$). Hence, for $r \geq r_c$, $\min\limits_{d \geq \bar{a}} \Gamma_2(d) = \Gamma_2(\bar{a})$. This is because that

$$\text{the minimum point } d_c \text{ of the parabola is to the left of } \bar{a} \text{ for all } r \in [0, \infty). \quad (5.6)$$

Moreover, $\min\limits_{0<d\leq \bar{a}} \Gamma_1(d) = \Gamma_1(\bar{a})$.

For in this situation, the minimum point of $\Gamma_1(d)$ is $\frac{1}{2a_1}$, which is to the right of $\bar{a}$. Consequently, for $r \geq r_c$,

$$\delta(a_1, a_2) = \Gamma_2^{\frac{1}{2}}(\bar{a}) = \Gamma_1^{\frac{1}{2}}(\bar{a}) = \frac{2r^2 - 2r + 1}{2r^2 + 2r - 1} \geq \frac{1}{3}.$$

The minimum of $\delta(a_1, a_2)$ occurs at $r = 1$ or $a_1 = a_2$. For $\frac{1}{2} \leq r < r_c$, we have, via (5.4), (5.5) and (5.6), that

$$\delta(a_1, a_2) = \Gamma_2^{\frac{1}{2}}(d_c) = \left(\frac{1}{1 + (1 + 2r)^2}\right)^{\frac{1}{2}} \geq \left(\frac{1}{1 + (1 + 2r_c)^2}\right)^{\frac{1}{2}} > \frac{1}{3}.$$

(**Case iii**) $0 < r < \frac{\sqrt{3} - 1}{2}$.

In this case, $\bar{a} > 0, r < r_c$ and $\bar{a} \leq d_c$. Thus,

$$\delta(a_1, a_2) = \Gamma_2^{\frac{1}{2}}(d_c) = \left(\frac{1}{1 + (1 + 2r)^2}\right)^{\frac{1}{2}} \geq \frac{1}{\sqrt{2}}.$$

Combining all three cases, we conclude that (5.3) holds true and that the minimum is achieved whenever two nonzero a_i are equal.

Remark 5.1. *Theorem 3.1 is amount to saying that for the class of real-valued coupling matrices of size 4 that are circulant but not all-to-all coupled, the equal coupling weights would yield the fastest synchronized rate. For these matrices with complex entries, our computation seems to suggest that the assertion in Theorem 3.1 holds as well, as seen in Table 5.1.*

(a_1, a_3)	(9+i, 1+i)	(8+i, 2+i)	(7+i, 3+i)	(6+i, 4+i)	(5+i, 5+i)
d	0.0610	0.0734	0.0688	0.0661	0.0654
δ	0.6201	0.5090	0.4092	0.3559	$\frac{1}{3}$

Table 5.1: The coupling matrix $\mathbf{G} \in \mathscr{A}_2$ with complex-valued entries. The computation seems to suggest that $\mathbf{G}$ with the equal weight yields the smallest δ, which is $\dfrac{1}{3}$.

The network configuration under consideration is the $2k$ nearest neighbor coupled network, which is a symmetric circulant matrix $\mathbf{G_k}$ of the following form:

$$\mathbf{G_k} = circ\left(-2\sum_{i=1}^{k} a_i, a_1, a_2, \ldots, a_k, 0, 0, \ldots, 0, a_k, \ldots, a_2, a_1\right).$$

The case with equal weights on coupling coefficients is considered. We assume that $a_i = 1, 1 \leq i \leq k$. Note that the dimension of the matrix is m and so $1 \leq k \leq \left[\dfrac{m-1}{2}\right]$. Then the spectrum of such $\mathbf{G_k}$ is

$$\sigma(\mathbf{G_k}) = \left\{-2k + 2\sum_{l=1}^{k} \cos\frac{2l(j-1)\pi}{m},\ j = 1, 2, \ldots, m\right\}. \tag{5.7}$$

For $k = 1$, $\sigma(\mathbf{G_1})$ reduces to $\left\{-2 + 2\cos\dfrac{2(j-1)\pi}{m},\ j = 1, 2, \ldots, m\right\}$.
Armed with the formula for eigenvalues of $\mathbf{G_k}$, Proposition3.1 and Theorem 3.1, we obtain the following tables of d_f and δ for various (m, k).

m	3	4	5	6	7	8
d_f	$\frac{1}{3}$	$\frac{1}{3}$	0.4000	0.4000	0.4391	0.4361
δ	0	$\frac{1}{3}$	0.4472	0.6000	0.6693	0.7445

Table 5.2: Let $k = 1$. For various size m of coupling matrix $\mathbf{G_1}$, the corresponding d_f and δ, as defined in (3.4a) and (3.4b), respectively, are listed in the table.

To obtain global synchronization theory, we need to have a bounded dissipative region for CMMLs (2.1).

Proposition 5.1. *Let* $\mathbf{G} = \mathbf{G_1}$. *Let* $f : J(f) \to J(f)$. *Suppose* $|z_i(0)| \le r$ *for some* $r > 0$. *Then, for all* $0 < d \le \frac{1}{2}$, $|z_i(n)| \le r$ *for all* $n \in \mathbb{N}$ *and all* $1 \le i \le m$. *Here,* $z_i(n)$ *are defined as in (2.1).*

Proof: We shall prove the proposition by the induction. It is clear, via the assumptions, that the assertion of proposition holds for $n = 0$. Suppose $|z_i(k)| \le r$ for all i. Then

$$|z_i(k+1)| \le (1-2d)|f(z_{i-1}(k))| + d|f(z_i(k))| + d|f(z_{i+1}(k))| \le r.$$

We have used the fact that f is invariant on $J(f)$ to justify the above inequality. The proof of the proposition is completed.

Theorem 5.2. *Consider CCMLs* (4.1) *with* $f(z) = f_c(z)$, *where* $|c| < \frac{1}{4}$, *and* $\mathbf{G} = \mathbf{G_1}$. *Then CCMLs* (2.1) *acquires global synchrony on its Julia set with* $m = 3$ *and* 4.

Proof: To acquire synchrony for (2.1), it suffices to show, via (2.5), that

$$\left(\min_{d\in\mathbb{C}} \max_{2\le j\le m} |1 + d\lambda_j|\right) \|\mathbf{D}(n)\| < 1. \tag{5.8}$$

Let $f(z) = f_c(z)$. Then

$$\mathbf{D}(n) = diag(z_1(n) + z_2(n), \ldots, z_{m-1}(n) + z_m(n)).$$

It then follows from Propositions 4.2 and 5.1 that

$$\|\mathbf{D}(n)\| \le 1 + \sqrt{2}.$$

Using Proposition 3.1, Theorem 3.1 and (5.7), we have that

$$\min_{d\in\mathbb{C}} \max_{2\le j\le m} |1 + d\lambda_j| = \delta = \frac{\lambda_{\overline{m}} - \lambda_2}{\lambda_{\overline{m}} + \lambda_2} = \begin{cases} \dfrac{2\cos^2\frac{\pi}{m} + \cos\frac{\pi}{m} - 1}{-2\cos^2\frac{\pi}{m} + \cos\frac{\pi}{m} + 3}, & m \text{ is odd,} \\ \dfrac{\cos^2\frac{\pi}{m}}{2 - 2\cos^2\frac{\pi}{m}}, & m \text{ is even.} \end{cases}$$

Clearly, for $m = 3$ and $m = 4$, the corresponding $\lambda_2 = -3 = \lambda_{\overline{m}}$, $\lambda_2 = -2$ and $\lambda_{\overline{m}} = -4$. Moreover, using Proposition 4.2-(i), we have that

$$\|\mathbf{D}(\mathbf{z})\| \le 1 + \sqrt{2}.$$

In both cases, (5.8) are satisfied, as evidently seen in Table 5.2. The proof of the assertion of the theorem is thus completed.

Theorem 5.3. *Consider* $f(z) = f_c(z)$*, and* $\mathbf{G} = \mathbf{G_1}$.

(i) *For* $c = 0$*, the local synchronization of* (2.1) *on its Julia set is lost with* $m \geq 6$*. On the other hand, the local synchronization of* (2.1) *on its Julia set is achieved with* $m = 3, 4, 5$.

(ii) *For* $c = -2$*, CCMLs* (2.1) *is locally synchronized on its Julia set with* $m = 3, 4$ *and* 5. *Its synchronization is lost whenever* $m \geq 6$.

Proof: Following Remark 4.1-(i), we have that (2.6) becomes

$$\mathbf{e}(n+1) = 2(\mathbf{I} + d\mathbf{G}^{+})\mathbf{e}(n).$$

Applying (5.7), we get that

$$\lambda_2 = -2 + 2\cos\frac{2\pi}{m}$$

and

$$\lambda_{\overline{m}} = \begin{cases} -2 - 2\cos\dfrac{\pi}{m}, & m \text{ is odd,} \\ -4, & m \text{ is even.} \end{cases}$$

In both cases, it is easily to verify that $\delta = \dfrac{\lambda_{\overline{m}} - \lambda_2}{\lambda_{\overline{m}} + \lambda_2}$ is increasing in m and approaches to one as m goes to infinity. Upon using Table 5.2, we have that $2\delta < 1$ if and only if $m = 3, 4$ and 5. We have just completed the first part of the theorem. To prove the second assertion of the theorem, we first note that the logistic map $f(x) = 4x(1-x)$ exists an invariant measure μ (see e.g., [13]), and is topological transitive. It then follows from the Birkhoff's Ergodic theorem that its Lyapunov exponent $\lambda(x)$ is constant almost everywhere (see e.g., [13]). In fact, $\lambda(x) = ln2$ for all $x \in [0, 1]$ except those whose orbit containing zero (see e.g., [13]). Upon using Remark 4.1-(ii), we conclude that $\mathbf{e}(n)$, as defined in (2.6), converges to zero provided that

$$h + ln\delta = ln2\delta < 0, \text{ or, equivalently, } 2\delta < 1,$$

where $h = ln2$ is the Lyapunov exponent of $f_{-2}(x) = x^2 - 2,\ -2 \leq x \leq 2$ and δ is defined as in (3.4b).

Remark 5.2. *For $f(z) = g_c(z)$, the corresponding $\mathbf{D}(n)$ has the form of*

$$\begin{aligned}\mathbf{D}(n) &= diag(c + z_1^2(n) + z_1(n)z_2(n) + z_2^2(n), c + z_2^2(n) + z_2(n)z_3(n) + z_3^2(n), \ldots, \\ &\quad c + z_{m-1}^2(n) + z_{m-1}(n)z_m(n) + z_m^2(n)).\end{aligned}$$

Moreover, we have, via Proposition 4.2-(ii)*, that $\|\mathbf{D}(n)\| \leq 7$. Using Table 5.3, we conclude that for $\mathbf{G_9}$ or $\mathbf{G_{10}}$ with the matrix size $m = 21$, the corresponding CCMLs is globally synchronized on its Julia set. Finally, we see, via Table 5.4, that, for $\mathbf{G}_{\frac{m}{2}-1}$ with the matrix size ≥ 17, global synchronization of the corresponding CCMLs on its Julia set can also be acquired.*

k	5	6	7	8	9	10
d_f	0.1139	0.0907	0.0713	0.0599	0.0527	0.0476
δ	0.5091	0.3877	0.3048	0.2078	0.1025	0

Table 5.3: Let $m = 21$ be fixed. d_f and δ for various $\mathbf{G_k}$ are listed in the table.

$\left(m, \dfrac{m-1}{2} - 1\right)$	(11, 4)	(13, 5)	(15, 6)	(17, 7)	(19, 8)	(21, 9)
d_f	0.1126	0.0916	0.0773	0.0669	0.0590	0.0527
δ	0.2027	0.1701	0.1462	0.1281	0.1139	0.1025

Table 5.4: Let $k = \dfrac{m-1}{2} - 1$. The table gives d_f and δ for various m.

6 Concluding Remarks

We shall conclude this thesis by mentioning the possible future work.

(1) For a real-valued coupling matrix, an efficient procedure was proposed in [7] to solve min max problem (3.2). It would be desirable to solve corresponding min max problem (3.2) efficiently provided that a complex-valued coupling matrix is given.

(2) Constructing a network under a certain constraints so as to give the fastest synchronized speed is both an interesting and challenging problem. Specifically, the problem of solving inf min max (5.1) for a class $\mathscr{A}$ of coupling matrices is worthwhile to pursue.

(3) For non-smooth Julia sets, the numerical verification of the synchronization seems to be a nontrivial problem. For instance, if $|c| < \frac{1}{4}$ and c is a complex number, how one can verify computationally the theoretical results provided in Theorem 5.2.

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