國立交通大學

應用數學系

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碩　士　論　文

在Ｅ型代數結構下之Ｎ相黎曼空間的

單擺運動之確切理論與數值運算

The Exact Theory and Numerical Computations of Pendulum Motions on Riemann Surface of Genus N with Cut-Structure of Type E

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中 華 民 國 一 零 二 年 六 月

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摘要

在此篇文章中，我們主要在探討理想的單擺運動。首先，藉由多項式去逼近，並探討相對應之方程式，在這之中我們發現黎曼空間的理論是必要的，因此接著介紹如何造出相對應的黎曼空間，並利用 Mathematica 幫助我們去計算相對應的黎曼空間上的路徑積分及方程式上相關之性質。

再來，介紹橢圓函數的基本性質，我們利用橢圓函數解出原微分方程的實際解、週期及相關性質。

中 華 民 國 一 零 二 年 六 月

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Abstract

In this paper, we study the ideal pendulum equation. First, we study the nonlinear approximation of the exact theory, and the Riemann surface theory is needed. So we study the Riemann surface of genus N in various algebraic cut-structures. We then apply Mathematica to evaluate path integrals on those Riemann surfaces.

Secondly, we study the classical Elliptic functions. From which, we are able to solve the exact solution and certain properties of the pendulum motions.

June 2013

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Chapter 1

Introduction

The simple pendulum is an idealized mathematical model of a pendulum. This is a bob on the end of a massless cord suspended from a pivot, with no friction in a closed system. As we give an initial push, it will swing at a constant amplitude back and forth.

In this paper, we consider a simple pendulum motion

$$u'' + \sin u = 0\,. \tag{1.0.1}$$

Multiply the equation (1.0.1) by u', then we have

$$u'u'' + u' \sin u = 0\,. \tag{1.0.2}$$

Integrate the equation (1.0.2) and compute it, we can get

$$\begin{aligned}
&\frac{1}{2}(u')^2 - \cos u = E\,, \text{ where } E \text{ is the integration constant.}\\
&\Rightarrow (u')^2 = 2(E + \cos u)\\
&\Rightarrow u' = \pm\sqrt{2(E + \cos u)}\\
&\Rightarrow \frac{du}{dt} = \pm\sqrt{2(E + \cos u)} \qquad (1.0.3)
\end{aligned}$$

The equation (1.0.3) can then be expressed as

$$\int \frac{1}{\sqrt{2(E + \cos u)}}\, du = \pm \int dt\,. \tag{1.0.4}$$

Indeed the equation (1.0.4) is not easy to solve.

At first, we can analyze the properties of the solution of the equation. We know that $\sin u$ can be expanded by Taylor series

$$\sin u = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} u^{2k+1}, \text{ for all values of } u.$$

Use the nonlinear approximation we can get

$$\sin u \approx \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)!} u^{2k+1}, \text{ for all positive integer values of } N.$$

Let

$$P_{2N+1}(u) = \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)!} u^{2k+1}$$

So the equation(1.0.1) becomes to

$$u'' + P_{2N+1}(u) = 0 .$$

As above, after computing we obtain the following integral equation

$$\int \frac{1}{\sqrt{2(E - P_{2N+2}(u))}} du = \pm \int dt , \text{ where } E \text{ is the integration constant.}$$

Since $2(E - P_{2N+2}(u))$ is a polynomial of u, it can be written as

$$\begin{aligned} 2(E - P_{2N+2}(u)) &= (u-u_1)(u-u_2)\cdots(u-u_{2N+2}) \\ &= \prod_{k=1}^{2N+2} (u-u_k) , \end{aligned}$$

where u_k's are the roots of the equation $2(E - P_{2N+2}(u)) = 0$.

Thus, the function theory of solutions u of the equation involves $\sqrt{\prod_{k=1}^{2N+2} (u-u_k)}$.

Where the space u reside is worth investigating.

Consider a function $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$ and it is not single-valued on the complex plane $\mathbb{C}$ that we will have more to say about later. We use algebra and analysis to develop a new surface such that f becomes a single-valued function on it. This surface is called a *Riemann Surface*.

But later, in order to get the exact solution of the original equation, we use the elliptic functions, which Chapter 4 and Chapter 5 will discuss it in detail.[4]

Chapter 2

Riemann Surface

2.1 Structures of the Riemann surfaces

In this section, we use the function $f(z) = \sqrt{z}$ to show how to construct the corresponding Riemann surface for $f(z)$. Using polar form, let $z = re^{i(\theta+2k\pi)}$, $r \neq 0$, $k \in \mathbb{Z}$. Then we have

$$\begin{aligned} f(z) &= \sqrt{r}e^{\frac{1}{2}i(\theta+2k\pi)} \\ &= \sqrt{r}e^{\frac{1}{2}i\theta}e^{k\pi i} \\ &= \begin{cases} \sqrt{r}e^{\frac{1}{2}i\theta} & \text{if k is even} \\ -\sqrt{r}e^{\frac{1}{2}i\theta} & \text{if k is odd} \end{cases} \end{aligned}$$

which is a two-valued function since it has different values as θ increases by 2π. Here we want to make $f(z)$ single-valued, so we modify its domain $\mathbb{C}$ to build the corresponding Riemann surface such that f becomes single-valued on it.

Beginning with $z = re^{i\theta}$, $r \neq 0$, we have $f(z) = \sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta}$. Holding r constant, and going along any closed path once around the origin so that θ increases by 2π, $f(z)$ changes to $\sqrt{r}e^{\frac{1}{2}i(\theta+2\pi)} = -\sqrt{r}e^{\frac{1}{2}i\theta}$ which is just the

negative of its original value. (Show in Figure 2.1.1.)
Continuing above way then θ increases by 2π and $f(z)$ returns to the original value.

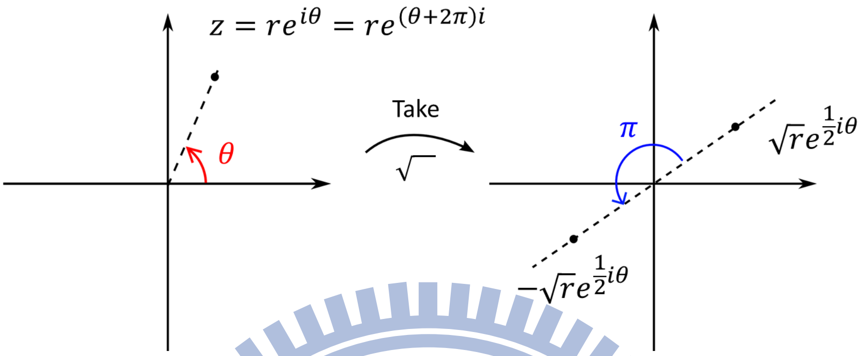


Figure 2.1.1. The idea of two sheets

We cut the plane along the negative real axis and restrict ourselves so as never to continue $f(z)$ over this cuts, then we get two single-valued branches of $f(z)$ when defined by the equations

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad \theta \in [-\pi, \pi) \qquad \text{(in sheet-I)}$$

and

$$f(z) = \sqrt{r}e^{\frac{1}{2}i\theta}, \quad \theta \in [\pi, 3\pi). \qquad \text{(in sheet-II)}$$

The cut on each sheet has two edges. We label the starting edge with "+" and the terminal edge with "−". (Show in Figure 2.1.2.) Imagine that the surface as two sheets lying over the complex plane, each cut along the negative real axis.

Moreover, we extend the complex plane with the "point" at infinity to constitute the *extended complex plane*. Use stereographic projection, we can think of the two sheets as spheres.

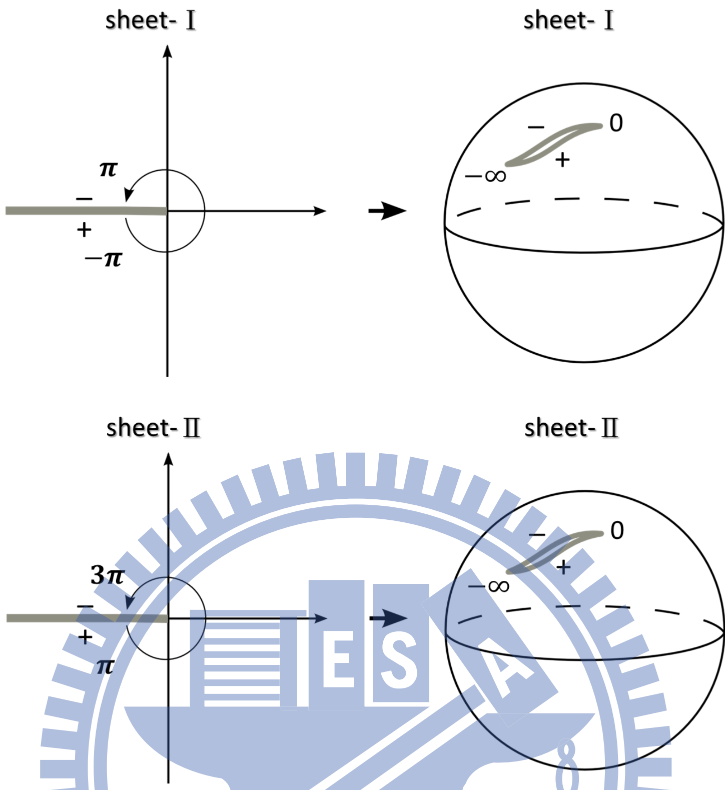
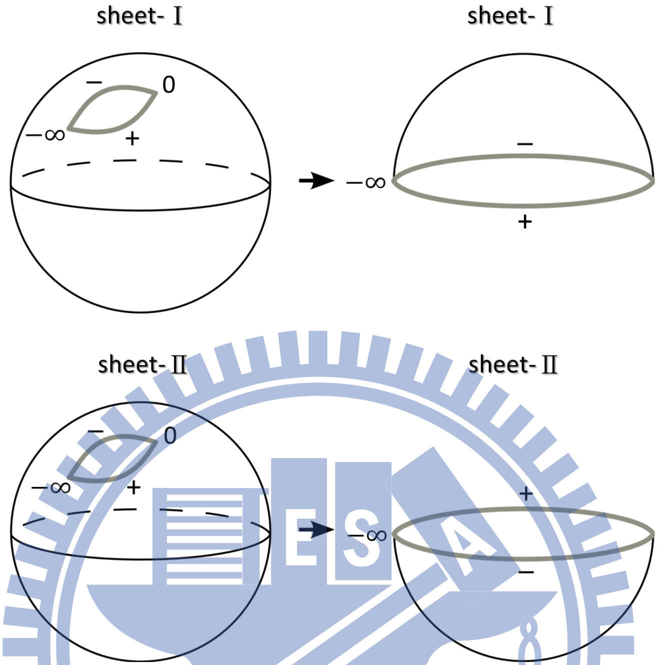


Figure 2.1.2. Two sheets and their corresponding spheres

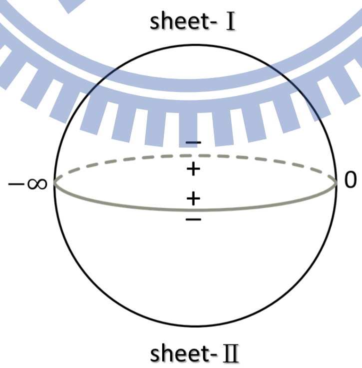
Now image that the spheres are made of rubber. By spreading the edges of the cuts, we can deform each sheet into a hemisphere.

Paste the hemispheres each other together (+)edge of sheet-I with (−)edge of sheet-II and (−)edge of sheet-I with (+)edge of sheet-II. Then we can derive a sphere, which is called the Riemann surface of genus 0 , denoted by R_0. (Show in Figure 2.1.3.)

Hence, (+)edge of sheet-I is equivalent to (−)edge of sheet-II and (−)edge of sheet-I is equivalent to (+)edge of sheet-II in the Riemann surface. As we cross the cut, we move around the other sheet.



(a) Place the cuts open and deform each sheet into a hemisphere



(b) Deform the sheets

Figure 2.1.3. Construct R_0

The following are two examples, we will construct whose corresponding Riemann surface in a similar way.

Example 2.1.

Construct the Riemann surface for $f(z) = \sqrt{\prod_{k=1}^{3}(z - z_k)} = \prod_{k=1}^{3}\sqrt{z - z_k}$, $z_k \in \mathbb{R}$ and $z_1 > z_2 > z_3$.

Solution.

First, we cut the plane starts from z_k to $-\infty$, $k = 1, 2, 3$. (Show in Figure 2.1.4.)

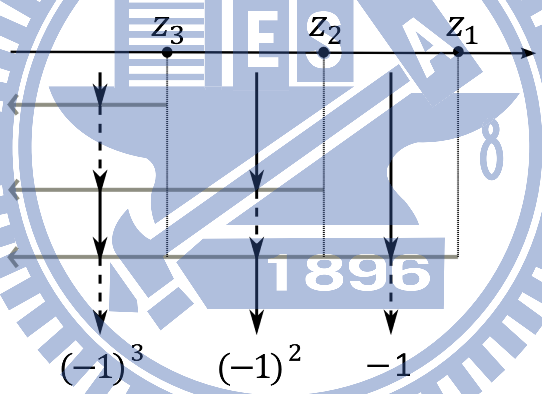


Figure 2.1.4. The cut from z_k to $-\infty$

Crossing one cut, we move around the other sheet, the argument of z increases by 2π then the argument of $f(z)$ increases by π which is just the negative of its original value. As we cross one cut, we need to change the sign by "-1". We find that crossing odd times will change the sign and even times will keep its same value. So there are branch cuts along $[-\infty, z_3]$ and $[z_2, z_1]$ as illustrated in the in Figure 2.1.5.

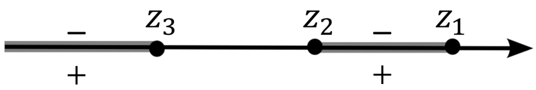
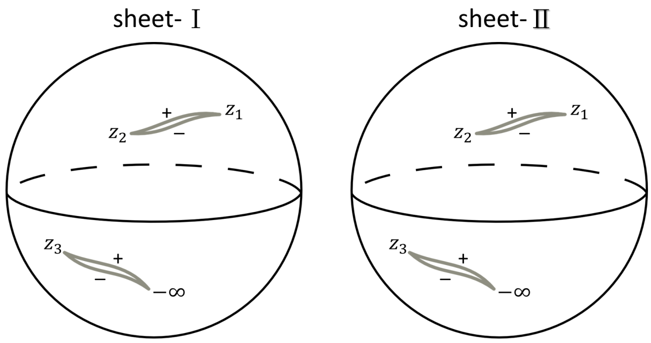
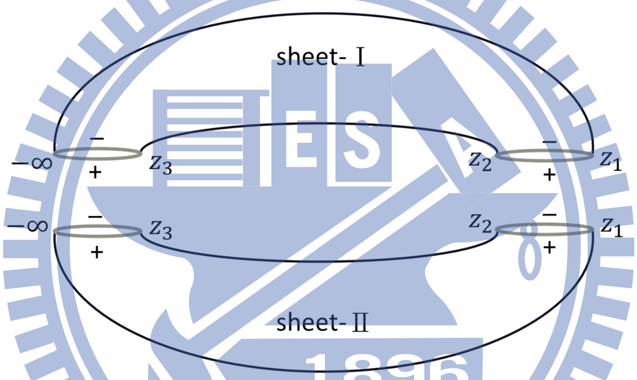


Figure 2.1.5. The cut structure

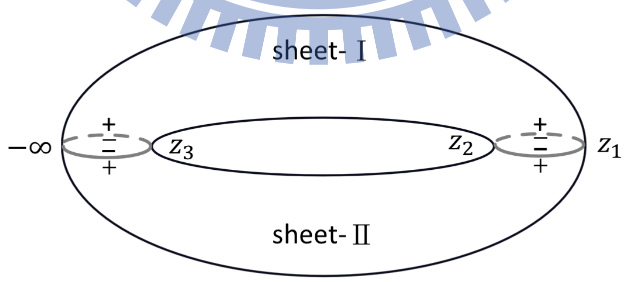
Second, placing the cuts open, pasting two sheets together (+)edge with (−)edge, and using the same idea as above. Then we obtain the corresponding Riemann surface of genus 1 for $f(z)$, denoted by R_1. (Show in Figure 2.1.6.)



(a) Cuts on the two spheres



(b) Deform the two sheets



(c) Construct R_1

Figure 2.1.6. Geometric structure for $f(z)$

Example 2.2.

Construct the Riemann surface for $f(z) = \sqrt{\prod_{k=1}^{4}(z - z_k)} = \prod_{k=1}^{4} \sqrt{z - z_k}$, $z_k \in \mathbb{R}$ and $z_1 > z_2 > z_3 > z_4$.

Solution.

As in Example 2.1, we cut the plane start from z_k to $-\infty$, $k = 1, \ldots, 4$. (Show in Figure 2.1.7.)

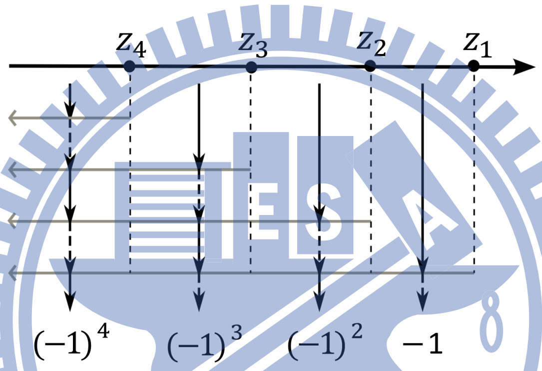


Figure 2.1.7. The cut from z_k to $-\infty$

Then there are branch cuts along $[z_4, z_3]$ and $[z_2, z_1]$. (Show in Figure 2.1.8.)

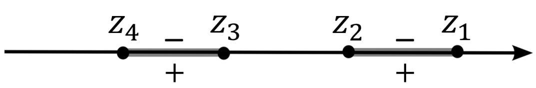


Figure 2.1.8. The cut structure

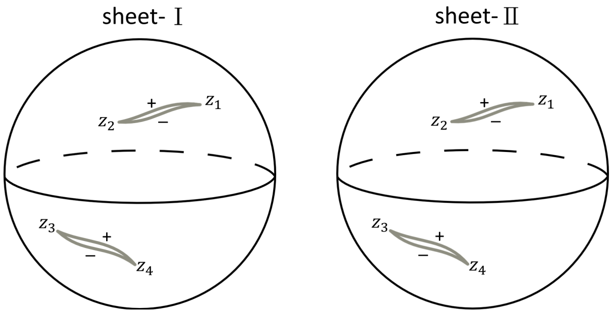


Figure 2.1.9. Construct cuts on two spheres

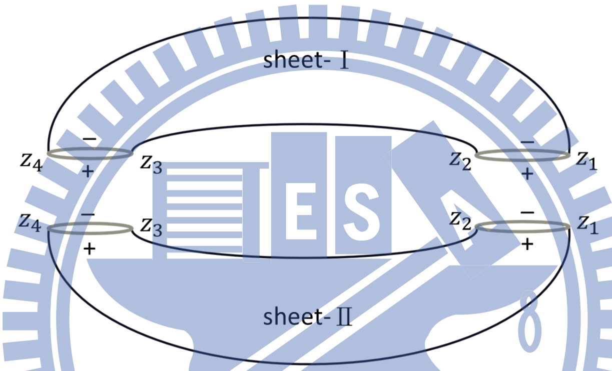


Figure 2.1.10. The deformation from the Figure 2.1.9.

Paste two sheets together (+)edge with (−)edge, then we obtain the corresponding Riemann surface of genus 1 for $f(z)$, denoted by R_1.

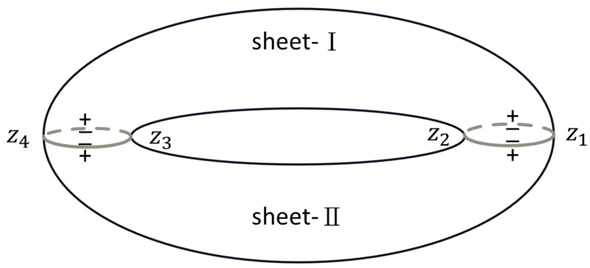


Figure 2.1.11. Construct R_1

We found $f(z)$ of 3 or 4 roots have different algebraic structures but same geometric graph with 1 holes. That is no matter 3 or 4 points, we can construct corresponding Riemann surface of genus 1.

Now we generalize the results from Example 2.1 and Example 2.2 by the following general example. Let $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$, where $z_k \in \mathbb{R}$ and $z_1 > z_2 > \cdots > z_n$. Using the same idea to construct the Riemann surface for $f(z)$. First, we cut the plane start from z_k to $-\infty$, $k = 1, \ldots, n$. (Show in Figure 2.1.12.)

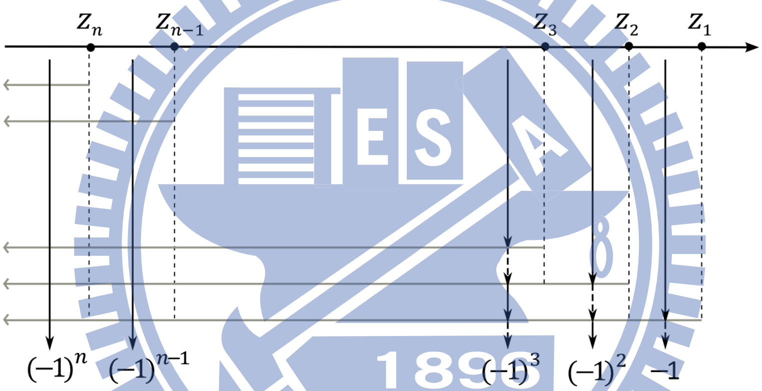


Figure 2.1.12. The cut from z_k to $-\infty$

Then we discuss the cuts structure in two cases according to n is odd or even.

Case i. If $n \in$ odd, denoted by $2N - 1$.

There are cuts along $[-\infty, z_{2N-1}], [z_{2N-2}, z_{2N-3}], \cdots, [z_{2j}, z_{2j-1}], \cdots, [z_4, z_3], [z_2, z_1]$.

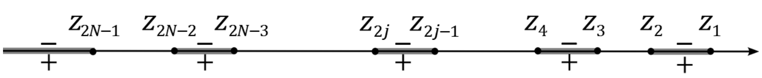


Figure 2.1.13. The cut structure as $n = 2N - 1$

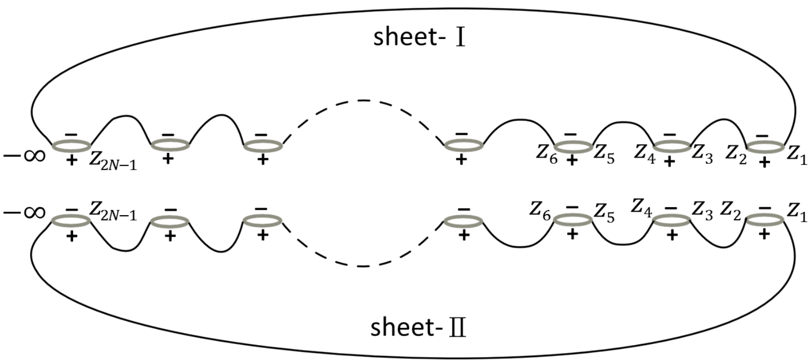


Figure 2.1.14. Together two sheets

Then we obtain the corresponding Riemann surface of genus $N-1$ for $f(z)$, denoted by R_{N-1}. (Show in Figure 2.1.15.)

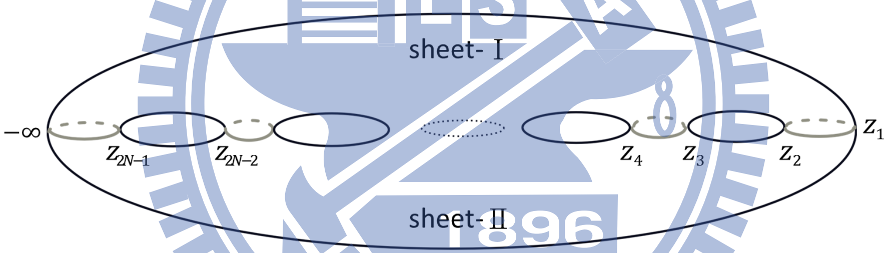


Figure 2.1.15. Geometric graph of R_{N-1}

Case ii. If $n \in$ even, denoted by $2N$.

There are cuts along $[z_{2N}, z_{2N-1}], [z_{2N-2}, z_{2N-3}], \cdots, [z_{2j}, z_{2j-1}], \cdots, [z_4, z_3], [z_2, z_1]$

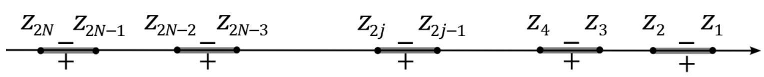


Figure 2.1.16. The cut structure as $n = 2N$

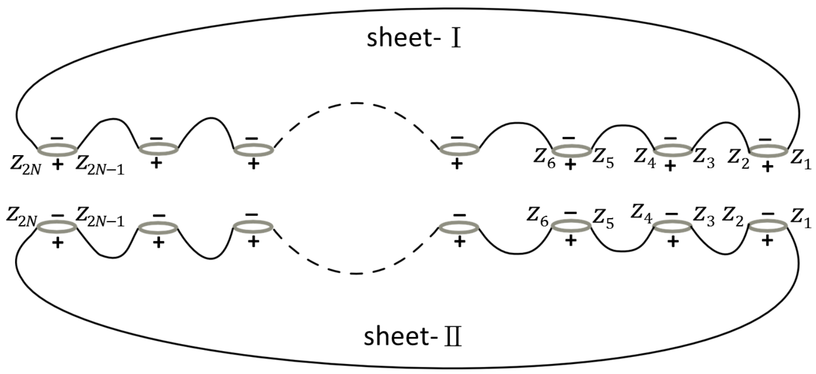


Figure 2.1.17. Together two sheets

Similarly, we obtain the corresponding Riemann surface of genus $N-1$ for $f(z)$ as the same as Case i. (Show in Figure 2.1.18.)

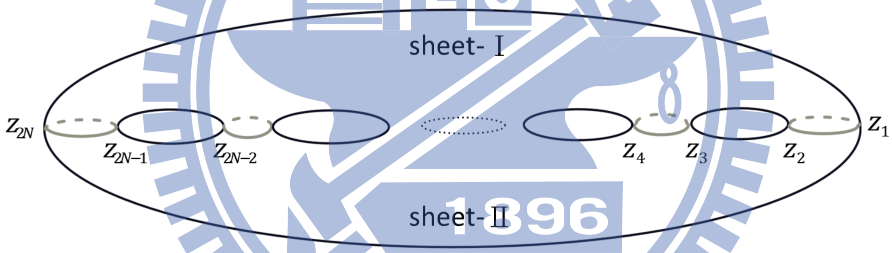


Figure 2.1.18. Geometric graph of R_{N-1}

Thus, no matter $f(z) = \sqrt{\prod_{k=1}^{2N-1} (z - z_k)}$ or $f(z) = \sqrt{\prod_{k=1}^{2N} (z - z_k)}$, there are N cuts and $N-1$ holes on the corresponding geometric structure R_{N-1} .

2.2 The contour in algebraic and geometric structure

We already comprehend the relation of algebraic and geometric structure for $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$ and how to construct the corresponding Riemann surface. In this section, we will show that the contour on the algebraic structure and its corresponding geometric structure.
Note that

1. In the algebraic structure, solid line means the contour in sheet-I and dash line means the contour in sheet-II.
2. In the geometric structure, solid line means the contour in overhead Riemann surface and dash line means the contour in ventral Riemann surface.

2.2.1 *a*- and *b*-cycles

Every closed curve on Riemann surface can be deformed into an integral combination of the loop-cut a_i and b_i, $i = 1, 2, ..., N$. So in this section, we will introduce a-,b-cycles, which can help us simplifying the computation.

- a-cycle is a simple closed curve enclosing a finite cut (the endpoint of cut is a finite number).
- b-cycle is a simple closed curve starting from (+)edge of a cut (it maybe finite cut or infinite cut) without enclosed by any a-cycle, to (+)edge of another cut enclosed by a a-cycle. Then the curve crosses through (−)edge of this cut and goes into sheet-II, and finally arrives to the (−)edge of the starting cut.

Each a-cycles are non-overlapping and each b-cycles are non-overlapping. Note that a- and b-cycles have the same amount.

Here we take $f(z) = \sqrt{z(z-1)(z-2)}$ for example to illustrate a- ,b-cycles on the cut plane and on the Riemann surface.

The a- and b-cycle are shown in Figure 2.2.1 and Figure 2.2.2.

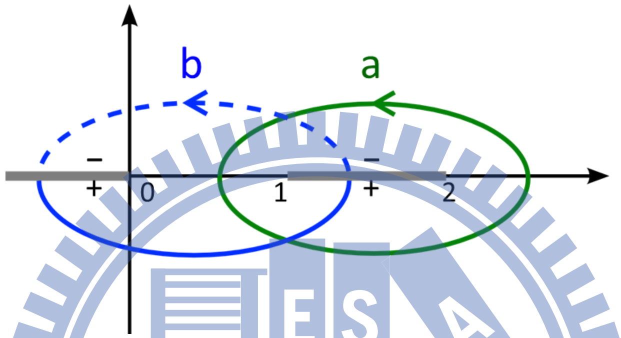
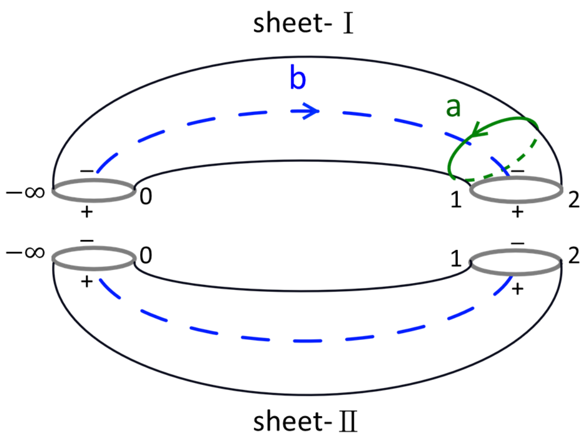


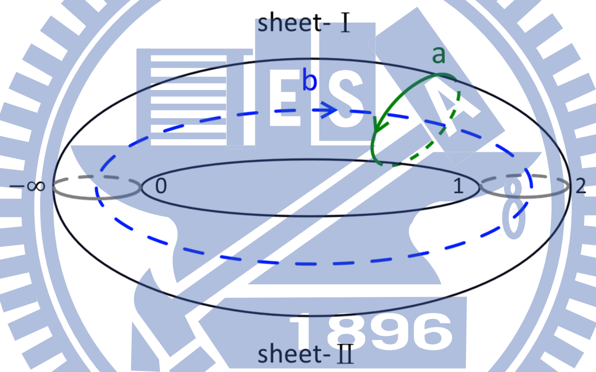
Figure 2.2.1. a- and b-cycle on the algebraic structure

In this case, $f(z)$ has three roots and then make two cuts and one a- ,b-cycle. Also, the number of a- and b-cycle are the same.

Then we get the corresponding Riemann surface of genus 1 with a- and b-cycle for $f(z)$.



(a) a- ,b-cycle on two sheets



(b) R_1 for $f(z)$ with a- ,b-cycle

Figure 2.2.2. a- and b-cycle on the corresponding geometric structure

Here we review some famous theorems.

Theorem 2.1. (Cauchy-Goursat Theorem)

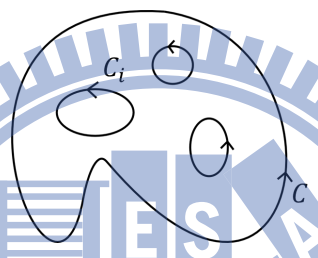
If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)dz = 0\,.$$

Theorem 2.2. (Cauchy Theorem)

Let C and $C_1, C_2, \ldots, C_n$ denote counterclockwise simple closed contours. Let all the contours C_i's be outside each other but inside C. If a function f is analytic in the closed and "holey" region consisting of those contours and all points between them, then

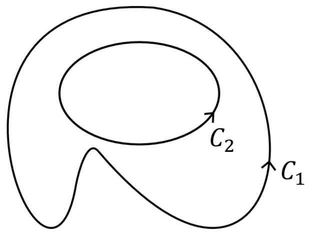
$$\int_C f(z)dz = \sum_{i=1}^{n} \int_{C_i} f(z)dz\,.$$



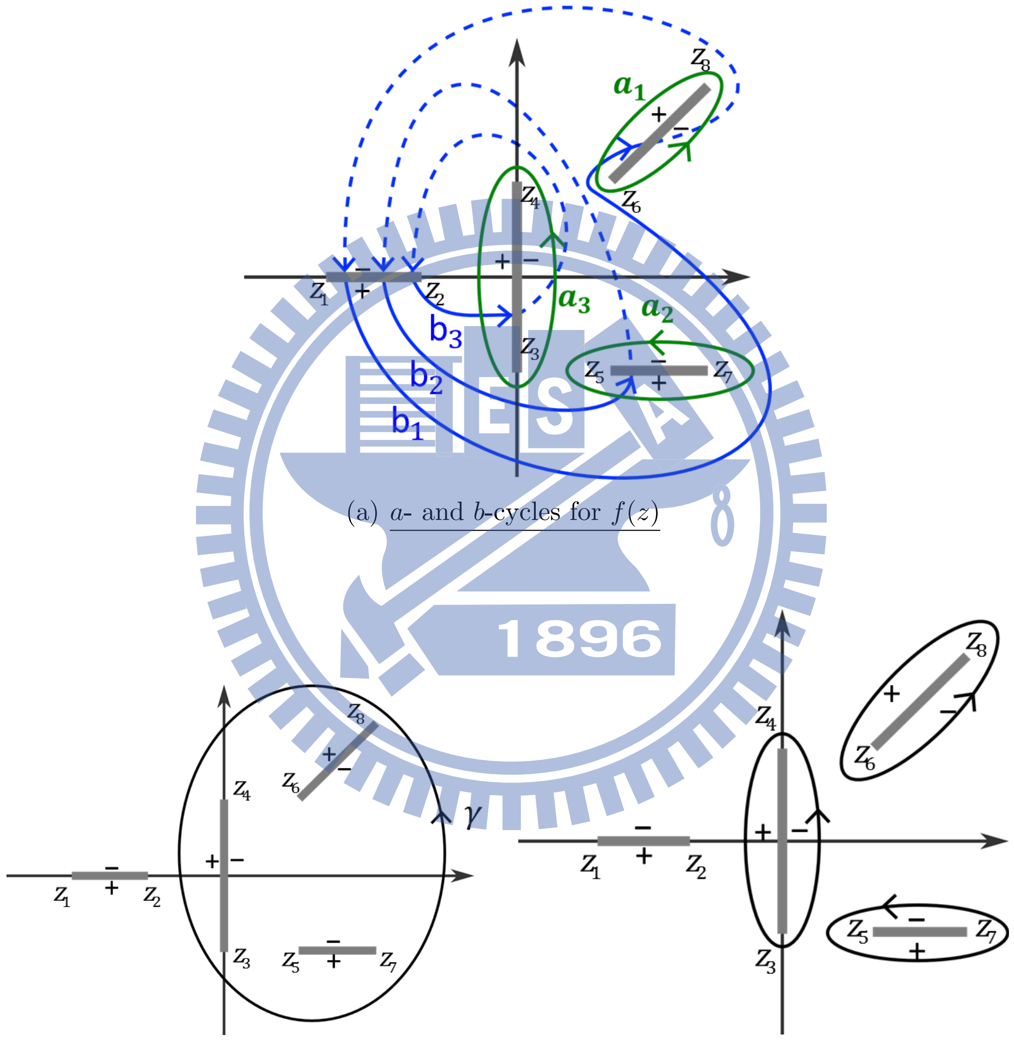
Corollary 2.1.

Let C_1 and C_2 denote counterclockwise simple closed contours, where C_2 is interior to C_1. If a function f is analytic in the closed and "holey" region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz\,.$$



Take another example, let $f(z) = \sqrt{\prod_{k=1}^{8}(z - z_k)}$, and make a_i and b_i cycles, $i = 1, 2, 3$. Consider a closed contour γ as shown in Figure 2.2.3.



(a) a- and b-cycles for $f(z)$

(b) A closed contour γ

(c) The equivalent path for γ

Figure 2.2.3. Deform γ into a combination of a-cycles

Using Cauchy Theorem, then γ can be deformed into a combination of a_i cycles, $i = 1, 2, 3$.

Any closed curve on the Riemann surface can be deformed into a combination of a- and b-cycles. Thus, in this paper, we will consider the integrals of $f(z)$ over a- and b-cycles that help us to evaluate the integrals easier.

2.2.2 The equivalent path

Sometimes the curves are difficult to write out their parametric representation, but straight lines are easy to write out their parametric representation. Thus, we can use the homotopic of curves to find the equivalent paths of curves. It helps us quicker and easier to evaluate the integrals over the curves.

From C_1 is homotopic to C_2, denotes $C_1 \approx C_2$. We have

$$\int_{C_1} \frac{1}{f(z)} dz = \int_{C_2} \frac{1}{f(z)} dz .$$

Take an example to explain, in Figure 2.2.4, $C_1 \approx C_2 \approx C_3$, and finally we compression the curve C_1 until we find the equivalent paths of curves $C_1 \approx r_1 \cup r_2$.

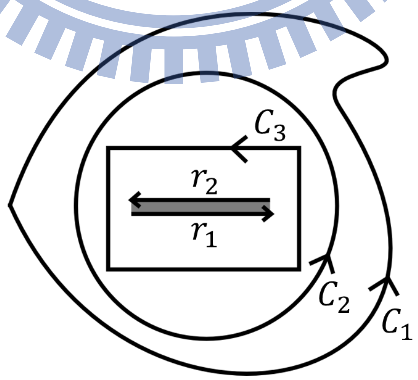


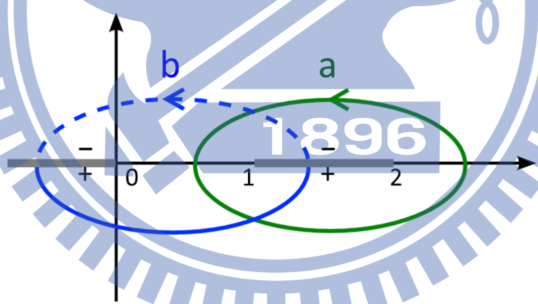
Figure 2.2.4. Equivalent path

Therefore we obtain

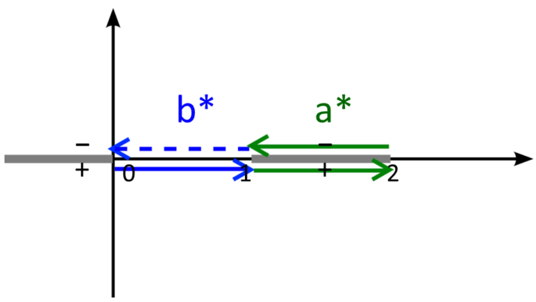
$$\begin{aligned}
&\int_{C_1} \frac{1}{f(z)} dz \\
&= \int_{C_2} \frac{1}{f(z)} dz \\
&= \int_{C_3} \frac{1}{f(z)} dz \\
&= \int_{r_1 \cup r_2} \frac{1}{f(z)} dz \\
&= \int_{r_1} \frac{1}{f(z)} dz + \int_{r_2} \frac{1}{f(z)} dz .
\end{aligned}$$

Hence we can replace the complicated path C_1 for simple path $r_1 \bigcup r_2$.

We take $f(z) = \sqrt{z(z-1)(z-2)}$ for example to show the equivalent path.



(a) a, b-cycle



(b) The equivalent path for a, b-cycle

In this paper, we will take the equivalent path by this way.

2.2.3 The integrals of $\frac{1}{f(z)}$ over a-, b-cycles

As in Section2.1, we consider the function $f(z) = \sqrt{z}$. Let $\theta_2 = \theta_1 + 2\pi$ and $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$, $r \neq 0$, where θ_1 denotes the argument in sheet-I and θ_2 denotes the argument in sheet-II. Then on the complex plane $z_1 = z_2$, but

$$\begin{aligned} f(z_2) &= \sqrt{z_2} = \sqrt{r}e^{\frac{1}{2}i\theta_2} = \sqrt{r}e^{\frac{1}{2}i(\theta_1+2\pi)} \\ &= \sqrt{r}e^{\frac{1}{2}i\theta_1}e^{i\pi} = -\sqrt{r}e^{\frac{1}{2}i\theta_1} = -\sqrt{z_1} = -f(z_1). \end{aligned}$$

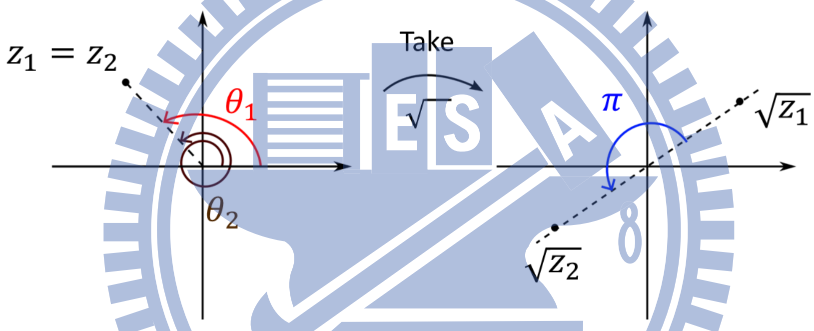


Figure 2.2.5. $f(z_1) \neq f(z_2)$

That is because the difference of argument between z in sheet-I and sheet-II is 2π, that is the difference between $f(z)|_{(I)}$ and $f(z)|_{(II)}$ is π. Hence, we have

$$f(z)|_{(I)} = -f(z)|_{(II)}$$

where $f(z)|_{(I)}$ denotes the computation of $f(z)$ in sheet-I and $f(z)|_{(II)}$ denotes the computation of $f(z)$ in sheet-II.

2.2.4 For horizontal cuts

- **The problem as using Mathematica**

We want to calculate $\int \frac{1}{f(z)}dz$, where $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$, by using Mathematica we can obtain the value. But when we compute the integrals by Mathematica, we found something uncommon.
We observe that the computation of $f(z)$ in sheet-I is not equivalent to the computation of $f(z)$ in Mathematica as the argument is $-\pi$. Take $\sqrt{-1}$ for example, in sheet-I the argument of -1 is $-\pi$ and the argument of $\sqrt{-1}$ is $-\frac{\pi}{2}$, then we have $\sqrt{-1} = e^{-\frac{\pi}{2}i} = -i$. But using Mathematica we obtain $\sqrt{-1} \overset{Math.}{=} i$, which we must multiply by -1 to get the correct value. The reason is in Mathematica the argument belongs to $(-\pi, \pi]$ which is different from in sheet-I the argument belongs to $[-\pi, \pi)$.

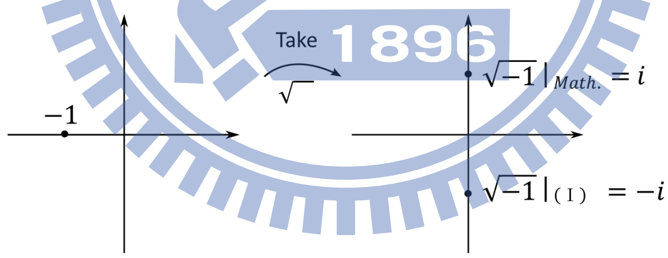


Figure 2.2.6. The value of $\sqrt{-1}$ in Mathematica and in theory

- **Modification**

For convenience, we denote $f(z) \overset{Math.}{=} -f(z)$, which signifies that the polynomial $f(z)$ in front of "$\overset{Math.}{=}$" is the value of $f(z)$ in theory and the polynomial $f(z)$ behind the "$\overset{Math.}{=}$" is the value of $f(z)$ in Mathematica.

Compare the value of $f(z)$ with z in sheet-I and in Mathematica. We then observed that sometimes the computation of Mathematica needed to modify, collating as following.

Lemma 2.1. If $\prod_{k=1}^{n}(z - z_k) = re^{i\theta}$ in sheet-I with horizontal cut, then

$$f(z)|_{(I)} = \begin{cases} f(z)|_{Math.} & \theta \in (-\pi, \pi) \\ -f(z)|_{Math.} & \theta = -\pi \end{cases}$$

where $f(z)|_{Math.}$ denotes the computation of $f(z)$ in Mathematica.

Proof.

Since $-\pi$ does not in $(-\pi, \pi]$, Mathematica will transform $re^{-i\pi}$ into $re^{i\pi}$, but $f(re^{-i\pi})$ and $f(re^{i\pi})$ are different.

In theory: $\quad -1 = e^{-i\pi} \quad \Rightarrow \sqrt{-1} = e^{-\frac{i\pi}{2}} = -i$

In Mathematica: $\quad -1 = e^{-i\pi} \overset{Math.}{=} e^{i\pi} \quad \Rightarrow \sqrt{-1} = e^{\frac{i\pi}{2}} = i$

So $f(z) \overset{Math.}{=} -f(z)$ if $\theta = -\pi$ in Mathematica. ∎

2.2.5 For vertical cuts

• The problem as using Mathematica

We cut the plane along the positive imaginary axis and define tha

$$z - z_k = re^{i\theta}\ ,\ \theta \in [-\frac{3}{2}\pi, \frac{1}{2}\pi) \qquad \text{(in sheet-I)}$$

and

$$z - z_k = re^{i\theta}\ ,\ \theta \in [\frac{1}{2}\pi, \frac{5}{2}\pi)\,. \qquad \text{(in sheet-II)}$$

The cut on each sheet has two edges. We label the starting edge with "+" and the terminal edge with "−" .

Suppose that $f(z) = \sqrt{z}\,, z = ri$.

$$\text{In sheet-I, } z = |z|e^{i\theta}\,, \theta \in [-\frac{3}{2}\pi, \frac{1}{2}\pi) \text{ then } \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}\,, \frac{\theta}{2} \in [-\frac{3}{4}\pi, \frac{1}{4}\pi)$$

$$\text{In sheet-II, } z = |z|e^{i\theta}\,, \theta \in [\frac{1}{2}\pi, \frac{5}{2}\pi) \text{ then } \sqrt{z} = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}\,, \frac{\theta}{2} \in [\frac{1}{4}\pi, \frac{5}{4}\pi)$$

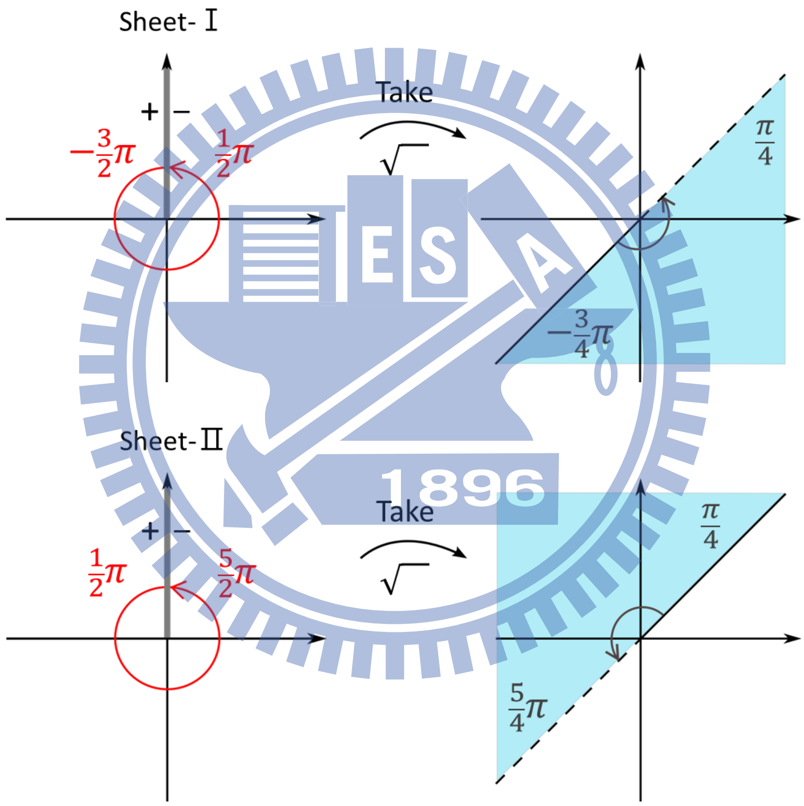


Figure 2.2.7. The domain and image on two sheets for vertical cut

• Modification

Compare the difference between the computation in theory and in Mathematica as illustrated in the in Figure 2.2.8.

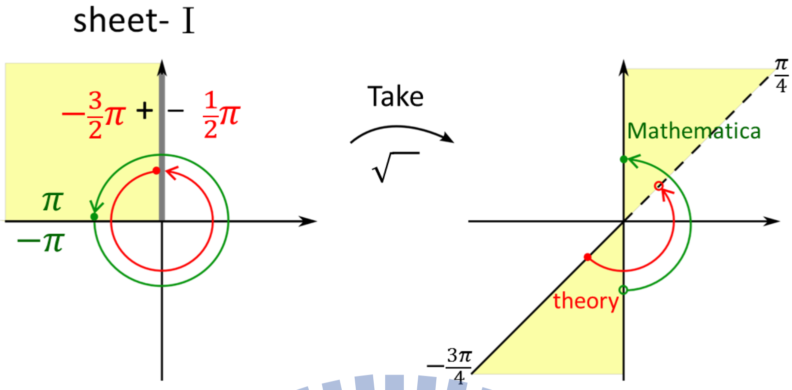


Figure 2.2.8. The difference between in theory and in Mathematica for vertical cut

So we need to modify the computation in Mathematica, collating as following.

Lemma 2.2. If z in sheet-I for vertical cut, then

$$\sqrt{z - z_k} \overset{Math.}{=} \begin{cases} -\sqrt{z - z_k} & arg(z - z_k) \in [-\frac{3}{2}\pi, -\pi] \\ \sqrt{z - z_k} & arg(z - z_k) \in (-\pi, \frac{1}{2}\pi) \end{cases}$$

where $arg(z - z_k)$ denotes the argument of $f(z)$.

Proof.

Let z in sheet-I and using polar form $z - z_k = re^{i\theta}$. When $\theta \in (-\pi, \frac{\pi}{2})$, the argument in theory or Mathematica is the same. When $\theta \in [-\frac{3\pi}{2}, -\pi]$, Mathematica will conversion θ into $\theta + 2\pi$ where $\theta + 2\pi \in [\frac{\pi}{2}, \pi]$ and $re^{i\theta} = re^{(\theta+2\pi)i}$, but

In theory: $\sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta}{2}i}$

In Mathematica: $\sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta+2\pi}{2}i} = -\sqrt{r}e^{\frac{\theta}{2}i}$

Thus, if $\theta \in [-\frac{3\pi}{2}, -\pi]$, $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$. ■

2.2.6 For slant cuts

• The problem as using Mathematica

We cut the plane along a straight line which slope is $m = \tan\alpha\,, 0 < \alpha \leq \pi$. Notice that the cut with α means the slope of the straight line where the cut on is $m = \tan\alpha\,, 0 < \alpha \leq \pi$. We define that

$$z = re^{i\theta}\ ,\ \theta \in [\alpha - 2\pi, \alpha) \qquad \text{(in sheet-I)}$$

and

$$z = re^{i\theta}\ ,\ \theta \in [\alpha, \alpha + 2\pi)\,. \qquad \text{(in sheet-II)}$$

The cut on each sheet has two edges. We label the starting edge with "+" and the terminal edge with "−" .

Suppose that $f(z) = \sqrt{z}\,, z \in \mathbb{C}$.

In sheet-I, $z = |z|e^{i\theta}\,, \theta \in [\alpha{-}2\pi, \alpha)$ then $\sqrt{z} = |z|^{\frac{1}{2}}e^{i\frac{\theta}{2}}\,, \frac{\theta}{2} \in [\frac{\alpha - 2\pi}{2}, \frac{\alpha}{2})$

In sheet-II, $z = |z|e^{i\theta}\,, \theta \in [\alpha, \alpha{+}2\pi)$ then $\sqrt{z} = |z|^{\frac{1}{2}}e^{i\frac{\theta}{2}}\,, \frac{\theta}{2} \in [\frac{\alpha}{2}, \frac{\alpha + 2\pi}{2})$

• Modification

Compare the difference between the computation in theory and in Mathematica as illustrated in the Figure 2.2.10.

So we need to modify the computation in Mathematica, collating as Lemma2.3.

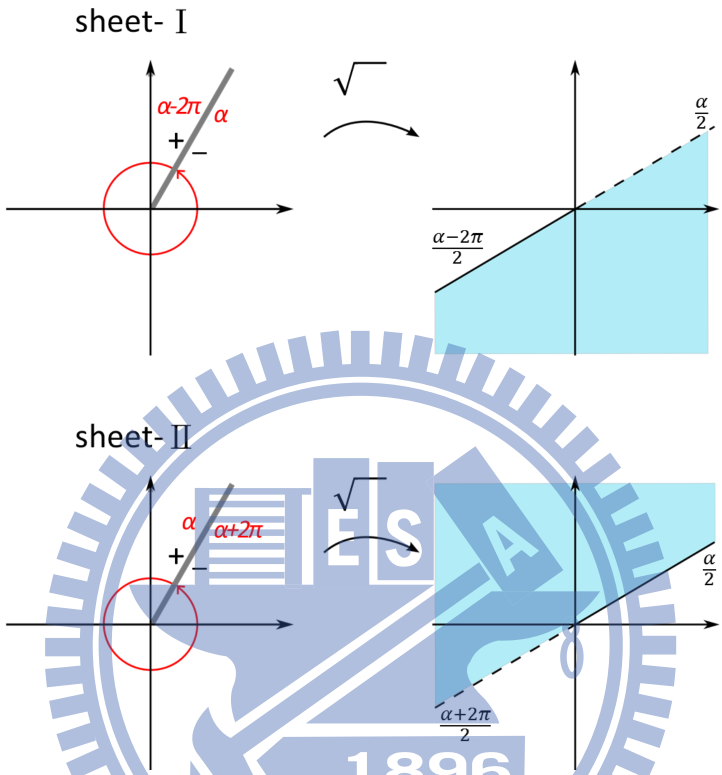


Figure 2.2.9. The domain and image on two sheets for the cut with α

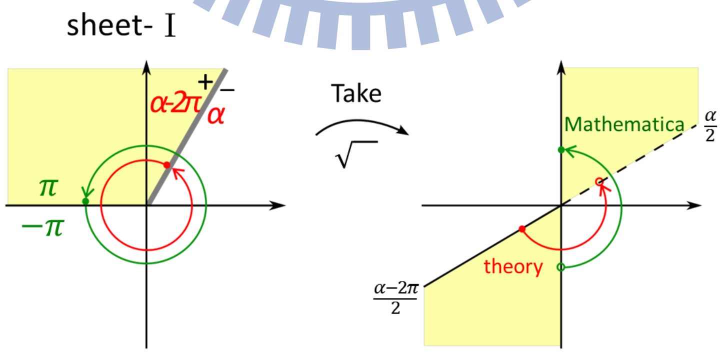


Figure 2.2.10. The difference between in theory and in Mathematica for the cut with α

Lemma 2.3. If the cut with z_k on the line where the slope of line is $m = \tan\alpha\,, 0 < \alpha \leq \pi$. and z in sheet-I, then

$$\sqrt{z - z_k} \overset{Math.}{=} \begin{cases} -\sqrt{z - z_k} & arg(z - z_k) \in [\alpha - 2\pi, -\pi] \\ \sqrt{z - z_k} & arg(z - z_k) \in (-\pi, \alpha) \end{cases}$$

Proof.

Let z in sheet-I and using polar form $z - z_k = re^{i\theta}$ where $arg(z - z_k) = \theta$. When $\theta \in (-\pi, \alpha)$, the argument in theory and in Mathematica is the same. When $\theta \in [\alpha - 2\pi, -\pi]$, Mathematica will transform θ into $\theta + 2\pi$ where $\theta + 2\pi \in [\alpha, \pi]$ and $re^{i\theta} = re^{(\theta+2\pi)i}$, but

In theory: $\sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta}{2}i}$

In Mathematica: $\sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta+2\pi}{2}i} = -\sqrt{r}e^{\frac{\theta}{2}i}$

So if $\theta \in [\alpha - 2\pi, -\pi]$,

$$\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}.$$

■

2.2.7 Summary

Because sometimes the form of integration is complex, if we could simplify the way about modify the sign of $f(z)$, it will help us to get right value easier. We divided domain $\mathbb{C}$ into many blocks to discuss way to modify on slant cuts. It can help us reduce the steps of modifying $f(z)$.

Definition 2.1. Any slant cut whose slope of line is $m = \tan\alpha\,, 0 < \alpha \leq \pi$ and the end points of cut are $z_k = x_k + iy_k$ and $z_{k+1} = x_{k+1} + iy_{k+1}$. Define the domain L as

$$L = (x, y) : y - y_k > \tan\alpha\,(x - x_k) \qquad when \tan\alpha > 0$$

$$L = (x, y) : y - y_k < \tan\alpha\,(x - x_k) \qquad when \tan\alpha < 0$$

Lemma 2.4. If $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}}$ for the cut with α. We divided domain $\mathbb{C}$ into 6 areas as illustrated in the Figure 2.2.11.

$(B_1) = \{\, (x,y) \,|\, (x,y) \in L \text{ and } y \geq y_{k+1} \,\}$
$(B_2) = \{\, (x,y) \,|\, (x,y) \in L \text{ and } y_k \leq y < y_{k+1} \,\}$
$(B_3) = \{\, (x,y) \,|\, (x,y) \in L \text{ and } y < y_k \,\}$
$$(M) = (B_4)\bigcup(B_5)\bigcup(B_6) = \{\, (x,y) \,|\, (x,y) \in \mathbb{C} \setminus L \,\} \bigcup \left\{\, (x,y) \,|\, \frac{y-y_k}{x-x_k} = \tan\alpha \,\right\}$$

then we have

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in (B_2)\bigcup\{\text{ the cut with } (+)\text{edge of sheet-I }\} \\ f(z) & \text{otherwise.} \end{cases}$$

Proof.

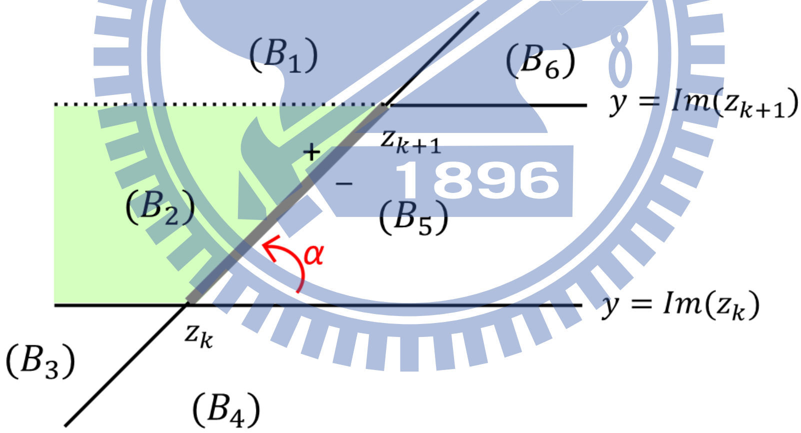


Figure 2.2.11. Divided domain C into 6 blocks

(1) $z \in (B_1) : arg(z-z_k),\, arg(z-z_{k+1}) \in [\alpha - 2\pi, -\pi]$
$\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_{k+1}}$
$f(z) \overset{Math.}{=} f(z)$

(2) $z \in (B_2) : arg(z-z_k) \in [\alpha - 2\pi, -\pi]$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$

$\sqrt{z-z_{k+1}} \in [-\pi,\pi]$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$
$f(z) \overset{Math.}{=} -f(z)$

(3) $z \in (B_3)\bigcup(M)$: $arg(z-z_k)$, $arg(z-z_{k+1}) \in [-\pi,\pi]$
$\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$
$f(z) \overset{Math.}{=} f(z)$

(4) $z \in$the cut with (+)edge of sheet-I:
$arg(z-z_k) = \alpha - 2\pi$ then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$
$arg(z-z_{k+1}) = \alpha - \pi$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$
$f(z) \overset{Math.}{=} -f(z)$

(5) $z \in$the cut with (−)edge of sheet-I:
$arg(z-z_k) = \alpha$ then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$
$arg(z-z_{k+1}) = \alpha - \pi$ then $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$
$f(z) \overset{Math.}{=} f(z)$

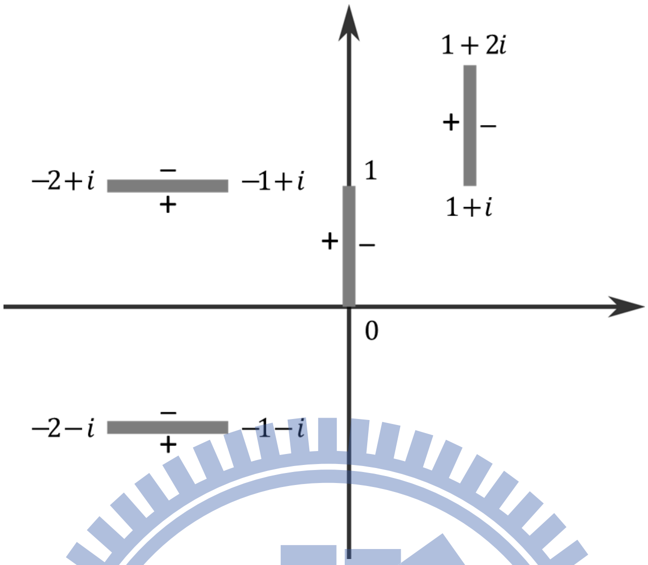
■

There are two examples we calculate the integrals in theory and in Mathematica respectively, and also draw the path on the the corresponding Riemann surface.

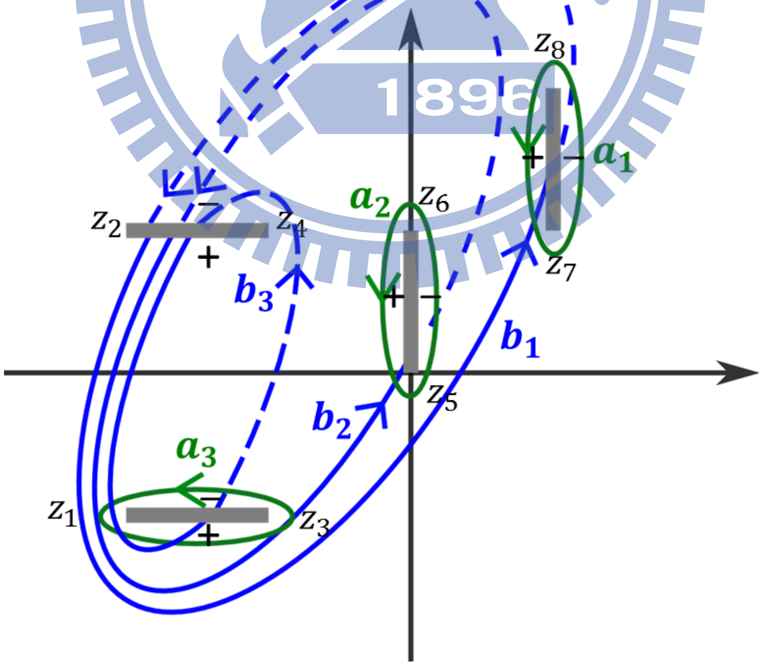
Example 2.3. Evaluate $\displaystyle\int \frac{1}{f(z)}dz$ over a_3, b_1 and b_2 cycles, where

$$f(z) = \sqrt{(z+2+i)(z+2-i)(z+1+i)(z+1-i)(z-0)(z-1)(z-1-i)(z-1-2i)}\,.$$

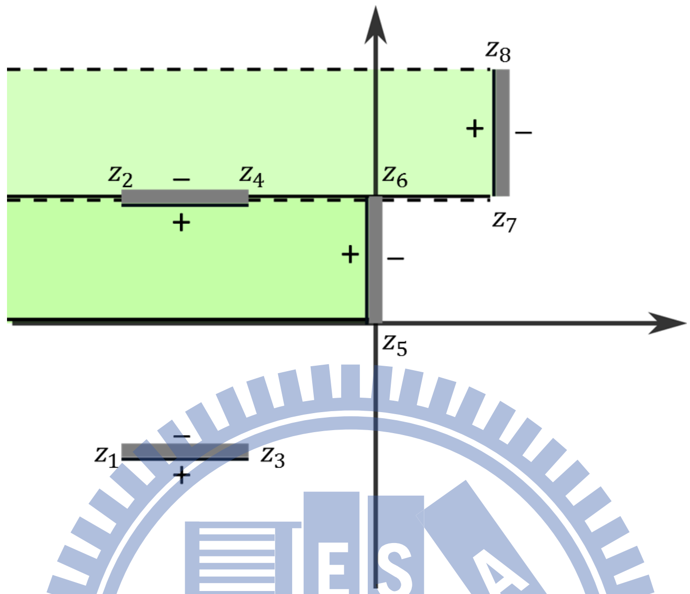
Let $z_1 = -2-i$, $z_2 = -2+i$, $z_3 = -1-i$, $z_4 = -1+i$, $z_5 = 0$, $z_6 = 1$, $z_7 = 1+i$, $z_8 = 1+2i$.



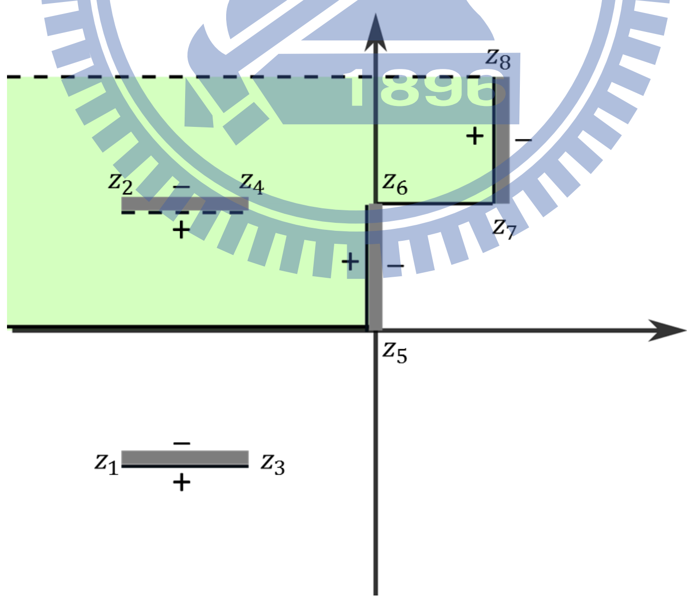
(a) The cut plane



(b) The a-,b-cycles



(c) The regions need changing sign of cuts



(d) (M): region of modify

Using region of modify to get result by Mathematica:
The region needs changing sign of $\sqrt{z-z_1}\sqrt{z-z_3}$ is { (+)edge of the cut $\overline{z_1 z_3}$ }. The region needs changing sign of $\sqrt{z-z_2}\sqrt{z-z_4}$ is { (+)edge of the cut $\overline{z_2 z_4}$ }. The region needs changing sign of $\sqrt{z-z_5}\sqrt{z-z_6}$ is { $z = x+iy \,|\, x<0\,,\, 0 \le y < 1$ } $\bigcup$ { (+) edge of the cut $\overline{z_2 z_4}$ }. The region needs changing sign of $\sqrt{z-z_7}\sqrt{z-z_8}$ is { $z = x+iy \,|\, x<1\,,\, 1 \le y < 2$ }. We let region of modify (M) = { $z = x+iy \,|\, x<0\,,\, 0 \le y < 1$ } $\bigcup$ { $z = x+iy \,|\, x<1\,,\, 1 \le y < 2$ } $\bigcup$ { (+)edge of the cut $\overline{z_1 z_3}$ } $\setminus$ { (+)edge of the cut $\overline{z_2 z_4}$ }.

1. Evaluate $\displaystyle\int_{a_3} \frac{1}{f(z)} dz$

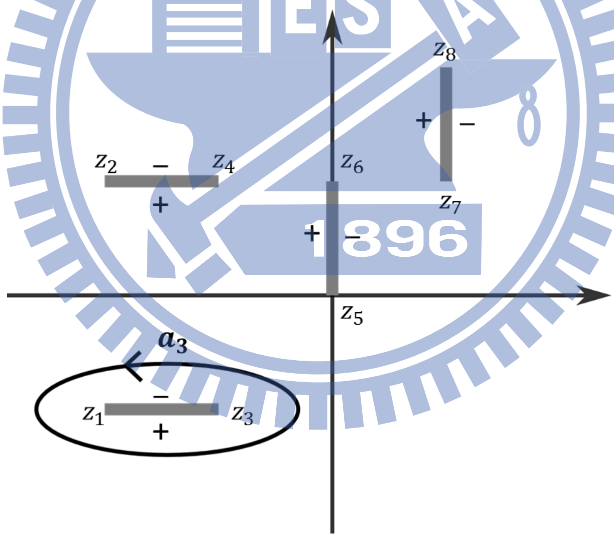


Figure 2.2.12. The contour a_3 in the cut plane

a_3 : Consider the equivalent path $a_3^* = a_{31}^* \bigcup a_{32}^*$, where
a_{31}^* = the path on the horizontal cut from $-2-i$ to $-1-i$ on (+)edge of sheet-I and
a_{32}^* = the path on the horizontal cut from $-1-i$ to $-2-i$ on (−)edge of sheet-I.

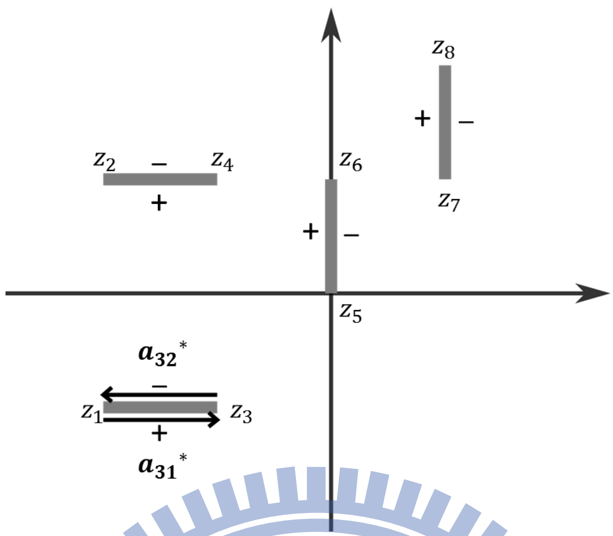


Figure 2.2.13. The equivalent path for a_3

(1) a_{31}^* : Let $z = r - i\,, r : -2 \to -1\,, dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{a_{31}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{-2}^{-1} \frac{1}{f(r-i)} dr$$

(2) a_{32}^* : Let $z = r - i\,, r : -1 \to -2\,, dz = dr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

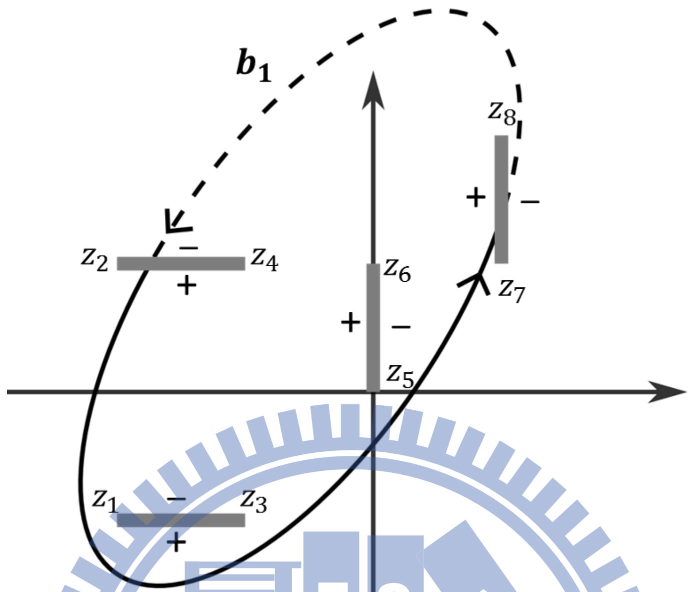
$$\int_{a_{32}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{-1}^{-2} \frac{1}{f(r-i)} dr$$

By (1),(2) and Cauchy Theorem we can obtain $\int_{a_3} \frac{1}{f(z)} dz$, which value is shown in the Appendix A.0.1.

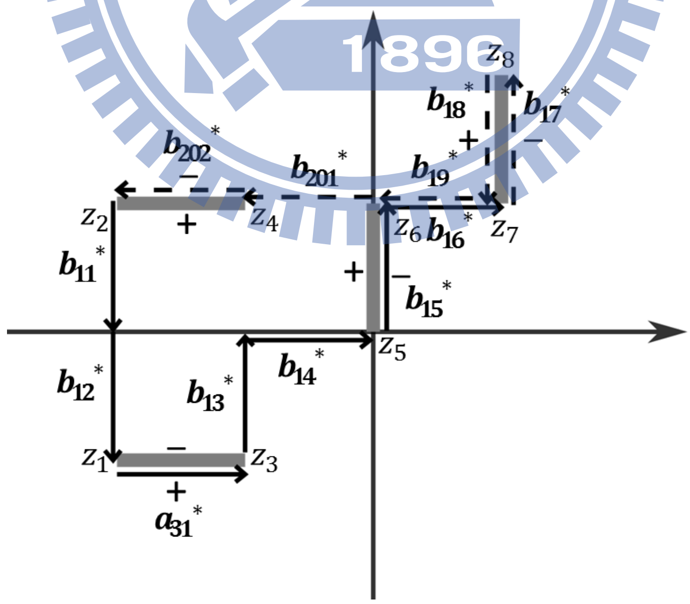
2. Evaluate $\int_{b_1} \frac{1}{f(z)} dz$

b_1 : Consider the equivalent path

$b_1^* = b_{11}^* \bigcup b_{12}^* \bigcup a_{31}^* \bigcup b_{13}^* \bigcup b_{14}^* \bigcup b_{15}^* \bigcup b_{16}^* \bigcup b_{17}^* \bigcup b_{18}^* \bigcup b_{19}^* \bigcup b_{201}^* \bigcup b_{202}^*$, where



(a) The contour b_3 in the cut plane



(b) The equivalent path for b_3

$b_{11}^* =$ the path on the vertical line from $-2+i$ to -2 on sheet-I,

$b_{12}^* =$ the path on the vertical line from -2 to $-2-i$ on sheet-I,

$b_{13}^* =$ the path on the vertical line from $-1-i$ to -1 on sheet-I,

$b_{14}^* =$ the path on the horizontal line from -1 to 0 on sheet-I,

$b_{15}^* =$ the path on the vertical cut from 0 to i on $(-)$edge of sheet-I,

$b_{16}^* =$ the path on the horizontal line from i to $1+i$ on sheet-I,

$b_{17}^* =$ the path on the vertical cut from $1+i$ to $1+2i$ on $(-)$edge of sheet-II,

$b_{18}^* =$ the path on the vertical cut from $1+2i$ to $1+i$ on $(+)$edge of sheet-II,

$b_{19}^* =$ the path on the horizontal line from $1+i$ to i on sheet-II,

$b_{201}^* =$ the path on the horizontal line from i to $-1+i$ on sheet-II, and

$b_{202}^* =$ the path on the horizontal line from $-1+i$ to $-2+i$ on $(-)$edge of sheet-II.

(1) b_{11}^* : Let $z = -2+ri\,, r : 1 \to 0\,, dz = idr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b_{11}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^0 \frac{i}{f(-2+ri)}dr$$

(2) b_{12}^* : Let $z = -2+ri\,, r : 0 \to -1\,, dz = idr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{b_{12}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^{-1} \frac{i}{f(-2+ri)}dr$$

(3) b_{13}^* : Let $z = -1+ri\,, r : -1 \to 0\,, dz = idr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{b_{13}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{-1}^0 \frac{i}{f(-1+ri)}dr$$

(4) b_{14}^* : Let $z = r\,, r : -1 \to 0\,, dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b_{14}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{-1}^0 \frac{1}{f(r)}dr$$

(5) b^*_{15} : Let $z = ri\,,\, r : 0 \to 1\,, dz = idr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{b^*_{15}} \frac{1}{f(z)} dz \overset{Math.}{=} \int_0^1 \frac{i}{f(ri)} dr$$

(6) b^*_{16} : Let $z = r + i\,,\, r : 0 \to 1\,, dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b^*_{16}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_0^1 \frac{1}{f(r+i)} dr$$

(7) $b^*_{17} \equiv$ the path on the vertical cut from $1+i$ to $1+2i$ on $(+)$edge of sheet-I,

so $z \in (M)$. Let $z = 1 + ri\,,\, r : 1 \to 2$ and $dz = idr$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b^*_{17}} \frac{1}{f(z)} dz = \int_{1+i \overset{-}{\dashrightarrow} 1+2i} \frac{1}{f(z)} dz = \int_{1+i \overset{+}{\rightarrow} 1+2i} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_1^2 \frac{i}{f(1+ri)} dr$$

(8) $b^*_{18} \equiv$ the path on the vertical cut from $1+2i$ to $1+i$ on $(-)$edge of sheet-I,

so $z \notin (M)$. Let $z = 1 + ri\,,\, r : 2 \to 1$ and $dz = idr$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{b^*_{18}} \frac{1}{f(z)} dz = \int_{1+2i \overset{+}{\dashrightarrow} 1+i} \frac{1}{f(z)} dz = \int_{1+2i \overset{-}{\rightarrow} 1+i} \frac{1}{f(z)} dz \overset{Math.}{=} \int_2^1 \frac{i}{f(1+ri)} dr$$

(9) b^*_{19} : Let $z = r + i\,,\, r : 1 \to 0\,, dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b^*_{39}} \frac{1}{f(z)} dz = \int_{i \dashleftarrow 1+i} \frac{1}{f(z)} dz = -\int_{i \leftarrow 1+i} \frac{1}{f(z)} dz \overset{Math.}{=} \int_1^0 \frac{1}{f(r+i)} dr$$

(10) b^*_{201} : Let $z = r + i\,,\, r : 0 \to -1\,, dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b^*_{201}} \frac{1}{f(z)} dz = \int_{-1+i \dashleftarrow i} \frac{1}{f(z)} dz = -\int_{-1+i \leftarrow i} \frac{1}{f(z)} dz \overset{Math.}{=} \int_0^{-1} \frac{1}{f(r+i)} dr$$

(11) $b^*_{202} \equiv$ the path on the horizontal cut from $-1+i$ to $-2+i$ on (+)edge of sheet-I.

Let $z = r+i\,,\, r : -1 \to -2\,, dz = dr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{b^*_{202}} \frac{1}{f(z)}dz = \int_{-2+i \overset{-}{\dashleftarrow} -1+i} \frac{1}{f(z)}dz = \int_{-2+i \overset{+}{\leftarrow} -1+i} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{-1}^{-2} \frac{1}{f(r+i)}dr$$

By (1)∼(11) and Cauchy Theorem we can obtain $\int_{b_1} \frac{1}{f(z)}dz$, which value is shown in the Appendix A.0.2.

3. Evaluate $\int_{b_2} \frac{1}{f(z)}dz$

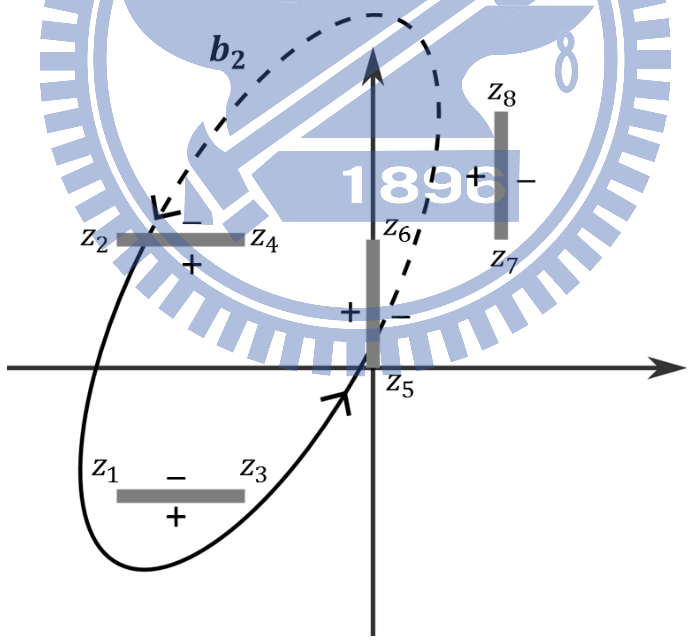


Figure 2.2.14. The contour b_2 in the cut plane

b_2 : Consider the equivalent path

$b^*_2 = b^*_{11} \bigcup b^*_{12} \bigcup a^*_{31} \bigcup b^*_{13} \bigcup b^*_{14} \bigcup b^*_{21} \bigcup b^*_{22} \bigcup b^*_{202}$, where

b^*_{21} = the path on the vertical line from 0 to i on sheet-II,
b^*_{22} = the path on the horizontal line from i to $-1+i$ on sheet-II.

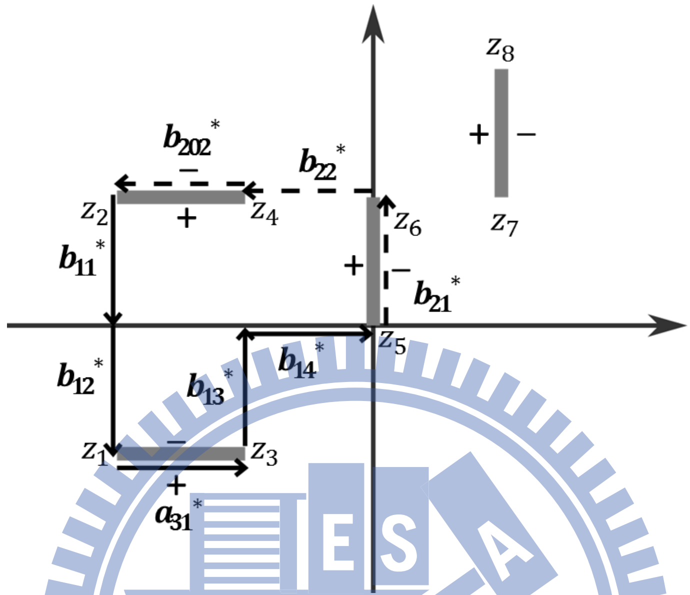


Figure 2.2.15. The equivalent path for b_2

(1) $b^*_{21} \equiv$ the path on the vertical cut from 0 to i on $(+)$edge of sheet-I,
so $z \in (M)$. Let $z = ri\,, r: 0 \to 1$ and $dz = idr$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b^*_{21}} \frac{1}{f(z)}dz = \int_{0 \overset{-}{\dashrightarrow} i} \frac{1}{f(z)}dz = \int_{0 \overset{+}{\to} i} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_0^1 \frac{i}{f(ri)}dr$$

(2) b^*_{22} : Let $z = r + i\,, r : 0 \to -1\,, dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b^*_{32}} \frac{1}{f(z)}dz = \int_{1+i \dashleftarrow i} \frac{1}{f(z)}dz = -\int_{1+i \leftarrow i} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^{-1} \frac{1}{f(r+i)}dr$$

By (1),(2) and Cauchy Theorem we can obtain $\int_{b_2} \frac{1}{f(z)}dz$, which value is shown in the Appendix A.0.3.

Example 2.4. Evaluate $\int \frac{1}{f(z)}dz$ over a_3, b_1 and b_2 cycles, where

$f(z) = \sqrt{(z+2+i)(z+2-i)(z+1+i)(z+1-i)(z-0)(z-1)(z-1-i)(z-1-2i)}$

.

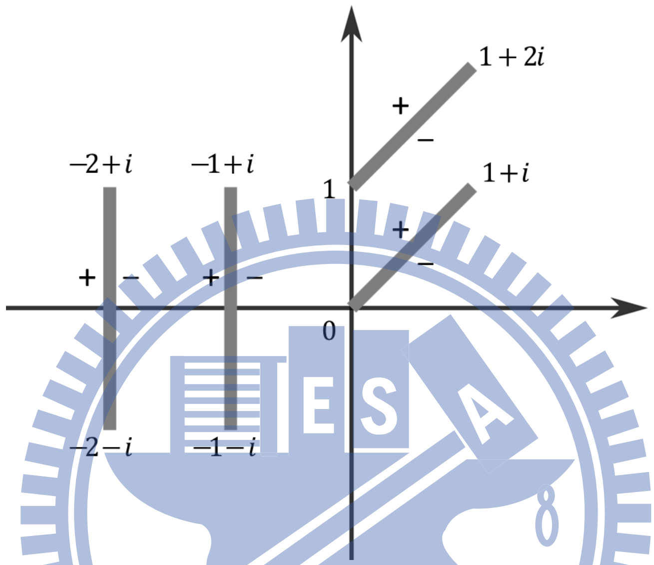


Figure 2.2.16. The cut plane

Let $z_1 = -2-i,\ z_2 = -2+i,\ z_3 = -1-i,\ z_4 = -1+i,\ z_5 = 0,\ z_6 = 1,\ z_7 = 1+i,\ z_8 = 1+2i$.

Using region of modify to get result by Mathematica:

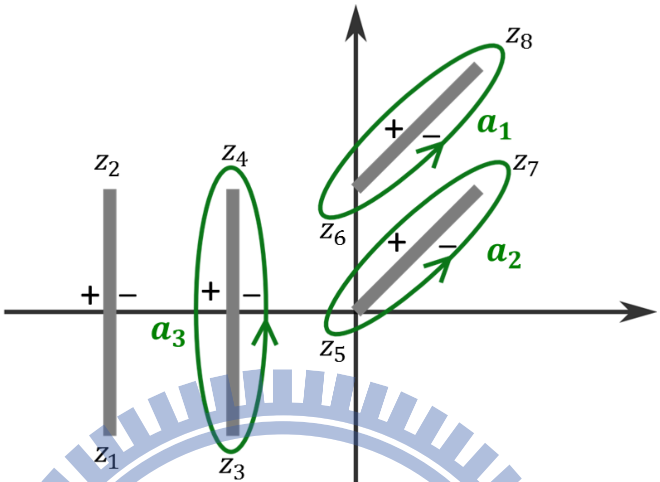
Let

(A)={ $\{\ z = x+iy \,|\, x-y < 0\,,\ 1 \le y < 2\ \} \bigcup \{$ (+) edge of the cut $\overline{z_6 z_8}$ } }

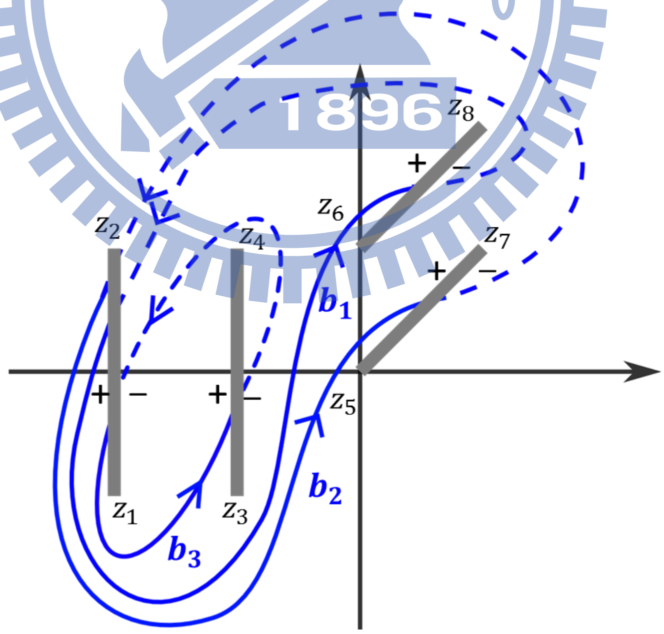
(B)={ $\{\ z = x+iy \,|\, x < -2\,,\ 0 \le y < 1\ \} \bigcup \{$ (+) edge of the cut from z_2 to $-2i$ } }

(C)={ $\{\ z = x+iy \,|\, x < -2\,,\ -1 \le y < 0\ \} \bigcup \{$ (+) edge of the cut from $-2i$ to z_1 } }

(D)={ $\{\ z = x+iy \,|\, x < -1\,,\ 0 \le y < 1\ \} \bigcup \{$ (+) edge of the cut from z_4 to $-i$ } }



(a) The a-cycles



(b) The b-cycles

(E)={ { $z = x + iy \,|\, x < -1\,,\, -1 \leq y < 0$ } $\bigcup$ { $(+)$ edge of the cut from $-i$ to z_3 } }

(F)={ { $z = x + iy \,|\, x - y < 0\,,\, 0 \leq y < 1$ } $\bigcup$ { $(+)$ edge of the cut $\overline{z_5 z_7}$ } }

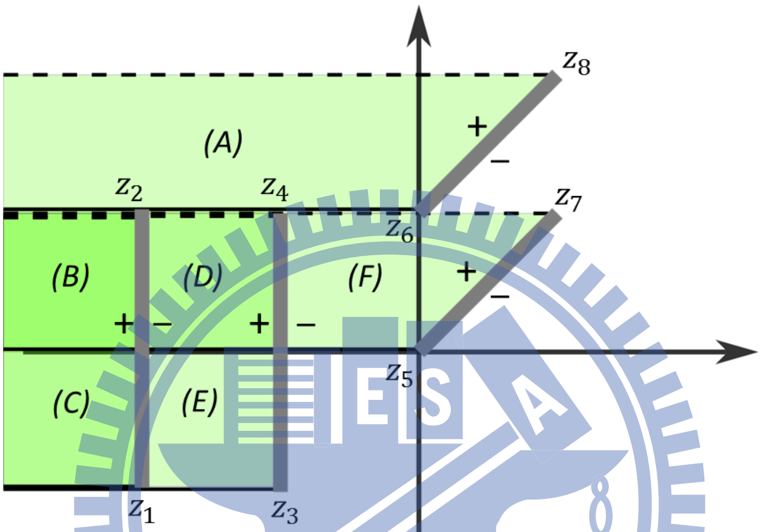


Figure 2.2.17. The regions need changing sign of cuts

The region needs changing sign of $\sqrt{z - z_1}\sqrt{z - z_2}$ is
$(B)\bigcup(C)$ ={ { $z = x + iy \,|\, x < -2\,,\, -1 \leq y < 1$ } $\bigcup$ { $(+)$edge of the cut $\overline{z_1 z_2}$ } }.

The region needs changing sign of $\sqrt{z - z_3}\sqrt{z - z_4}$ is
$(B)\bigcup(C)\bigcup(D)\bigcup(E)$ ={ { $z = x + iy \,|\, x < -1\,,\, -1 \leq y < 1$ } $\bigcup$ { $(+)$edge of the cut $\overline{z_3 z_4}$ } }.

The region needs changing sign of $\sqrt{z - z_5}\sqrt{z - z_7}$ is
$(B)\bigcup(D)\bigcup(F)$ ={ { $z = x + iy \,|\, x - y < 0\,,\, 0 \leq y < 1$ } $\bigcup$ { $(+)$ edge of the cut $\overline{z_5 z_7}$ } }.

The region needs changing sign of $\sqrt{z - z_6}\sqrt{z - z_8}$ is (A).

The region (B) changes the sign three times, so need to change here. The region (C) and (D) change the sign two times, so no change here. We let region of modify $(M) = (A)\bigcup(B)\bigcup(E)\bigcup(F)$.

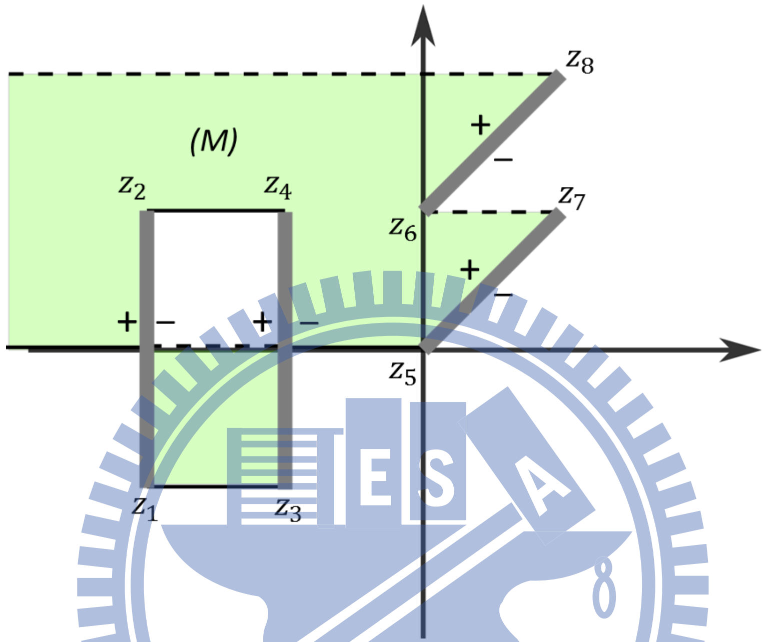


Figure 2.2.18. (M): region of modify

1. Evaluate $\displaystyle\int_{a_3} \frac{1}{f(z)}dz$

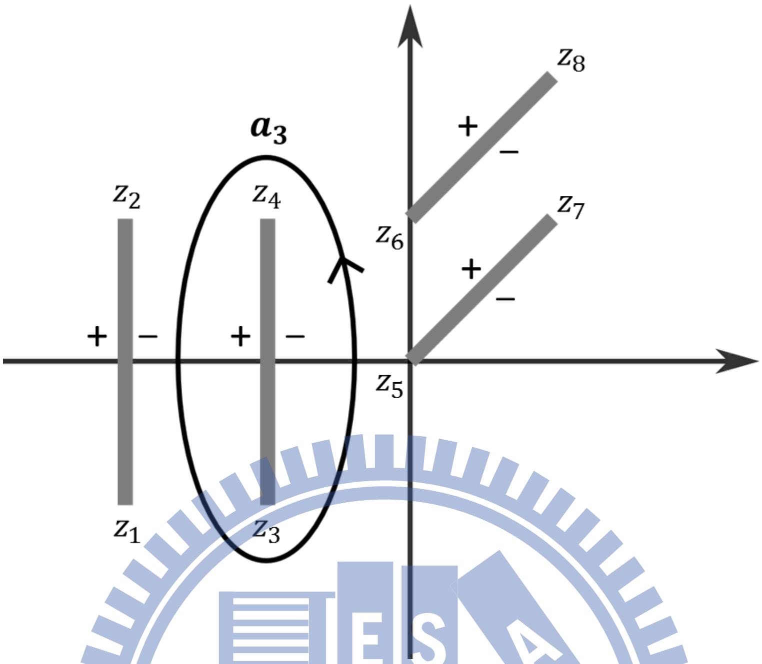
   a_3 : Consider the equivalent path $a_3^* = a_{31}^* \bigcup a_{32}^* \bigcup a_{33}^* \bigcup a_{34}^*$, where

   a_{31}^* = the path on the vertical cut from $-1+i$ to -1 on $(+)$edge of sheet-I,

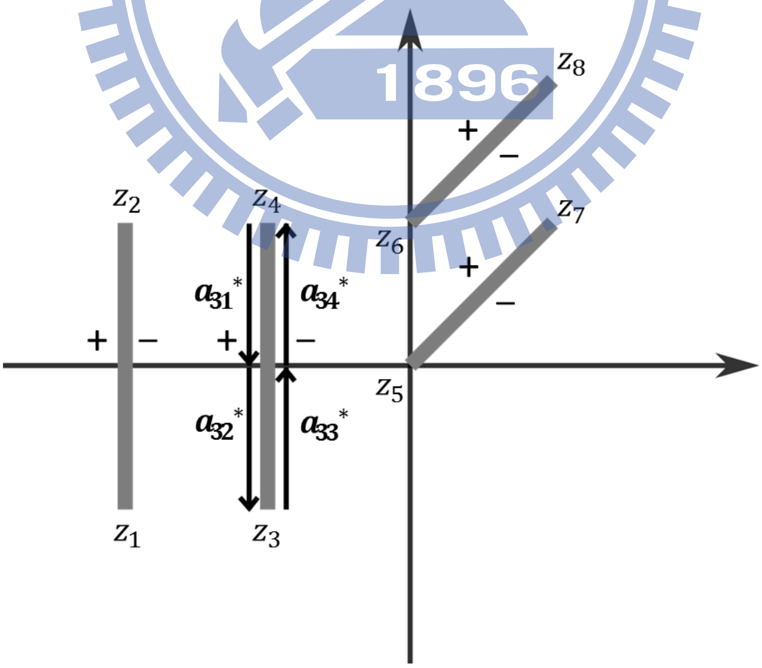
   a_{32}^* = the path on the vertical cut from -1 to $-1-i$ on $(+)$edge of sheet-I,

   a_{33}^* = the path on the vertical cut from $-1-i$ to -1 on $(-)$edge of sheet-I,

   a_{34}^* = the path on the vertical cut from -1 to $-1+i$ on $(-)$edge of sheet-I.



(a) The contour a_3 in the cut plane



(b) The equivalent path for a_3

(1) a_{31}^* : Let $z = -1 + ri\,,\, r : 1 \to 0\,, dz = idr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{a_{31}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_1^0 \frac{i}{f(-1+ri)} dr$$

(2) a_{32}^* : Let $z = -1 + ri\,,\, r : 0 \to -1\,, dz = idr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{a_{32}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_0^{-1} \frac{i}{f(-1+ri)} dr$$

(3) a_{33}^* : Let $z = -1 + ri\,,\, r : -1 \to 0\,, dz = idr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{a_{33}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{-1}^0 \frac{i}{f(-1+ri)} dr$$

(4) a_{34}^* : Let $z = -1 + ri\,,\, r : 0 \to 1\,, dz = idr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{a_{34}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_0^1 \frac{i}{f(-1+ri)} dr$$

By (1)∼(4) and Cauchy Theorem we can obtain $\int_{a_3} \frac{1}{f(z)} dz$, which value is shown in the Appendix A.0.4.

2. Evaluate $\int_{b_2} \frac{1}{f(z)} dz$

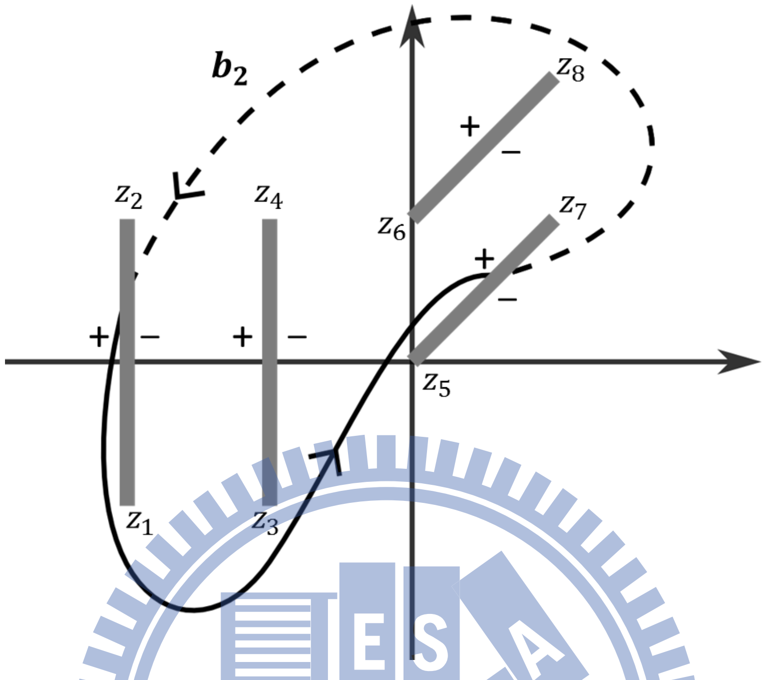
b_2 : Consider the equivalent path

$b_2^* = b_{21}^* \bigcup b_{22}^* \bigcup b_{23}^* \bigcup a_{33}^* \bigcup b_{24}^* \bigcup b_{25}^* \bigcup b_{26}^* \bigcup b_{27}^* \bigcup b_{28}^* \bigcup b_{29}^*$, where

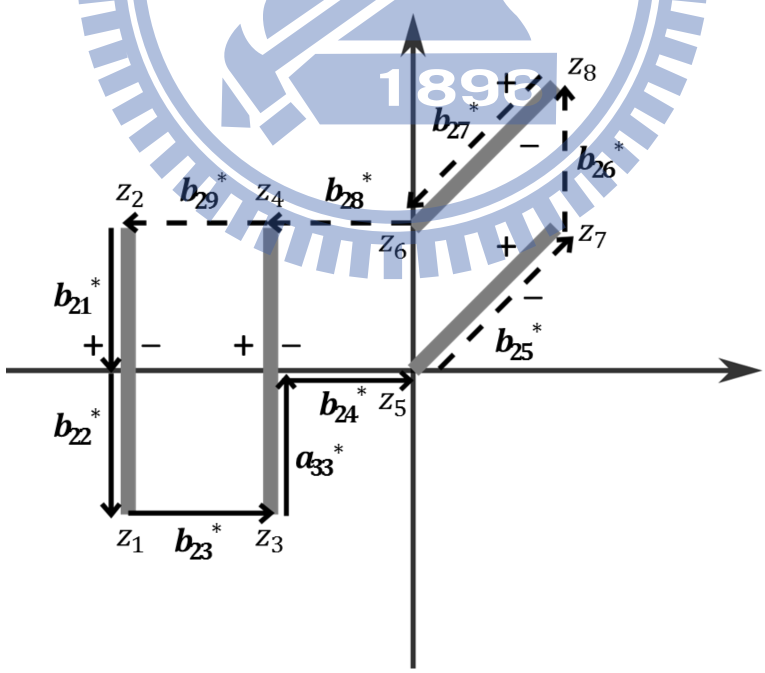
b_{21}^* = the path on the vertical cut from $-2+i$ to -2 on (+)edge of sheet-I,

b_{22}^* = the path on the vertical cut from -2 to $-2-i$ on (+)edge of sheet-I,

b_{23}^* = the path on the horizontal line from $-2-i$ to $-1-i$ on sheet-I,



(c) The contour b_2 in the cut plane



(d) The equivalent path for b_2

$b_{24}^* =$ the path on the horizontal line from -1 to 0 on sheet-I,
$b_{25}^* =$ the path on the cut with $\frac{\pi}{4}$ from 0 to $1+i$ on $(-)$edge of sheet-II,
$b_{26}^* =$ the path on the vertical line from $1+i$ to $1+2i$ on sheet-II,
$b_{27}^* =$ the path on the cut with $\frac{\pi}{4}$ from $1+2i$ to i on $(+)$edge of sheet-II,
$b_{28}^* =$ the path on the horizontal line from i to $-1+i$ on sheet-II,
$b_{29}^* =$ the path on the horizontal line from $-1+i$ to $-2+i$ on sheet-II.

(1) b_{21}^* : Let $z = -2+ri\,, r : 1 \to 0\,, dz = idr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b_{21}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^0 \frac{i}{f(-2+ri)}dr$$

(2) b_{22}^* : Let $z = -2+ri\,, r : 0 \to -1\,, dz = idr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{b_{22}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^{-1} \frac{i}{f(-2+ri)}dr$$

(3) b_{23}^* : Let $z = r-i\,, r : -2 \to -1\,, dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b_{23}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{-2}^{-1} \frac{1}{f(r-i)}dr$$

(4) b_{24}^* : Let $z = r\,, r : -1 \to 0\,, dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b_{24}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{-1}^{0} \frac{1}{f(r)}dr$$

(5) $b_{25}^* \equiv$ the path on the cut with $\frac{\pi}{4}$ from 0 to $1+i$ on $(+)$edge of sheet-I.

Let $z = r(1+i)\,, r : 0 \to 1\,, dz = (1+i)dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b_{25}^*} \frac{1}{f(z)}dz = \int_{0 \overset{-}{\dashrightarrow} 1+i} \frac{1}{f(z)}dz = \int_{0 \overset{+}{\to} 1+i} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_0^1 \frac{1+i}{f(r(1+i))}dr$$

(6) b_{26}^{*} : Let $z = 1 + ri \, , r : 1 \to 2 \, , dz = idr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

$$\begin{aligned}\int_{b_{26}^{*}} \frac{1}{f(z)} dz &= \int_{1+i \dashrightarrow 1+2i} \frac{1}{f(z)} dz \\ &= - \int_{1+i \to 1+2i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} - \int_{1}^{2} \frac{i}{f(1+ri)} dr\end{aligned}$$

(7) $b_{27}^{*} \equiv$ the path on the cut with $\frac{\pi}{4}$ from $1+2i$ to i on $(-)$edge of sheet-I.

Let $z = i + r(1+i) \, , r : 1 \to 0 \, , dz = (1+i)dr$ and $z \notin (M)$ then $f(z) \overset{Math.}{=} f(z)$

$$\int_{b_{27}^{*}} \frac{1}{f(z)} dz = \int_{1+2i \overset{+}{\dashrightarrow} i} \frac{1}{f(z)} dz = \int_{1+2i \overset{-}{\to} i} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{1}^{0} \frac{1+i}{f(i + r(1+i))} dr$$

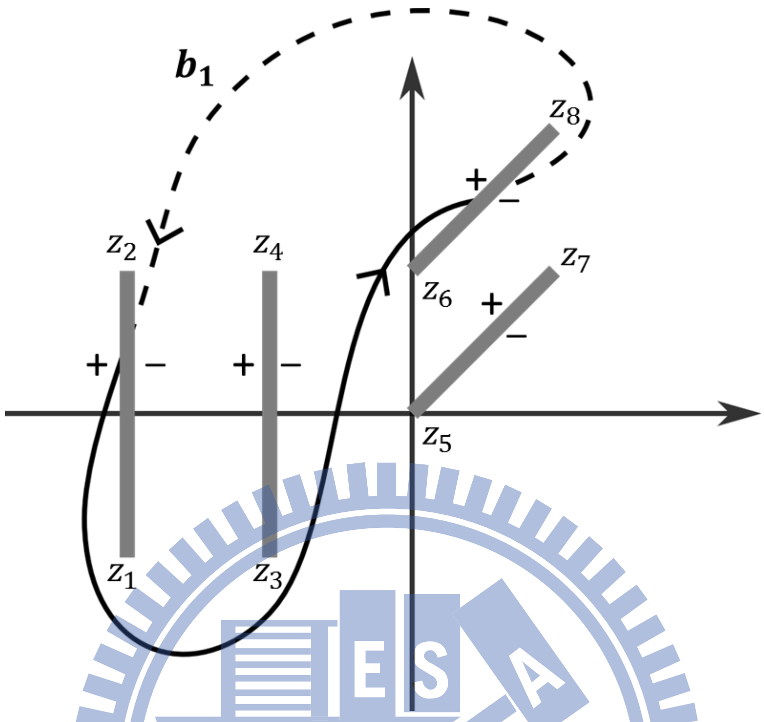
(8) b_{28}^{*} : Let $z = i + r \, , r : 0 \to -1 \, , dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b_{28}^{*}} \frac{1}{f(z)} dz = \int_{-1+i \leftarrow\!-\!- i} \frac{1}{f(z)} dz = - \int_{-1+i \leftarrow i} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{0}^{-1} \frac{1}{f(i+r)} dr$$

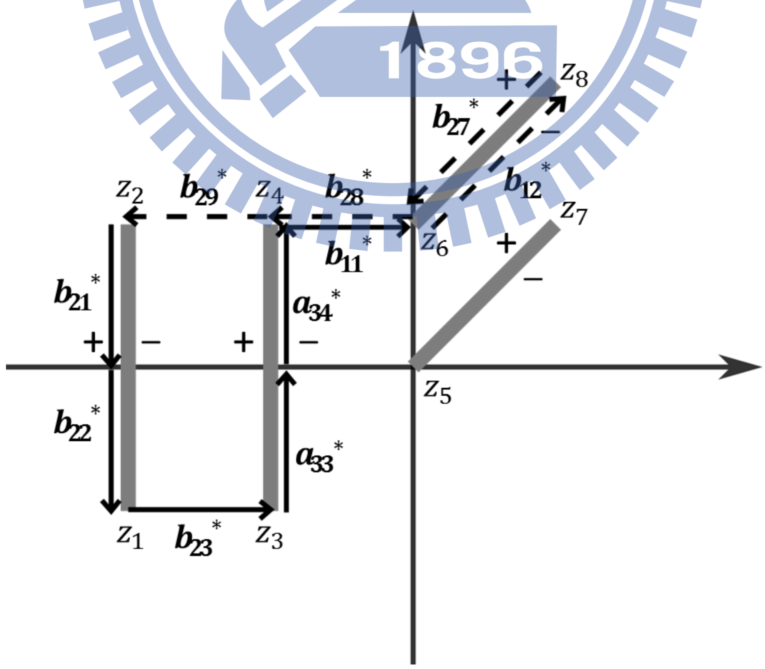
(9) b_{29}^{*} : Let $z = -1 + i + r \, , r : 0 \to -1 \, , dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\begin{aligned}\int_{b_{29}^{*}} \frac{1}{f(z)} dz &= \int_{-2+i \leftarrow\!-\!-\!- -1+i} \frac{1}{f(z)} dz \\ &= - \int_{-2+i \leftarrow -1+i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{0}^{-1} \frac{1}{f(-1+i+r)} dr\end{aligned}$$

By (1)∼(9) and Cauchy Theorem we can obtain $\int_{b_2} \frac{1}{f(z)} dz$, which value is shown in the Appendix A.0.5.



(e) The contour b_1 in the cut plane



(f) The equivalent path for b_1

3. Evaluate $\int_{b_1} \frac{1}{f(z)} dz$

b_1 : Consider the equivalent path

$b_1^* = b_{21}^* \bigcup b_{22}^* \bigcup b_{23}^* \bigcup a_{33}^* \bigcup b_{34}^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{27}^* \bigcup b_{28}^* \bigcup b_{29}^*$, where

b_{11}^* = the path on the horizontal line from $-1+i$ to i on sheet-I,

b_{12}^* = the path on the cut with $\frac{\pi}{4}$ from i to $1+2i$ on $(-)$edge of sheet-II.

(1) b_{11}^* : Let $z = r + i$, $r : -1 \to 0$, $dz = dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\int_{b_{11}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{-1}^{0} \frac{1}{f(r+i)} dr$$

(2) $b_{12}^* \equiv$ the path on the cut with $\frac{\pi}{4}$ from i to $1+2i$ on $(+)$edge of sheet-I.

Let $z = i + r(1+i)$, $r : 0 \to 1$, $dz = (1+i)dr$ and $z \in (M)$ then $f(z) \overset{Math.}{=} -f(z)$

$$\begin{aligned}\int_{b_{12}^*} \frac{1}{f(z)} dz &= \int_{i \overset{-}{\dashrightarrow} 1+2i} \frac{1}{f(z)} dz \\ &= \int_{i \overset{+}{\to} 1+2i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} -\int_{0}^{1} \frac{1+i}{f(i + r(1+i))} dr\end{aligned}$$

By (1),(2) and Cauchy Theorem we can obtain $\int_{b_1} \frac{1}{f(z)} dz$, which value is shown in the Appendix A.0.6.

Note that these are not the only choices of branch cuts. We could have branch cuts from the two branch points go off to infinity in any direction, and they don't even need to be straight lines. What above gave are just convenient choices.

Chapter 3

Nonlinear Approximations on Riemann Surfaces of the Pendulum Equation

As in Chapter1, the pendulum equation can be written as

$$u'' + \sin u = 0\,. \tag{3.0.1}$$

We know that $\sin u$ can be expanded by Taylor series

$$\sin u = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} u^{2k+1}, \text{ for all values of } u.$$

Take the first eight terms to approximate $\sin u$ as

$$\sin u \approx u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!} - \frac{u^{11}}{11!} + \frac{u^{13}}{13!} - \frac{u^{15}}{15!}\,.$$

Let

$$P_{15}(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!} - \frac{u^{11}}{11!} + \frac{u^{13}}{13!} - \frac{u^{15}}{15!}\,.$$

So the equation 3.0.1 becomes to

$$u'' + P_{15}(u) = 0\,.$$

As in section **??**, we derived that

$$\frac{1}{2}(u')^2 + P_{16}(u) = E\,,$$

where E is the integration constant and

$$P_{16}(u) = \frac{u^2}{2} - \frac{u^4}{4!} + \frac{u^6}{6!} - \frac{u^8}{8!} + \frac{u^{10}}{10!} - \frac{u^{12}}{12!} + \frac{u^{14}}{14!} - \frac{u^{16}}{16!}\,.$$

Then, we obtain the following integral equation

$$\int \frac{1}{\sqrt{2(E - P_{16}(u))}}\, du = \pm \int dt\,.$$

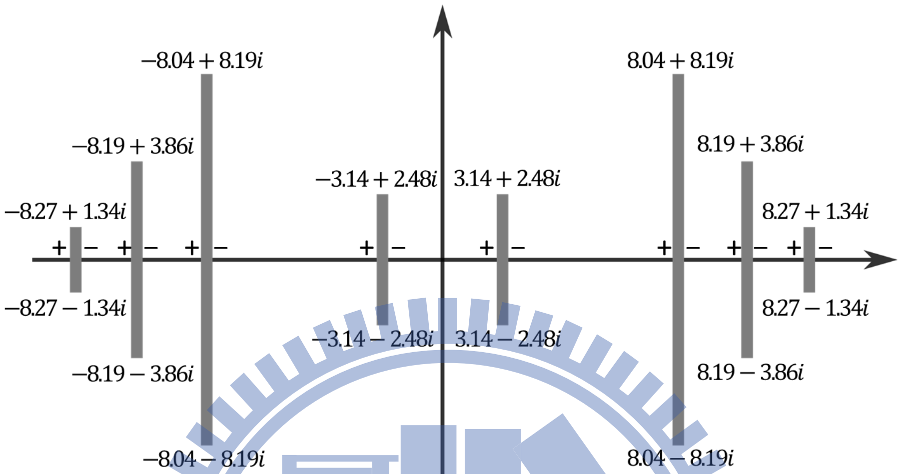
Since $2(E - P_{16}(u))$ is a polynomial of u, it can be written as

$$\begin{aligned} 2(E - P_{16}(u)) &= (u - u_1)(u - u_2)\cdots(u - u_{16}) \\ &= \prod_{k=1}^{16}(u - u_k)\,,\ \text{where } u_k\text{'s are the roots of the equation } 2(E - P_{16}(u)) = 0\,. \end{aligned}$$

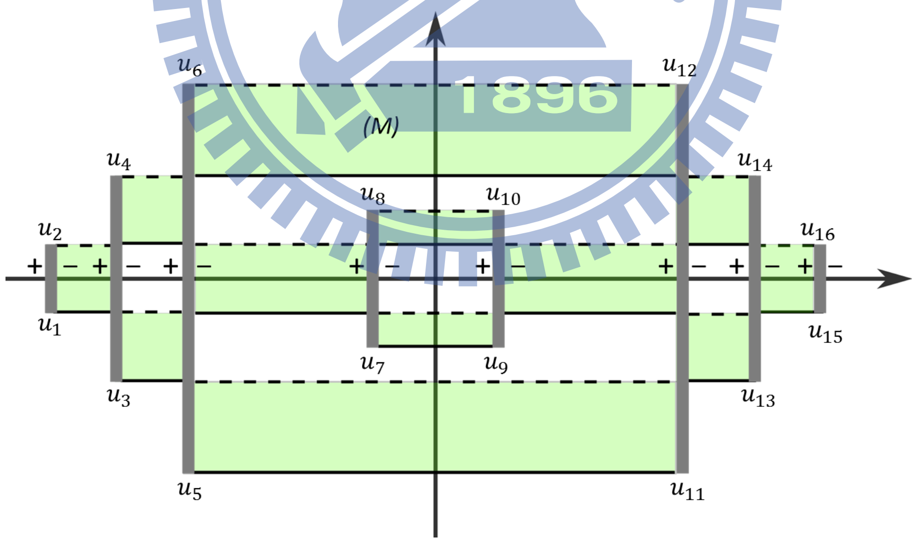
Thus, the function theory of solutions u of the equation involves $\sqrt{\prod_{k=1}^{16}(u - u_k)}$.

Let $f(u) = \sqrt{2(E - P_{16}(u))}$, and compute $\int \frac{1}{f(u)} du$ over a,b cycles.
Given $E = 7$, we have $u_1 = -8.27 - 1.34i\,,\ u_2 = -8.27 + 1.34i\,,\ u_3 = -8.19 - 3.86i\,,\ u_4 = -8.19 + 3.86i\,,\ u_5 = -8.04 - 8.19i\,,\ u_6 = -8.04 + 8.19i\,,\ u_7 = -3.14 - 2.48i\,,\ u_8 = -3.14 + 2.48i\,,\ u_9 = 3.14 - 2.48i\,,\ u_{10} = 3.14 + 2.48i\,,\ u_{11} = 8.04 - 8.19i\,,\ u_{12} = 8.04 + 8.19i\,,\ u_{13} = 8.19 - 3.86i\,,\ u_{14} = 8.19 + 3.86i\,,\ u_{15} = 8.27 - 1.34i\,,\ u_{16} = 8.27 + 1.34i\,.$

We let region of modify for $f(u)$ is (M) as illustrated in the in Figure **??**.



(g) The cut plane



(h) (M): region of modify

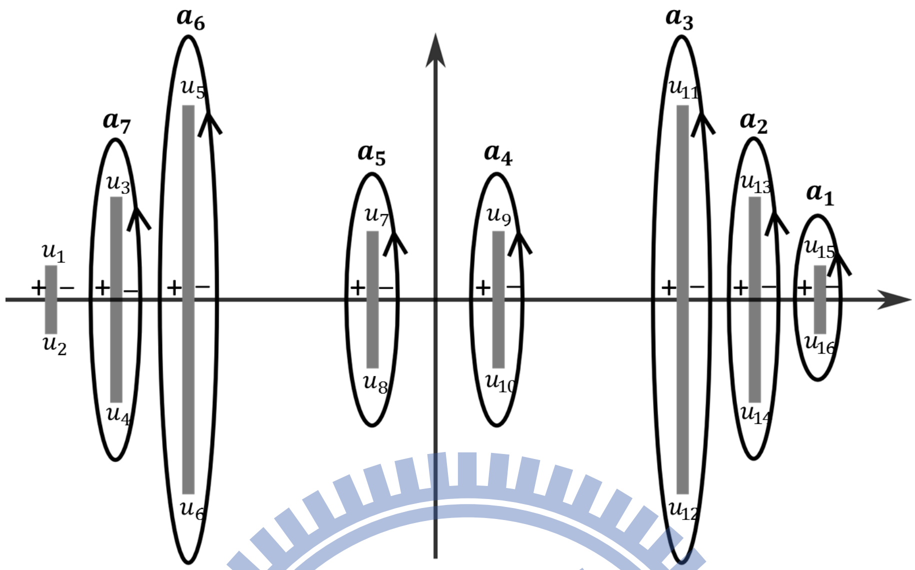


Figure 3.0.1. a-cycles

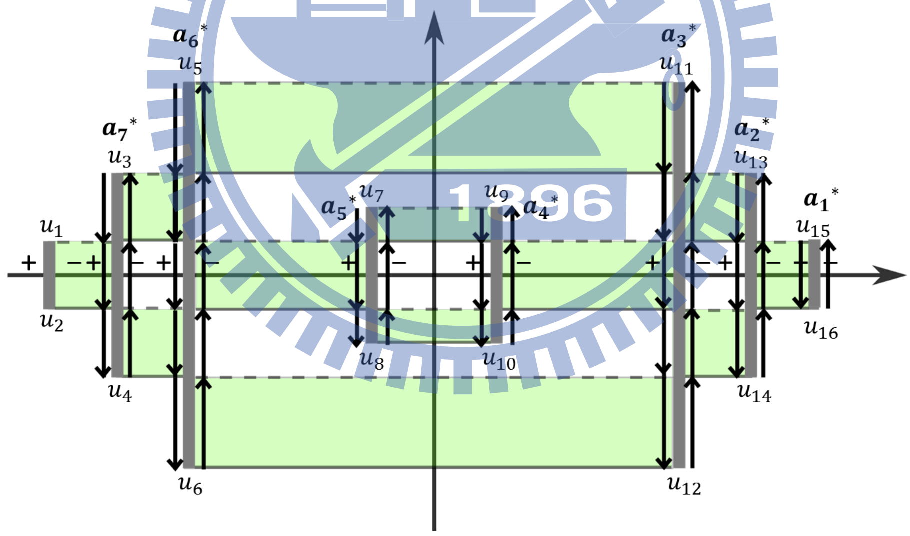


Figure 3.0.2. The equivalent path for a-cycles

1. Evaluate $\displaystyle\int_{a_1} \frac{1}{f(u)}du$, a_1 : Consider the equivalent path $a_1^* = a_{11}^* \bigcup a_{12}^*$, where

   a_{11}^* = the path on the vertical cut from $8.27 + 1.34i$ to $8.27 - 1.34i$ on

(+)edge of sheet-I, $a_{12}^* =$ the path on the vertical cut from $8.27 - 1.34i$ to $8.27 + 1.34i$ on (−)edge of sheet-I.

(1) a_{11}^* : Let $u = 8.27 + ri$, $r : 1.34 \to -1.34$, $du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{11}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{1.34}^{-1.34} \frac{i}{f(8.27 + ri)} dr$$

(2) a_{12}^* : Let $u = 8.27 + ri$, $r : -1.34 \to 1.34$, $du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{12}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{-1.34}^{1.34} \frac{i}{f(8.27 + ri)} dr$$

By (1),(2) and Cauchy Theorem we can obtain $\int_{a_1} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.7.

2. Evaluate $\int_{a_2} \frac{1}{f(u)} du$

a_2 : Consider the equivalent path $a_2^* = a_{21}^* \bigcup a_{22}^* \bigcup a_{23}^* \bigcup a_{24}^* \bigcup a_{25}^* \bigcup a_{26}^*$, where

$a_{21}^* =$ the path on the vertical cut from $8.19 + 3.86i$ to $8.19 + 1.34i$ on (+)edge of sheet-I, $a_{22}^* =$ the path on the vertical cut from $8.19 + 1.34i$ to $8.19 - 1.34i$ on (+)edge of sheet-I, $a_{23}^* =$ the path on the vertical cut from $8.19 - 1.34i$ to $8.19 - 3.86i$ on (+)edge of sheet-I, $a_{24}^* =$ the path on the vertical cut from $8.19 - 3.86i$ to $8.19 - 1.34i$ on (−)edge of sheet-I, $a_{25}^* =$ the path on the vertical cut from $8.19 - 1.34i$ to $8.19 + 1.34i$ on (−)edge of sheet-I, $a_{26}^* =$ the path on the vertical cut from $8.19 + 1.34i$ to $8.19 + 3.86i$ on (−)edge of sheet-I.

(1) a_{21}^* : Let $u = 8.19 + ri$, $r : 3.86 \to 1.34$, $du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{21}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{3.86}^{1.34} \frac{i}{f(8.19 + ri)} dr$$

(2) a^*_{22} : Let $u = 8.19 + ri$, $r : 1.34 \to -1.34$, $du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{22}} \frac{1}{f(u)} du \overset{Math.}{=} \int_{1.34}^{-1.34} \frac{i}{f(8.19 + ri)} dr$$

(3) a^*_{23} : Let $u = 8.19 + ri$, $r : -1.34 \to -3.86$, $du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{23}} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{-1.34}^{-3.86} \frac{i}{f(8.19 + ri)} dr$$

(4) a^*_{24} : Let $u = 8.19 + ri$, $r : -3.86 \to -1.34$, $du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{24}} \frac{1}{f(u)} du \overset{Math.}{=} \int_{-3.86}^{-1.34} \frac{i}{f(8.19 + ri)} dr$$

(5) a^*_{25} : Let $u = 8.19 + ri$, $r : -1.34 \to 1.34$, $du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{25}} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{-1.34}^{1.34} \frac{i}{f(8.19 + ri)} dr$$

(6) a^*_{26} : Let $u = 8.19 + ri$, $r : 1.34 \to 3.86$, $du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{26}} \frac{1}{f(u)} du \overset{Math.}{=} \int_{1.34}^{3.86} \frac{i}{f(8.19 + ri)} dr$$

By (1)∼(6) and Cauchy Theorem we can obtain $\int_{a_2} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.8.

3. Evaluate $\int_{a_3} \frac{1}{f(u)} du$

a_3 : Consider the equivalent path $a^*_3 = a^*_{31} \bigcup a^*_{32} \bigcup a^*_{33} \bigcup a^*_{34} \bigcup a^*_{35} \bigcup a^*_{36}$

$\bigcup a_{37}^* \bigcup a_{38}^* \bigcup a_{39}^* \bigcup a_{40}^*$, where a_{31}^* = the path on the vertical cut from $8.04 + 8.19i$ to $8.04 + 3.86i$ on $(+)$edge of sheet-I, a_{32}^* = the path on the vertical cut from $8.04 + 3.86i$ to $8.04 + 1.34i$ on $(+)$edge of sheet-I, a_{33}^* = the path on the vertical cut from $8.04 + 1.34i$ to $8.04 - 1.34i$ on $(+)$edge of sheet-I, a_{34}^* = the path on the vertical cut from $8.04 - 1.34i$ to $8.04 - 3.86i$ on $(+)$edge of sheet-I, a_{35}^* = the path on the vertical cut from $8.04 - 3.86i$ to $8.04 - 8.19i$ on $(+)$edge of sheet-I, a_{36}^* = the path on the vertical cut from $8.04 - 8.19i$ to $8.04 - 3.86i$ on $(-)$edge of sheet-I, a_{37}^* = the path on the vertical cut from $8.04 - 3.86i$ to $8.04 - 1.34i$ on $(-)$edge of sheet-I, a_{38}^* = the path on the vertical cut from $8.04 - 1.34i$ to $8.04 + 1.34i$ on $(-)$edge of sheet-I, a_{39}^* = the path on the vertical cut from $8.04 + 1.34i$ to $8.04 + 3.86i$ on $(-)$edge of sheet-I, a_{40}^* = the path on the vertical cut from $8.04 + 3.86i$ to $8.04 + 8.19i$ on $(-)$edge of sheet-I.

(1) a_{31}^* : Let $u = 8.04 + ri\,,\, r : 8.19 \to 3.86\,, du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{31}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{8.19}^{3.86} \frac{i}{f(8.04 + ri)} dr$$

(2) a_{32}^* : Let $u = 8.04 + ri\,,\, r : 3.86 \to 1.34\,, du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{32}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{3.86}^{1.34} \frac{i}{f(8.04 + ri)} dr$$

(3) a_{33}^* : Let $u = 8.04 + ri\,,\, r : 1.34 \to -1.34\,, du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{33}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{1.34}^{-1.34} \frac{i}{f(8.04 + ri)} dr$$

(4) a_{34}^* : Let $u = 8.04 + ri\,,\, r : -1.34 \to -3.86\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{34}^*} \frac{1}{f(u)}du \overset{Math.}{=} \int_{-1.34}^{-3.86} \frac{i}{f(8.04+ri)}dr$$

(5) a_{35}^* : Let $u = 8.04 + ri$, $r : -3.86 \to -8.19$, $du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{35}^*} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{-3.86}^{-8.19} \frac{i}{f(8.04+ri)}dr$$

(6) a_{36}^* : Let $u = 8.04 + ri$, $r : -8.19 \to -3.86$, $du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{36}^*} \frac{1}{f(u)}du \overset{Math.}{=} \int_{-8.19}^{-3.86} \frac{i}{f(8.04+ri)}dr$$

(7) a_{37}^* : Let $u = 8.04 + ri$, $r : -3.86 \to -1.34$, $du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{37}^*} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{-3.86}^{-1.34} \frac{i}{f(8.04+ri)}dr$$

(8) a_{38}^* : Let $u = 8.04 + ri$, $r : -1.34 \to 1.34$, $du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{38}^*} \frac{1}{f(u)}du \overset{Math.}{=} \int_{-1.34}^{1.34} \frac{i}{f(8.04+ri)}dr$$

(9) a_{39}^* : Let $u = 8.04 + ri$, $r : 1.34 \to 3.86$, $du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{39}^*} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{1.34}^{3.86} \frac{i}{f(8.04+ri)}dr$$

(10) a_{40}^* : Let $u = 8.04 + ri$, $r : 3.86 \to 8.19$, $du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{40}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{3.86}^{8.19} \frac{i}{f(8.04 + ri)} dr$$

By (1)∼(10) and Cauchy Theorem we can obtain $\int_{a_3} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.9.

4. Evaluate $\int_{a_4} \frac{1}{f(u)} du$

a_4 : Consider the equivalent path $a_4^* = a_{41}^* \bigcup a_{42}^* \bigcup a_{43}^* \bigcup a_{44}^* \bigcup a_{45}^* \bigcup a_{46}^*$, where

a_{41}^* = the path on the vertical cut from $3.14 + 2.48i$ to $3.14 + 1.34i$ on (+)edge of sheet-I, a_{42}^* = the path on the vertical cut from $3.14 + 1.34i$ to $3.14 - 1.34i$ on (+)edge of sheet-I, a_{43}^* = the path on the vertical cut from $3.14 - 1.34i$ to $3.14 - 2.48i$ on (+)edge of sheet-I, a_{44}^* = the path on the vertical cut from $3.14 - 2.48i$ to $3.14 - 1.34i$ on (−)edge of sheet-I, a_{45}^* = the path on the vertical cut from $3.14 - 1.34i$ to $3.14 + 1.34i$ on (−)edge of sheet-I, a_{46}^* = the path on the vertical cut from $3.14 + 1.34i$ to $3.14 + 2.48i$ on (−)edge of sheet-I.

(1) a_{41}^* : Let $u = 3.14 + ri$, $r : 2.48 \to 1.34$, $du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{41}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{2.48}^{1.34} \frac{i}{f(3.14 + ri)} dr$$

(2) a_{42}^* : Let $u = 3.14 + ri$, $r : 1.34 \to -1.34$, $du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{42}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{1.34}^{-1.34} \frac{i}{f(3.14 + ri)} dr$$

(3) a_{43}^* : Let $u = 3.14 + ri\,, r : -1.34 \to -2.48\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{43}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{-1.34}^{-2.48} \frac{i}{f(3.14+ri)} dr$$

(4) a_{44}^* : Let $u = 3.14 + ri\,, r : -2.48 \to -1.34\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{44}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{-2.48}^{-1.34} \frac{i}{f(3.14+ri)} dr$$

(5) a_{45}^* : Let $u = 3.14 + ri\,, r : -1.34 \to 1.34\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{45}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{-1.34}^{1.34} \frac{i}{f(3.14+ri)} dr$$

(6) a_{46}^* : Let $u = 3.14 + ri\,, r : 1.34 \to 2.48\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{46}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{1.34}^{2.48} \frac{i}{f(3.14+ri)} dr$$

By (1)∼(6) and Cauchy Theorem we can obtain $\int_{a_4} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.10.

5. Evaluate $\int_{a_5} \frac{1}{f(u)} du$

a_5 : Consider the equivalent path $a_5^* = a_{51}^* \bigcup a_{52}^* \bigcup a_{53}^* \bigcup a_{54}^* \bigcup a_{55}^* \bigcup a_{56}^*$, where

a_{51}^* = the path on the vertical cut from $-3.14+2.48i$ to $-3.14+1.34i$ on (+)edge of sheet-I, a_{52}^* = the path on the vertical cut from $-3.14+1.34i$ to $-3.14-1.34i$ on (+)edge of sheet-I, a_{53}^* = the path on the vertical cut from $-3.14-1.34i$ to $-3.14-2.48i$ on $(+)$edge of sheet-I, a^*_{54} = the path on the vertical cut from $-3.14-2.48i$ to $-3.14-1.34i$ on $(-)$edge of sheet-I, a^*_{55} = the path on the vertical cut from $-3.14-1.34i$ to $-3.14+1.34i$ on $(-)$edge of sheet-I, a^*_{56} = the path on the vertical cut from $-3.14+1.34i$ to $-3.14+2.48i$ on $(-)$edge of sheet-I.

(1) a^*_{51} : Let $u=-3.14+ri\,,\ r: 2.48 \to 1.34\,, du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{51}} \frac{1}{f(u)}du \overset{Math.}{=} \int_{2.48}^{1.34} \frac{i}{f(-3.14+ri)}dr$$

(2) a^*_{52} : Let $u=-3.14+ri\,,\ r: 1.34 \to -1.34\,, du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{52}} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{1.34}^{-1.34} \frac{i}{f(-3.14+ri)}dr$$

(3) a^*_{53} : Let $u=-3.14+ri\,,\ r: -1.34 \to -2.48\,, du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{53}} \frac{1}{f(u)}du \overset{Math.}{=} \int_{-1.34}^{-2.48} \frac{i}{f(-3.14+ri)}dr$$

(4) a^*_{54} : Let $u=-3.14+ri\,,\ r: -2.48 \to -1.34\,, du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{54}} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{-2.48}^{-1.34} \frac{i}{f(-3.14+ri)}dr$$

(5) a^*_{55} : Let $u=-3.14+ri\,,\ r: -1.34 \to 1.34\,, du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{55}} \frac{1}{f(u)}du \overset{Math.}{=} \int_{-1.34}^{1.34} \frac{i}{f(-3.14+ri)}dr$$

(6) a_{56}^* : Let $u = -3.14 + ri\,,\; r : 1.34 \to 2.48\,, du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{56}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{1.34}^{2.48} \frac{i}{f(-3.14 + ri)} dr$$

By (1)~(6) and Cauchy Theorem we can obtain $\int_{a_5} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.11.

6. Evaluate $\int_{a_6} \frac{1}{f(u)} du$

a_6 : Consider the equivalent path $a_6^* = a_{61}^* \bigcup a_{62}^* \bigcup a_{63}^* \bigcup a_{64}^* \bigcup a_{65}^* \bigcup a_{66}^* \bigcup a_{67}^* \bigcup a_{68}^* \bigcup a_{69}^* \bigcup a_{70}^*$, where a_{61}^* = the path on the vertical cut from $-8.04 + 8.19i$ to $-8.04 + 3.86i$ on (+)edge of sheet-I, a_{62}^* = the path on the vertical cut from $-8.04 + 3.86i$ to $-8.04 + 1.34i$ on (+)edge of sheet-I, a_{63}^* = the path on the vertical cut from $-8.04 + 1.34i$ to $-8.04 - 1.34i$ on (+)edge of sheet-I, a_{64}^* = the path on the vertical cut from $-8.04 - 1.34i$ to $-8.04 - 3.86i$ on (+)edge of sheet-I, a_{65}^* = the path on the vertical cut from $-8.04 - 3.86i$ to $-8.04 - 8.19i$ on (+)edge of sheet-I, a_{66}^* = the path on the vertical cut from $-8.04 - 8.19i$ to $-8.04 - 3.86i$ on (−)edge of sheet-I, a_{67}^* = the path on the vertical cut from $-8.04 - 3.86i$ to $-8.04 - 1.34i$ on (−)edge of sheet-I, a_{68}^* = the path on the vertical cut from $-8.04 - 1.34i$ to $-8.04 + 1.34i$ on (−)edge of sheet-I, a_{69}^* = the path on the vertical cut from $-8.04 + 1.34i$ to $-8.04 + 3.86i$ on (−)edge of sheet-I, a_{70}^* = the path on the vertical cut from $-8.04 + 3.86i$ to $-8.04 + 8.19i$ on (−)edge of sheet-I.

(1) a_{61}^* : Let $u = -8.04 + ri\,,\; r : 8.19 \to 3.86\,, du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{61}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{8.19}^{3.86} \frac{i}{f(-8.04 + ri)} dr$$

(2) a^*_{62} : Let $u = -8.04 + ri$, $r : 3.86 \to 1.34$, $du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{62}} \frac{1}{f(u)} du \overset{Math.}{=} - \int_{3.86}^{1.34} \frac{i}{f(-8.04 + ri)} dr$$

(3) a^*_{63} : Let $u = -8.04 + ri$, $r : 1.34 \to -1.34$, $du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{63}} \frac{1}{f(u)} du \overset{Math.}{=} \int_{1.34}^{-1.34} \frac{i}{f(-8.04 + ri)} dr$$

(4) a^*_{64} : Let $u = -8.04 + ri$, $r : -1.34 \to -3.86$, $du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{64}} \frac{1}{f(u)} du \overset{Math.}{=} - \int_{-1.34}^{-3.86} \frac{i}{f(-8.04 + ri)} dr$$

(5) a^*_{65} : Let $u = -8.04 + ri$, $r : -3.86 \to -8.19$, $du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{65}} \frac{1}{f(u)} du \overset{Math.}{=} \int_{-3.86}^{-8.19} \frac{i}{f(-8.04 + ri)} dr$$

(6) a^*_{66} : Let $u = -8.04 + ri$, $r : -8.19 \to -3.86$, $du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{66}} \frac{1}{f(u)} du \overset{Math.}{=} - \int_{-8.19}^{-3.86} \frac{i}{f(-8.04 + ri)} dr$$

(7) a^*_{67} : Let $u = -8.04 + ri$, $r : -3.86 \to -1.34$, $du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{67}} \frac{1}{f(u)} du \overset{Math.}{=} \int_{-3.86}^{-1.34} \frac{i}{f(-8.04 + ri)} dr$$

(8) a_{68}^* : Let $u = -8.04 + ri\,,\, r : -1.34 \to 1.34\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{68}^*} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{-1.34}^{1.34} \frac{i}{f(-8.04+ri)}dr$$

(9) a_{69}^* : Let $u = -8.04+ri\,,\, r : 1.34 \to 3.86\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{69}^*} \frac{1}{f(u)}du \overset{Math.}{=} \int_{1.34}^{3.86} \frac{i}{f(-8.04+ri)}dr$$

(10) a_{70}^* : Let $u = -8.04+ri\,,\, r : 3.86 \to 8.19\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{70}^*} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{3.86}^{8.19} \frac{i}{f(-8.04+ri)}dr$$

By (1)∼(10) and Cauchy Theorem we can obtain $\int_{a_6} \frac{1}{f(u)}du$, which value is shown in the Appendix A.0.12.

7. Evaluate $\int_{a_7} \frac{1}{f(u)}du$

a_7 : Consider the equivalent path $a_7^* = a_{71}^* \bigcup a_{72}^* \bigcup a_{73}^* \bigcup a_{74}^* \bigcup a_{75}^* \bigcup a_{76}^*$, where

a_{71}^* = the path on the vertical cut from $-8.19+3.86i$ to $-8.19+1.34i$ on (+)edge of sheet-I, a_{72}^* = the path on the vertical cut from $-8.19+1.34i$ to $-8.19-1.34i$ on (+)edge of sheet-I, a_{73}^* = the path on the vertical cut from $-8.19-1.34i$ to $-8.19-3.86i$ on (+)edge of sheet-I, a_{74}^* = the path on the vertical cut from $-8.19-3.86i$ to $-8.19-1.34i$ on (−)edge of sheet-I, a_{75}^* = the path on the vertical cut from $-8.19-1.34i$ to $-8.19+1.34i$ on (−)edge of sheet-I, a_{76}^* = the path on the vertical cut from $-8.19+1.34i$ to $-8.19+3.86i$ on (−)edge of sheet-I.

(1) a^*_{71} : Let $u = -8.19 + ri\,,\, r : 3.86 \to 1.34\,, du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{71}} \frac{1}{f(u)}du \overset{Math.}{=} \int_{3.86}^{1.34} \frac{i}{f(-8.19+ri)}dr$$

(2) a^*_{72} : Let $u = -8.19 + ri\,,\, r : 1.34 \to -1.34\,, du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{72}} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{1.34}^{-1.34} \frac{i}{f(-8.19+ri)}dr$$

(3) a^*_{73} : Let $u = -8.19 + ri\,,\, r : -1.34 \to -3.86\,, du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{73}} \frac{1}{f(u)}du \overset{Math.}{=} \int_{-1.34}^{-3.86} \frac{i}{f(-8.19+ri)}dr$$

(4) a^*_{74} : Let $u = -8.19 + ri\,,\, r : -3.86 \to -1.34\,, du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{74}} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{-3.86}^{-1.34} \frac{i}{f(-8.19+ri)}dr$$

(5) a^*_{75} : Let $u = -8.19 + ri\,,\, r : -1.34 \to 1.34\,, du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{75}} \frac{1}{f(u)}du \overset{Math.}{=} \int_{-1.34}^{1.34} \frac{i}{f(-8.19+ri)}dr$$

(6) a^*_{76} : Let $u = -8.19 + ri\,,\, r : 1.34 \to 3.86\,, du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{76}} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{1.34}^{3.86} \frac{i}{f(-8.19+ri)}dr$$

By (1)∼(6) and Cauchy Theorem we can obtain $\int_{a_7} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.13.

1. Evaluate $\int_{b_7} \frac{1}{f(u)} du$

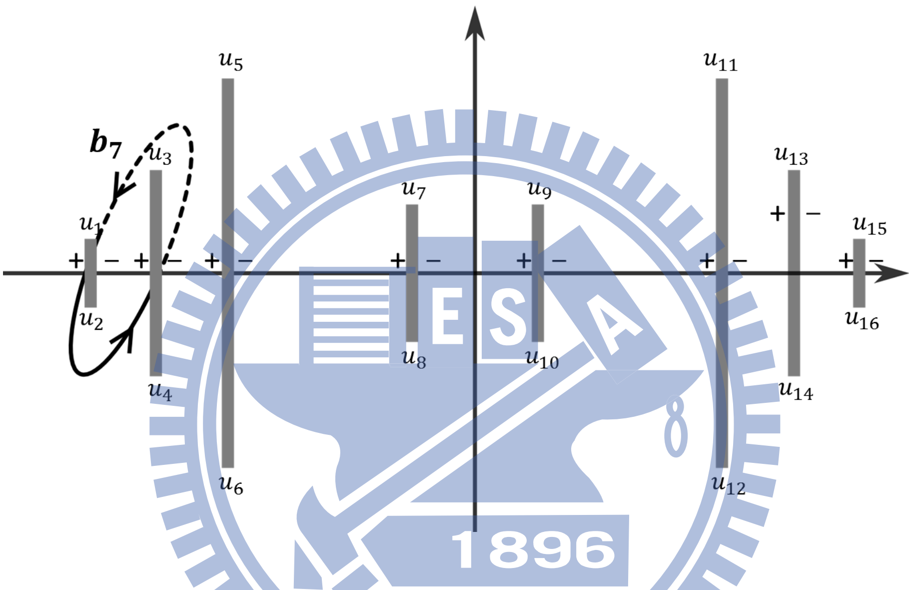


Figure 3.0.3. The contour b_7 in the cut plane

b_7 : Consider the equivalent path

$b_7^* = b_{71}^* \bigcup b_{72}^* \bigcup b_{73}^* \bigcup b_{74}^* \bigcup b_{75}^* \bigcup a_{76}^* \bigcup b_{77}^* \bigcup b_{78}^*$, where

b_{71}^* = the path on the vertical cut from -8.27 to $-8.27 - 1.34i$ on (+)edge of sheet-I,

b_{72}^* = the path on the vertical cut from $-8.27 - 1.34i$ to -8.27 on (−)edge of sheet-I,

b_{73}^* = the path on the horizontal line from -8.27 to -8.19 on sheet-I,

b_{74}^* = the path on the vertical cut from -8.19 to $-8.19 + 1.34i$ on (−)edge of sheet-II,

b_{75}^* = the path on the vertical cut from $-8.19 + 1.34i$ to $-8.19 + 3.86i$ on (−)edge of sheet-II,

b^*_{76} = the path on the vertical cut from $-8.19+3.86i$ to $-8.19+1.34i$ on $(+)$edge of sheet-II,

b^*_{77} = the path on the vertical cut from $-8.19+1.34i$ to -8.19 on $(+)$edge of sheet-II,

b^*_{78} = the path on the horizontal line from -8.19 to -8.27 on sheet-II.

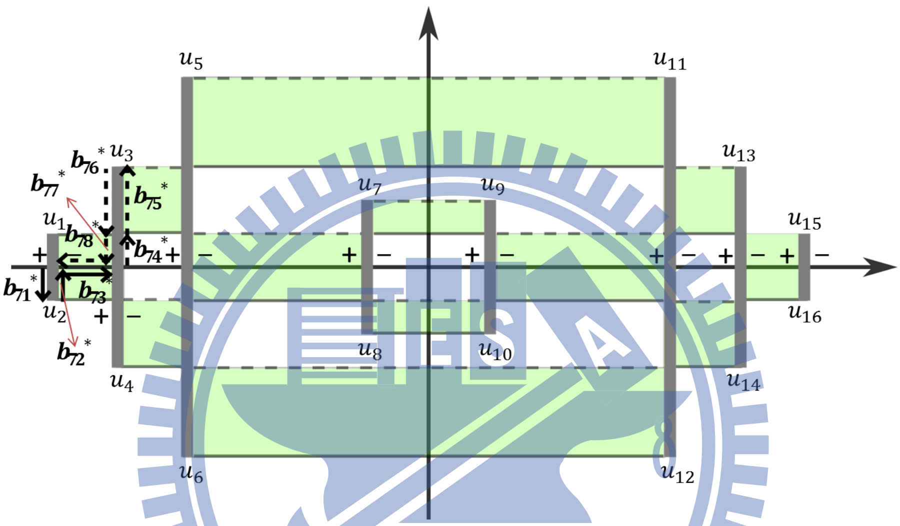


Figure 3.0.4. The equivalent path for b_7

(1) b^*_{71} : Let $u=-8.27-ri\,,\, r:0\to 1.34\,, du=-idr$ and $u\notin(M)$ then $f(u)\overset{Math.}{=}f(u)$

$$\int_{b^*_{71}}\frac{1}{f(u)}du \overset{Math.}{=} -\int_0^{1.34}\frac{i}{f(-8.27-ri)}dr$$

(2) b^*_{72} : Let $u=-8.27-ri\,,\, r:1.34\to 0\,, du=-idr$ and $u\in(M)$ then $f(u)\overset{Math.}{=}-f(u)$

$$\int_{b^*_{72}}\frac{1}{f(u)}dz \overset{Math.}{=} \int_{1.34}^{0}\frac{i}{f(-8.27-ri)}dr$$

(3) b^*_{73} : Let $u=r\,,\, r:-8.27\to -8.19\,, du=dr$ and $u\in(M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b_{73}^*} \frac{1}{f(u)} dz \overset{Math.}{=} -\int_{-8.27}^{-8.19} \frac{1}{f(r)} dr$$

(4) $b_{74}^* \equiv$ the path on the vertical cut from -8.19 to $-8.19+1.34i$ on (+)edge of sheet-I.

Let $u = -8.19 + ri\,,\ r : 0 \to 1.34\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b_{74}^*} \frac{1}{f(u)} du = \int_{-8.19 \overset{-}{\dashrightarrow} -8.19+1.34i} \frac{1}{f(u)} du = \int_{-8.19 \overset{+}{\to} -8.19+1.34i} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{0}^{1.34} \frac{i}{f(-8.19+ri)} dr$$

(5) $b_{75}^* \equiv$ the path on the vertical cut from $-8.19+1.34i$ to $-8.19+3.86i$ on (+)edge of sheet-I.

Let $u = -8.19 + ri\,,\ r : 1.34 \to 3.86\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b_{75}^*} \frac{1}{f(u)} du = \int_{-8.19+1.34i \overset{-}{\dashrightarrow} -8.19+3.86i} \frac{1}{f(u)} du$$
$$= \int_{-8.19+1.34i \overset{+}{\to} -8.19+3.86i} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} \int_{1.34}^{3.86} \frac{i}{f(-8.19+ri)} dr$$

(6) $b_{76}^* \equiv$ the path on the vertical cut from $-8.19+3.86i$ to $-8.19+1.34i$ on (−)edge of sheet-I.

Let $u = -8.19 + ri\,,\ r : 3.86 \to 1.34\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b_{76}^*} \frac{1}{f(u)} du = \int_{-8.19+3.86i \overset{+}{\dashrightarrow} -8.19+1.34i} \frac{1}{f(u)} du$$
$$= \int_{-8.19+3.86i \overset{-}{\rightarrow} -8.19+1.34i} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} - \int_{3.86}^{1.34} \frac{i}{f(-8.19+ri)} dr$$

(7) $b_{77}^* \equiv$ the path on the vertical cut from $-8.19+1.34i$ to -8.19 on $(-)$edge of sheet-I.

Let $u = -8.19 + ri$, $r : 1.34 \to 0$, $du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b_{77}^*} \frac{1}{f(u)} du = \int_{-8.19+1.34i \overset{+}{\dashrightarrow} -8.19} \frac{1}{f(u)} du$$
$$= \int_{-8.19+1.34i \overset{-}{\rightarrow} -8.19} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} \int_{1.34}^{0} \frac{i}{f(-8.19+ri)} dr$$

(8) b_{78}^* : Let $u = r$, $r : -8.19 \to -8.27$, $du = dr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b_{78}^*} \frac{1}{f(u)} du = \int_{-8.19 \dashrightarrow -8.27} \frac{1}{f(u)} du$$
$$= - \int_{-8.19 \rightarrow -8.27} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} \int_{-8.19}^{-8.27} \frac{1}{f(r)} dr$$

By (1)∼(8) and Cauchy Theorem we can obtain $\int_{b_7} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.14.

2. Evaluate $\int_{b_6} \frac{1}{f(u)} du$

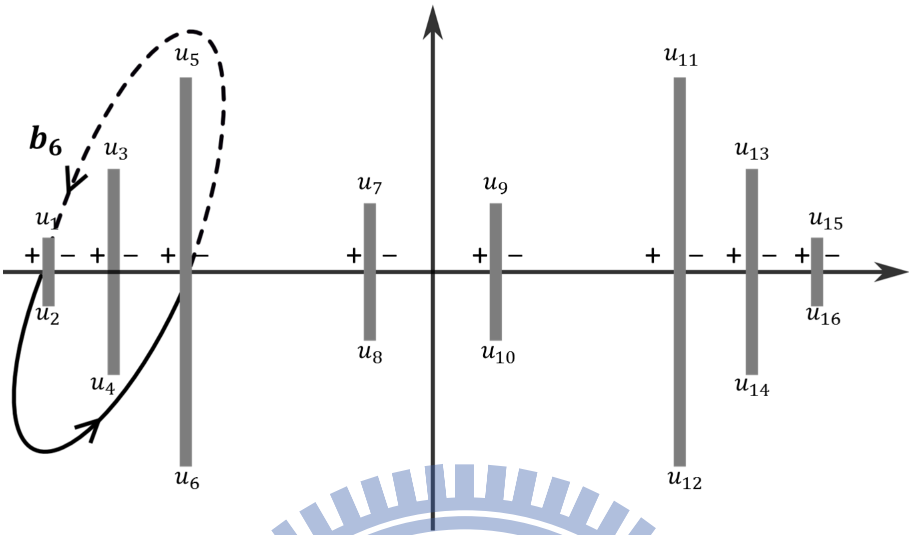


Figure 3.0.5. The contour b_6 in the cut plane

b_6 : Consider the equivalent path

$b_6^* = b_7^* \bigcup a_{72'}^* \bigcup a_{73}^* \bigcup a_{74}^* \bigcup a_{75'}^* \bigcup b_{61}^* \bigcup b_{62}^* \bigcup b_{63}^* \bigcup b_{64}^*$

$\bigcup b_{65}^* \bigcup b_{66}^* \bigcup b_{67}^* \bigcup b_{68}^*$, where

$a_{72'}^*$ = the path on the vertical cut from -8.19 to $-8.19 - 1.34i$ on (+)edge of sheet-I, $a_{75'}^*$ = the path on the vertical cut from $-8.19-1.34i$ to -8.19 on (−)edge of sheet-I, b_{61}^* = the path on the horizontal line from -8.19 to -8.04 on sheet-I,

b_{62}^* = the path on the vertical cut from -8.04 to $-8.04 + 1.34i$ on (−)edge of sheet-II,

b_{63}^* = the path on the vertical cut from $-8.04+1.34i$ to $-8.04+3.86i$ on (−)edge of sheet-II,

b_{64}^* = the path on the vertical cut from $-8.04+3.86i$ to $-8.04+8.19i$ on (−)edge of sheet-II,

b_{65}^* = the path on the vertical cut from $-8.04+8.19i$ to $-8.04+3.86i$ on (+)edge of sheet-II,

b_{66}^* = the path on the vertical cut from $-8.04+3.86i$ to $-8.04+1.34i$ on (+)edge of sheet-II,

b_{67}^* = the path on the vertical cut from $-8.04+1.34i$ to -8.04 on (+)edge of sheet-II,

b_{68}^* = the path on the horizontal line from -8.04 to -8.19 on sheet-II.

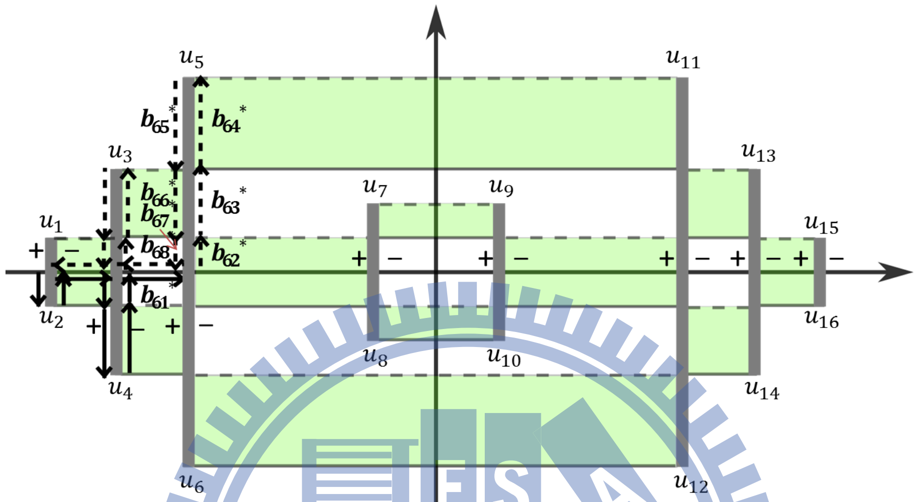


Figure 3.0.6. The equivalent path for b_6

(1) $a_{72'}^*$: Let $u=-8.19+ri\,,\, r:0\to -1.34\,, du=idr$ and $u\in(M)$ then $f(u)\overset{Math.}{=}-f(u)$

$$\int_{a_{72'}^*}\frac{1}{f(u)}du\overset{Math.}{=}-\int_0^{-1.34}\frac{i}{f(-8.19+ri)}dr$$

(2) $a_{75'}^*$: Let $u=-8.19+ri\,,\, r:-1.34\to 0\,, du=idr$ and $u\notin(M)$ then $f(u)\overset{Math.}{=}f(u)$

$$\int_{a_{75'}^*}\frac{1}{f(u)}du\overset{Math.}{=}\int_{-1.34}^{0}\frac{i}{f(-8.19+ri)}dr$$

(3) b_{61}^* : Let $u=r\,,\, r:-8.19\to -8.04\,, du=dr$ and $u\notin(M)$ then $f(u)\overset{Math.}{=}f(u)$

$$\int_{b_{61}^*}\frac{1}{f(u)}du\overset{Math.}{=}\int_{-8.19}^{-8.04}\frac{1}{f(r)}dr$$

(4) $b^*_{62} \equiv$ the path on the vertical cut from -8.04 to $-8.04 + 1.34i$ on (+)edge of sheet-I.

Let $u = -8.04 + ri\,,\ r : 0 \to 1.34\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b^*_{62}} \frac{1}{f(u)} dz = \int_{-8.04 \overset{-}{\dashrightarrow} -8.04+1.34i} \frac{1}{f(u)} du$$
$$= \int_{-8.04 \overset{+}{\to} -8.04+1.34i} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} \int_0^{1.34} \frac{i}{f(-8.04 + ri)} dr$$

(5) $b^*_{63} \equiv$ the path on the vertical cut from $-8.04 + 1.34i$ to $-8.04 + 3.86i$ on (+)edge of sheet-I.

Let $u = -8.04 + ri\,,\ r : 1.34 \to 3.86\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b^*_{63}} \frac{1}{f(u)} du = \int_{-8.04+1.34i \overset{-}{\dashrightarrow} -8.04+3.86i} \frac{1}{f(u)} du$$
$$= \int_{-8.04+1.34i \overset{+}{\to} -8.04+3.86i} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} -\int_{1.34}^{3.86} \frac{i}{f(-8.04 + ri)} dr$$

(6) $b^*_{64} \equiv$ the path on the vertical cut from $-8.04 + 3.86i$ to $-8.04 + 8.19i$ on (+)edge of sheet-I.

Let $u = -8.04 + ri\,,\ r : 3.86 \to 8.19\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b^*_{64}} \frac{1}{f(u)} du = \int_{-8.04+3.86i \overset{-}{\dashrightarrow} -8.04+8.19i} \frac{1}{f(u)} du$$
$$= \int_{-8.04+3.86i \overset{+}{\to} -8.04+8.19i} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} \int_{3.86}^{8.19} \frac{i}{f(-8.04 + ri)} dr$$

(7) $b^*_{65} \equiv$ the path on the vertical cut from $-8.04 + 8.19i$ to $-8.04 + 3.86i$ on (−)edge of sheet-I.

Let $u = -8.04 + ri\,,\ r : 8.19 \to 3.86\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}\int_{b_{65}^*} \frac{1}{f(u)}du &= \int_{-8.04+8.19i \overset{+}{\dashrightarrow} -8.04+3.86i} \frac{1}{f(u)}du \\ &= \int_{-8.04+8.19i \overset{-}{\rightarrow} -8.04+3.86i} \frac{1}{f(u)}du \\ &\overset{Math.}{=} -\int_{8.19}^{3.86} \frac{i}{f(-8.04+ri)}dr\end{aligned}$$

(8) $b_{66}^* \equiv$ the path on the vertical cut from $-8.04 + 3.86i$ to $-8.04 + 1.34i$ on $(-)$edge of sheet-I.

Let $u = -8.04 + ri\,,\ r : 3.86 \to 1.34\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b_{66}^*} \frac{1}{f(u)}du &= \int_{-8.04+3.86i \overset{+}{\dashrightarrow} -8.04+1.34i} \frac{1}{f(u)}du \\ &= \int_{-8.04+3.86i \overset{-}{\rightarrow} -8.04+1.34i} \frac{1}{f(u)}du \\ &\overset{Math.}{=} \int_{3.86}^{1.34} \frac{i}{f(-8.04+ri)}dr\end{aligned}$$

(9) $b_{67}^* \equiv$ the path on the vertical cut from $-8.04 + 1.34i$ to -8.04 on $(-)$edge of sheet-I.

Let $u = -8.04 + ri\,,\ r : 1.34 \to 0\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}\int_{b_{67}^*} \frac{1}{f(u)}du &= \int_{-8.04+1.34i \overset{+}{\dashrightarrow} -8.04} \frac{1}{f(u)}du \\ &= \int_{-8.04+1.34i \overset{-}{\rightarrow} -8.04} \frac{1}{f(u)}du \\ &\overset{Math.}{=} -\int_{1.34}^{0} \frac{i}{f(-8.04+ri)}dr\end{aligned}$$

(10) b_{68}^* : Let $u = r\,,\ r : -8.04 \to -8.19\,, du = dr$ and $u \notin (M)$ then

$f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b_{68}^*} \frac{1}{f(u)} dz &= \int_{-8.04 \dashrightarrow -8.19} \frac{1}{f(u)} du \\ &= -\int_{-8.04 \to -8.19} \frac{1}{f(u)} du \\ &\overset{Math.}{=} -\int_{-8.04}^{-8.19} \frac{1}{f(r)} dr\end{aligned}$$

By (1)∼(10) and Cauchy Theorem we can obtain $\int_{b_6} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.15.

3. Evaluate $\int_{b_5} \frac{1}{f(u)} du$

b_5 : Consider the equivalent path

$b_5^* = b_6^* \bigcup a_{63'}^* \bigcup a_{64}^* \bigcup a_{65}^* \bigcup a_{66}^* \bigcup a_{67}^* \bigcup a_{68'}^* \bigcup b_{51}^* \bigcup b_{52}^* \bigcup b_{53}^* \bigcup b_{54}^* \bigcup b_{55}^* \bigcup b_{56}^*$, where

$a_{63'}^*$ = the path on the vertical cut from -8.04 to $-8.04 - 1.34i$ on (+)edge of sheet-I,

$a_{68'}^*$ = the path on the vertical cut from $-8.04 - 1.34i$ to -8.04 on (−)edge of sheet-I,

b_{51}^* = the path on the horizontal line from -8.04 to -3.14 on sheet-I,

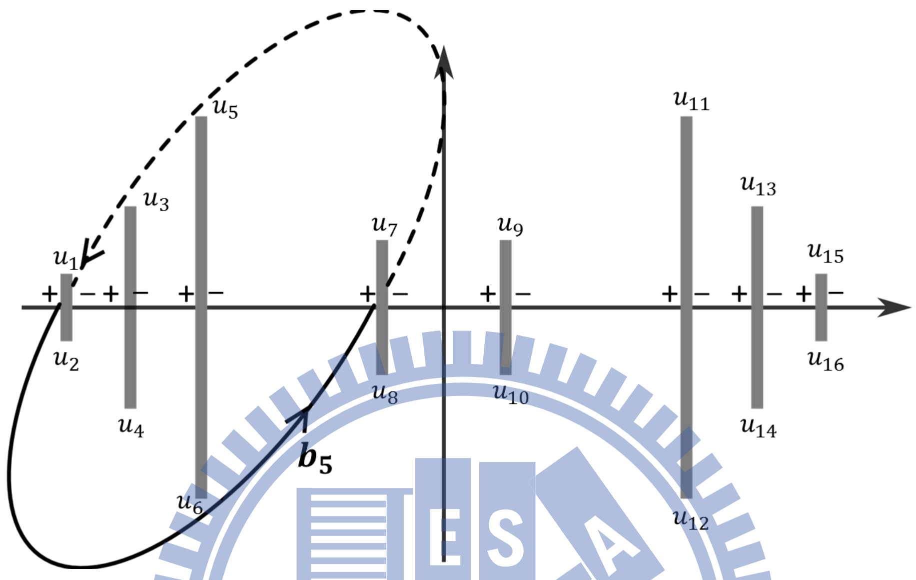
b_{52}^* = the path on the vertical cut from -3.14 to $-3.14 + 1.34i$ on (−)edge of sheet-II,

b_{53}^* = the path on the vertical cut from $-3.14 + 1.34i$ to $-3.14 + 2.48i$ on (−)edge of sheet-II,

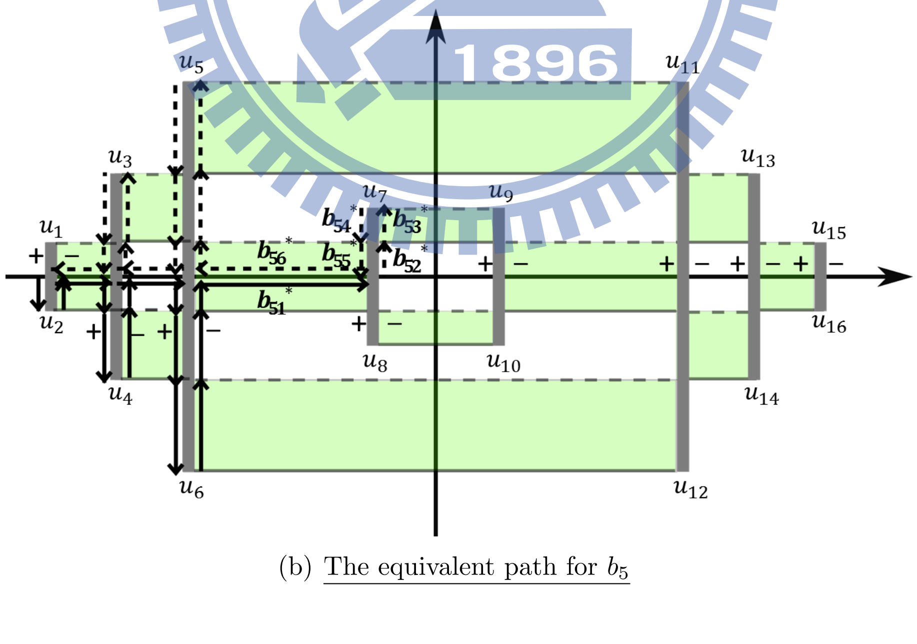
b_{54}^* = the path on the vertical cut from $-3.14 + 2.48i$ to $-3.14 + 1.34i$ on (+)edge of sheet-II,

b_{55}^* = the path on the vertical cut from $-3.14 + 1.34i$ to -3.14 on (+)edge of sheet-II,

b_{56}^* = the path on the horizontal line from -3.14 to -8.04 on sheet-II.



(a) The contour b_5 in the cut plane



(b) The equivalent path for b_5

(1) $a^*_{63'}$: Let $u = -8.04 + ri\,,\ r : 0 \to -1.34\,, du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{63'}} \frac{1}{f(u)} du \overset{Math.}{=} \int_0^{-1.34} \frac{i}{f(-8.04 + ri)} dr$$

(2) $a^*_{68'}$: Let $u = -8.04 + ri\,,\ r : -1.34 \to 0\,, du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{68'}} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{-1.34}^{0} \frac{i}{f(-8.04 + ri)} dr$$

(3) b^*_{51} : Let $u = r\,,\ r : -8.04 \to -3.14\,, du = dr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b^*_{51}} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{-8.04}^{-3.14} \frac{1}{f(r)} dr$$

(4) $b^*_{52} \equiv$ the path on the vertical cut from -3.14 to $-3.14 + 1.34i$ on (+)edge of sheet-I.
Let $u = -3.14 + ri\,,\ r : 0 \to 1.34\,, du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}\int_{b^*_{52}} \frac{1}{f(u)} dz &= \int_{-3.14 \overset{-}{\dashrightarrow} -3.14+1.34i} \frac{1}{f(u)} du \\ &= \int_{-3.14 \overset{+}{\to} -3.14+1.34i} 7\frac{1}{f(u)} du \\ &\overset{Math.}{=} -\int_0^{1.34} \frac{i}{f(-3.14 + ri)} dr\end{aligned}$$

(5) $b^*_{53} \equiv$ the path on the vertical cut from $-3.14 + 1.34i$ to $-3.14 + 2.48i$ on (+)edge of sheet-I.
Let $u = -3.14 + ri\,,\ r : 1.34 \to 2.48\,, du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b^*_{53}} \frac{1}{f(u)} dz &= \int_{-3.14+1.34i \overset{-}{\dashrightarrow} -3.14+2.48i} \frac{1}{f(u)} du \\ &= \int_{-3.14+1.34i \overset{+}{\to} -3.14+2.48i} \frac{1}{f(u)} du \\ &\overset{Math.}{=} \int_{1.34}^{2.48} \frac{i}{f(-3.14 + ri)} dr\end{aligned}$$

(6) $b_{54}^{*} \equiv$ the path on the vertical cut from $-3.14+2.48i$ to $-3.14+1.34i$ on $(-)$edge of sheet-I.

Let $u=-3.14+ri\,,\; r: 2.48 \to 1.34\,, du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}
\int_{b_{54}^{*}} \frac{1}{f(u)}dz &= \int_{-3.14+2.48i \overset{+}{\dashrightarrow} -3.14+1.34i} \frac{1}{f(u)}du \\
&= \int_{-3.14+2.48i \overset{-}{\to} -3.14+1.34i} \frac{1}{f(u)}du \\
&\overset{Math.}{=} -\int_{2.48}^{1.34} \frac{i}{f(-3.14+ri)}dr
\end{aligned}$$

(7) $b_{55}^{*} \equiv$ the path on the vertical cut from $-3.14+1.34i$ to -3.14 on $(-)$edge of sheet-I.

Let $u=-3.14+ri\,,\; r: 1.34 \to 0\,, du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}
\int_{b_{55}^{*}} \frac{1}{f(u)}dz &= \int_{-3.14+1.34i \overset{+}{\dashrightarrow} -3.14} \frac{1}{f(u)}du \\
&= \int_{-3.14+1.34i \overset{-}{\to} -3.14} \frac{1}{f(u)}du \\
&\overset{Math.}{=} \int_{1.34}^{0} \frac{i}{f(-3.14+ri)}dr
\end{aligned}$$

(8) b_{56}^{*} : Let $u=r\,,\; r: -3.14 \to -8.04\,, du = dr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}
\int_{b_{56}^{*}} \frac{1}{f(u)}dz &= \int_{-3.14 \dashrightarrow -8.04} \frac{1}{f(u)}du \\
&= -\int_{-3.14 \to -8.04} \frac{1}{f(u)}du \\
&\overset{Math.}{=} \int_{-3.14}^{-8.04} \frac{1}{f(r)}dr
\end{aligned}$$

By (1)∼(8)and Cauchy Theorem we can obtain $\int_{b_5} \frac{1}{f(u)}du$, which value is shown in the Appendix A.0.16.

4. Evaluate $\displaystyle\int_{b_4} \frac{1}{f(u)}du$

b_4 : Consider the equivalent path

$b_4^* = b_5^* \bigcup a_{52'}^* \bigcup a_{53}^* \bigcup a_{54}^* \bigcup a_{55'}^* \bigcup b_{41}^* \bigcup b_{42}^* \bigcup b_{43}^* \bigcup b_{44}^* \bigcup b_{45}^* \bigcup b_{46}^*$,

where

$a_{52'}^*$ = the path on the vertical cut from -3.14 to $-3.14 - 1.34i$ on (+)edge of sheet-I,

$a_{55'}^*$ = the path on the vertical cut from $-3.14 - 1.34i$ to -3.14 on (−)edge of sheet-I,

b_{41}^* = the path on the horizontal line from -3.14 to 3.14 on sheet-I,

b_{42}^* = the path on the vertical cut from 3.14 to $3.14 + 1.34i$ on (−)edge of sheet-II,

b_{43}^* = the path on the vertical cut from $3.14 + 1.34i$ to $3.14 + 2.48i$ on (−)edge of sheet-II,

b_{44}^* = the path on the vertical cut from $3.14 + 2.48i$ to $3.14 + 1.34i$ on (+)edge of sheet-II,

b_{45}^* = the path on the vertical cut from $3.14 + 1.34i$ to 3.14 on (+)edge of sheet-II,

b_{46}^* = the path on the horizontal line from 3.14 to -3.14 on sheet-II.

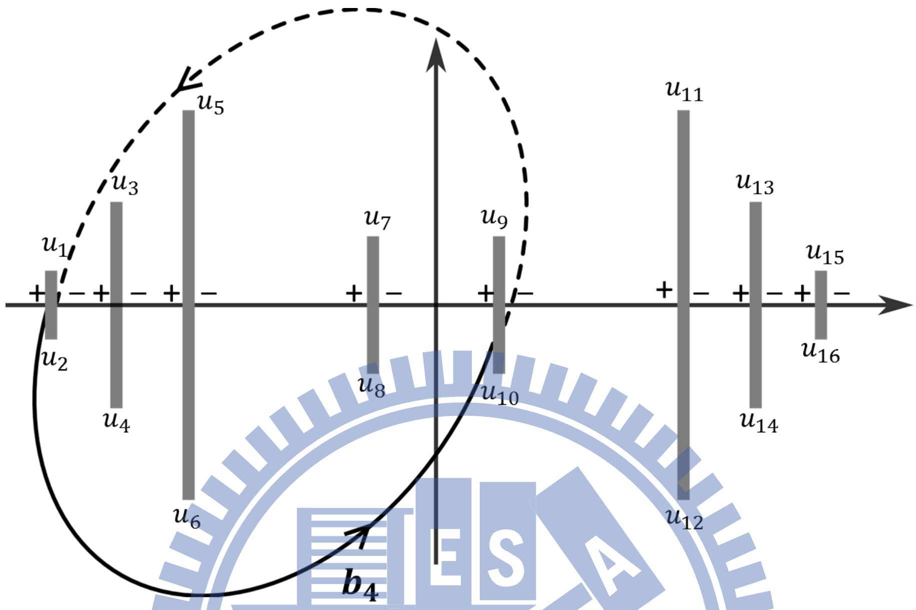
(1) $a_{52'}^*$: Let $u = -3.14 + ri$, $r : 0 \to -1.34$, $du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{52'}^*} \frac{1}{f(u)}du \overset{Math.}{=} -\int_0^{-1.34} \frac{i}{f(-3.14 + ri)}dr$$

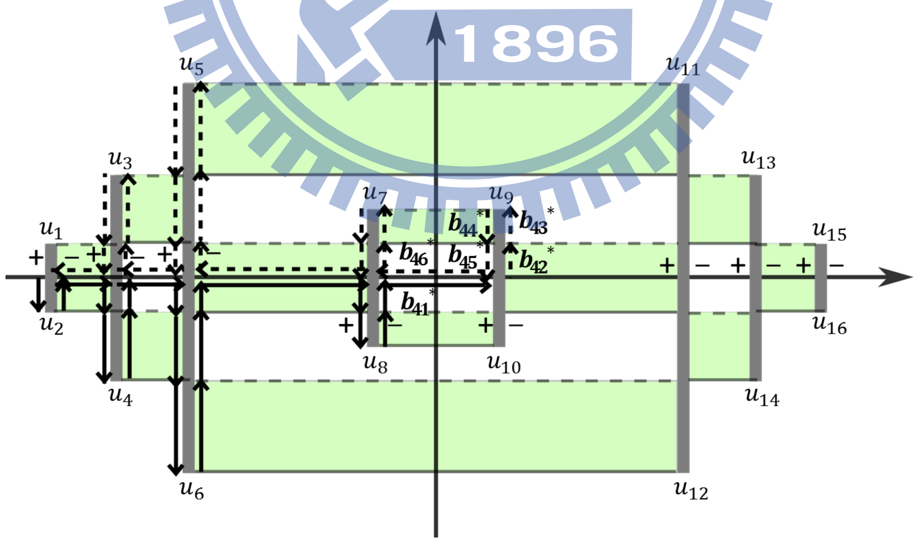
(2) $a_{55'}^*$: Let $u = -3.14 + ri$, $r : -1.34 \to 0$, $du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{55'}^*} \frac{1}{f(u)}du \overset{Math.}{=} \int_{-1.34}^{0} \frac{i}{f(-3.14 + ri)}dr$$

(3) b_{41}^* : Let $u = r$, $r : -3.14 \to 3.14$, $du = dr$ and $u \notin (M)$



(c) The contour b_4 in the cut plane



(d) The equivalent path for b_4

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b_{41}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_{-3.14}^{3.14} \frac{1}{f(r)} dr$$

(4) $b_{42}^* \equiv$ the path on the vertical cut from 3.14 to $3.14 + 1.34i$ on (+)edge of sheet-I.

Let $u = 3.14 + ri\,,\ r: 0 \to 1.34\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b_{42}^*} \frac{1}{f(u)} du &= \int_{3.14 \overset{-}{\dashrightarrow} 3.14+1.34i} \frac{1}{f(u)} du \\ &= \int_{3.14 \overset{+}{\to} 3.14+1.34i} \frac{1}{f(u)} du \\ &\overset{Math.}{=} \int_{0}^{1.34} \frac{i}{f(3.14+ri)} dr\end{aligned}$$

(5) $b_{43}^* \equiv$ the path on the vertical cut from $3.14+1.34i$ to $3.14+2.48i$ on (+)edge of sheet-I.

Let $u = 3.14 + ri\,,\ r: 1.34 \to 2.48\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}\int_{b_{43}^*} \frac{1}{f(u)} du &= \int_{3.14+1.34i \overset{-}{\dashrightarrow} 3.14+2.48i} \frac{1}{f(u)} du \\ &= \int_{3.14+1.34i \overset{+}{\to} 3.14+2.48i} \frac{1}{f(u)} du \\ &\overset{Math.}{=} -\int_{1.34}^{2.48} \frac{i}{f(3.14+ri)} dr\end{aligned}$$

(6) $b_{44}^* \equiv$ the path on the vertical cut from $3.14+1.34i$ to $3.14+2.48i$ on (+)edge of sheet-I.

Let $u = 3.14 + ri\,,\ r: 1.34 \to 2.48\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b_{44}^*} \frac{1}{f(u)}du &= \int_{3.14+2.48i \overset{+}{\dashrightarrow} 3.14+1.34i} \frac{1}{f(u)}du \\ &= \int_{3.14+2.48i \overset{-}{\rightarrow} 3.14+1.34i} \frac{1}{f(u)}du \\ &\overset{Math.}{=} \int_{2.48}^{1.34} \frac{i}{f(3.14+ri)}dr\end{aligned}$$

(7) $b_{45}^* \equiv$ the path on the vertical cut from $3.14+1.34i$ to 3.14 on (+)edge of sheet-I.

Let $u = 3.14 + ri\,,\, r : 1.34 \to 0\,,\, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}\int_{b_{45}^*} \frac{1}{f(u)}du &= \int_{3.14+1.34i \overset{+}{\dashrightarrow} 3.14} \frac{1}{f(u)}du \\ &= \int_{3.14+1.34i \overset{-}{\rightarrow} 3.14} \frac{1}{f(u)}du \\ &\overset{Math.}{=} -\int_{1.34}^{0} \frac{i}{f(3.14+ri)}dr\end{aligned}$$

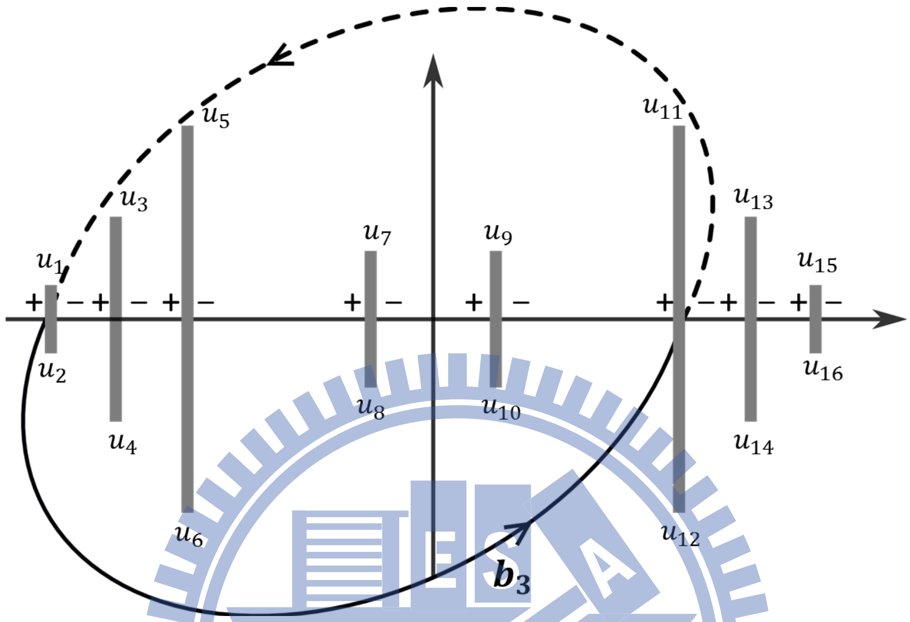
(8) b_{46}^* : Let $u = r\,,\, r : 3.14 \to -3.14\,,\, du = dr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

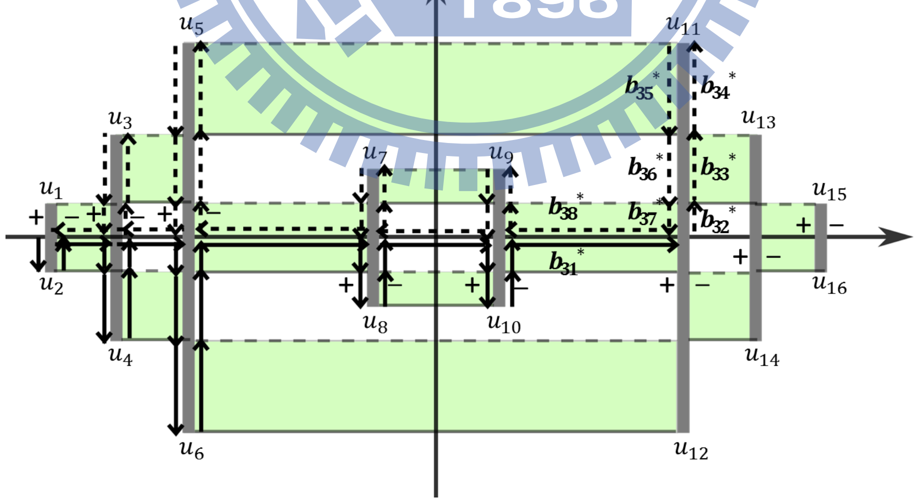
$$\begin{aligned}\int_{b_{46}^*} \frac{1}{f(u)}dz &= \int_{3.14 \dashrightarrow -3.14} \frac{1}{f(u)}du \\ &= -\int_{3.14 \rightarrow -3.14} \frac{1}{f(u)}du \\ &\overset{Math.}{=} -\int_{3.14}^{-3.14} \frac{1}{f(r)}dr\end{aligned}$$

By (1)$\sim$(8) and Cauchy Theorem we can obtain $\displaystyle\int_{b_4} \frac{1}{f(u)}du$, which value is shown in the Appendix A.0.17.

5. Evaluate $\displaystyle\int_{b_3} \frac{1}{f(u)}du$



(e) The contour b_3 in the cut plane



(f) The equivalent path for b_3

b_3 : Consider the equivalent path

$b_3^* = b_4^* \bigcup a_{42'}^* \bigcup a_{43}^* \bigcup a_{44}^* \bigcup a_{45'}^* \bigcup b_{31}^* \bigcup b_{32}^* \bigcup b_{33}^* \bigcup b_{34}^* \bigcup b_{35}^* \bigcup b_{36}^*$
$\bigcup b_{37}^* \bigcup b_{38}^*$, where

$a_{42'}^*$ = the path on the vertical cut from 3.14 to $3.14 - 1.34i$ on (+)edge of sheet-I,

$a_{45'}^*$ = the path on the vertical cut from $3.14 - 1.34i$ to 3.14 on (−)edge of sheet-I,

b_{31}^* = the path on the horizontal line from 3.14 to 8.04 on sheet-I,

b_{32}^* = the path on the vertical cut from 8.04 to $8.04 + 1.34i$ on (−)edge of sheet-II,

b_{33}^* = the path on the vertical cut from $8.04 + 1.34i$ to $8.04 + 3.86i$ on (−)edge of sheet-II,

b_{34}^* = the path on the vertical cut from $8.04 + 3.86i$ to $8.04 + 8.19i$ on (−)edge of sheet-II,

b_{35}^* = the path on the vertical cut from $8.04 + 8.19i$ to $8.04 + 3.86i$ on (+)edge of sheet-II,

b_{36}^* = the path on the vertical cut from $8.04 + 3.86i$ to $8.04 + 1.34i$ on (+)edge of sheet-II,

b_{37}^* = the path on the vertical cut from $8.04 + 1.34i$ to 8.04 on (+)edge of sheet-II,

b_{38}^* = the path on the horizontal line from 8.04 to 3.14 on sheet-II.

(1) $a_{42'}^*$: Let $u = 3.14 + ri$, $r : 0 \to -1.34$, $du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{42'}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_0^{-1.34} \frac{i}{f(3.14 + ri)} dr$$

(2) $a_{45'}^*$: Let $u = 3.14 + ri$, $r : -1.34 \to 0$, $du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a_{45'}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{-1.34}^{0} \frac{i}{f(3.14 + ri)} dr$$

(3) b_{31}^* : Let $u = r\,,\, r : 3.14 \to 8.04\,, du = dr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b_{31}^*} \frac{1}{f(u)} du \overset{Math.}{=} -\int_{3.14}^{8.04} \frac{1}{f(r)} dr$$

(4) $b_{32}^* \equiv$ the path on the vertical cut from 8.04 to $8.04 + 1.34i$ on (+)edge of sheet-I.

Let $u = 8.04 + ri\,,\, r : 0 \to 1.34\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}\int_{b_{32}^*} \frac{1}{f(u)} du &= \int_{8.04 \overset{-}{\dashrightarrow} 8.04+1.34i} \frac{1}{f(u)} du \\ &= \int_{8.04 \overset{+}{\to} 8.04+1.34i} \frac{1}{f(u)} du \\ &\overset{Math.}{=} -\int_{0}^{1.34} \frac{i}{f(8.04+ri)} dr\end{aligned}$$

(5) $b_{33}^* \equiv$ the path on the vertical cut from $8.04+1.34i$ to $8.04+3.86i$ on (+)edge of sheet-I.

Let $u = 8.04 + ri\,,\, r : 1.34 \to 3.86\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b_{33}^*} \frac{1}{f(u)} du &= \int_{8.04+1.34i \overset{-}{\dashrightarrow} 8.04+3.86i} \frac{1}{f(u)} du \\ &= \int_{8.04+1.34i \overset{+}{\to} 8.04+3.86i} \frac{1}{f(u)} du \\ &\overset{Math.}{=} \int_{1.34}^{3.86} \frac{i}{f(8.04+ri)} dr\end{aligned}$$

(6) $b_{34}^* \equiv$ the path on the vertical cut from $8.04+3.86i$ to $8.04+8.19i$ on (+)edge of sheet-I.

Let $u = 8.04 + ri\,,\, r : 3.86 \to 8.19\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b_{34}^*} \frac{1}{f(u)} du = \int_{8.04+3.86i \overset{-}{\dashrightarrow} 8.04+8.19i} \frac{1}{f(u)} du$$
$$= \int_{8.04+3.86i \overset{+}{\rightarrow} 8.04+8.19i} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} -\int_{3.86}^{8.19} \frac{i}{f(8.04+ri)} dr$$

(7) $b_{35}^* \equiv$ the path on the vertical cut from $8.04+8.19i$ to $8.04+3.86i$ on (+)edge of sheet-I.

Let $u = 8.04+ri$, $r: 8.19 \to 3.86$, $du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b_{35}^*} \frac{1}{f(u)} du = \int_{8.04+8.19i \overset{+}{\dashrightarrow} 8.04+3.86i} \frac{1}{f(u)} du$$
$$= \int_{8.04+8.19i \overset{-}{\rightarrow} 8.04+3.86i} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} \int_{8.19}^{3.86} \frac{i}{f(8.04+ri)} dr$$

(8) $b_{36}^* \equiv$ the path on the vertical cut from $8.04+3.86i$ to $8.04+1.34i$ on (+)edge of sheet-I.

Let $u = 8.04+ri$, $r: 3.86 \to 1.34$, $du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b_{36}^*} \frac{1}{f(u)} du = \int_{8.04+3.86i \overset{+}{\dashrightarrow} 8.04+1.34i} \frac{1}{f(u)} du$$
$$= \int_{8.04+3.86i \overset{-}{\rightarrow} 8.04+1.34i} \frac{1}{f(u)} du$$
$$\overset{Math.}{=} -\int_{3.86}^{1.34} \frac{i}{f(8.04+ri)} dr$$

(9) $b_{37}^* \equiv$ the path on the vertical cut from $8.04+1.34i$ to 8.04 on (−)edge of sheet-I.

Let $u = 8.04+ri$, $r: 1.34 \to 0$, $du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b_{37}^*} \frac{1}{f(u)}du &= \int_{8.04+1.34i \overset{+}{\dashrightarrow} 8.04} \frac{1}{f(u)}du \\ &= \int_{8.04+1.34i \overset{-}{\rightarrow} 8.04} \frac{1}{f(u)}du \\ &\overset{Math.}{=} \int_{1.34}^{0} \frac{i}{f(8.04+ri)}dr\end{aligned}$$

(10) b_{38}^* : Let $u = r$, $r : 8.04 \to 3.14$, $du = dr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}\int_{b_{38}^*} \frac{1}{f(u)}dz &= \int_{8.04 \dashrightarrow 3.14} \frac{1}{f(u)}du \\ &= -\int_{8.04 \rightarrow 3.14} \frac{1}{f(u)}du \\ &\overset{Math.}{=} \int_{8.04}^{3.14} \frac{1}{f(r)}dr\end{aligned}$$

By (1)∼(10) and Cauchy Theorem we can obtain $\displaystyle\int_{b_3} \frac{1}{f(u)}du$, which value is shown in the Appendix A.0.18.

6. Evaluate $\displaystyle\int_{b_2} \frac{1}{f(u)}du$

b_2 : Consider the equivalent path

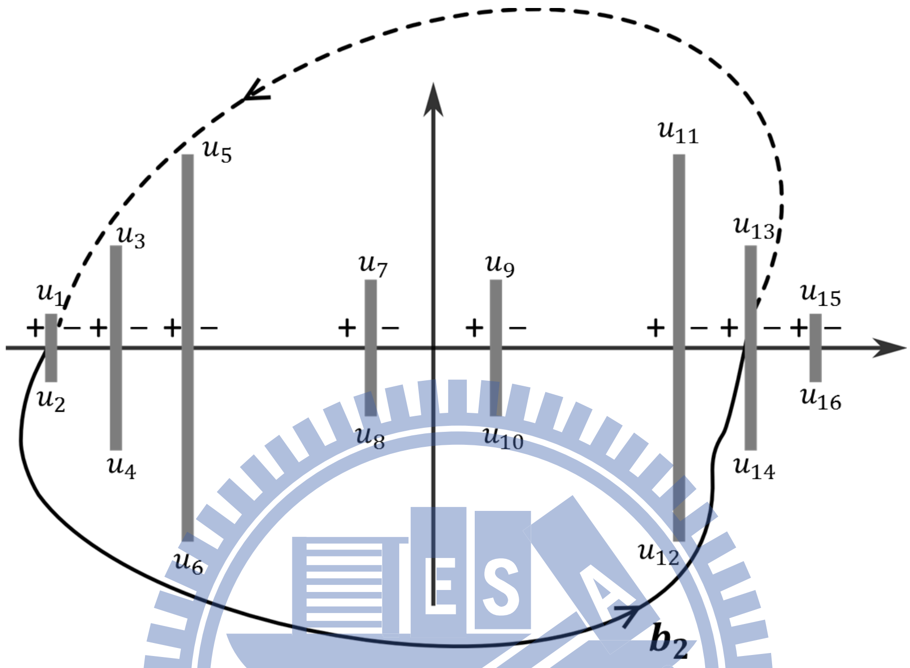
$b_2^* = b_3^* \bigcup a_{33'}^* \bigcup a_{34}^* \bigcup a_{35}^* \bigcup a_{36}^* \bigcup a_{37}^* \bigcup a_{38'}^* \bigcup b_{21}^* \bigcup b_{22}^* \bigcup b_{23}^* \bigcup b_{24}^* \bigcup b_{25}^* \bigcup b_{26}^*$,

where

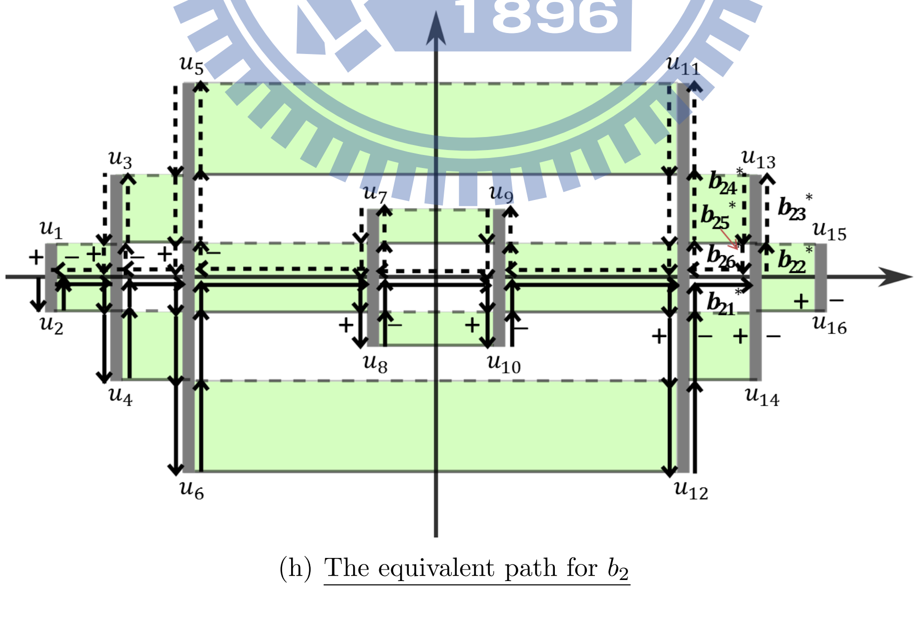
$a_{33'}^*$ = the path on the vertical cut from 8.04 to $8.04 - 1.34i$ on (+)edge of sheet-I, $a_{38'}^*$ = the path on the vertical cut from $8.04 - 1.34i$ to 8.04 on (−)edge of sheet-I, b_{21}^* = the path on the horizontal line from 8.04 to 8.19 on sheet-I,

b_{22}^* = the path on the vertical cut from 8.19 to $8.19 + 1.34i$ on (−)edge of sheet-II,

b_{23}^* = the path on the vertical cut from $8.19 + 1.34i$ to $8.19 + 3.86i$ on



(g) The contour b_2 in the cut plane



(h) The equivalent path for b_2

(−)edge of sheet-II,

b^*_{24} = the path on the vertical cut from $8.19+3.86i$ to $8.19+1.34i$ on (+)edge of sheet-II,

b^*_{25} = the path on the vertical cut from $8.19+1.34i$ to 8.19 on (+)edge of sheet-II,

b^*_{26} = the path on the horizontal line from 8.19 to 8.04 on sheet-II.

(1) $a^*_{33'}$: Let $u = 8.04 + ri\,,\ r : 0 \to -1.34\,, du = idr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{33'}} \frac{1}{f(u)}du \overset{Math.}{=} -\int_0^{-1.34} \frac{i}{f(8.04+ri)}dr$$

(2) $a^*_{38'}$: Let $u = 8.04 + ri\,,\ r : -1.34 \to 0\,, du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a^*_{38'}} \frac{1}{f(u)}du \overset{Math.}{=} \int_{-1.34}^{0} \frac{i}{f(8.04+ri)}dr$$

(3) b^*_{21} : Let $u = r\,,\ r : 8.04 \to 8.19\,, du = dr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b^*_{21}} \frac{1}{f(u)}du \overset{Math.}{=} \int_{8.04}^{8.19} \frac{1}{f(r)}dr$$

(4) b^*_{22} ≡ the path on the vertical cut from 8.19 to $8.19+1.34i$ on (+)edge of sheet-I.
Let $u = 8.19 + ri\,,\ r : 0 \to 1.34\,, du = idr$ and $u \notin (M)$
then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b^*_{22}} \frac{1}{f(u)}dz &= \int_{8.19 \xrightarrow{-} 8.19+1.34i} \frac{1}{f(u)}du \\ &= \int_{8.19 \xrightarrow{+} 8.19+1.34i} \frac{1}{f(u)}du \\ &\overset{Math.}{=} \int_0^{1.34} \frac{i}{f(8.19+ri)}dr\end{aligned}$$

(5) $b^*_{23} \equiv$ the path on the vertical cut from $8.19+1.34i$ to $8.19+3.86i$ on $(+)$edge of sheet-I.

Let $u = 8.19 + ri\,,\ r : 1.34 \to 3.86\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b^*_{23}} \frac{1}{f(u)}du = \int_{8.19+1.34i \overset{-}{\dashrightarrow} 8.19+3.86i} \frac{1}{f(u)}du$$

$$= \int_{8.19+1.34i \overset{+}{\to} 8.19+3.86i} \frac{1}{f(u)}du$$

$$\overset{Math.}{=} -\int_{1.34}^{3.86} \frac{i}{f(8.19+ri)}dr$$

(6) $b^*_{24} \equiv$ the path on the vertical cut from $8.19+3.86i$ to $8.19+1.34i$ on $(-)$edge of sheet-I.

Let $u = 8.19 + ri\,,\ r : 3.86 \to 1.34\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\int_{b^*_{24}} \frac{1}{f(u)}du = \int_{8.19+3.86i \overset{+}{\dashrightarrow} 8.19+1.34i} \frac{1}{f(u)}du$$

$$= \int_{8.19+3.86i \overset{-}{\to} 8.19+1.34i} \frac{1}{f(u)}du$$

$$\overset{Math.}{=} \int_{3.86}^{1.34} \frac{i}{f(8.19+ri)}dr$$

(7) $b^*_{25} \equiv$ the path on the vertical cut from $8.19 + 1.34i$ to 8.19 on $(-)$edge of sheet-I.

Let $u = 8.19 + ri\,,\ r : 1.34 \to 0\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b^*_{67}} \frac{1}{f(u)}du = \int_{8.19+1.34i \overset{+}{\dashrightarrow} 8.19} \frac{1}{f(u)}du$$

$$= \int_{8.19+1.34i \overset{-}{\to} 8.19} \frac{1}{f(u)}du$$

$$\overset{Math.}{=} -\int_{1.34}^{0} \frac{i}{f(8.19+ri)}dr$$

(8) b_{26}^* : Let $u = r\,, r : 8.19 \to 8.04\,, du = dr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b_{26}^*} \frac{1}{f(u)} dz &= \int_{8.19 \dashrightarrow 8.04} \frac{1}{f(u)} du \\ &= -\int_{8.19 \to 8.04} \frac{1}{f(u)} du \\ &\overset{Math.}{=} -\int_{8.19}^{8.04} \frac{1}{f(r)} dr\end{aligned}$$

By (1)∼(8) and Cauchy Theorem we can obtain $\int_{b_2} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.19.

7. Evaluate $\int_{b_1} \frac{1}{f(u)} du$

b_1 : Consider the equivalent path

$b_1^* = b_2^* \bigcup a_{22'}^* \bigcup a_{23}^* \bigcup a_{24}^* \bigcup a_{25'}^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{14}^*$, where

$a_{22'}^*$ = the path on the vertical cut from 8.19 to $8.19 - 1.34i$ on (+)edge of sheet-I, $a_{25'}^*$ = the path on the vertical cut from $8.19 - 1.34i$ to 8.19 on (−)edge of sheet-I, b_{11}^* = the path on the horizontal line from 8.19 to 8.27 on sheet-I,

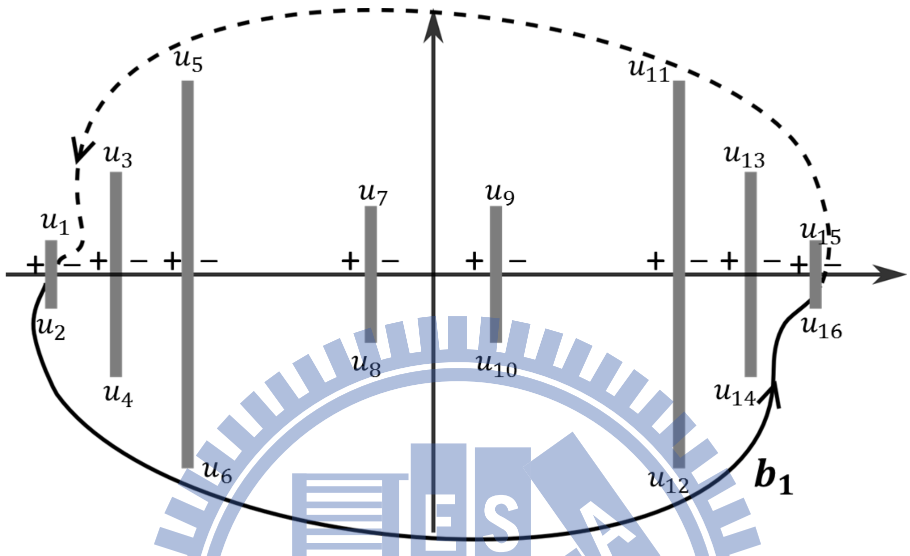
b_{12}^* = the path on the vertical cut from 8.27 to $8.27 + 1.34i$ on (−)edge of sheet-II,

b_{13}^* = the path on the vertical cut from $8.27 + 1.34i$ to 8.27 on (+)edge of sheet-II,

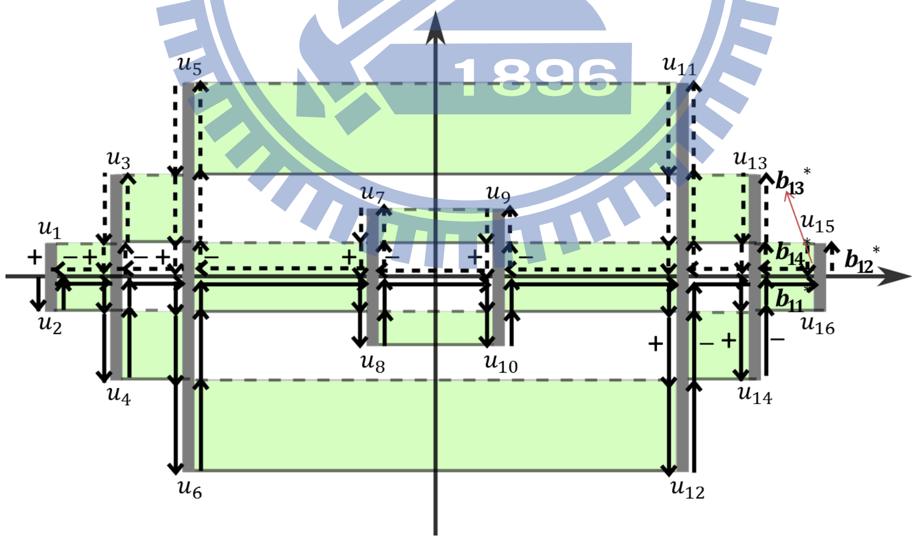
b_{14}^* = the path on the horizontal line from 8.27 to 8.19 on sheet-II.

(1) $a_{22'}^*$: Let $u = 8.19 + ri\,, r : 0 \to -1.34\,, du = idr$ and $u \notin (M)$ then $f(u) \overset{Math.}{=} f(u)$

$$\int_{a_{22'}^*} \frac{1}{f(u)} du \overset{Math.}{=} \int_0^{-1.34} \frac{i}{f(8.19 + ri)} dr$$



(i) The contour b_1 in the cut plane



(j) The equivalent path for b_1

(2) $a^*_{25'}$: Let $u = 8.19 + ri\,,\ r : -1.34 \to 0\,, du = idr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{a^*_{25'}} \frac{1}{f(u)}du \overset{Math.}{=} -\int_{-1.34}^{0} \frac{i}{f(8.19+ri)}dr$$

(3) b^*_{11} : Let $u = r\,,\ r : 8.19 \to 8.27\,, du = dr$ and $u \in (M)$ then $f(u) \overset{Math.}{=} -f(u)$

$$\int_{b^*_{11}} \frac{1}{f(u)}dz \overset{Math.}{=} -\int_{8.19}^{8.27} \frac{1}{f(r)}dr$$

(4) $b^*_{12} \equiv$ the path on the vertical cut from 8.27 to $8.27 + 1.34i$ on $(+)$edge of sheet-I.

Let $u = 8.27 + ri\,,\ r : 0 \to 1.34\,, du = idr$ and $u \in (M)$

then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}\int_{b^*_{12}} \frac{1}{f(u)}du &= \int_{8.27 \overset{-}{\dashrightarrow} 8.27+1.34i} \frac{1}{f(u)}du \\ &= \int_{8.27 \overset{+}{\to} 8.27+1.34i} \frac{1}{f(u)}du \\ &\overset{Math.}{=} -\int_{0}^{1.34} \frac{i}{f(8.27+ri)}dr\end{aligned}$$

(5) $b^*_{13} \equiv$ the path on the vertical cut from $8.27 + 1.34i$ to 8.27 on $(-)$edge of sheet-I.

Let $u = 8.27 + ri\,,\ r : 1.34 \to 0\,, du = idr$ and $u \notin (M)$

then $f(u) \overset{Math.}{=} f(u)$

$$\begin{aligned}\int_{b^*_{13}} \frac{1}{f(u)}du &= \int_{8.27+1.34i \overset{+}{\dashrightarrow} 8.27} \frac{1}{f(u)}du \\ &= \int_{8.27+1.34i \overset{-}{\to} 8.27} \frac{1}{f(u)}du \\ &\overset{Math.}{=} \int_{1.34}^{0} \frac{i}{f(8.27+ri)}dr\end{aligned}$$

(6) b^*_{14} : Let $u = r$, $r : 8.27 \to 8.19$, $du = dr$ and $u \in (M)$
then $f(u) \overset{Math.}{=} -f(u)$

$$\begin{aligned}
\int_{b^*_{14}} \frac{1}{f(u)} du &= \int_{8.27 \dashrightarrow 8.19} \frac{1}{f(u)} du \\
&= -\int_{8.27 \to 8.19} \frac{1}{f(u)} du \\
&\overset{Math.}{=} \int_{8.27}^{8.19} \frac{1}{f(r)} dr
\end{aligned}$$

By (1)∼(6) and Cauchy Theorem we can obtain $\int_{b_1} \frac{1}{f(u)} du$, which value is shown in the Appendix A.0.20.

Chapter 4

Elliptic Functions

In this chapter, we will introduce some definitions, theorems and properties taken from book [4].

4.1 Basic concepts about the elliptic function

It is necessary for us to study some basic concepts about the elliptic function.

Definition 4.1.
A function f is called periodic function if there is a number ω, such that $f(z+\omega) = f(z)$ for all values of z for which $f(z)$ exists.

Remark 4.1.
If f is a periodic function with period ω, then $f(z+k\omega) = f(z)$ for all values of z and any integer k.

Definition 4.2.
The smallest period of a periodic function f is called the fundamental period of f.

4.1.1 Doubly-periodic function

Lemma 4.1.
Let ω and ω' be a pair of fundamental period of the function f. Then $f(z) = f(z + m\omega + n\omega')$ for all values of z and for all integer values of m, n. That implies $m\omega + n\omega'$ represents the all periods of f.

Definition 4.3.
Let $2\omega_1$ and $2\omega_2$ be any two numbers (real or complex) whose ratio is not purely real. If a function f satisfies the equations

$$f(z + 2\omega_1) = f(z)\,,\ f(z + 2\omega_2) = f(z)\,,$$

for all values of z for which $f(z)$ exists, then it is called a **doubly-periodic function** of z with a pair of fundamental periods $2\omega_1$ and $2\omega_2$.

Definition 4.4.
If a doubly-periodic function f is analytic (except at poles) and has no singularities other than poles in the finite part of the complex plane, then f is called an **elliptic function**.

4.1.2 Period-parallelograms

Remark 4.2.
If there is no point ω inside or on the boundary of the parallelogram (except the vertices) such that $f(z+\omega) = f(z)$, for all values of z, then the parallelogram constructed by z, $z+2\omega_1$, $z+2\omega_1+2\omega_2$, $z+2\omega_2$ is called a **fundamental period-parallelogram** for an elliptic function with periods $2\omega_1, 2\omega_2$.

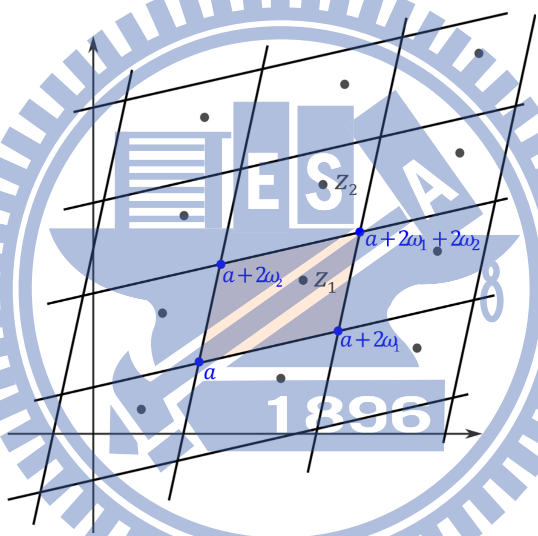


Figure 4.1.1. Take $z = a$, for some $a \in \mathbb{C}$

It is clear that the complex plane can be covered with a network of parallelograms equal to the fundamental period-parallelogram and similarly situated, each of the points $z+2m\omega_1+2n\omega_2$, $\forall m,n \in \mathbb{Z}$, $z \in \mathbb{C}$ being a vertex of four parallelograms. (Show in Figure 4.1.1.)
The points z, $z+2\omega_1$, $z+2\omega_2$, $\ldots$ $z+2m\omega_1+2n\omega_2$, $\ldots$ will have the same value after transferring by f since $2\omega_1$ and $2\omega_2$ are periods. In other words, all periods of f can be represented of the form $2m\omega_1+2n\omega_2$, $\forall m,n \in \mathbb{Z}$. Thus, any pair of such points are said to be **congruent** to one another.

Generally if the difference between two points z_1, z_2 is $2m\omega_1 + 2n\omega_2$, then z_1, z_2 are said to be congruent. The congruence of two points z_1, z_2 is denoted by $z_1 \equiv z_2(mod.\, 2\omega_1,\, 2\omega_2)$.

We know it is inconvenient to deal with the parallelograms if they have singularities of the integrand on their boundaries. So in order to avoid there are poles on the integral path, we can take $z = a$, for some $a \in \mathbb{C}$ by translating the parallelograms such that none of the poles of the integrand considered are on the sides of the parallelogram. Such a parallelogram is called a **cell**.

The set of poles of an elliptic function in any given cell is called an irreducible set.

4.1.3 Simple properties of the elliptic function

We already know the definitions of the elliptic function, and then we can derive some simple properties of the elliptic function as following.

1. The rational functions of an elliptic function are also elliptic functions of the same kind.

2. **Liouville's theorem**
   There are four theorems about elliptic functions known as Liouville[1]'s theorems.

   - *Liouville's first theorem*
     An elliptic function ,$f(z)$, with no poles in a cell is merely a constant.

[1] Joseph Liouville (1809-1882)

- *Liouville's second theorem*

  The sum of the residue of an elliptic function, $f(z)$, at its poles in any cell is zero.

- *Liouville's third theorem*

  The number of poles is the same as the number of zeros of an elliptic function in any cell.

- *Liouville's fourth theorem*

  The difference between the sum of poles and the sum of zeros is a period.

4.1.4 The order of an elliptic function

Definition 4.5.
Let c be a constant and $f(z)$ be an elliptic function. The order of the elliptic function is the number of the roots of the equation $f(z) = c$ which lies in any cell depends only on $f(z)$.

Remark 4.3.
a. The order of $f(z)$ is the number of poles in the cell.
b. The order of an elliptic function is ≥ 2.
c. The simplest elliptic function could be divided into two classes.
One is the elliptic functions which have a single irreducible double pole with residue $= 0$. The other is the elliptic functions which have two single poles and the sum of their residues is 0.

4.2 Weierstrass elliptic function

The Weierstrass elliptic function $\wp(z)$ is defined as

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{m,n\neq 0}\left\{\frac{1}{(z-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2}\right\} \\ &= \frac{1}{z^2} + \sum_{\Omega_{m,n}\neq 0}\left\{\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}\right\}, \end{aligned} \tag{4.2.1}$$

where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_2$, $\forall m,n \in \mathbb{Z}$ (except $m = n = 0$)

Remark 4.4.

1. When m,n such that $|\Omega_{m,n}|$ is large, the general terms of the series defining $\wp(z)$ is $O(|\Omega_{m,n}|^{-3})$. So $\wp(z)$ converges absolutely and uniformly.
2. $\wp(z)$ is analytic except the poles, namely the points $\Omega_{m,n}$ and the points $\Omega_{m,n}$ are all double poles.

We proceed to introduce some properties and theorems about $\wp(z)$ and $\wp'(z)$ as following.

4.2.1 Periodicity and other properties of $\wp(z)$

Since $\wp(z)$ is uniformly convergent series of analytic function, we could differentiate it term-by-term. And we obtain

$$\wp'(z) = \frac{d}{dz}\wp(z) = -2\sum_{m,n}\frac{1}{(z-\Omega_{m,n})^3} \tag{4.2.2}$$

and

$$\wp'(-z) = -2\sum_{m,n}\frac{1}{(-z-\Omega_{m,n})^3} = 2\sum_{m,n}\frac{1}{(z+\Omega_{m,n})^3} \quad . \tag{4.2.3}$$

According to equation(4.2.2) and (4.2.3), we can get

$$\wp'(-z) = -\wp'(z)\,. \tag{4.2.4}$$

This means that $\wp'(z)$ is an odd function.

We know that $\wp'(z)$ is analytic except at poles and which has no singularity other than poles. Moreover, it is clear that $2\omega_1, 2\omega_2$ are periods of $\wp'(z)$.

Thus $\wp'(z)$ is also an elliptic function, but it is different to $\wp(z)$. Compare $\wp(z)$ with $\wp'(z)$ as following table:

	Definition	Poles	Periods	Parity
$\wp(z)$	equation(4.2.1)	$\Omega_{m,n}$	$2\omega_1, 2\omega_2$	even
$\wp'(z)$	equation(4.2.2)	$\Omega_{m,n}$	$2\omega_1, 2\omega_2$	odd

Table 4.1. The difference between $\wp(z)$ and$\wp'(z)$

Corollary 4.1.
If $f(z)$ is an elliptic function, then its derivative $f'(z)$ is also an elliptic function. And $f^{(n)}(z), \forall n \in \mathbb{N}$ is an elliptic function of the same kind. Moreover, if $f(z)$ and $g(z)$ are elliptic function with the same periods, then

$$f(z) \pm g(z)\ ,\ f(z)g(z)\ ,\ \frac{f(z)}{g(z)}$$

are also elliptic functions of the same kind.

4.2.2 The differential equation satisfied by $\wp(z)$

From equation(4.2.2), we can let $S(z) = \wp(z) - \dfrac{1}{z^2}$. Then $S(z)$ could be represented as

$$S(z) = \sum_{m,n\neq 0} \left\{ \left(\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}\right) \right\}. \tag{4.2.5}$$

then $S(z)$ is analytic at $z = 0$, and it is an even function.

Do Taylor expansion of $S(z)$ for $|z| \to 0$. The equation(4.2.5) can be derived as

$$\wp(z) - \frac{1}{z^2} = \frac{z^2}{20} g_2 + \frac{z^4}{28} g_3 + O(z^6)\,, \qquad (4.2.6)$$

where $g_2 = \sum_{m,n \neq 0} 60\Omega_{m,n}^{-4}$ and $g_3 = \sum_{m,n \neq 0} 140\Omega_{m,n}^{-6}$.

By equation(4.2.6), $\wp(z)$ can be written as

$$\wp(z) = \frac{1}{z^2} + \frac{z^2}{20} g_2 + \frac{z^4}{28} g_3 + O(z^6)\,. \qquad (4.2.7)$$

Differentiating the equation, we get

$$\wp'(z) = \frac{-2}{z^3} + \frac{z}{10} g_2 + \frac{z^3}{7} g_3 + O(z^5)\,. \qquad (4.2.8)$$

From the equation(4.2.7) and (4.2.8), we can derive

$$\wp^3(z) = \frac{1}{z^6} + \frac{3}{20z^2} g_2 + \frac{3}{28} g_3 + O(z^2)\,, \qquad (4.2.9)$$

$$[\wp'(z)]^2 = \frac{4}{z^6} - \frac{2}{5z^2} g_2 + \frac{4}{7} g_3 + O(z^2)\,. \qquad (4.2.10)$$

Hence, we obtain

$$[\wp'(z)]^2 - 4\wp^3(z) + \wp(z) g_2 + g_3 = O(z^2)\,. \qquad (4.2.11)$$

For convenient, we define $H(z) = [\wp'(z)]^2 - 4\wp^3(z) + \wp(z)g_2 + g_3 = O(z^2)$. Since $H(z)$ is an elliptic function and it is analytic at the origin, the all congruent points of 0 are also analytic. This means that $H(z)$ is an elliptic function with no singularities.

This implies that $H(z) = c$ where c is a constant by Liouville's Theorem. Let $z \to 0$, we derive that the constant c is zero. So the function $\wp(z)$ satisfies $[\wp'(z)]^2 = 4\wp^3(z) - \wp(z)g_2 - g_3 = O(z^2)$, where $g_2 = \sum_{m,n \neq 0} 60\Omega_{m,n}^{-4}$ and

$$g_3 = \sum_{m,n \neq 0} 140\Omega_{m,n}^{-6}\,.$$

Furthermore, given the equation $(\frac{dy}{dz})^2 = 4y^3 - g_2 y - g_3$. If ω_1, ω_2 can be determined such that $g_2 = \sum_{m,n \neq 0} 60\Omega_{m,n}^{-4}$ and $g_3 = \sum_{m,n \neq 0} 140\Omega_{m,n}^{-6}$′ then the general solution of the differential equation is $y(z) = \wp(\pm z + \alpha)$, where α is the constant of integration. Since $\wp(z)$ is an even function, the solution can be written as $y(z) = \wp(z + \alpha)$.

4.2.3 The integral formula for $\wp(z)$

Here we consider the integral equation

$$z = \int_{\xi}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt \qquad (4.2.12)$$

where the path of integration may be any curve which does not pass through a zero of $4t^3 - g_2 t - g_3$.

By the equation (4.2.12), we differentiate z with respect to ξ, and get

$$(\frac{d\xi}{dz})^2 = 4\xi^3 - g_2 \xi - g_3\,. \qquad (4.2.13)$$

And by the previous result, we know that $\xi = \wp(z+\alpha)$, where α is a constant. Let $\xi \to \infty$, then $z \to 0$. This implies that α is a pole of $\wp(z)$. In other words, $\alpha \in \Omega_{m,n}$ and

$$\xi = \wp(z + \Omega_{m,n}) = \wp(z)\,. \qquad (4.2.14)$$

So the equation (4.2.12) is called the integral formula for $\wp(z)$ and we sometimes write the equation in the form

$$z = \int_{\wp(z)}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt\,. \qquad (4.2.15)$$

4.3 The theta function

The theta-function $\vartheta(z,q)$ is defined as

$$\vartheta(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} \tag{4.3.16}$$

$$= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos(2nz)\,, \tag{4.3.17}$$

where $q = e^{\pi i\tau}$ with $|q| < 1$, and $\tau \in \mathbb{C}$ is constant whose imaginary part is positive.

By the equation (4.3.16), it is obvious that

$$\vartheta(z+\pi,q) = \vartheta(z,q)\,,$$

and

$$\begin{aligned}
\vartheta(z+\pi\tau,q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} e^{2ni\pi\tau} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} q^{2n} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1)^2-1} e^{2(n+1)iz-2iz} \\
&= -q^{-1}e^{-2iz} \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{(n+1)^2} e^{2(n+1)iz} \\
&= -q^{-1}e^{-2iz}\vartheta(z,q)
\end{aligned}$$

Hence we called $\vartheta(z,q)$ is a **quasi-doubly periodic function** of z whose multipliers or periodicity factors associated with periods π and $\pi\tau$ are 1 and $-q^{-1}e^{-2iz}$ respectively.

And if $\vartheta(z_0) = 0$, then $\vartheta(z_0 + m\pi + n\pi\tau) = 0$, for all integral values of m and n.

4.3.1 The four types of theta-function

The four types of theta-function are defined as

$$\begin{cases} \vartheta_1(z,q) = -ie^{iz+\frac{1}{4}\pi i\tau}\vartheta_4(z+\frac{1}{2}\pi\tau,q) \\ \vartheta_2(z,q) = \vartheta_1(z+\frac{1}{2}\pi,q) \\ \vartheta_3(z,q) = \vartheta_4(z+\frac{1}{2}\pi,q) \\ \vartheta_4(z,q) = \vartheta(z,q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2niz} \end{cases} \tag{4.3.18}$$

Equations (4.3.18) can be written in another form:

$$\begin{cases} \vartheta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z \\ \vartheta_2(z,q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(2n+1)z \\ \vartheta_3(z,q) = 1+2\sum_{n=0}^{\infty} q^{n^2}\cos(2nz) \\ \vartheta_4(z,q) = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz) \end{cases} \tag{4.3.19}$$

We are just interested in the parameter z, so $\vartheta_i(z,q)$ is denoted as $\vartheta_i(z)$. Moreover, the notation ϑ_i is represented by $\vartheta_i(0)$, for $i = 1,2,3,4$.

And then we introduce some properties of theta-function.

1. By (4.3.19) and parity of trigonometric functions, we have

function	$\vartheta_1(z)$	$\vartheta_2(z)$	$\vartheta_3(z)$	$\vartheta_4(z)$
parity	odd	even	even	even

2. We get relations between four types of theta-function by equations (4.3.19) and simple computation.

$$\vartheta_1(z) = -\vartheta_2(z+\frac{1}{2}\pi) = -i\mathcal{K}\vartheta_3(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = -i\mathcal{K}\vartheta_4(z+\frac{1}{2}\pi\tau)$$
$$\vartheta_2(z) = \mathcal{K}\vartheta_3(z+\frac{1}{2}\pi\tau) = \mathcal{K}\vartheta_4(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = \vartheta_1(z+\frac{1}{2}\pi)$$
$$\vartheta_3(z) = \vartheta_4(z+\frac{1}{2}\pi) = \mathcal{K}\vartheta_1(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = \mathcal{K}\vartheta_2(z+\frac{1}{2}\pi\tau)$$
$$\vartheta_4(z) = -i\mathcal{K}\vartheta_1(z+\frac{1}{2}\pi\tau) = i\mathcal{K}\vartheta_2(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = \vartheta_3(z+\frac{1}{2}\pi)$$

where $\mathcal{K} = q^{\frac{1}{4}}e^{iz}$.

3. We can get the periodicity factors of the four theta-functions associated with periods π and $\pi\tau$, which are shown in the following table:

	$\vartheta_1(z)$	$\vartheta_2(z)$	$\vartheta_3(z)$	$\vartheta_4(z)$
π	-1	-1	1	1
$\pi\tau$	$-\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$-\mathcal{L}$

where $\mathcal{L} = q^{-1}e^{-2iz}$.

4. The theta-function $\vartheta_i(z)$ and $\vartheta_i'(z)$ satisfy the following equations

$$\frac{\vartheta_i'(z+\pi)}{\vartheta_i(z+\pi)} = \frac{\vartheta_i'(z)}{\vartheta_i(z)}$$
$$\frac{\vartheta_i'(z+\pi\tau)}{\vartheta_i(z+\pi\tau)} = -2i + \frac{\vartheta_i'(z)}{\vartheta_i(z)}.$$

where $i = 1,2,3,4$ and $\vartheta_i'(z)$ is the derivative of $\vartheta_i(z)$ with respect to z.

Theorem 4.1.

Let C be a cell with corners t, $t+\pi$, $t+\pi\tau$, $t+\tau$.

$\vartheta_i(z)$ has only one zero inside C, for $i = 1,2,3,4$.

From equations (4.3.18), it is manifest that 0 is the zero of $\vartheta_1(z)$. And use relation between the theta-functions, we can find out that the zero of

$\vartheta_2(z)$, $\vartheta_3(z)$, $\vartheta_4(z)$ are 0, $\frac{1}{2}\pi$, $\frac{1}{2}\pi + \frac{1}{2}\pi\tau$, $\frac{1}{2}\tau$, respectively. The result can be summarized as:

	zeros	relation
$\vartheta_1(z)$	$z = 0 \; mod(\pi, \pi\tau)$	$2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z$
$\vartheta_2(z)$	$z = \frac{1}{2}\pi \; mod(\pi, \pi\tau)$	$\vartheta_1(z) = -\vartheta_2(z + \frac{1}{2}\pi)$
$\vartheta_3(z)$	$z = \frac{1}{2}\pi + \frac{1}{2}\pi\tau \; mod(\pi, \pi\tau)$	$\vartheta_1(z) = -iq^{\frac{1}{4}}e^{iz}\vartheta_3(z + \frac{1}{2}\pi + \frac{1}{2}\pi\tau)$
$\vartheta_4(z)$	$z = \frac{1}{2}\pi\tau \; mod(\pi, \pi\tau)$	$\vartheta_1(z) = -iq^{\frac{1}{4}}e^{iz}\vartheta_4(z + \frac{1}{2}\pi\tau)$

Next, we can obtain the relation between these theta-functions. Since $\vartheta_i(z)$, for $i = 1, 2, 3, 4$, are analytic and have periodicity factors 1, $-q^{-1}e^{-2\pi\tau}$ associated with periods $\pi, \pi\tau$. It is obvious that $\vartheta_i^2(z)$, for$i = 1, 2, 3, 4$, are analytic and have periodicity factors 1, $-q^{-2}e^{-4\pi\tau}$ with periods $\pi, \pi\tau$.

Choosing suitable constants a, b, a, b, then

$$\frac{a\vartheta_1^2(z) + b\vartheta_4^2(z)}{\vartheta_2^2(z)} \tag{4.3.20}$$

and

$$\frac{a'\vartheta_1^2(z) + b'\vartheta_4^2(z)}{\vartheta_3^2(z)} \tag{4.3.21}$$

will become doubly-periodic function with periods π and $\pi\tau$.

Since each of theta-functions $\vartheta_i(z)$, for$i = 1, 2, 3, 4$, has a double zero (and no other zeros) in any cell. So (4.3.20) and (4.3.21) just have a simple pole in any cell where the constants a, b, a, b are suitably chosen. But the order of an elliptic function is never less than 2, otherwise such a function is merely a constant. And we choose the constant is 1.

Hence the equations (4.3.20) and (4.3.21) will become

$$\begin{cases} \vartheta_2^2(z) = a\vartheta_1^2(z) + b\vartheta_4^2(z) \\ \vartheta_3^2(z) = a'\vartheta_1^2(z) + b'\vartheta_4^2(z) \end{cases} \tag{4.3.22}$$

Given z the special values 0 and $\frac{1}{2}\pi\tau$, then we get

$$\vartheta_2^2 = b\vartheta_4^2 \ , \ \vartheta_3^2 = b'\vartheta_4^2 \ , \ \vartheta_3^2 = -a\vartheta_4^2 \ , \ \vartheta_2^2 = -a'\vartheta_4^2 \ . \tag{4.3.23}$$

Thus we have the following relation:

$$\begin{cases} \vartheta_4^2\vartheta_1^2(z) = \vartheta_2^2\vartheta_3^2(z) - \vartheta_3^2\vartheta_2^2(z) \\ \vartheta_4^2\vartheta_2^2(z) = \vartheta_2^2\vartheta_4^2(z) - \vartheta_3^2\vartheta_1^2(z) . \end{cases} \tag{4.3.24}$$

Additionally, if we replace z by $z + \frac{1}{2}\pi$, we have

$$\begin{cases} \vartheta_4^2\vartheta_1^2(z) = \vartheta_2^2\vartheta_3^2(z) - \vartheta_3^2\vartheta_2^2(z) \\ \vartheta_4^2\vartheta_4^2(z) = \vartheta_3^2\vartheta_3^2(z) - \vartheta_2^2\vartheta_2^2(z) . \end{cases} \tag{4.3.25}$$

Remark 4.5.
If $z = 0$, the last relation in (4.3.25) will become $\vartheta_3^4 - \vartheta_2^4 = \vartheta_4^4$.

In order to get some relation between the theta-functions easily, we can represent the theta-functions as infinite products. The result is derived by Jacobi[2]. Let

$$f(z) = \prod_{n=1}^{\infty}(1 - q^{2n-1}e^{2iz})\prod_{n=1}^{\infty}(1 - q^{2n-1}e^{-2iz}) .$$

We know $\frac{\vartheta_4(z)}{f(z)}$ has neither poles nor zeros since $f(z)$ has the same zeros as $\vartheta_4(z)$. And it is obvious that

$$f(z+\pi) = f(z) \ , \ f(z+\pi\tau) = -q^{-1}e^{-2iz}f(z) .$$

[2] Carl Gustav Jakob Jacobi (1804-1851)

Thus $f(z)$ has the same periodicity factors as $\vartheta_4(z)$. This means that $\dfrac{\vartheta_4(z)}{f(z)}$ is a doubly-periodic function with no poles. By Liouville's theorem, $\dfrac{\vartheta_4(z)}{f(z)}$ is a constant. Hence, $\vartheta_4(z)$ can be represented by

$$\begin{aligned}\vartheta_4(z) &= \nu f(z)\\ &= \nu \prod_{n=1}^{\infty}(1-q^{2n-1}e^{2iz})\prod_{n=1}^{\infty}(1-q^{2n-1}e^{-2iz})\\ &= \nu \prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2z+q^{4n-2})\,.\end{aligned}$$

Moreover, we get other relations as

$$\begin{cases}\vartheta_1(z) = 2\nu q^{\frac{1}{4}}\sin z \displaystyle\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2z+q^{4n})\\ \vartheta_2(z) = 2\nu q^{\frac{1}{4}}\cos z \displaystyle\prod_{n=1}^{\infty}(1+2q^{2n}\cos 2z+q^{4n})\\ \vartheta_3(z) = \nu \displaystyle\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos 2z+q^{4n-2})\end{cases} \tag{4.3.26}$$

Remark 4.6. (A relation between theta-functions for $z=0$)
By the expression of infinite product form and given $z=0$. We can derive the relation

$$\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4\,. \tag{4.3.27}$$

Remark 4.7. (The value of the constant ν)
Using the relation (4.3.27), the constant ν can be determined as

$$\nu = \prod_{n=1}^{\infty}(1-q^{2n})\,. \tag{4.3.28}$$

Remark 4.8. (The differential equation satisfied by the quotient of theta-functions)
By the table of periodicity factors, we know that $\frac{\vartheta_1(z)}{\vartheta_4(z)}$ and $\frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_4^2(z)}$ have the same periodicity factors -1 and 1 with respect to π , $\pi\tau$ respectively. And

$$(\frac{\vartheta_1(z)}{\vartheta_4(z)})' = \frac{\vartheta_1'(z)\vartheta_4(z) - \vartheta_4'(z)\vartheta_1(z)}{\vartheta_4^2(z)}$$

also has the same periodicity factors -1 and 1 with respect to π , $\pi\tau$ respectively. Define

$$\varphi(z) = \frac{(\frac{\vartheta_1(z)}{\vartheta_4(z)})'}{\frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_4^2(z)}} = \frac{\vartheta_1'(z)\vartheta_4(z) - \vartheta_4'(z)\vartheta_1(z)}{\vartheta_2(z)\vartheta_3(z)} \,. \tag{4.3.29}$$

By Liouville's theorem, it shows that $\varphi(z) = c$ where c is a constant since there is no poles of $\varphi(z) = 0$ in any cell. Let $z \to 0$, we can determine $c = \vartheta_4^2$. Then we get

$$(\frac{\vartheta_1(z)}{\vartheta_4(z)})' = \vartheta_4^2 \frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_4^2(z)} \,. \tag{4.3.30}$$

Let $\xi = \frac{\vartheta_1(z)}{\vartheta_4(z)}$, and using the relations (4.3.24). Then (4.3.30) will become

$$(\frac{d\xi}{dz})^2 = (\vartheta_2^2 - \vartheta_3^2\xi^2)(\vartheta_3^2 - \vartheta_2^2\xi^2) \,. \tag{4.3.31}$$

The function $\frac{\vartheta_1(z)}{\vartheta_4(z)}$ is a solution of the equation (4.3.31).

Remark 4.9.

By the same discussion, we could also find that:

$$(\frac{\vartheta_2(z)}{\vartheta_4(z)})' = -\vartheta_3^2 \frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_4^2(z)} \,, \tag{4.3.32}$$

$$(\frac{\vartheta_3(z)}{\vartheta_4(z)})' = -\vartheta_2^2 \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_4^2(z)} \,. \tag{4.3.33}$$

4.4 The Jacobian elliptic function

In this section, we introduce the Jacobian elliptic function. Starting from the equation (4.3.31):

$$(\frac{d\xi}{dz})^2 = (\vartheta_2^2 - \vartheta_3^2\xi^2)(\vartheta_3^2 - \vartheta_2^2\xi^2)\,.$$

Let $y = \frac{\vartheta_3}{\vartheta_2}\xi$ and $u = \vartheta_3^2 z$. Then the equation (4.3.31) can be written as

$$(\frac{dy}{du})^2 = (1 - y^2)(1 - \kappa^2 y^2)\,, \tag{4.4.34}$$

where κ is defined by $\kappa = (\frac{\vartheta_2}{\vartheta_3})^2$ and it is called the modulus.

And one solution of equation (4.4.34) is

$$y = \frac{\vartheta_3}{\vartheta_2}\xi = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)}\,. \tag{4.4.35}$$

Furthermore, (4.4.34) can be written as the integral form

$$u = \int_0^y (1 - t^2)^{-\frac{1}{2}}(1 - \kappa^2 t^2)^{-\frac{1}{2}} dt\,, \tag{4.4.36}$$

Since it is customary to consider y the function of u, and y is defined by the quotient of theta-functions. We denote y as $y = sn(u,\kappa)$ or simply $y = sn(u)$.

Thus (4.4.34) can be represented as

$$(\frac{dsn(u)}{du})^2 = (1 - sn^2(u))(1 - \kappa^2 sn^2(u))\,, \tag{4.4.37}$$

where $\kappa = (\frac{\vartheta_2}{\vartheta_3})^2$.

Remark 4.10.

It is easy to see that $sn(u)$ is a quasi-doubly periodic function which has periodicity factors $-1\,,1$ with periods $\vartheta_3^2\pi$ and $\vartheta_3^2\pi\tau$ respectively. This also implies that $sn(u)$ is a quasi-doubly periodic function with periods $2\vartheta_3^2\pi$ and $\vartheta_3^2\pi\tau$. We usually write the periods $\vartheta_3^2\pi$ and $\vartheta_3^2\pi\tau$ as $2K$ and $2iK'$, so that $sn(u)$ has periods $4K$ and $2iK'$.

Definition 4.6. (Jacobian elliptic functions)

The Jacobian elliptic functions $sn(u)$, $cn(u)$, $dn(u)$ are defined as following

$$\begin{aligned} sn(u) &= \frac{\vartheta_3\,\vartheta_1(\vartheta_3^{-2}u)}{\vartheta_2\,\vartheta_4(\vartheta_3^{-2}u)} \\ cn(u) &= \frac{\vartheta_4\,\vartheta_2(\vartheta_3^{-2}u)}{\vartheta_2\,\vartheta_4(\vartheta_3^{-2}u)} \\ dn(u) &= \frac{\vartheta_4\,\vartheta_3(\vartheta_3^{-2}u)}{\vartheta_3\,\vartheta_4(\vartheta_3^{-2}u)} \end{aligned}$$

And there are some properties and relation between the three Jacobian elliptic functions as following:

1. From (4.4.38), we can get some results:

$$\begin{aligned} &sn^2(u) + cn^2(u) = 1 \\ &\kappa^2 sn^2(u) + dn^2(u) = 1 \\ &cn(0) = dn(0) = 1 \end{aligned}$$

2. The derivatives of $sn(u)$, $cn(u)$, and $dn(u)$ are as following:

$$\begin{aligned} \frac{d}{du}sn(u) &= cn(u)dn(u) \\ \frac{d}{du}cn(u) &= -sn(u)dn(u) \\ \frac{d}{du}dn(u) &= -\kappa^2 sn(u)cn(u) \end{aligned}$$

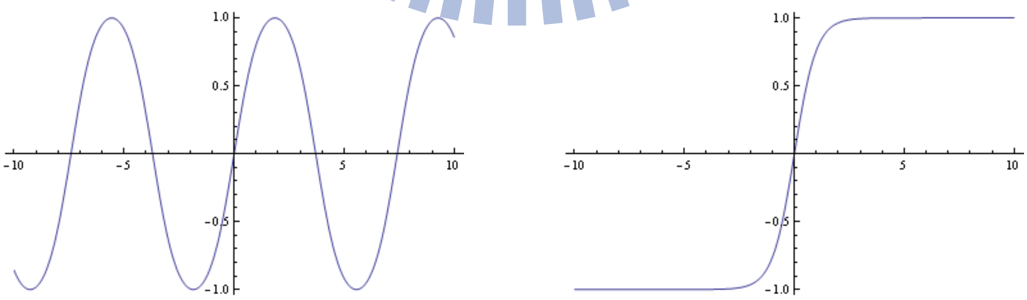
3. The parity of $sn(u)$, $cn(u)$, and $dn(u)$ are as following:

$$sn(-u) = -sn(u)$$
$$cn(-u) = cn(u)$$
$$dn(-u) = dn(u)$$

4. By the properties of theta-functions, we could find their periods, parity, poles, and zeros respectively, which can be summarized as:

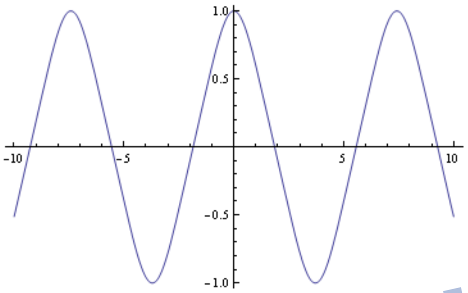
	$sn(u)$	$cn(u)$	$dn(u)$
`Periods`	$4K\ ,\ 2iK'$	$4K\ ,\ 2K+2iK'$	$2K\ ,\ 4iK'$
`Parity`	odd	even	even
`Poles`	$iK'\ ,\ 2K+iK'$	$iK'\ ,\ 2K+iK'$	$iK'\ ,\ 3K'$
	$mod(4K\ ,\ 2iK')$	$mod(4K\ ,\ 2K+2iK')$	$mod(2K\ ,\ 4iK')$
`Zeros`	0	K	$K+iK'$
	$mod(2K\ ,\ 2iK')$	$mod(2K\ ,\ 2iK')$	$mod(2K\ ,\ 2iK')$
`Derivative`	$cn(u)dn(u)$	$-sn(u)dn(u)$	$-\kappa^2 cn(u)cn(u)$

In the end of the chapter, we gives graphs of these Jacobian elliptic functions with different κ:

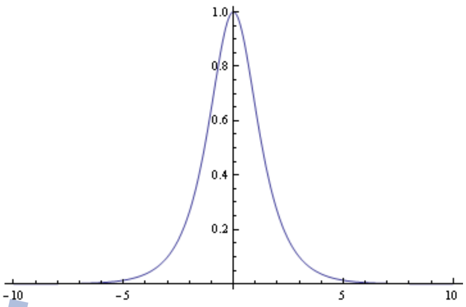


(a) $sn(u,\kappa)$ with $\kappa = \frac{1}{2}$ (b) $sn(u,\kappa)$ with $\kappa = 1$

Figure 4.4.1. $sn(u,\kappa)$

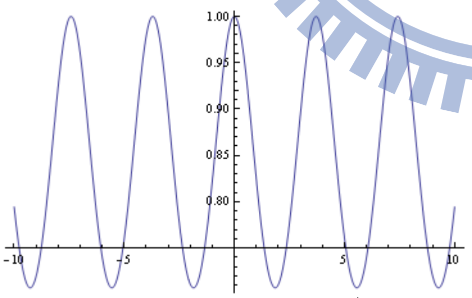


(a) $cn(u, \kappa)$ with $\kappa = \frac{1}{2}$

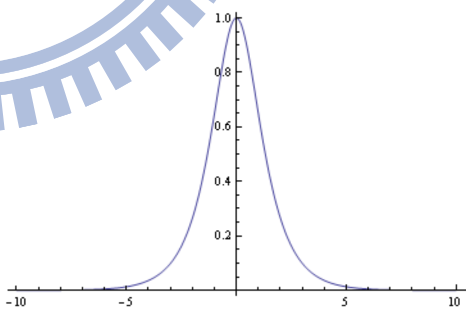


(b) $cn(u, \kappa)$ with $\kappa = 1$

Figure 4.4.2. $cn(u, \kappa)$



(a) $dn(u, \kappa)$ with $\kappa = \frac{1}{2}$



(b) $dn(u, \kappa)$ with $\kappa = 1$

Figure 4.4.3. $dn(u, \kappa)$

Chapter 5

The Exact Theory of the Pendulum Motion

5.1 Introduction of the simple pendulum motion

Here we use two ways, by Newton's second law and conservation of energy, to derive the differential equation

$$u'' + \sin u = 0\,. \tag{5.1.1}$$

(1) According to Newton's second law:

Consider Newton's second law,

$$F = ma$$

where F is the sum of forces on the object, m is mass, and a is the acceleration.

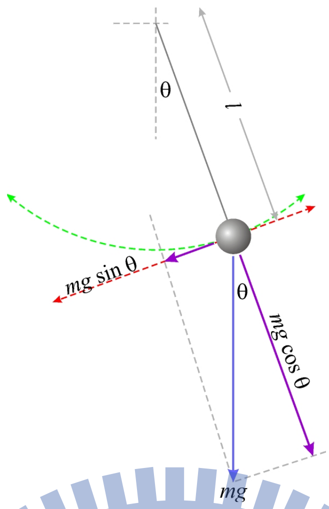


Figure 5.1.1. Using Newton's second law

Because the bob is forced to stay in a circular path, we apply equation,5.1 to the tangential axis only,

$$F = -mg\sin\theta = ma$$

$$a = -g\sin\theta$$

where g is the acceleration due to gravity near the surface of the earth. The negative sign on the right hand side implies that θ and a always point in opposite directions. This makes sense because when a pendulum swings further to the left, we would expect it to accelerate back toward the right.

This linear acceleration a can be related to the change in angle θ by the arc length formulas; l is the length of the pendulum and s is the arc length:

$$s = l\theta \tag{5.1.2}$$

$$v = \frac{ds}{dt} = l\,\frac{d\theta}{dt} \tag{5.1.3}$$

$$a = \frac{d^2s}{dt^2} = l\,\frac{d^2\theta}{dt^2} \tag{5.1.4}$$

Thus

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{5.1.5}$$

(2) According to conservation of energy:

Any object falling a vertical distance h would acquire kinetic energy equal to that which it lost to the fall. That is to say, gravitational potential energy is converted into kinetic energy.

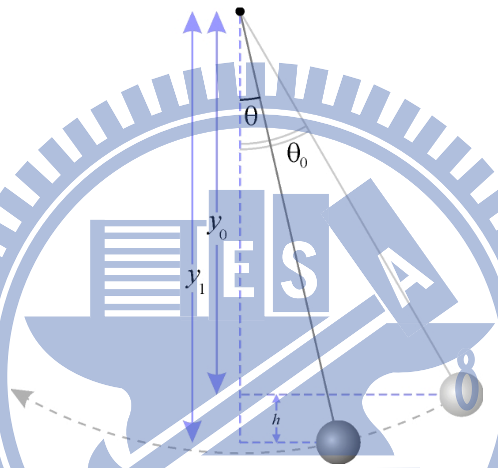


Figure 5.1.2. Using conservation of energy

Change in potential energy is given by

$$\Delta U = mgh,$$

and change in kinetic energy is given by

$$\Delta K = \frac{1}{2}mv^2.$$

As a result of the conservation of energy, no energy is lost, those two must be equal:

$$\frac{1}{2}mv^2 = mgh$$
$$\Rightarrow v = \sqrt{2gh}.$$

Using the equation (5.1.3), this equation can be rewritten as

$$v = l\,\frac{d\theta}{dt} = \sqrt{2gh}$$
$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{l}\sqrt{2gh}$$

where h is the vertical distance the pendulum fell.

See the Figure 5.1.2, which presents the trigonometry of a simple pendulum. If the pendulum starts its swing from some initial angle θ_0, then y_0, the vertical distance from the screw, is given by

$$y_0 = l\cos\theta_0\,.$$

Similarly, for y_1, we have

$$y_1 = l\cos\theta,$$

then h is the difference of the two

$$h = l(\cos\theta - \cos\theta_0).$$

In terms of $\frac{d\theta}{dt}$ gives

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}.$$

We can differentiate with respect to time, then obtain

$$\begin{aligned}
\frac{d^2\theta}{dt^2} &= \frac{d}{dt}\frac{d\theta}{dt} \\
&= \frac{d}{dt}\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)} \\
&= \frac{1}{2}\frac{\frac{2g}{l}(-\sin\theta)}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}}\frac{d\theta}{dt} \\
&= \frac{1}{2}\frac{\frac{2g}{l}(-\sin\theta)}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}}\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)} \\
&= -\frac{g}{l}\sin\theta.
\end{aligned}$$

Thus

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{5.1.6}$$

No matter which idea for derivation, there are the same results as (5.1.5) and (5.1.6). For convenience, we let $\frac{g}{l} = 1$, and there is (5.1.1).
After the above pre-work, the following contents will recall the conclusion in Chapter 4 and get the exact theory of the simple pendulum motion.

5.2 The exact theory

The ordinary differential equation

$$u_{tt} + \sin u = 0\,. \tag{5.2.7}$$

Multiplying u_t to (5.5.29) and integrating it with respect to t, then we obtain

$$\frac{1}{2}u_t^2 - \cos u = E\,,\ \text{where } E \text{ is a constant.} \tag{5.2.8}$$

Adding 1 to both sides yields

$$\frac{1}{2}u_t^2 + (1 - \cos u) = E + 1.$$

In the idea of energy, we can regard $\frac{1}{2}u_t^2$ as kinetic energy, $(1 - \cos u)$ as potential energy, and $E + 1$ as the total energy of this system.

Since $(1 - \cos u)$ is regarded as the potential energy, $0 \le (1 - \cos u) \le 2$, and the kinetic energy $\frac{1}{2}u_t^2 \ge 0$, the total energy $E+1$ must be greater than or equal to 0. Additionally, when the potential energy reaches the maximum 2, it also means that the pendulum is right at the highest position in the circular path. Thus, the total energy $E+1=2$ will be the key factor of different types of the pendulum motion.

$$\begin{aligned} 0 < E+1 < 2 &\Rightarrow -1 < E < 1 \\ E+1 = 2 &\Rightarrow E = 1 \\ E+1 > 2 &\Rightarrow E > 1 \end{aligned}$$

Consider the equation 5.2.8, then the square roots of u_t are $\pm\sqrt{2(E+\cos u)}$, here "±" denotes the bob's trajectory. We only focus on the positive sign,

$$u_t = \sqrt{2(E+\cos u)}\,. \tag{5.2.9}$$

Using the relation of trigonometric function that $\cos(2\theta) = 1 - 2\sin^2\theta$. Then we have

$$u_t = \sqrt{2(E+1-2\sin^2\frac{u}{2})} = \sqrt{2(E+1) - 4\sin^2\frac{u}{2}}\,. \tag{5.2.10}$$

Therefore, we can obtain

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2(E+1) - 4\sin^2\frac{u}{2}}}\,du\,. \tag{5.2.11}$$

We have transferred the initial problem into solving the equation (5.2.11). Now, our purpose is to solve (5.2.11). That is, we have to find the representation of $U(t)$ in terms of t. We discuss it in three different cases according to different E.

Case I. $-1 < E < 1$

If the constant $E \in (-1, 1)$, the equation (5.2.11) becomes

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2(E+1) - 4\sin^2(\frac{u}{2})}}\, du \tag{5.2.12}$$

$$= \sqrt{\frac{2}{E+1}} \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1 - \frac{2}{E+1}\sin^2(\frac{u}{2})}}\, d(\frac{u}{2}). \tag{5.2.13}$$

Let $x = \sqrt{\frac{2}{E+1}}\sin(\frac{u}{2})$, then $d(\frac{u}{2}) = \frac{1}{\sqrt{\frac{2}{E+1} - x^2}}\, dx$.

And (5.2.12) can be represented as

$$t = \int_0^{\sqrt{\frac{2}{E+1}}\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1 - (\frac{E+1}{2})x^2}}\, dx. \tag{5.2.14}$$

Let $\kappa = \sqrt{\frac{E+1}{2}}$, then the equation (5.2.14) becomes

$$t = \int_0^{\kappa^{-1}\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-\kappa^2 x^2}}\, dx. \tag{5.2.15}$$

By Jacobian elliptic function $sn(u, \kappa)$, the equation (5.2.15) signifies that

$$sn(t, \kappa) = \kappa^{-1}\sin(\frac{U(t)}{2}).$$

This means

$$U(t) = 2\sin^{-1}(\kappa\, sn(t, \kappa)), \text{ where } \kappa = \sqrt{\frac{E+1}{2}}. \tag{5.2.16}$$

Remark 5.1. In this case, $E \in (-1, 1)$, so $\sqrt{\frac{E+1}{2}} \in (0, 1)$. That is, $0 < \kappa < 1$. Besides, $\kappa \propto E$.

Case II. $E = 1$

As the constant $E = 1$, the equation (5.2.11) can be written as

$$t = \int_0^{U(t)} \frac{1}{\sqrt{4 - 4\sin^2(\frac{u}{2})}}\, du \tag{5.2.17}$$

$$= \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1 - \sin^2(\frac{u}{2})}}\, d(\frac{u}{2}) \,. \tag{5.2.18}$$

Let $x = \sin(\frac{u}{2})$, then $d(\frac{u}{2}) = \frac{1}{\sqrt{1-x^2}}\, dx$.

And (5.2.17) can be represented as

$$t = \int_0^{\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2}}\, dx \,. \tag{5.2.19}$$

By Jacobian elliptic function $sn(u, \kappa)$, the equation (5.2.19) implies that

$$sn(t, 1) = \sin(\frac{U(t)}{2}) \,.$$

This means

$$U(t) = 2\sin^{-1}(sn(t, 1)) \,. \tag{5.2.20}$$

Remark 5.2. In this case, if we do not use Jacobian elliptic function, we can also get the solution by Calculus. The solution is

$$U(t) = 2\sin^{-1}(tanh(t)) \,.$$

Case III. $E > 1$

The discussion of this case is similar to the first case. The different is on the modulus κ. From the equation (5.2.11), we have

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2(E+1) - 4\sin^2(\frac{u}{2})}}\, du \tag{5.2.21}$$

$$= \sqrt{\frac{2}{E+1}} \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1 - \frac{2}{E+1}\sin^2(\frac{u}{2})}}\, d(\frac{u}{2})\,. \tag{5.2.22}$$

Let $\kappa = \sqrt{\dfrac{2}{E+1}}$, then the equation (5.2.21) becomes

$$t = \kappa \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1 - \kappa^2 \sin^2(\frac{u}{2})^2}}\, d(\frac{u}{2})\,. \tag{5.2.23}$$

And let $x = \sin(\dfrac{u}{2})$, then $d(\dfrac{u}{2}) = \dfrac{1}{\sqrt{1-x^2}}\, dx$.

Then (5.2.23) can be represented as

$$t = \kappa \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-\kappa^2 x^2}}\, dx\,. \tag{5.2.24}$$

By Jacobian elliptic function $sn(u,\kappa)$, the equation (5.2.24) implies that

$$sn(\kappa^{-1}t, \kappa) = \sin(\frac{U(t)}{2})\,.$$

This means

$$U(t) = 2\sin^{-1}(sn(\kappa^{-1}t,\kappa))\,,\ \text{where } \kappa = \sqrt{\frac{2}{E+1}}\,. \tag{5.2.25}$$

Remark 5.3. In this case, $E > 1$, so $\sqrt{\dfrac{2}{E+1}}$ is smaller than 1. That is, $0 < \kappa < 1$. Besides, $\kappa \propto E^{-1}$.

5.3 The periods

We had found the solutions for the ordinary differential equation in the form of Jacobian elliptic function with different constant E. Now we want to find out the period of solution if it is a periodic solution. The idea is to find the rest position $U(t_0)$ and t^* denotes the time of the particle moves from $U(0)$ to $U(t_0)$. Then the period is the four times t^* .

Case I. $-1 < E < 1$

The solution of this case is (5.2.16) as

$$U(t) = 2\sin^{-1}(\kappa\, sn(t,\kappa))\,, \text{ where } \kappa = \sqrt{\frac{E+1}{2}}\,.$$

By (5.2.9), we could get the velocity of the particle is

$$U_t = \sqrt{2(E+1) - 4\sin^2(\frac{U}{2})}\,. \tag{5.3.26}$$

If the equation (5.3.26) equal to 0, then

$$\frac{U(t)}{2} = \pm\sin^{-1}(\kappa)\,, \text{ where } \kappa = \sqrt{\frac{E+1}{2}}\,.$$

As a result, by (5.2.15), we know that the period is

$$\begin{aligned} T &= 4\,t^* \\ &= 4\int_0^{\kappa^{-1}\sin(\sin^{-1}(\kappa))} \frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-\kappa^2x^2}}\,dx \\ &= 4\int_0^1 \frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-\kappa^2x^2}}\,dx \\ &= 4K\,. \end{aligned}$$

Then we find the period for this case.

Remark 5.4.

(1) The constant K here is defined as

$$K = \int_0^1 \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-\kappa^2 x^2}}\, dx\,,$$

which is the same value to the Remark 4.10.

(2) The constant $K \propto \kappa$, which signifies that the period $T \propto \kappa$. And it is easy to calculate that $T = 2\pi$ if $\kappa = 0$. This means that the period $T > 2\pi$, for all $\kappa \in (0, 1)$. Additionally, this also takes on that if $U(t) = 2\sin^{-1}(\kappa) < 2\sin^{-1}(1) = \pi$, it is a periodic solution with period $4K$.

Case II. $E = 1$

The solution of this case is (5.2.20) as

$$U(t) = 2\sin^{-1}(sn(t, 1))\,.$$

By (5.3.26), and given $E = 1$ we could get

$$U_t = \sqrt{4 - 4\sin^2(\frac{U}{2})}\,. \tag{5.3.27}$$

If the equation (5.3.27) equal to 0, then

$$\frac{U(t)}{2} = \pm\frac{\pi}{2}\,.$$

Then using (5.2.19), the period can be calculated

$$\begin{aligned} T &= 4\,t^* \\ &= 4\int_0^{\sin(\frac{\pi}{2})} \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2}}\, dx \\ &= 4\int_0^1 \frac{1}{1-x^2}\, dx \\ &= \infty\,. \end{aligned}$$

The period of this case could be regarded as ∞ although it is not a periodic solution. This means that if we release the particle at the position $-\pi$, it needs infinity time to approach the position π.

Case III. $E > 1$

By the equation (5.3.26), we know that the velocity is always positive for this case $E > 1$. This means that at each position $U(t)$, the pendulum always has velocity, so the pendulum will never stop. This implies that it has no periodicity.

5.4 The phase portraits

The ordinary differential equation we had discussed is the mathematical model of ideal pendulum. Now we try to plot the relation between $U(t)$ and U_t and the graph is called phase portrait. Before drawing the phase portrait, we see back to the equation (5.2.8) as

$$\frac{1}{2}u_t^2 - \cos u = E\,, \text{ where } E \text{ is a constant.}$$

It shows that $\frac{1}{2}u_t^2 - \cos u$ is a constant. It can be regarded as a conservation law in the viewpoint of mathematics since $-\cos u$ is not always larger than 0.

This constant and the former part $\frac{1}{2}u_t^2$ can be regarded as kinetic energy and the latter part $-\cos u$ can be regarded as potential energy.

We will discuss the potential energy and phase portrait with different cases.

Case I. $-1 < E < 1$

We set $E = 0$ to analyze this case. By the equation (5.2.8), we have the equation $u_t = \pm\sqrt{2\cos u}\,.$

The following graphs are potential energy and phase portrait respectively. This means that they are the relation between u and $\cos u$ and the relation between u and u_t .

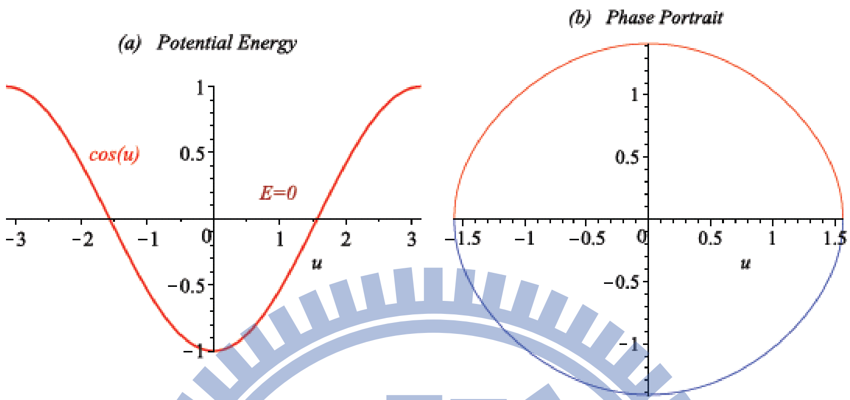


Figure 5.4.1. The potential energy and phase portrait for $E = 0$

Remark 5.5.

(1) From the graph of the phase portrait, the red curve means that the velocity at those position are positive and the blue curve means that the velocity at those position are negative. The positive velocity is defined by rotating counterclockwise and the negative velocity is defined by rotating clockwise.

(2) By the graph of potential energy, we can find out that the maximum of amplitude, $u(t)$, for the pendulum is $\frac{\pi}{2}$ and it oscillates forth and back.

Case II. $E = 1$

Now we focus on the case with $E = 1$. By the equation (5.2.8), we have $u_t = \pm\sqrt{2(1 + \cos u)}$. We see the potential energy and phase portrait as following.

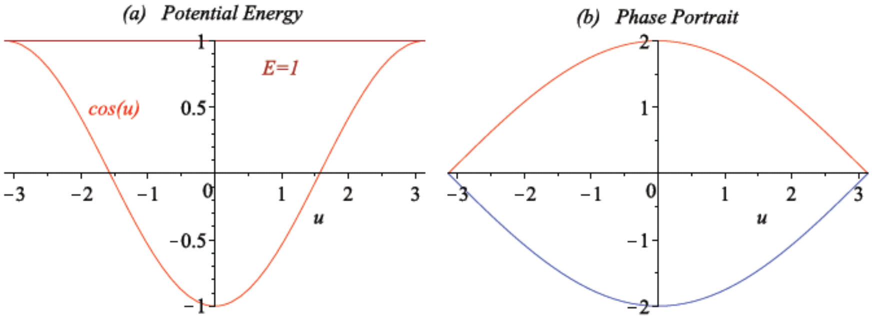


Figure 5.4.2. The potential energy and phase portrait for $E = 1$

By the graph of potential energy, we can find out that the maximum of amplitude, $u(t)$, for the pendulum is π. If we release the pendulum at position π, the particle will approach to the position $-\pi$ after infinite time.

Case III. $E > 1$

Last, we see the case $E > 1$ with $E = \dfrac{3}{2}$. By the equation (5.2.8), we have $u_t = \pm\sqrt{2(\frac{3}{2} + \cos u)}$. We see the potential energy and phase portrait as following.

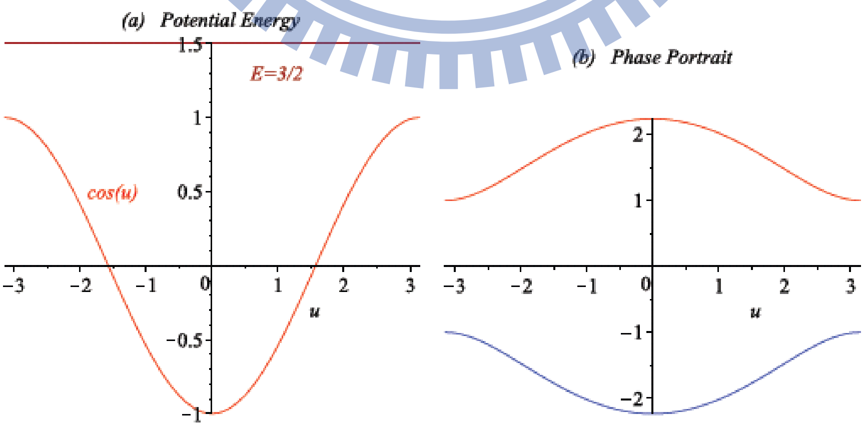


Figure 5.4.3. The potential energy and phase portrait for $E = \dfrac{3}{2}$

Remark 5.6.

(1) From the graph of the phase portrait, we know that the pendulum of this case will never stop since the phase portrait has no intersection with the u-axis.

(2) By the graph of potential energy, we observe that the kinetic energy is never equal 0. This implies that the case has no periodic solution and the result is corresponded to the property which we had discussed.

By our discussion, there are three kinds of the phase portraits. Before finishing the section, we combine the three phase portraits and the vector field together.

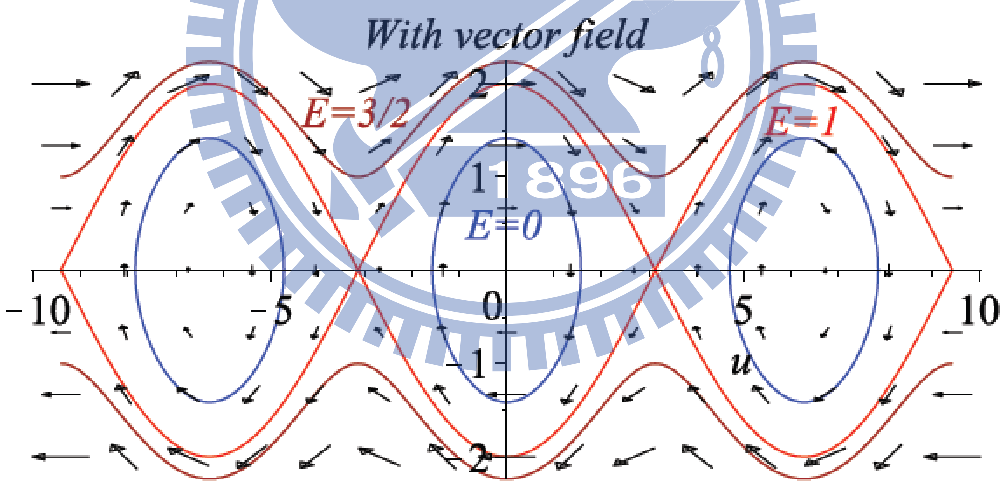


Figure 5.4.4. Global phase portrait

5.5 Related knowledge

We can connect with a partial differential equation

$$u_{tt} - u_{xx} + \sin u = 0\,, \tag{5.5.28}$$

which is called sine-Gordon equation.

Firstly, we can simplify the equation (5.5.28). Assume that $\theta = kx - \omega t$ with $\omega^2 - k^2 = 1$.

And by chain rule, we have

$$u_t = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial t} = (-\omega) u_\theta$$

$$u_x = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = k\, u_\theta$$

Using the same way, we get

$$u_{tt} = \omega^2 u_{\theta\theta}$$

$$u_{xx} = k^2 u_{\theta\theta}$$

Then (5.5.28) can be transferred to be an ordinary differential equation

$$u_{\theta\theta} + \sin(u) = 0\,. \tag{5.5.29}$$

That is the pendulum motion we discussed.

Chapter 6

Conclusion

In this paper, we study the ideal pendulum equation $u'' + \sin u = 0$, which can be translated into integral form

$$\int \frac{1}{\sqrt{2(E+\cos u)}}\, du = \pm \int dt$$

where E is the integration constant. The integrator involve $\sqrt{2(E+\cos u)}$ where $E+\cos u$ is a transcendental function and it has infinitely many zeros, so u resides on Riemann surface of genus ∞.
Hence, we study its nonlinear approximation, namely

$$u'' + P_{2N+1}(u) = 0\,,$$

where $P_{2N+1}(u)$ is the {2N+1}-th Taylor expansion of $\sin u$.
Then this O.D.E. has the integral form

$$\int \frac{1}{\sqrt{2(E-P_{2N+2}(u))}}\, du\,,$$

where u now resides on Riemann surface of genus N, then we can analyze and compute it.

We then study the classical elliptic function and apply to analyze the exact theory of pendulum motions with a table given as follows:

Energy E	$-1 < E < 1$	$E = 1$	$E > 1$
Solution $U(t)$	$2\sin^{-1}(\kappa sn(t,\kappa))$	$2\sin^{-1}(sn(t,1))$	$2\sin^{-1}(sn(\kappa^{-1}t,\kappa))$
Modulus κ	$\sqrt{\dfrac{E+1}{2}}$	1	$\sqrt{\dfrac{2}{E+1}}$
Periods T	$4K$	∞	No periodicity

For further study, we may consider the more complicated sine-Gorden equation

$$u_{tt} - u_{xx} + \sin u = 0\,,$$

where $u'' + \sin u = 0$ is just a special case.

Appendix A

The process of computation

A.0.1 $\int_{a_3} \frac{1}{f(z)} dz$

$$\int_{a_3} \frac{1}{f(z)} dz = \int_{a_3^*} \frac{1}{f(z)} dz \overset{Math.}{=} -2 \int_{-2}^{-1} \frac{1}{f(r-i)} dr = -0.084919 + 0.36914\, i$$

A.0.2 $\int_{b_1} \frac{1}{f(z)} dz$

$$\begin{aligned}
\int_{b_1} \frac{1}{f(z)} dz = \int_{b_1^*} \frac{1}{f(z)} dz& \\
\overset{Math.}{=} -\int_{1}^{0} \frac{i}{f(-2+ri)} dr + \int_{0}^{-1} \frac{i}{f(-2+ri)} dr - \int_{-2}^{-1} \frac{1}{f(r-i)} dr& \\
+ \int_{-1}^{0} \frac{i}{f(-1+ri)} dr - \int_{-1}^{0} \frac{1}{f(r)} dr + \int_{0}^{1} \frac{i}{f(ri)} dr - \int_{0}^{1} \frac{1}{f(r+i)} dr& \\
- \int_{1}^{2} \frac{i}{f(1+ri)} dr + \int_{2}^{1} \frac{i}{f(1+ri)} dr + \int_{1}^{0} \frac{1}{f(r+i)} dr + \int_{0}^{-1} \frac{1}{f(r+i)} dr& \\
+ \int_{-1}^{-2} \frac{1}{f(r+i)} dr = -0.804295 + 0.615335\, i&
\end{aligned}$$

A.0.3 $\int_{b_2} \frac{1}{f(z)} dz$

$$\begin{aligned}
\int_{b_2} \frac{1}{f(z)} dz &= \int_{b_2^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} -\int_1^0 \frac{i}{f(-2+ri)} dr + \int_0^{-1} \frac{i}{f(-2+ri)} dr - \int_{-2}^{-1} \frac{1}{f(r-i)} dr \\
&+ \int_{-1}^0 \frac{i}{f(-1+ri)} dr - \int_{-1}^0 \frac{1}{f(r)} dr - \int_0^1 \frac{i}{f(ri)} dr \\
&+ \int_0^{-1} \frac{1}{f(r+i)} dr + \int_{-1}^{-2} \frac{1}{f(r+i)} dr \\
&= -0.768165 + 0.242221\, i
\end{aligned}$$

A.0.4 $\int_{a_3} \frac{1}{f(z)} dz$

$$\begin{aligned}
\int_{a_3} \frac{1}{f(z)} dz &= \int_{a_3^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} -2\int_0^1 \frac{i}{f(-1+ri)} dr + 2\int_{-1}^0 \frac{i}{f(-1+ri)} dr \\
&= 0.170019 - 0.268168\, i
\end{aligned}$$

A.0.5 $\int_{b_2} \frac{1}{f(z)} dz$

$$
\begin{aligned}
\int_{b_2} \frac{1}{f(z)} dz &= \int_{b_2^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} -\int_1^0 \frac{i}{f(-2+ri)} dr + \int_0^{-1} \frac{i}{f(-2+ri)} dr - \int_{-2}^{-1} \frac{1}{f(r-i)} dr \\
&+ \int_{-1}^0 \frac{i}{f(-1+ri)} dr - \int_{-1}^0 \frac{1}{f(r)} dr - \int_0^1 \frac{1+i}{f(r(1+i))} dr \\
&+ \int_1^0 \frac{1+i}{f(i+r(1+i))} dr - \int_1^2 \frac{i}{f(1+ri)} dr + \int_0^{-1} \frac{1}{f(i+r)} dr \\
&+ \int_0^{-1} \frac{1}{f(-1+i+r)} dr \\
&= -0.963521 + 0.24317\, i
\end{aligned}
$$

A.0.6 $\int_{b_1} \frac{1}{f(z)} dz$

$$
\begin{aligned}
\int_{b_1} \frac{1}{f(z)} dz &= \int_{b_1^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} -\int_1^0 \frac{i}{f(-2+ri)} dr + \int_0^{-1} \frac{i}{f(-2+ri)} dr - \int_{-2}^{-1} \frac{1}{f(r-i)} dr \\
&+ \int_{-1}^0 \frac{i}{f(-1+ri)} dr - \int_0^1 \frac{i}{f(-1+ri)} dr - \int_{-1}^0 \frac{1}{f(r+i)} dr \\
&- 2\int_0^1 \frac{1+i}{f(i+r(1+i))} dr + \int_0^{-1} \frac{1}{f(i+r)} dr + \int_0^{-1} \frac{1}{f(-1+i+r)} dr \\
&= -0.604473 + 0.635889\, i
\end{aligned}
$$

A.0.7 $\int_{a_1} \frac{1}{f(u)}du$

$$\int_{a_1} \frac{1}{f(u)}du = \int_{a_1^*} \frac{1}{f(u)}du \overset{Math.}{=} 2\int_{-1.34}^{1.34} \frac{i}{f(8.27+ri)}dr$$

Here the computation by Mathematica is too huge to appear.

A.0.8 $\int_{a_2} \frac{1}{f(u)}du$

$$\begin{aligned}\int_{a_2} \frac{1}{f(u)}du &= \int_{a_2^*} \frac{1}{f(u)}du \\ &\overset{Math.}{=} 2\int_{1.34}^{3.86} \frac{i}{f(8.19+ri)}dr - 2\int_{-1.34}^{1.34} \frac{i}{f(8.19+ri)}dr \\ &+ 2\int_{-3.86}^{-1.34} \frac{i}{f(8.19+ri)}dr\end{aligned}$$

Here the computation by Mathematica is too huge to appear.

A.0.9 $\int_{a_3} \frac{1}{f(u)}du$

$$\begin{aligned}\int_{a_3} \frac{1}{f(u)}du &= \int_{a_3^*} \frac{1}{f(u)}du \\ &\overset{Math.}{=} 2\int_{3.86}^{8.19} \frac{i}{f(8.04+ri)}dr - 2\int_{1.34}^{3.86} \frac{i}{f(8.04+ri)}dr \\ &+ 2\int_{-1.34}^{1.34} \frac{i}{f(8.04+ri)}dr - 2\int_{-3.86}^{-1.34} \frac{i}{f(8.04+ri)}dr \\ &+ 2\int_{-8.19}^{-3.86} \frac{i}{f(8.04+ri)}dr\end{aligned}$$

Here the computation by Mathematica is too huge to appear.

A.0.10 $\int_{a_4} \frac{1}{f(u)} du$

$$
\begin{aligned}
\int_{a_4} \frac{1}{f(u)} du = \int_{a_4^*} \frac{1}{f(u)} du \\
&\overset{Math.}{=} 2\int_{1.34}^{2.48} \frac{i}{f(3.14+ri)} dr - 2\int_{-1.34}^{1.34} \frac{i}{f(3.14+ri)} dr \\
&+ 2\int_{-2.48}^{-1.34} \frac{i}{f(3.14+ri)} dr
\end{aligned}
$$

Here the computation by Mathematica is too huge to appear.

A.0.11 $\int_{a_5} \frac{1}{f(u)} du$

$$
\begin{aligned}
\int_{a_5} \frac{1}{f(u)} du = \int_{a_5^*} \frac{1}{f(u)} du \\
&\overset{Math.}{=} -2\int_{1.34}^{2.48} \frac{i}{f(-3.14+ri)} dr + 2\int_{-1.34}^{1.34} \frac{i}{f(-3.14+ri)} dr \\
&- 2\int_{-2.48}^{-1.34} \frac{i}{f(-3.14+ri)} dr
\end{aligned}
$$

Here the computation by Mathematica is too huge to appear.

A.0.12 $\int_{a_6} \frac{1}{f(u)} du$

$$
\begin{aligned}
\int_{a_6} \frac{1}{f(u)} du = & \int_{a_6^*} \frac{1}{f(u)} du \\
\overset{Math.}{=} & -2\int_{3.86}^{8.19} \frac{i}{f(-8.04+ri)} dr + 2\int_{1.34}^{3.86} \frac{i}{f(-8.04+ri)} dr \\
& -2\int_{-1.34}^{1.34} \frac{i}{f(-8.04+ri)} dr + 2\int_{-3.86}^{-1.34} \frac{i}{f(-8.04+ri)} dr \\
& -2\int_{-8.19}^{-3.86} \frac{i}{f(-8.04+ri)} dr
\end{aligned}
$$

Here the computation by Mathematica is too huge to appear.

A.0.13 $\int_{a_7} \frac{1}{f(u)} du$

$$
\begin{aligned}
\int_{a_7} \frac{1}{f(u)} du = & \int_{a_7^*} \frac{1}{f(u)} du \\
\overset{Math.}{=} & -2\int_{1.34}^{3.86} \frac{i}{f(-8.19+ri)} dr + 2\int_{-1.34}^{1.34} \frac{i}{f(-8.19+ri)} dr \\
& -2\int_{-3.86}^{-1.34} \frac{i}{f(-8.19+ri)} dr
\end{aligned}
$$

Here the computation by Mathematica is too huge to appear.

A.0.14 $\displaystyle\int_{b_7} \frac{1}{f(u)}du$

$$\begin{aligned}\int_{b_7} \frac{1}{f(u)}du &= \int_{b_7^*} \frac{1}{f(u)}du \\ &\overset{Math.}{=} -2\int_0^{1.34} \frac{i}{f(-8.27-ri)}dr - 2\int_{-8.27}^{-8.19} \frac{1}{f(r)}dr \\ &- 2\int_0^{1.34} \frac{i}{f(-8.19+ri)}dr + 2\int_{1.34}^{3.86} \frac{i}{f(-8.19+ri)}dr\end{aligned}$$

Here the computation by Mathematica is too huge to appear.

A.0.15 $\displaystyle\int_{b_6} \frac{1}{f(u)}du$

$$\begin{aligned}\int_{b_6} \frac{1}{f(u)}du &= \int_{b_6^*} \frac{1}{f(u)}du \\ &\overset{Math.}{=} \int_{b_7^*} \frac{1}{f(u)}du - 2\int_{-3.86}^{-1.34} \frac{i}{f(-8.19+ri)}dr 2\int_{-1.34}^{0} \frac{i}{f(-8.19+ri)}dr \\ &+ 2\int_{-8.19}^{-8.04} \frac{1}{f(r)}dr + 2\int_0^{1.34} \frac{i}{f(-8.04+ri)}dr - 2\int_{1.34}^{3.86} \frac{i}{f(-8.04+ri)}dr \\ &+ 2\int_{3.86}^{8.19} \frac{i}{f(-8.04+ri)}dr\end{aligned}$$

Here the computation by Mathematica is too huge to appear.

A.0.16 $\int_{b_5} \frac{1}{f(u)} du$

$$\int_{b_5} \frac{1}{f(u)} du = \int_{b_5^*} \frac{1}{f(u)} du$$

$$\overset{Math.}{=} \int_{b_6^*} \frac{1}{f(u)} du + 2 \int_{-3.86}^{-1.34} \frac{i}{f(-8.04+ri)} dr - 2 \int_{-8.19}^{-3.86} \frac{i}{f(-8.04+ri)} dr$$

$$- 2 \int_{-1.34}^{0} \frac{i}{f(-8.04+ri)} dr - 2 \int_{-1.34}^{0} \frac{i}{f(-8.04+ri)} dr$$

$$- 2 \int_{-8.04}^{-3.14} \frac{1}{f(r)} dr - 2 \int_{0}^{1.34} \frac{i}{f(-3.14+ri)} dr$$

$$+ 2 \int_{1.34}^{2.48} \frac{i}{f(-3.14+ri)} dr$$

Here the computation by Mathematica is too huge to appear.

A.0.17 $\int_{b_4} \frac{1}{f(u)} du$

$$\int_{b_4} \frac{1}{f(u)} du = \int_{b_4^*} \frac{1}{f(u)} du$$

$$\overset{Math.}{=} \int_{b_5^*} \frac{1}{f(u)} du - 2 \int_{-2.48}^{-1.34} \frac{i}{f(-3.14+ri)} dr + 2 \int_{-1.34}^{0} \frac{i}{f(-3.14+ri)} dr$$

$$+ 2 \int_{-3.14}^{3.14} \frac{1}{f(r)} dr + 2 \int_{0}^{1.34} \frac{i}{f(3.14+ri)} dr - 2 \int_{1.34}^{2.48} \frac{i}{f(3.14+ri)} dr$$

Here the computation by Mathematica is too huge to appear.

A.0.18 $\int_{b_3} \frac{1}{f(u)} du$

$$
\begin{aligned}
\int_{b_3} \frac{1}{f(u)} du &= \int_{b_3^*} \frac{1}{f(u)} du \\
&\overset{Math.}{=} \int_{b_4^*} \frac{1}{f(u)} du - 2\int_{-1.34}^{0} \frac{i}{f(3.14+ri)} dr + 2\int_{-2.48}^{-1.34} \frac{i}{f(3.14+ri)} dr \\
&- 2\int_{3.14}^{8.04} \frac{1}{f(r)} dr - 2\int_{0}^{1.34} \frac{i}{f(8.04+ri)} dr \\
&+ 2\int_{1.34}^{3.86} \frac{i}{f(8.04+ri)} dr + 2\int_{3.86}^{8.19} \frac{i}{f(8.04+ri)} dr
\end{aligned}
$$

Here the computation by Mathematica is too huge to appear.

A.0.19 $\int_{b_2} \frac{1}{f(u)} du$

$$
\begin{aligned}
\int_{b_2} \frac{1}{f(u)} du &= \int_{b_2^*} \frac{1}{f(u)} du \\
&\overset{Math.}{=} \int_{b_3^*} \frac{1}{f(u)} du - 2\int_{-3.86}^{-1.34} \frac{i}{f(8.04+ri)} dr + 2\int_{-8.19}^{-3.86} \frac{i}{f(8.04+ri)} dr \\
&- 2\int_{-1.34}^{0} \frac{i}{f(8.04+ri)} dr + 2\int_{8.04}^{8.19} \frac{1}{f(r)} dr \\
&+ 2\int_{0}^{1.34} \frac{i}{f(8.19+ri)} dr - 2\int_{1.34}^{3.86} \frac{i}{f(8.19+ri)} dr
\end{aligned}
$$

Here the computation by Mathematica is too huge to appear.

A.0.20 $\int_{b_1} \frac{1}{f(u)} du$

$$\int_{b_1} \frac{1}{f(u)} du = \int_{b_1^*} \frac{1}{f(u)} du$$
$$\stackrel{Math.}{=} \int_{b_2^*} \frac{1}{f(u)} du - 2\int_{-1.34}^{0} \frac{i}{f(8.19+ri)} dr$$
$$+ 2\int_{-3.86}^{-1.34} \frac{i}{f(8.19+ri)} dr - 2\int_{8.19}^{8.27} \frac{1}{f(r)} dr - 2\int_{0}^{1.34} \frac{i}{f(8.27+ri)} dr$$

Here the computation by Mathematica is too huge to appear.

Bibliography

[1] Paul Blanchard, Robert L. Devaney, Glen R. Hall, and Jong-Eao Lee, Differential Equations: A Contemporary Approach, Thomson Learning, 2007.

[2] Yun-Ting Wu, Theory of Riemann Surfaces and Its Applications to DifferentialEquations, Master thesis, NCTU, 2010.

[3] 洪維恩, 數學運算大師 - Mathematica 5, 旗標, 2009.

[4] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, 1927.

[5] 沈璿, 橢圓函數概論, 國立編譯館, 1982.

[6] Wen-Yu Chien, The Exact Theory and Perturbation of the Pendulum Motions, Master thesis, NCTU, 2009.

[7] Jian-Wei Chang, The Exact Theory and Numerical Computations of Pendulum Motions on Riemann Surface of Genus N with Cut-Structure of Type A, Master thesis, NCTU, 2012.