國立交通大學

應用數學系

碩士論文

三角形棒棒糖圖的無號拉普拉斯矩陣之特徵值探討

Signless laplacian spectrum of a lollipop graph with a triangle

研 究 生：徐志杰

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碩 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

in partial Fulfillment of the Requirements

for the Degree of

Master

in

Applied Mathematics

June 2013

Hsinchu, Taiwan, Republic of China

中 華 民 國 一 零 二 年 六 月

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摘 要

假設 G 是一個由點 1、2、…、n 所構成的簡單圖，其中每個點相對應的價數分別為 d_1、d_2、…、d_n。設 A(G) 是 G 的 (0,1)-鄰接矩陣，D(G) 是一個對角矩陣，其對角線上分別是 d_1、d_2、…、d_n。矩陣 L(G)=D(G)-A(G) 稱為 G 的拉普拉斯矩陣，矩陣 |L|(G)=D(G)+A(G) 稱為 G 的無號拉普拉斯矩陣。A(G)，L(G)，|L|(G) 的特徵值給了我們很多訊息去了解 G 的構造。在這個論文中，我們研究一種圖形叫做三角形棒棒糖圖，其由一個三個點的完全圖與一個路徑圖共用一點而接起來。我們探討三角形棒棒糖圖的無號拉普拉斯矩陣的特徵值、特徵多項式及它們的相關比較問題。

關鍵詞：棒棒糖圖、無號拉普拉斯矩陣、特徵值。

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Abstract

Let G be a simple graph with vertices $1,\cdots,n$ of degrees $d_1,\cdots,d_n$ respectively. Let $A(G)$ be the $(0,1)$-adjacency matrix of G, and let $D(G)$ be the diagonal matrix $\text{diag}(d_1,\cdots,d_n)$. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of G, while $|L|(G) = D(G) + A(G)$ is called the signless laplacian matrix of G. The eigenvalues of $A(G)$, $L(G)$, and $|L|(G)$ give many hints to the structure of G. In this thesis we study a class of graphs, called lollipop graph with a triangle, which are obtained from paths by adding a new vertex to a path and adding two edges from the new vertex to one end of the path and to the neighbor of this end, forming a triangle K_3. We study the signless Laplacian eigenvalues and characteristic polynomial of lollipop graphs with K_3.

Keywords: lollipop graph, signless laplacian matrix, eigenvalue.

誌　　謝

來到交大已經兩年了，這兩年有許多美好的回憶，也很幸運的能夠準時完成學業，這一路上要感謝很多人。其中最感謝的是我的指導教授翁志文，沒有老師的耐心，是沒辦法完成這篇論文的，真的很感謝老師這兩年來的幫助。

再來要謝謝康明軒老師、傅恆霖老師、符麥克老師，每一位老師的課程都讓我收穫豐富，感謝陳秋媛老師常來我們研究室關心我們的近況，還有舉辦很多很棒的活動，感謝同門的王秉鈞在生活上的幫忙，在研究上的幫助，感謝李光祥學長對我論文上的建議以及幫助，劉家安學長抽空陪我們模擬口試，也要感謝王紹鈞、林立庭等組合學的同學們，跟你們一起真的很開心，讓我留下了美好的回憶，也是我一生中一段蠻值得回憶的時光。

感謝我的父母親以及阿嬤，每次回家都這麼的關心我，給我這麼多關懷以及支持，讓我在求學之路上可以毫無顧忌，感謝我的女友，沒有妳我是沒辦法在這裡打這篇誌謝的。

要感謝的人太多了，人生無不散的宴席，能留下的只有心中的回憶。

Contents

1 Introduction

Let G be a simple graph with vertices $1, \cdots, n$ of degrees $d_1, \cdots, d_n$ respectively. Let $A(G)$ be the $(0,1)$-adjacency matrix of G, and let $D(G)$ be the diagonal matrix diag$(d_1, \cdots, d_n)$. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of G, while $|L|(G) = D(G) + A(G)$ is called the signless laplacian matrix of G. The eigenvalues of $A(G)$, $L(G)$, and $|L|(G)$ give many hints to the structure of G. The least eigenvalue of $L(G)$ is always zero, and the second least is known as the algebraic connectivity of G, which is related to the connectivity of G in some sense[1]. It is well-known that the numbers of distinct eigenvalues of A, $L(A)$, and $|L|(G)$ respectively are at least one plus the diameter of G[4].

A bipartite graph is a graph whose vertices can be divided into two disjoint sets **U** and **V** such that every edge connects a vertex in **U** to one in **V**; that is, **U** and **V** are each independent sets[2, 3]. For the case that G is bipartite, the eigenvalue of $L(G)$ are that of $|L|(G)$. We are interested in the determination of eigenvalue of $|L|(G)$ for a non-bipartite graph G.

The simplest connected graphs which are not bipartite are trees with one more edge. If we add an edge to a tree to make a graph G with an odd-length cycle, then the least eigenvalue of $|L|(G)$ is not zero. Besides this, to let G have the longest diameter, we study the graph G which is obtained from a path by adding one more vertex with two neighbors: one end of the path and the neighbor of this end. The graph G is called a **lollipop graph with** K_3 of order n, denote by $L_{3,n-3}$. The paper [5] tells us that signless Laplacian matrix of a lollipop graph with K_3 has the least Laplacian eigenvalue among all non-bipartite connected graphs of order n. We are interested about the eigenvalues of $|L|(L_{3,n-3})$, and we try to find other properties about the eigenvalues of $|L|(L_{3,n-3})$.

Thesis is organized as following. First we give some definitions in graph theory and matrix theory in Section 2. In Section 3, some propositions which will be used in the thesis are given. Our main results are given in Section 4. By using a simple example, we introduce a method to compare the least Laplacian eigenvalues of two graphs with the same number of edges. We study the upper bound of the largest Laplacian eigenvalues of lollipop graphs with K_3 by using known results and using computer software Mathematica. We compute the characteristic polynomial $P_n(\lambda)$ of $|L|(L_{3,n-3})$ and find a three-term recurrence relation of $P_n(\lambda)$. Then we find 1 is a common eigenvalue of $L_{3,n-3}$ and determine its multiplicity.

2 Preliminaries

In this section, we introduce notations which we will use in this thesis.

2.1 Graphs

A graph G considered in the thesis is finite, undirected, and connected, without loops or multiple edges. We use $V(G)$ to denote the vertex set and $E(G)$ to denote the edge set of G, usually $V(G) = [n] = \{1,2,\dots,n\}$. The cardinality $|V(G)|$ is called the **order** of G. The following special graphs with vertex set $[n]$ and their corresponding symbols are used in the thesis.

1. The **complete graph** K_n: $E(K_n) = \{ij \mid 1 \le i < j \le n\}$.

2. The **path** P_n: $E(P_n) = \{i\,i+1 \mid 1 \le i \le n-1\}$.

3. The $(m,n-m)$-**lollipop** $L_{m,n-m}$: $E(L_{m,n-m}) = \{ij \mid 1 \le i < j \le m\} \cup \{i\,i+1 \mid m \le i \le n-1\}$, where $m \le n$. See Figure 1 for $L_{3,n-3}$.

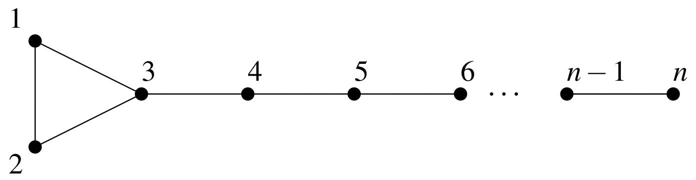

Figure 1. The graph $L_{3,n-3}$.

2.2 Matrices

Definition 2.1. Let M be a $n \times n$ square matrix, if there is a vector $v \in \mathbb{R}^n$ such that

$$Mv = \lambda v$$

for some scalar λ. Then λ is called the **eigenvalue** of M corresponding to v, and the vector v is called the **eigenvector** of λ.

Let G be a graph of order n. The matrices considered in the thesis are all symmetric over the real number field $\mathbb{R}$ whose rows and columns are indexed by $V(G)$. Let $D(G)$ denote the diagonal matrix with rows and columns indexed by vertices of G such that $D(G)_{xx} = d(x)$ which the $d(x)$ is degree of x in G. Then the **adjacency matrix** $A(G)$, **Laplace matrix** $L(G)$, **signless laplacian matrix** $|L|(G)$ are defined as follows.

(i) $A(G)_{xy} = \begin{cases} 1, & \text{if } xy \in E(G); \\ 0, & \text{otherwise.} \end{cases}$

(ii) $L(G) = D(G) - A(G)$,

(iii) $|L|(G) = D(G) + A(G)$.

In the thesis, we only study the signless lpalacian matrix $|L|(G)$ of a graph G.

Let $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$ be the **eigenvalues** of $|L|(G)$, and we refer to this sequence as the **spectrum** of $|L|(G)$, or that of G for short. If the graph G is clear, we might delete the symbol G in a notation $\ell(G)$ and write it as ℓ.

We use the symbol $G \setminus e$ to denote the graph with the same vertex set as $V(G)$ and obtained by deleting the edge e of G.

2.3 Characteristic polynomial

The **characteristic polynomial** of a square matrix M is the polynomial $\det(\lambda I - M)$. It is well-known that the eigenvalues of M are the roots of the characteristic polynomial of M.

The following example will be uesed in Lemma 4.1 and 4.6

Example 2.2.

$$|L|(L_{3,1}) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, |L|(L_{3,2}) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then

$$\det(\lambda I - |L|(L_{3,1})) = \lambda^4 - 8\lambda^3 + 19\lambda^2 - 16\lambda + 4,$$

$$\det(\lambda I - |L|(L_{3,2})) = \lambda^5 - 10\lambda^4 + 34\lambda^3 - 48\lambda^2 + 27\lambda - 4.$$

The spectrum of $|L|(L_{3,1})$ is $\{\frac{5-\sqrt{17}}{2}, 1, 2, \frac{5+\sqrt{17}}{2}\}$, and the spectrum of $|L|(L_{3,2})$ is $\{0.2243, 1, 1.4108, 2.7237, 4.6412\}$, computed by Mathematica.

2.4 Interlacing of two sequences

For $m < n$, a sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ of real numbers is said to **interlace** another sequence $q_1 \geq q_2 \geq \cdots \geq q_n$ of real numbers if

$$q_i \geq \lambda_i \geq q_{n-m+i} \quad \text{for} \quad 1 \leq i \leq m.$$

3 Basic properties

In this section, we shall review a few basic properties in matrix theory and some previous results in the study of spectrum of a graph. For completeness, we shall provide the proofs of some properties.

3.1 Rayleigh's principle

It is well-known that the largest eigenvalue λ_1 and the least eigenvalue λ_n of a symmetric matrix M or order n satisfy

$$\lambda_1 = \max_{0 \neq x \in \mathbb{R}^n} \frac{x^\top M x}{x^\top x}, \qquad \lambda_n = \min_{0 \neq x \in \mathbb{R}^n} \frac{x^\top M x}{x^\top x}.$$

The following proposition generalizes this property.

Proposition 3.1. *Let M be a real symmetric matrix of order n with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and respective orthonormal eigenvectors $u_1, u_2, \cdots, u_n$. Then*

(i) $\dfrac{u^\top Mu}{u^\top u} \geq \lambda_i$ *for any $u \in \mathrm{Span}(u_1, u_2, \cdots, u_i)$, and equality holds iff u is an eigenvector of M corresponding to λ_i,*

(ii) $\dfrac{u^\top Mu}{u^\top u} \leq \lambda_{i+1}$ *for any $u \in \mathrm{Span}(u_1, u_2, \cdots, u_i)^\perp$, and equality holds iff u is an eigenvector of M corresponding to λ_{i+1}.*

Proof. (i) Write $u = c_1u_1 + \cdots + c_iu_i$ for some $c_j \in \mathbb{R}$, $j = 1, 2, \cdots, i$. Then

$$\frac{u^\top Mu}{u^\top u} = \frac{c_1^2\lambda_1 + c_2^2\lambda_2 + \cdots + c_i^2\lambda_i}{c_1^2 + c_2^2 + \cdots + c_i^2} \geq \lambda_i.$$

If u is an eigenvector of M corresponding to λ_i, then by the definition of eigenvector,

$$\begin{aligned} Mu &= \lambda_i u, \\ u^\perp Mu &= u^\perp \lambda_i u, \\ u^\perp Mu &= \lambda_i u^\perp u, \\ \frac{u^\perp Mu}{u^\perp u} &= \lambda_i. \end{aligned}$$

Hence the equality holds. If the equality holds, then $\lambda_j = \lambda_i$ if $c_j \neq 0$, where $j \leq i$. Hence u is an eigenvector of M corresponding to λ_i.

(ii) Similar to the above proof except here we use

$$\mathrm{Span}(u_1, u_2, \cdots, u_i)^\perp = \mathrm{Span}(u_{i+1}, u_{i+2}, \cdots, u_n).$$

□

3.2 Interlacing property for edge deleting

We shall show that deleting an edge from a graph G does not increase any value of the spectrum of G.

Definition 3.2. An $m\times m$ matrix P is a **principal submatrix** of an $n\times n$ matrix M, where $m<n$, if P is obtained from M by removing any $n-m$ rows and the same $n-m$ columns.

The following lemma describes the relation between eigenvalues of a symmetric matrix and that of its principal submatrix.

Lemma 3.3. *If an $m\times m$ matrix P is a principal submatrix of an $n\times n$ symmetric matrix M, where $m<n$, then the eigenvalues of P interlace those of M.*

Proof. To simplify the notation, we may assume P is obtained from M by removing the last $n-m$ rows and columns. Then we can write $P=S^\top MS$, where S is an $n\times m$ matrix of the form

$$\begin{pmatrix} I_m \\ 0_{(n-m)\times m} \end{pmatrix},$$

and I_m is the $m\times m$ identity matrix. In particular $S^\top S=I_m$.

Let $u_1,u_2,\cdots,u_n$ be orthonormal eigenvectors of M corresponding to eigenvalues $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$ respectively and $v_1,v_2,\cdots,v_m$ be orthonormal eigenvectors of P corresponding to eigenvalues $\eta_1\ge\eta_2\ge\cdots\ge\eta_m$ respectively. Note that

$$\dim\left(\mathrm{Span}(v_1,v_2,\cdots,v_i)\cap\mathrm{Span}(S^\top u_1,S^\top u_2,\cdots,S^\top u_{i-1})^\perp\right)\ge i+(n-i+1)-n=1.$$

Hence there exists a nonzero vector $s_i\in\mathrm{Span}(v_1,v_2,\cdots,v_i)\cap\mathrm{Span}(S^\top u_1,S^\top u_2,\cdots,S^\top u_{i-1})^\perp$. Note that $(Ss_i)^\top u_j=s_i^\top S^\top u_j=0$ for $1\le j\le i-1$, hence $Ss_i\in\mathrm{Span}(u_1,u_2,\cdots,u_{i-1})^\perp$ and by

Rayleigh's principle,

$$\lambda_i \geq \frac{(Ss_i)^\top M(Ss_i)}{(Ss_i)^\top (Ss_i)} = \frac{(s_i)^\top P(s_i)}{(s_i)^\top (s_i)} \geq \eta_i.$$

By applying the above inequality to $-M$ and $-P$ we get $\lambda_{n-m+i} \leq \eta_i$. Hence

$$\lambda_{n-m+i} \leq \eta_i \leq \lambda_i.$$

□

Definition 3.4. Let M denote the (vertex-edge) **incidence matrix** of G, i.e. M is a matrix with rows indexed by vertices and columns indexed by edges, such that for $x \in V(G)$ and $e \in E(G)$,

$$M_{xe} = \begin{cases} 1, & x \in e; \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma describes the relation between incidence matrix and signless lpalacian matrix.

Lemma 3.5. *Let M be the incidence matrix of G. Then $|L|(G) = MM^\top$.*

Proof. Note that for $x, y \in V(G)$,

$$(MM^\top)_{xy} = \sum_e M_{xe}(M^\top)_{ey} = \begin{cases} 1, & \text{if } xy \in E(G); \\ d(x), & \text{if } x = y; \\ 0, & \text{otherwise,} \end{cases} = (|L|(G))_{xy},$$

where $d(x)$ is the degree of x. □

The following lemma indicates the relation between two matrices which have the same eigenvalues.

Lemma 3.6. *Let N be an $n \times m$ matrix. Then there exists a one-one correspondence between the nonzero eigenvalues of $NN^\top$ and $N^\top N$.*

Proof. Suppose q is a nonzero eigenvalue of $NN^\top$ with corresponding eigenvector u. Then $NN^\top u = qu \neq 0$. In particular $N^\top u \neq 0$. Since $N^\top NN^\top u = qN^\top u$, $N^\top u$ is an eigenvector of $N^\top N$ corresponding to the eigenvalue q. Suppose q has multiplicity m as an eigenvalue of $NN^\top$. Let $u_1, u_2, \cdots, u_m$ be the corresponding orthogonal eigenvectors. If $c_1 N^\top u_1 + \cdots + c_m N^\top u_m = 0$ then

$$0 = N(c_1 N^\top u_1 + \cdots + c_m N^\top u_m) = q(c_1 u_1 + \cdots + c_m u_m),$$

and hence $c_1 = c_2 = \cdots = c_m = 0$. This proves that the multiplicity of q in $NN^\top$ is no larger than that in $N^\top N$. Similarly for the other side, so the two multiplicities are the same. □

The following proposition demonstrates the interlacing property for edge deleting.

Proposition 3.7.

$$q_{i+1}(G) \leq q_i(G \setminus e) \leq q_i(G) \quad \text{for} \quad 1 \leq i \leq n-1.$$

Proof. Let M denote the vertex-edge incidence matrix of G and recall that $|L|(G) = MM^\top$ by Lemma 3.5. Note that the incidence matrix M' of $G \setminus e$ is obtained from M be deleting the column associated with e. Hence $M'^\top M'$ is a principal submatrix of $M^\top M$. By interlacing property in Lemma 3.3, the sequence of eigenvalues of $(n-1) \times (n-1)$ matrix $M'^\top M'$ interlaces that of $n \times n$ matrix $M^\top M$. Since $M^\top M$ and $MM^\top$ have same nonzero eigenvalues by Lemma 3.6, we have

$$q_{i+1}(G) \leq q_i(G \setminus e) \leq q_i(G) \quad \text{for} \quad 1 \leq i \leq n-1.$$

□

3.3 Bounds of the largest signless laplacian eigenvalue

We shall provide some known bounds of the largest eigenvalue $q_1(G)$ of G. For $v \in V(G)$, denote the neighbor of v by $N(v)$, and define

$$m(v) := \sum_{u \in N(v)} \frac{d(u)}{d(v)},$$

the average of the degrees of the vertices adjacent to v. Let $\Delta(G)$ be the maximal degree of G and N_i be the set of the neighbor of the vertex v_i. The following proposition is well-known.

Proposition 3.8. *[6]*

(i) $q_1(G) \leq max\{d(v_i) + d(v_j) : v_i v_j \in E(G)\}$,

(ii) $q_1(G) \leq max\{m(v_i) + d(v_i) : v_i \in V(G)\}$,

(iii) $q_1(G) \leq max\{d(v_i) + d(v_j) - |N_i \cap N_j| : 1 \leq i < j \leq n, v_i v_j \in E(G)\}$,

(iv) $q_1(G) \geq \Delta(G) + 1$. □

4 Main Results

Since we are mainly concerned with the graph $(3, n-3)$-lollipop $L_{3,n-3}$, we will use the symbol $|L|_n$ to denote the signless laplacian matrix of the graph $L_{3,n-3}$. We study the spectrum of $|L|_n$ in this section.

4.1 The least eigenvalue of $L_{3,2}$

From edge-interlacing in Proposition 3.7, we know that

$$q_5(L_{3,1}) \le q_5(L_{3,2}) \le q_5(L_{3,3}) \le \cdots,$$

since $L_{3,n-3}$ and the graph obtained by removing the edge $n\ n+1$ from $L_{3,n-2}$ have the same spectrum. However if two graphs have the same number of edges, it is impossible to use the edge-interlacing in Proposition 3.7 to compare their spectra.

Consider the graph $L_{3,2}$ in Figure 2, and the graph G_5 in Figure 3. Their corresponding signless laplician matrices $|L|_5$ and $|L|(G_5)$ as shown below.

$$|L|_5 = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad |L|(G_5) = \begin{pmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ 1 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

$L_{3,2}$ and G_5 have the same number of edges. We present another method to compare the least eigenvalue of $L_{3,2}$ and G_5.

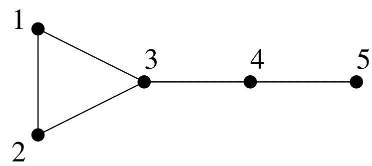

Figure 2. The graph $L_{3,2}$.

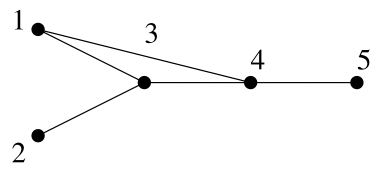

Figure 3. The graph G_5.

Let

$$A = |L|_5 - |L|(G_5) = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $x = (x_1, x_2, \cdots, x_5)^\top$ be the eigenvector of $|L|(G_5)$ corresponding to $q_5(G_5)$, and $x^\top Ax = (x_1 + x_2)^2 - (x_1 + x_4)^2$, and by Rayleigh's principle,

$$\frac{x^\top Ax}{x^\top x} = \frac{x^\top(|L|_5)x}{x^\top x} - q_5(G_5) \geq q_5(L_{3,2}) - q_5(G_5).$$

If we can prove

$$\frac{x^\top Ax}{x^\top x} \leq 0,$$

then $q_5(L_{3,2}) - q_5(G_5) \leq 0$, so $q_5(L_{3,2}) \leq q_5(G_5)$. Thus we need to find the eigenvector of $|L|(G_5)$ corresponding to $q_5(G_5)$. This is not a good idea, because in this case computing by Mathematica,

$$x = (0, -0.6015, 0.3717, -0.3717, 0.6015),$$

and

$$\frac{x^\top A x}{x^\top x} = \frac{x^\top (|L|_5) x}{x^\top x} - q_5(G_5) = 0.6056 - 0.3820 = 0.2236 > 0.$$

Now we try to change the symbols of vertices of $L_{3,2}$, denoted by $L'_{3,2}$. We switch the indices 2 and 4 as shown in Figure 4, so the new matrix is

$$|L'|_5 = \begin{pmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

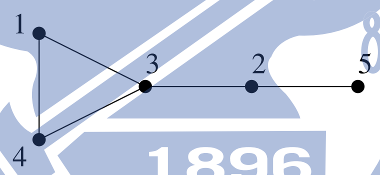

Figure 4. The graph $L'_{3,2}$

First we need a lemma.

Lemma 4.1. *Let* $x = (x_1, x_2, \cdots, x_5)^\top$ *be the eigenvector of* $|L|(G_5)$ *corresponding to* $q_5(G_5)$. *Then* $|x_2| = |x_5| > |x_3| = |x_4|$, *and* $x_1 = 0$.

Proof. Let q_5 be the least eigenvalue of $|L|(G_5)$, where G_5 is defined in Figure 3. Since deleting the edge 45 in the graph G_5 yields $L_{3,1}$, we have

$$q_5 \le q_4(L_{3,1}) = \frac{5 - \sqrt{17}}{2} < 1$$

by edge-interlacing in Proposition 3.7 and Example 2.2. By the definition of eigenvalue and eigenvector,

$$0 = (|L|(G_5) - q_5 I)x = \begin{pmatrix} (2-q_5)x_1 + 0 + x_3 + x_4 + 0 \\ 0 + (1-q_5)x_2 + x_3 + 0 + 0 \\ x_1 + x_2 + (3-q_5)x_3 + x_4 + 0 \\ x_1 + 0 + x_3 + (3-q_5)x_4 + x_5 \\ 0 + 0 + 0 + x_4 + (1-q_5)x_5 \end{pmatrix}.$$

Considering the second and fifth entries, and since $q_5 < 1$, we have

$$|x_2| = |\frac{x_3}{q_5 - 1}| > |x_3|, \quad |x_5| = |\frac{x_4}{q_5 - 1}| > |x_4|, \quad x_2 x_4 = x_3 x_5.$$

Notice that any one of x_2, x_3, x_4, x_5 is zero will imply $x = 0$, a contradiction. Considering the third and fourth entries, we have the following equations, step after step:

$$\begin{aligned} x_1 + (3-q_5)x_3 + x_4 &= -x_2, \\ x_1 + x_3 + (3-q_5)x_4 &= -x_5, \\ \frac{x_1 + (3-q_5)x_3 + x_4}{x_1 + x_3 + (3-q_5)x_4} &= \frac{x_2}{x_5} = \frac{x_3}{x_4}, \\ x_1x_4 + x_4^2 + (3-q_5)x_3x_4 &= x_1x_3 + x_3^2 + (3-q_5)x_3x_4, \\ x_1x_4 - x_1x_3 &= x_3^2 - x_4^2, \\ x_1(x_4 - x_3) &= (x_3 - x_4)(x_3 + x_4), \\ -x_1 &= (x_3 + x_4). \end{aligned}$$

Considering the first entry, $(2-q_5)x_1 + x_3 + x_4 = 0$, and by $-x_1 = (x_3 + x_4)$, we have $(2 - q_5)x_1 - x_1 = 0$, and then $(1-q_5)x_1 = 0$. Since $(1-q_5) \neq 0$, we conclude that $x_1 = 0$. This implies that $(x_3 + x_4) = -x_1 = 0$, so $x_2/x_5 = x_3/x_4 = -1$. $\square$

Let

$$A' = |L'|_5 - |L|({}_5G) = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and as before let $x = (x_1, x_2, \cdots, x_5)^\top$ be the eigenvector of $|L|(G_5)$ corresponding to $q_5(G_5)$. Then

$$\frac{x^\top(|L'|_5)x}{x^\top x} - q_5(G_5) = \frac{x^\top A' x}{x^\top x} = \frac{(x_1+x_4)^2 - (x_1+x_2)^2}{x^\top x}.$$

Hence

$$0 > \frac{(x_4)^2 - (x_2)^2}{x^\top x} = \frac{(x_1+x_4)^2 - (x_1+x_2)^2}{x^\top x} = \frac{x^\top(|L'|_5)x}{x^\top x} - q_5(G_5) \geq q_5(L'_{3,2}) - q_5(G_5).$$

Since $q_5(L_{3,2}) = q_5(L'_{3,2})$, we have the following Lemma.

Lemma 4.2. $q_5(L_{3,2}) < q_5(G_5)$. □

If we extend the definition of G_5 to the graph G_n of order n by adding more vertices $6, 7, \ldots, n$ and edges 56, 67, …, $n-1$ n, then generalized the above arguments, one can show that $q_n(L_{3,n-3}) < q_n(G_n)$. Because the matrix $B = |L|_n - |L|(G_n)$ is the principal submatrix of A, $x^\top Bx$ is the same of above result, so $q_n(L_{3,n-3}) < q_n(G_n)$.

4.2 The largest eigenvalue

We shall study the largest signless laplacian eigenvalue $q_1(L_{3,n-3})$ of $L_{3,n-3}$ in this section. We use proposition 3.8 to compute the upper bound of the graphs $L_{3,n-3}$ for $n \geq 4$.

(i) $q_1(L_{3,n-3}) \leq max\{d(v_i)+d(v_j) : v_iv_j \in E(L_{3,n-3})\}$:

$$\begin{aligned}
d(v_1)+d(v_2) &= 4, \\
d(v_2)+d(v_3) &= 5 = d(v_1)+d(v_3), \\
d(v_3)+d(v_4) &= 5, \\
d(v_4)+d(v_5) &= 5 = d(v_i)+d(v_{i+1}) \ for\ 4 \leq i \leq n-2, \\
d(v_{n-2})+d(v_{n-1}) &= 4, \\
d(v_{n-1})+d(v_n) &= 3.
\end{aligned}$$

So $q_1(L_{3,n-3}) \leq 5$.

(ii) $q_1(L_{3,n-3}) \leq max\{m(v_i)+d(v_i) : v_i \in V(L_{3,n-3})\}$:

Since $m(v) = \sum_{u \in N(v)} \frac{d(u)}{d(v)}$

$$\begin{aligned}
m(v_1)+d(v_1) &= \frac{d(v_2)}{d(v_1)} + \frac{d(v_3)}{d(v_1)} + d(v_1) = \frac{5}{2}+2 = \frac{9}{2} = m(v_2)+d(v_2), \\
m(v_3)+d(v_3) &= \frac{d(v_1)}{d(v_3)} + \frac{d(v_2)}{d(v_3)} + \frac{d(v_4)}{d(v_3)} + d(v_3) = 2+3 = 5, \\
m(v_4)+d(v_4) &= \frac{d(v_3)}{d(v_4)} + \frac{d(v_5)}{d(v_4)} + d(v_4) = \frac{5}{2}+2 = \frac{9}{2}, \\
m(v_5)+d(v_5) &= \frac{d(v_4)}{d(v_5)} + \frac{d(v_6)}{d(v_5)} + d(v_5) = 2+2 = 4, \\
m(v_5)+d(v_5) &= m(v_i)+d(v_i) \ for\ 5 \leq i \leq n-2, \\
m(v_{n-1})+d(v_{n-1}) &= \frac{d(v_{n-2})}{d(v_{n-1})} + \frac{d(v_n)}{d(v_{n-1})} + d(v_{n-1}) = \frac{3}{2}+2 = \frac{7}{2}, \\
m(v_n)+d(v_n) &= \frac{d(v_{n-1})}{d(v_n)} + d(v_n)+1 = 2+1 = 3.
\end{aligned}$$

So $q_1(L_{3,n-3}) \leq 5$.

(iii) $q_1(L_{3,n-3}) \leq max\{d(v_i)+d(v_j)-|N_i \cap N_j| : 1 \leq i < j \leq n, v_iv_j \in E(L_{3,n-3})\}$:

$$\begin{aligned}
d(v_1)+d(v_2)-|N_1 \cap N_2| &= 4-1=3,\\
d(v_2)+d(v_3)-|N_2 \cap N_3| &= 4 = d(v_1)+d(v_3)-|N_1 \cap N_3|,\\
d(v_3)+d(v_4)-|N_3 \cap N_4| &= 5,\\
d(v_4)+d(v_5)-|N_4 \cap N_5| &= 5 = d(v_i)+d(v_{i+1})-|N_i \cap N_{i+1}| \; for \; 4 \leq i \leq n-2,\\
d(v_{n-2})+d(v_{n-1})-|N_{n-2} \cap N_{n-1}| &= 4,\\
d(v_{n-1})+d(v_n)-|N_{n-1} \cap N_n| &= 3.
\end{aligned}$$

So $q_1(L_{3,n-3}) \leq 5$.

(iv) $q_1(G) \geq \Delta(G)+1$: We have $q_1(L_{3,n-3}) \geq \Delta(L_{3,n-3})+1 = 4$ in this case.

From the above discussing, we conclude that $4 \leq q_1(L_{3,n-3}) \leq 5$. By using edge-interlacing in Proposition 3.7, a better lower bound of the least eigenvalue of $|L_n|$ will be found. Since $q_1(L_{3,1}) = \frac{5+\sqrt{17}}{2}$ by example 2.2, we have

$$\frac{5+\sqrt{17}}{2} \leq q_1(L_{3,n-3}) \leq 5.$$

To the end of this part, we use Mathematica to compute the $q_1(L_{3,n-3})$ for $4 \leq n \leq 10$ as follows.

$q_1(L_{3,1})$	$q_1(L_{3,2})$	$q_1(L_{3,3})$	$q_1(L_{3,4})$	$q_1(L_{3,5})$	$q_1(L_{3,6})$	$q_1(L_{3,7})$
4.5616	4.6412	4.6554	4.6582	4.6588	4.6589	4.6590

These numbers are close to 4.66.

4.3 Characteristic polynomial

One way to study eigenvalues of a matrix is to compute the characteristic polynomial of the matrix and determine its roots. Let

$$|L|_n = \begin{pmatrix} 2 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \ddots & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, B_n = \begin{pmatrix} 2 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \ddots & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

be an $n \times n$ matrix for $n \geq 3$. Note that $B_n = |L|_n + E_{nn}$, where E_{nn} is the binary matrix with a unique 1 in the nn-th position. We need B_n to compute the determinant of $|L|_n$.

Example 4.3.

$$B_3 := \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}, B_4 = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, B_5 = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Let $P_n(\lambda)$ and $F_n(\lambda)$ be the characteristic polynomial of $|L|_n$ and B_n, respectively.

Lemma 4.4. *$P_n(\lambda) = (\lambda - 1)F_{n-1}(\lambda) - F_{n-2}(\lambda)$ for $n \geq 5$.*

Proof. Note that

$$P_n(\lambda) = \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda-1 \end{pmatrix}_{n\times n}.$$

We expand about the determinant according to the nth column:

$$P_n(\lambda) = (\lambda - 1) \quad \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda-2 \end{pmatrix}_{n-1\times n-1}$$

$$-(-1) \quad \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{n-1\times n-1}$$

$$= (\lambda - 1) \quad \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda-2 \end{pmatrix}_{n-1\times n-1}$$

$$- \quad \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda-2 \end{pmatrix}_{n-2\times n-2} .$$

Hence we have

$$P_n(\lambda) = (\lambda - 1)F_{n-1}(\lambda) - F_{n-2}(\lambda).$$

□

Next we derive a recurrence relation for $F_n(\lambda)$

Lemma 4.5. $F_n(\lambda) = (\lambda - 2)F_{n-1}(\lambda) - F_{n-2}(\lambda)$ *for* $n \geq 5$.

Proof. Note that

$$F_n(\lambda) = \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda-2 \end{pmatrix}_{n\times n}.$$

We expand about the determinant according to the nth column:

$$F_n(\lambda) = (\lambda-2) \quad \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda-2 \end{pmatrix}_{n-1\times n-1}$$

$$
-(-1)\quad \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{n-1\times n-1}
$$

$$
= (\lambda-2)\quad \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda-2 \end{pmatrix}_{n-1\times n-1}
$$

$$
-\quad \det \begin{pmatrix} \lambda-2 & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda-2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & \lambda-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & \lambda-2 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda-2 \end{pmatrix}_{n-2\times n-2} .
$$

Hence we have

$$
F_n(\lambda) = (\lambda-2)F_{n-1}(\lambda) - F_{n-2}(\lambda).
$$

□

Lemma 4.6.

$$P_n(\lambda) = (\lambda - 2)P_{n-1}(\lambda) - P_{n-2}(\lambda) \;\; \textit{for } n \geq 6,$$

where initial functions are

$$\begin{aligned} P_4(\lambda) &= \lambda^4 - 8\lambda^3 + 19\lambda^2 - 16\lambda + 4, \\ P_5(\lambda) &= \lambda^5 - 10\lambda^4 + 34\lambda^3 - 48\lambda^2 + 27\lambda - 4. \end{aligned}$$

Proof. The initial functions are computed in Example 2.2. In general for $n \geq 6$,

$$\begin{aligned} P_n(\lambda) =& (\lambda - 1)F_{n-1}(\lambda) - F_{n-2}(\lambda) \qquad \text{(Lemma 4.4)} \\ =& (\lambda - 1)[(\lambda - 2)F_{n-2}(\lambda) - F_{n-3}(\lambda)] - [(\lambda - 2)F_{n-3}(\lambda) - F_{n-4}(\lambda)] \quad \text{(Lemma 4.5)} \\ =& (\lambda - 2)[(\lambda - 1)F_{n-2}(\lambda) - F_{n-3}(\lambda)] - [(\lambda - 1)F_{n-3}(\lambda) - F_{n-4}(\lambda)] \\ =& (\lambda - 2)P_{n-1}(\lambda) - P_{n-2}(\lambda) \qquad \text{(Lemma 4.4).} \end{aligned}$$

□

From the above recurrence relation, we obtain the following two theorems.

Theorem 4.7.

(i) 1 *is an eigenvalue of* $|L|_n$ *for* $n \geq 4$.

(ii) 2 *is an eigenvalue of* $|L|_n$ *for even* $n \geq 4$.

Proof. We prove by induction.

(i) This follows from $P_4(1) = P_5(1) = 0$ and the recurrence of $P_n(x)$ in Lemma 4.6.

(ii) This follows from $P_4(2) = 0$ and the recurrence of $P_n(x)$ in Lemma 4.6.

□

Lemma 4.8. *For* $4 \leq n \equiv 0 \pmod 3$, 1 *is an eigenvalue of* $|L|_n$ *with multiplicity at least* 2*, and for* $4 \leq n \not\equiv 0 \pmod 3$, 1 *is a simple eigenvalue of* $|L|_n$.

Proof. Computing the derivatives of $P_4(\lambda)$, $P_5(\lambda)$ and $P_n(\lambda)$ in Lemma 4.6,

$$\begin{aligned} P_4'(\lambda) &= 4\lambda^3 - 24\lambda^2 + 38\lambda - 16, \\ P_5'(\lambda) &= 5\lambda^4 - 40\lambda^3 + 102\lambda^2 - 96\lambda + 27, \\ P_n'(\lambda) &= P_{n-1}(\lambda) + (\lambda - 2)P_{n-1}'(\lambda) - P_{n-2}'(\lambda). \end{aligned}$$

Since $P_{n-1}(1) = 0$, we have

$$\begin{aligned} P_5'(1) &= -2, \\ P_4'(1) &= 2, \\ P_n'(1) &= -P_{n-1}'(1) - P_{n-2}'(1), \qquad (1) \\ P_6'(1) &= -P_5'(1) - P_4'(1) = -(-2) - 2 = 0. \end{aligned}$$

We prove by induction on $k \geq 2$ that

$$\begin{aligned} P_{3k-1}'(1) &= -P_{3k-2}'(1) \neq 0, \\ P_{3k}'(1) &= 0. \end{aligned}$$

This is true for $k = 2$. Suppose $k > 2$. Then by (1) and induction,

$$P'_{3k-2}(1) = -P'_{3k-3}(1) - P'_{3k-4}(1) = -P'_{3k-4}(1) \neq 0,$$

$$P'_{3k-1}(1) = -P'_{3k-2}(1) - P'_{3k-3}(1) = -P'_{3k-2}(1) \neq 0,$$

$$P'_{3k}(1) = -P'_{3k-1}(1) - P'_{3k-2}(1) = 0.$$

□

Theorem 4.9. *For $n \equiv 0$ mod 3, $n \geq 4$, $|L|_n$ has $n-1$ distinct eigenvalues, and the eigenvalue 1 has multiplity exactly 2.*

Proof. For any $n \geq 4$, since the diameter of $L_{3,n-3}$ is $n-2$, it has at least $n-2+1 = n-1$ distinct eigenvalues[4]. In the case $n \equiv 0$ *mod* 3, the eigenvalue 1 of $|L|_n$ has multiplicity at least two by the above theorem, so $|L|_n$ has exactly $n-1$ distinct eigenvalues, and the eigenvalue 1 has multiplity exactly 2. □

From Theorem 4.9, the following problem is raised.

Problem 4.10. *Determine the integer $n \geq 4$ such that the graph $L_{3,n-3}$ has n distinct signless laplcian eigenvalues.*

Example 4.11. Compute by Mathematica, we have the spectrum of $|L|_n$ for $4 \leq n \leq 7$:

$$|L|_4 \;:\; \{\frac{5-\sqrt{17}}{2}, 1, 2, \frac{5+\sqrt{17}}{2}\};$$

$$|L|_5 \;:\; \{0.2243, 1, 1.4108, 2.7237, 4.6412\};$$

$$|L|_6 \;:\; \{0.1338, 1, 1, 2, 3.2108, 4.6554\};$$

$$|L|_7 \;:\; \{0.0884, 0.7147, 1, 1.5710, 2.4798, 3.4877, 4.6582\}.$$

One might expect the answer of Problem 4.10 is $n \not\equiv 0 \pmod 3$. We leave the proof or disproof of this problem to successors.

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