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The Minimum Rank of a Mountain

山狀圖的最小秩

研究生：王稟鈞

指導教授：翁志文 教授

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Student: Ping-Chun Wang　　Advisor: Chih-wen Weng

研究生: 王稟鈞　　指導教授：翁志文教授

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研 究 生: 王 稟 鈞　　指 導 教 授: 翁 志 文 教授

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摘 要

對一以 1、2、…、n 爲點的簡單圖 G 而言， 當一大小爲 n 的實對稱矩陣滿足性質： 此矩陣第 ij 位置非零 若且唯若 i 與 j 在圖 G 上有邊， 則我們稱此矩陣與 G 相對應。一張圖的最小秩爲其相對應的所有矩陣之最小的秩。在此論文中我們定義一種與一介於 2 與 $n-1$ 間的數 m 有關的 圖，命名爲座落在 m 的山狀圖。 山狀圖含一 m 個點的路徑，其它 $n-m$ 個點之間沒有邊相連，且它們連接到路徑的方式將路徑分成一些只在端點重疊的小段，每一小段爲一點所獨佔，且該點至少有兩邊連接此小段。 在此論文中，我們將證明一個座落在 m 的山狀圖其最小秩爲 $m-1$。

關鍵詞：圖、最小秩、山狀圖。

The Minimum Rank of a Mountain

Student：Pingchun Wang   Advisor：Dr. Chihwen Weng

Department of Applied Mathematics

National Chiao Tung University

Hsinchu 300, *Taiwan, R.O.C.*

Abstract

Let G be a simple graph with vertex set $V(G) = [n] = \{1,2,\ldots,n\}$ and edge set $E(G)$. The **minimum rank** $m(G)$ of G is the minimum possible rank of an n by n symmetric matrix A whose ij-th entry is not zero if and only if $ij \in E(G)$, where i, j are distinct. For $m < n$, a graph G with vertex set $[n]$ is called a **mountain based on** $[m]$ if G satisfies

(i) the subgraph of G induced on $\{1,2,\ldots,m\}$ is a path which is partitioned into a few closed segments;

(ii) each segment is assigned a unique vertex in $[n]\backslash[m]$ which has at least two neighbors in the closed segment; and

(iii) all edges of G are either described in (i) or in (ii).

In the thesis we show that a mountain based on $[m]$ has minimum rank $m-1$.

Keywords: graph, minimum rank, mountain.

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1 Introduction

The study of matrices associated with a graph G gives a connection between Linear Algebra and Graph Theory, in which many mathematical theories have their combinatorial realizations and vice versa. The thesis studies ranks of matrices associated with graphs and their combinatorial interpretations.

All the graphs considered in this thesis are simple of order n. For a graph G, we use $E(G)$ as its edge set and $V(G)$ as its vertex set, usually $V(G) = [n] = \{1, 2, \cdots n\}$. For an $n \times n$ real symmetric matrix A, $\Gamma(A)$ represents the graph such that $ij \in E(\Gamma(A))$ if and only if the ij-th entry of A is not zero, where $i \neq j$. A real symmetric matrix A is then said to be **associated** with the graph $\Gamma(A)$. The **minimum rank** of G, denoted by $m(G)$, is defined to be the integer

$$m(G) = \min\{\text{rank}(A) : \Gamma(A) = G\},$$

where the minimum is taking for all $n \times n$ symmetric matrices A.

The minimum rank of G is related to the **maximum nullity** of G, denoted by

$$M(G) = \max\{\text{nullity}(A) : \Gamma(A) = G\}.$$

It's clear that the following equation holds for any graph G:

$$m(G) + M(G) = n. \tag{1}$$

Since $\Gamma(A) = \Gamma(A + \lambda I) = G$, $M(G)$ is also the maximum multiplicity of eigenvalues of a matrix associated with G.

The number $m(G)$ also has combinatorial meanings. In [1], Ping-Hong Wei, Chih-wen Weng showed that if G is a **tree**, which is a connected graph satisfies $|V(G)|-1=|E(G)|$, then $|E(G)|-m(G)$ is equal to the minimum size of edge subset S whose deletion will yield a graph with each vertex of degree 1 or 2. In [8], the AIM Minimum Rank - Special Graphs Work Group defined **color-change rule**, **derived coloring** and **zero-forcing set** of a graph:

(i) *color-change rule*: let G be a graph with each vertex colored either white or black. If u is a black vertex of G, and it has exactly one neighbor v which is white, then change the color of v to black.

(ii) *derived coloring*: for a coloring of G, derived coloring is the result of applying the color-change rule until that no more vertex u is a black vertex of G with exactly one neighbor white.

(iii) $zero\text{-}forcing\ set$: a zero-forcing set Z for a graph G is a subset of $V(G)$ such that if initially the vertices in Z are colored black and the remaining vertices are colored white, then the derived coloring is all black.

The minimum size of a zero-forcing set of G is denoted by $Z(G)$. Also in [8], they showed that $M(G) \leq Z(G)$.

We will compute the minimum rank of a special class of graphs which have order n and there exists an integer $2 \leq m < n$, such that the induced subgraph on vertex set $[m]$, $[n]\backslash[m]$ has edge set $\{i(i+1) \mid 1 \leq i \leq m-1\}$, $\emptyset$, respectively. Moreover, there exists a sequence of integers $1 = t_0 < t_1 < t_2 < \ldots < t_{k-1} < t_k = m$ and a function $f : [k] \mapsto [n] \backslash [m]$ such that for $1 \leq j < i \leq k$, $f(j) \neq f(j+1)$, $|[t_{i-1}, t_i] \cap G(f(i))| \geq 2$ and $\sum_{i=1}^{k} |[t_{i-1}, t_i] \cap G(f(i))| = |E(G)| - m + 1$, where $G(f(i))$ be the set of neighbors of $f(i)$. See Definitions 4.4, 4.7 for a detailed description. In the end of this thesis, we show that a mountain G based on $[m]$ has minimum rank $m(G) = m - 1$ and minimum size $Z(G) = n - m + 1$ of a zero-forcing set of G.

The thesis is organized as follows. In the second section, **Preliminaries**, we define the notations, operations for graphs, matrices which we will use in the thesis. The third section, **Known Results** introduces the known theorems and the provide their proof. The difference of minimum rank between a graph G and a graph obtained from G by deleting a vertex is investigated. At the end of this section, we prove an inequality about the maximum rank and zero-forcing set of a graph. In the last section **Our Results**, we introduce a class of graphs, called mountains. We also compute the minimum rank and the minimum size of a zero-forcing set of a mountain based on $[m]$ in this section, where m is a positive integer. At the end of the thesis, we give examples about the construction of a matrix associated with a fixed mountain, which has rank equal to the minimum rank of the mountain.

2 Preliminaries

In this section, we introduce notations we will use in this thesis.

2.1 Graphs

The three graphs K_n, P_n and C_n with vertex set $[n] = \{1,2,\dots,n\}$ and edge set defined in the following table will be used implicitly throughout the thesis.

Graphs	Notation	Edge set
Complete Graph	K_n	$\{ij \mid 1 \leq i < j \leq n\}$
Path	P_n	$\{i\,(i+1) \mid 1 \leq i \leq n-1\}$
Cycle	C_n	$\{i\,(i+1) \mid 1 \leq i \leq n-1\} \cup \{1\,n\}$

Let $G,\ G'$ be two vertex-disjoint graphs and $x,z \in V(G), y \in V(G')$. We adopt the following graph operations.

1. $x \sim z$ means x is a neighbor of z.
2. $G(x)$ denotes the set of the neighbors of the vertex x in $V(G)$.
3. $G - x$ denotes the induced subgraph of G with vertex set $V(G)\backslash\{x\}$.
4. $G +_{xy} G'$ denotes the **coalescence** of G and G' through the vertices x, y respectively, which by definition is a simple graph obtained from the disjoint union of G and G' by identifying the vertex x from G and the vertex y from G'.

2.2 Matrices

The matrices considered in the thesis are all symmetric over the real number field $\mathbb{R}$. We use the following notations with a square matrix of size $n \times n$ A, a column vector $x \in \mathbb{R}^n$, and subsets X, Y of $\mathbb{N}$.

1. $\{e_1^{(k)}, e_2^{(k)}, \ldots, e_k^{(k)}\}$ is the standard basis of $\mathbb{R}^k$, where $k \in \mathbb{N}$. If $k = n$, then we just write as $\{e_1, e_2 \ldots, e_n\}$.

2. $\mathrm{supp}(x) := \{i \in \mathbb{N} \mid \text{the } i\text{-th entry of } x \text{ is not zero}\}$.

3. $\mathrm{rank}(A)$ is the rank of A.

4. $\mathrm{cs}(A) := \{Au \mid u \in \mathbb{R}^n\}$ is the column space of A.

5. $\mathrm{rs}(A) := \{u^T A \mid u \in \mathbb{R}^n\}$ is the row space of A.

6. $\mathrm{ns}(A) := \{x \mid x \in \mathbb{R}^n, Ax = 0\}$ is the nullspace of A.

7. $A(X|Y)$ denotes the submatrix of A obtained by deleting the p-th row and the q-th column of A, for all $p \in X$, $q \in Y$. If $X = Y$, then we just write $A(X)$.

8. $A[X|Y]$, $A(X|Y]$ and $A[X|Y)$ denote the submatrices $A([n]\backslash X|[n]\backslash Y)$, $A(X|[n]\backslash Y)$ and $A([n]\backslash X|Y)$ respectively and replacing X, Y by "$-$" means that we don't delete any row or column respectively, i.e. $A[-|Y] = A(\emptyset|Y]$.

For simple illustration, A has the form

$$A = \begin{bmatrix} A[1] & A[1|1) \\ A(1|1] & A(1) \end{bmatrix}.$$

It is easy to see that

$$\operatorname{rank}(A) - 2 \leq \operatorname{rank}(A(1)) \leq \operatorname{rank}(A).$$

2.3 Matrices associated with graph G

For an $n \times n$ symmetric matrix $A = (a_{ij})$, $\Gamma(A)$ is the graph with vertex set $[n]$ such that for distinct i and j, $ij \in E(\Gamma(A))$ if and only if $a_{ij} \neq 0$. We said that A is associated with $\Gamma(A)$.

Example 2.1. The 4×4 matrix A in Figure 1 is associated with $\Gamma(A)$.

$$A = \begin{bmatrix} 1 & 1/5 & -1 & 0 \\ 1/5 & 0 & 1 & 0 \\ -1 & 1 & -4 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$\Gamma(A)$:

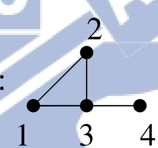

Figure 1. A symmetric matrix and its associated graph.

Note that the diagonal entries of A do not need to be 0.

3 Known Results

We shall introduce known properties of symmetric matrices in this section for later use.

3.1 Minimum ranks of the paths

Lemma 3.1. *Let A be an $n \times n$ symmetric matrix. If $\Gamma(A) = P_n$, then* $\operatorname{rank}(A) = n$ *or* $n-1$*. Moreover the following are equivalent.*

(i) $\operatorname{rank}(A) = n-1$;

(ii) $e_1^T \notin \operatorname{rs}(A)$;

(iii) $\operatorname{rank}(A) = \operatorname{rank}(A(1))$.

Proof. Let A be an n by n symmetric tridiagonal matrix such that $\Gamma(A) = P_n$ as the following

$$A = \begin{bmatrix} a_1 & b_1 & & & 0 \\ b_1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & a_{n-1} & b_{n-1} \\ 0 & & & b_{n-1} & a_n \end{bmatrix}.$$

Thus b_i's are non-zero real numbers. Then $\operatorname{rank}(A) \geq n-1$, since the first $n-1$ columns are linearly independent.

For (i) implying (ii), suppose $e_1^T \in \operatorname{rs}(A)$. Since the first row of A is $a_1 e_1^T + b_1 e_2^T$ with $b_1 \neq 0$, we have $e_2^T \in \operatorname{rs}(A)$. As we prove $e_1^T, e_2^T, \ldots, e_i^T \in \operatorname{rs}(A)$, we have $e_{i+1}^T \in \operatorname{rs}(A)$ for $2 \leq i \leq n-1$, since the i-th row of A is $b_{i-1} e_{i-1}^T + a_i e_i^T + b_i e_{i+1}^T$ and $b_i \neq 0$, This proves $\operatorname{Span}(e_1^T, e_2^T, \ldots, e_n^T) \subseteq \operatorname{rs}(A)$. Hence $\operatorname{rank}(A) = n$.

For (ii) implying (iii), if $e_1^T \notin \operatorname{rs}(A)$, then $\operatorname{rank}(A) \neq n$. Thus $\operatorname{rank}(A) = n-1$. Since b_i's are non-zero real numbers, $\mathrm{rank}(A(1|-)) = \mathrm{rank}(A) = n-1$. This implies $e_1^{(n-1)} \in \mathrm{cs}(A)$ and so $\mathrm{rank}(A(1|-)) = \mathrm{rank}(A(1))$.

For (iii) implying (i), $\mathrm{rank}(A) = n-1$, since $n-1 \leq \mathrm{rank}(A) = \mathrm{rank}(A(1)) \leq n-1$. □

Example 3.2. The following two matrices A_n, A'_n satisfy $\Gamma(A_n) = \Gamma(A') = P_n$:

$$A_n = \begin{bmatrix} 1 & 1 & & & 0 \\ 1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ 0 & & & 1 & 1 \end{bmatrix}_{n\times n}, \quad A'_n = \begin{bmatrix} 2 & 1 & & & 0 \\ 1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ 0 & & & 1 & 1 \end{bmatrix}_{n\times n} \tag{2}$$

Let $v = [1, -1, 1, \ldots, (-1)^{n-1}]^T$. Then $A_n v = 0$, $A'_n v = e_1$. This implies $\mathrm{rank}(A_n) = n-1$, and $\mathrm{rank}(A'_n) = n$ by Lemma 3.1.

Lemma 3.3. *If H is an induced subgraph of G, then $m(H) \leq m(G)$.*

Proof. Since for any matrix A such that $\Gamma(A) = G$, the submatrix $A[V(H)]$ is associated with H and $\mathrm{rank}(A[V(H)]) \leq \mathrm{rank}(A)$. This implies $m(H) \leq m(G)$. □

The following Theorem shows that P_n is the unique graph with minimum rank $n-1$ among graphs of order n.

Theorem 3.4. *([3, Theorem 2.8.]) Let A be a symmetric matrix of order n. Then the following (i)-(ii) are equivalent.*

(i) $\text{rank}(A+D) \geq n-1$ *for any diagonal matrix D;*

(ii) $\Gamma(A) = P_n$. □

Now we can determine the minimum rank of a graph which has an induced subgraph P_{n-1}.

Proposition 3.5. *If G is not a path and G contains an induced subgraph P_{n-1}, then $m(G) = n-2$.*

Proof. Since P_{n-1} is an induced subgraph of G, $m(G) \geq m(P_{n-1}) = n-2$. By theorem 3.4, $m(G) \leq n-2$. Thus $m(G) = n-2$. □

3.2 Compute the minimum rank by deleting a cut vertex

For a graph, if the edge set of the graph is as small as possible, then the matrices associated with the graph have more zero entries, and it may be easier to determine the minimum rank of the graph. Now we consider a connected graph G with a **cut vertex**, which by definition is a vertex whose deletion will make two or more components. Then the matrix associated with the graph is of the following form:

$$\begin{bmatrix} A & a & 0 \\ a^T & c & b^T \\ 0 & b & B \end{bmatrix}, \tag{3}$$

where A, B are real symmetric matrices, a, b are column vectors with proper sizes, and $c \in \mathbb{R}$. Then we may discuss the rank of the matrix by discussing the ranks of the submatrices $\begin{bmatrix} A & a \\ a^T & c \end{bmatrix}$ and $\begin{bmatrix} c & b^T \\ b & B \end{bmatrix}$.

Proposition 3.6. *Let G, H be two graphs, and $x \in V(G), y \in V(H)$. Then the following are equivalent.*

(i) $m(G +_{xy} H) = m(G - x) + m(H - y)$;

(ii) $m(H) = m(H - y)$ and $m(G) = m(G - x)$.

Proof. If $m(H) = m(H - y)$ and $m(G) = m(G - x)$, then there exists a matrix A_G with rank $m(G) = \text{rank}(A_G) = \text{rank}(A_G(x))$, also for graph H, vertex y and matrix A_H. Then we have $m(G +_{xy} H) \leq m(G - x) + m(H - y)$. Now let B be a matrix with rank less than $m(G - x) + m(H - y)$ and $\Gamma(B) = G +_{xy} H$. Then

$\text{rank}(B) < m(G +_{xy} H) \leq m(G) + m(H)$, implies $\text{rank}(B[V(G)]) < m(G)$ or $\text{rank}(B[V(H)]) < m(H)$, two contradictions. Thus $m(G +_{xy} H) = m(G - x) + m(H - y)$. For (i) implying (ii), let C be the matrix with rank $m(G +_{xy} H)$, which is associated with $G +_{xy} H$. Then C is of the form in (3), where $\Gamma(A) = G - x$ and $\Gamma(B) = H - y$.

Note that

$$
\begin{aligned}
& m(G-x)+m(H-y) \\
=& m(G+_{xy}H)=\text{rank}(C) \\
\geq& \text{rank}(A)+\text{rank}(B) \\
\geq& m(G-x)+m(H-y)
\end{aligned}
$$

Hence $m(H)=m(H-y)$ and $m(G)=m(G-x)$ by Lemma 3.3. $\square$

Example 3.7. Consider the graph C_n, where $n \geq 3$. Proposition 3.5 showed that for any $v \in V(C_n)$, the minimum rank of C_n is equal to $C_n - v$, which is also a path P_{n-1}. By Proposition 3.6, the following graph $G = C_n +_{vu} C_m$ has minimum rank

$$m(P_{n-1}) + m(P_{m-1}) = n+m-4,$$

where $u \in V(C_m)$ and $m \geq 3$.

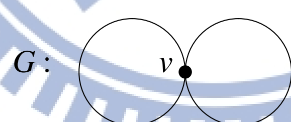

Figure 2. A graph $G = C_n +_{uv} C_m$.

Proposition 3.8. *([4, Proposition 4.1.]) Let G be a graph, and $y \in P_t$ be a vertex of degree 2, where $t \geq 3$. For any vertex $x \in G$,*

$$m(G+_{xy}P_t) = m(G-x)+t-1$$

Proof. Without loss of generality, let $x = |V(G)|$, y be a vertex in $V(G)$, $V(P_t)$, respectively, where y is of degree 2 and the induced subgraph of the vertex set $V(P_t)\backslash y$ is equal to $P_i \cup P_j$. Then the following matrix D, which is associated with $G +_{xy} P_t$ is of the form:

$$D = \begin{bmatrix} B & b & 0 & 0 \\ b^T & a & c_i^T & c_j^T \\ 0 & c_i & C_i & 0 \\ 0 & c_j & 0 & C_j \end{bmatrix},$$

where $c_i \in \mathbb{R}^i$, $c_j \in \mathbb{R}^j$ and B, C_i, C_j is the matrix associated with $G - x, P_i, P_j$, respectively. Now suppose $\text{rank}(D) < m(G-x)+t-1 = m(G-x)+i+j$. Then we consider three cases as the following:

(i) $\text{rank}(C_i)+\text{rank}(C_j) = i+j$.

Then

$$\text{rank}(D) = \text{rank}\begin{bmatrix} B & b^T \\ b & a' \end{bmatrix} + i + j < m(G-x)+i+j.$$

(ii) $\text{rank}(C_i)+\text{rank}(C_j) = i+j-1$.

Without loss of generality, suppose $\text{rank}(C_i) = i-1,\ \text{rank}(C_j) = j$. Then

$$\text{rank}(D) = \text{rank}\begin{bmatrix} B & b^T & 0 \\ b & a' & c_i^T \\ 0 & c_i & C_i \end{bmatrix} + j = \text{rank}(B)+i-1+2+j < m(G-x)+i+j.$$

(iii) $\text{rank}(C_i)+\text{rank}(C_j) = i+j-2$.

Then

$$\text{rank}(D) = \text{rank}(B) + i - 1 + j - 1 + 2 < m(G - x) + i + j.$$

All the three cases imply that $\text{rank}(B) < m(G-x)$, a contradiction. Thus we have $\text{rank}(D) \geq m(G-x) + t - 1$.

Now we choose B as a matrix associated with $G-x$ with rank $m(G-x)$ and C_i, C_j is of rank $i-1,\ j-1$, respectively. Then D has rank $m(G-x)+t-1$, since $c_i \notin \text{cs}(\text{C}_\text{i})$,$c_j \notin \text{cs}(\text{C}_\text{j})$ by lemma 3.1. This implies $m(G +_{xy} P_t) \leq m(G-x)+t-1$. Thus we conclude that $m(G +_{xy} P_t) = m(G-x)+t-1$. □

The propositions 3.6 and 3.8 are special cases of the following theorem.

Theorem 3.9. *([4, theorem 2.3]) Let G be a coalescence at vertex v of graphs* $G_1, \cdots, G_t$. *Then*

$$m(G) - m(G-v) = \min\left\{\sum_{i=1}^{t} m(G_i) - m(G_i - v), 2\right\}$$

□

A graph G is 2**-connected** if $G-v$ connected for any $v \in V(G)$. From Theorem 3.9, we may assume that G is 2-connected in determining the minimum rank $m(G)$ of G in the algorithmic aspect. However it is interesting to compute $m(G)$ or find its combinatorial meaning in general.

3.3 An inequality for the zero-forcing set and maximum nullity of the graphs

We are going to prove the inequality $M(G) \leq Z(G)$ for any graph G.

Lemma 3.10. *Let A be any $n \times n$ matrix such that* $\operatorname{rank}(A) < n-k$*, for some integer* $1 \leq k < n$*. Then for any k-subset S of $[n]$, there exists $x \in \operatorname{ns}(A)$ such that* $\operatorname{supp}(\mathrm{x}) = [\mathrm{n}] \backslash \mathrm{S}$. □

Proof. Applying Gaussian elimination to a $(k+1) \times n$ matrix whose rows are $k+1$ linearly independent vectors in the nullspace of A will yield the last row with zeros in any k designated position. □

The following lemma describes why we call $Z(G)$ a zero-forcing set for a graph G.

Lemma 3.11. *Let G be a graph with zero-forcing set Z. If $x \in \operatorname{ns}(A)$ and $Z \subseteq [n] \backslash \operatorname{supp}(x)$, then x is a zero vector.*

Proof. Let Z be a zero-forcing set of a graph G and $A = (a_{ij})$ be a matrix associated with G, where $\operatorname{rank}(A) = m(G)$. Suppose $x = (x_1, x_2, \cdots, x_n)^T \in \operatorname{ns}(A)$ and $Z \subseteq [n] \backslash \operatorname{supp}(x)$ with $[n] \backslash \operatorname{supp}(x)$ maximum. Since Z is a zero-forcing set, $[n] \backslash \operatorname{supp}(x)$ is also too. By interpreting 1 as white and 0 as black, there exists v with $x_v = 0$ and v has a unique neighbor $u \in \operatorname{supp}(x)$ with $x_u = 0$.

Then

$$0 = (Ax)_v = \sum_{i \in [n]} a_{vi} x_i = x_v + \sum_{i \sim v} a_{vi} x_i = a_{vu} x_u$$

The third equality holds since $a_{vi} \neq 0$ only when $i \sim v$. The last equality holds since v has only one neighbor $u \neq Z$ such that x_u may not be zero. Since u is a neighbor of v, $a_{uv} \neq 0$. Thus $x_u = 0$, a contradiction to $u \in \text{supp}(x)$. □

Now we can prove the inequality.

Theorem 3.12. *Let G be any graph. Then $M(G) \leq Z(G)$.*

Proof. Let G be a graph and Z be a zero-forcing set of G. If $M(G) > |Z|$, then there exists a matrix A associated with G such that $\text{nullity}(A) = M(G) > |Z|$.
By Lemma 3.10, there exists a non-zero vector $x \in \text{ns(A)}$ such that $Z \subseteq \text{supp}(x)$. This implies x is a zero vector by Lemma 3.11, a contradiction. Thus $M(G) \leq Z(G)$. □

4 Our Results

In this section, we will introduce a class of the graphs and compute the minimum rank, minimum size of zero-forcing set of the graphs.

4.1 A matrix associated with a path

Lemma 4.1. *For all $n \in \mathbb{N}$, let $A_n = (a_{ij})$ be the n by n symmetric matrix defined in (1), i.e.*

$$a_{ij} = \begin{cases} 2, & \text{if } i = j \text{ and } i, j \notin \{1, n\}; \\ 1, & \text{if } |i-j| = 1 \text{ or } i = j \in \{1, n\}; \\ 0, & \text{if } |i-j| \geq 2. \end{cases}$$

Then for any subset $S \subseteq [n]$ with $|S| > 1$, there exists a vector u such that

$\mathrm{supp}(u) \subseteq [\max S - 1]$ *and* $\mathrm{supp}(Au) = S$.

Proof. For integers $1 \le i < j \le n$, define $b_i,\ c_{i,j}$ as the following:

$$b_i := (-1)^0 e_i + (-1)^1 e_{i-1} + \cdots + (-1)^{i-1} e_1, \tag{4}$$

$$c_{i,j} := (-1)^0 b_i + (-1)^1 b_{i+1} + \cdots + (-1)^{j-i-1} b_{j-1}. \tag{5}$$

Then

$$Ab_i = e_i + e_{i+1}$$

and

$$\begin{aligned} &Ac_{i,j} \\ =&A[(-1)^0 b_i + (-1)^1 b_{i+1} + \cdots (-1)^{j-i-1} b_{j-1} \\ =&(e_i + e_{i+1}) + (-1)^1 (e_{i+1} + e_{i+2}) + \cdots + (-1)^{j-i} (e_{j-1} + e_j) \\ =&e_i + (-1)^{j-i-1} e_j. \end{aligned}$$

Now suppose $S = \{t_1, t_2 \ldots, t_k\} \subseteq [n]$, where $k \ge 2$ and $t_1 < t_2 < \cdots < t_k$. Choose

$$u = c_{t_1,t_2} + c_{t_1,t_3} + \cdots c_{t_1,t_k}.$$

This implies

$$Au = ke_{t_1} + (-1)^{t_2-t_1-1} e_{t_2} + (-1)^{t_3-t_1-1} e_{t_3} + \cdots + (-1)^{t_k-t_1-1} e_{t_k}.$$

Then we have that $\mathrm{supp}(Au) = S$ and $\mathrm{supp}(u) \subseteq [t_k - 1] = [\max S - 1]$.

□

We provide matrices A satisfy $\operatorname{rank}(A) = n-2$ and that $\Gamma(A)$ is the graph described in Proposition 3.5.

Example 4.2. Let $G = P_{n-1} +_{i1} P_2$ and A_i, A_{n-1-i} are defined as the matrix A_n at Lemma 4.1, where $1 < i < n$, the following matrix satisfies $\Gamma(A) = G$ and $\operatorname{rank}(A) = n-2$.

$$A = \begin{bmatrix} A_{i-1} & e_{i-1}^{(i-1)} & 0 & 0 \\ (e_{i-1}^{(i-1)})^T & 0 & (e_1^{(n-1-i)})^T & 1 \\ 0 & e_1^{(n-1-i)} & A_{n-1-i} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}_{n\times n}$$

Note that we have $\operatorname{rank}(A_{i-1}) = i-2, \operatorname{rank}(A_{n-1-i}) = n-2-i$. Since the both of n-th column and row are linearly independent, $\operatorname{rank}(A) = n-2$. Then $m(P_{n-1} +_{i1} P_2) = n-2$, by Lemma 3.3.

Example 4.3. Let G be a graph of order n such that the induced subgraph of G on $[n-1]$ is P_{n-1} and the vertex n has neighbors $S = \{t_1, t_2, \ldots t_k\}$, where $k \geq 2$. Let A_{n-1} be the matrix defined as the amtrix A_n at Lemma 4.1. Then there exists a vector $u \in \mathbb{R}^n$ such that $\operatorname{supp}(A_{n-1}u) = S$. The following matrix A satisfies $\operatorname{rank}(A) = n-2$ and $\Gamma(A) = G$:

$$A = \begin{bmatrix} A_{n-1} & A_{n-1}u \\ u^T A_{n-1} & u^T A_{n-1} u \end{bmatrix}_{n\times n}.$$

4.2 Mountains

In Example 4.3, we proved that joining a vertex to some internal node of a path doesn't increase the minimum rank. We are going to add 2 or more vertices to a path.

Definition 4.4. A sequence $T_1, T_2, \ldots, T_t$ of subsets of $[m]$ is said to be **separated** on $[m]$ with respect to the sequence of integers $1 = t_0 < t_1 < t_2 < \ldots < t_{k-1} < t_k = m$ if there exists a function $f : [k] \mapsto [t]$ such that for $1 \leq j < i \leq k$, $f(j) \neq f(j+1)$, $|[t_{i-1}, t_i] \cap T_{f(i)}| \geq 2$ and $\sum_{\ell=1}^{k} |[t_{\ell-1}, t_\ell] \cap T_{f(\ell)}| = \sum_{\ell=1}^{t} |T_\ell|$.

Example 4.5. The following subsets T_1, T_2 are separated on $[m]$ with respect to the specified sequence $(t_0, t_1, \ldots, t_k)$ and function $f = (f(1), f(2), \ldots, f(k))$.

(i) $T_1 = \{1,2,5,6\}$ and $T_2 = \{3,4\}$ with $(1,3,4,6)$ and $f = (1,2,1)$;

(ii) $T_1 = \{1,3,5,6\}$ and $T_2 = \{3,5\}$ with $(1,3,5,6)$ and $f = (1,2,1)$;

(iii) $T_1 = \{1,4,5,7,9\}$ and $T_2 = \{5,6,7\}$ with $(1,5,7,9)$ and $f = (1,2,1)$;

(iv) $T_1 = \{1,2,5,6,7\}$ and $T_2 = \{3,4,8,10\}$ with $(1,3,5,8,10)$ and $f = (1,2,1,2)$.

(v) $T_1 = \{1,2,3,4,5,6\}$ and $T_2 = \{2,3,4,5\}$ with $(1,2,3,4,5,6)$ and $f = (1,2,1,2,1)$.

Example 4.6. The following subsets $T_1,\ T_2,\ T_3$ are separated on $[m]$ with respect to the specified sequence $(t_0, t_1, \ldots, t_k)$.

(i) $T_1 = \{1,2,3,5,6,7\}$, $T_2 = \{3,4,9,10\}$,

$T_3 = \{7,8,9\}$ with $(1,3,4,7,9,10)$ and $f = (1,2,1,3,2)$;

(ii) $T_1 = \{3,4,5,10,11,12\}$, $T_2 = \{5,6,9,10\}$,

$T_3 = \{1,3,6,8,13,14\}$ with $(1,3,5,6,8,10,12,14)$ and $f = (3,1,2,3,2,1,3)$.

Definition 4.7. For integers $m < n$, let $M_{m,n}$ be the class of graphs G with vertex set $V(G) = [n]$ satisfying the following axioms.

(i) The subgraph of G induced on $[m]$ has edge set $\{i(i+1) \mid 1 \leq i \leq m-1\}$, and the subgraph of G induced on $\{m+1, m+2, \ldots, n\}$ has no edges.

(ii) The sequence $G(m+1)$, $G(m+2), \cdots$, $G(n)$ separated on $[m]$.

The graph $G \in M_{m,n}$ is called a **Mountain based on** $[m]$.

Theorem 4.8. *If $G \in M_{m,n}$, then $m(G) = m-1$.*

Proof. Choose a sequence $t_0 = 1 < t_1 < t_2 < \ldots < t_{k-1} < t_k = m$ and a function $f : [k] \mapsto [n] \backslash [m]$ with respect to which the sets $G(m+1), \ldots, G(n)$ are separated on $[m]$. For $1 \leq i \leq k$, choose a column vector $u_i \in \mathbb{R}^m$ such that

$\mathrm{supp}(u_i) \subseteq [(\max\ G(f(i)) \cap [t_{i-1}, t_i]) - 1]$ and $\mathrm{supp}(A_m u_i) = G(f(i)) \cap [t_{i-1}, t_i]$, where $A_m = (a'_{ij})$ is defined in Lemma 4.1. Notice that from the construction, $u_j^T A_m u_i = 0$ if $j < i$, and indeed for $j \neq i$ since A_m is symmetric. For $m+1 \leq \ell \leq n$, let

$S_\ell = \{i \in [k] \ : \ f(i) = \ell\}$, and define the column vector $v_\ell := \sum_{i\in S_\ell} u_i$. Note that for $m+1 \le \ell \le n$, $\ell = f(i)$ for any $i \in S_\ell$, so

$$\operatorname{supp}(A_m v_\ell) = \operatorname{supp}(A_m \sum_{i\in S_\ell} u_i) = G(\ell). \tag{6}$$

Also for $p \in [n]\backslash[m]$ with $p \neq \ell$, we have $S_\ell \cap S_p = \emptyset$, so

$$v_\ell^T A_m v_p = \sum_{i\in S_\ell}\sum_{j\in S_p} u_i^T A_m u_j = 0. \tag{7}$$

We now define the n by n symmetric matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} a'_{ij}, & \text{if } 1 \le i \le j \le m; \\ v_i^T A_m v_j, & \text{if } m+1 \le i \le j \le n; \\ (e_i^{(m)})^T A_m v_j, & \text{if } 1 \le i < m+1 \le j \le n. \end{cases} \tag{8}$$

From the above construction in (8) and (6), (7), one can easily check that $\Gamma(A) = G$. For all $m+1 \le \ell \le n$, $A[[m]|\ell] = A_m v_\ell$. Hence

$$\operatorname{rank}(A[[m]|-]) = \operatorname{rank}(A[-|[m]]) = \operatorname{rank}(A_m) = m-1.$$

For $m+1 \le i, j \le n$, $A[i|\ell] = v_i^T A_m v_j$. Hence the column of A is a linear combination of the first m columns. Thus the rank of A is equal to $m-1$. This proves $m(G) \le m-1$. Since G contains induce subgraph P_m, we have that $m(G) = m-1$ by Lemma 3.3. □

Example 4.9. Let G be a mountain of order 12 based on $[10]$ such that $G(11)$ and $G(12)$ are separated with respect to the sequence $(1,3,5,8,10)$ and $f = (11,12,11,12)$ as showing in the figure 3. We will give a matrix A associated with G and the rank of A is 9.

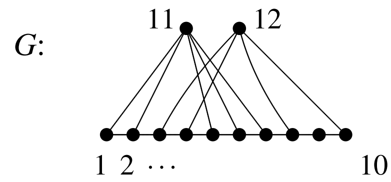

Figure 3. A mountain mased on [10] of order 12.

Since the sequence $G(11),\ G(12)$ is separated on [10] with $(1,3,5,8,10)$ and $f=(1,2,1,2)$, let A_{10} be the matrix defined at Lemma 4.1 and we choose

$$u_1=[1,0,0,0,0,0,0,0,0,0]^T,\ u_2=[1,-1,1,0,0,0,0,0,0,0]^T;$$

$$u_3=[-3,3,-3,3,-3,2,0,0,0,0]^T,\ u_4=[2,-2,2,-2,2,-2,2,-2,1,0]^T$$

such that

$$\mathrm{supp}(A_{10}u_1)=\{1,2\}=G(11)\cap[1,3],\ \mathrm{supp}(A_{10}u_2)=\{3,4\}=G(12)\cap[3,5];$$

$$\mathrm{supp}(A_{10}u_3)=\{5,6,7\}=G(11)\cap[5,8],\ \mathrm{supp}(A_{10}u_4)=\{8,10\}=G(12)\cap[8,10].$$

Choose $v_{11}=u_1+u_3$, $v_{12}=u_2+u_4$ such that $\mathrm{supp}(v_{11})=G(11)$ and $\mathrm{supp}(v_{12})=G(12)$. Then the following matrix A is associated with G and $\mathrm{rank}(A)=9$.

$$A=\begin{bmatrix} A_{10} & A_{10}v_{11} & A_{10}v_{12} \\ v_{11}^T A_{10} & 6 & 0 \\ v_{12}^T A_{10} & 0 & 3 \end{bmatrix}.$$

This implies $m(G)\le 9$, and it's clear that $m(G)\ge m(P_{10})=9$. Thus $m(G)=9$.

Example 4.10. There exists a matrix A associated with the following graph G with rank 9.

G:

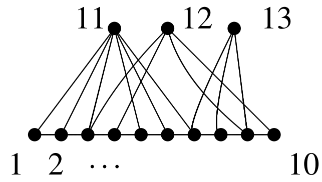

Figure 3. A mountain based on [10] of order 13.

The sequence $G(11) = \{1,2,3,5,6,7\}$, $G(12) = \{3,4,9,10\}$ and $G(13) = \{7,8,9\}$, is separated on $[10]$ with $(1,3,5,7,9,10)$and $f = (11,12,11,13,12)$. We choose

$$u_1 = [-3,2,0,0,0,0,0,0,0,0]^T,\ u_2 = [1,-1,1,0,0,0,0,0,0,0]^T;$$

$$u_3 = [-3,3,-3,3,-3,2,0,0,0,0]^T,\ u_4 = [-3,3,-3,3,-3,3,-3,2,0,0]^T;$$

$$u_5 = [0,0,0,0,0,0,0,0,0,1]^T.$$

Then we have

$$\text{supp}(A_{10}u_1) = \{1,2,3\} = [1,3] \cap G(11),\ \text{supp}(A_{10}u_2) = \{3,4\} = [3,5] \cap G(12);$$

$$\text{supp}(A_{10}u_3) = \{5,6,7\} = [5,7] \cap G(11),\ \text{supp}(A_{10}u_4) = \{7,8,9\} = [7,9] \cap G(13)$$

$$\text{and } \text{supp}(A_{10}u_5) = \{9,10\} = [9,10] \cap G(12).$$

Choose $v_{11} = u_1 + u_3$, $v_{12} = u_2 + u_5$ and $v_{13} = u_4$. The following matrix A is associated with G and $\text{rank}(A) = 9$.

$$A = \begin{bmatrix} A_{10} & A_{10}v_{11} & A_{10}v_{12} & A_{10}v_{13} \\ v_{11}^T A_{10} & 10 & 0 & 0 \\ v_{12}^T A_{10} & 0 & 3 & 0 \\ v_{13}^T A_{10} & 0 & 0 & 5 \end{bmatrix}.$$

Corollary 4.11. *If $G \in M_{m,n}$ then $M(G) = Z(G) = n - m + 1$.*

Proof. We have known $M(G) \leq Z(G)$ by Theorem 3.12, and $M(G) = n - m(G) = n - m + 1$ by Theorem 4.8 and equation (1). Initially by coloring the set $S = [n] \backslash [m-1]$ in black, one can check that applying color changing rules along the sequence of vertices $m-1, m-2, \ldots, 1$, eventually every vertex is black. Hence $Z(G) \leq n - m + 1$, and indeed $M(G) = Z(G) = n - m + 1$. □

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