國 立 交 通 大 學

電信工程研究所

碩 士 論 文

流動介質中的粒子通訊:在傳輸端有平均及最大延遲的限制下,相加性反高斯雜訊通道容量界線

Molecular Communication in a Liquid: Bounds on the Capacity of the Additive Inverse Gaussian Channel with Average and Peak Delay Constraints

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中文摘要

在本篇論文中，我們探討一個相當新的通道模型，此通道是藉由原子在液體中的交換來傳輸信號。而我們假設原子在傳輸過程中，是在一維的空間做移動。像我們將奈米級的儀器放入血管中，而此儀器在人體內和其他儀器交換訊息就是一個很典型的通訊應用。在此應用中，我們不再使用電磁波來傳輸我們的信號，相反地，我們將訊息放在原子從傳輸端釋放的時間點上。一旦原子被釋放在液體中，將會在液體中進行布朗運動，進而造成我們無法預估原子到達傳輸端的時間，換句話說，布朗運動造成接收時間的不確定性，而此不確定性就是我們的雜訊，而我們用反高斯(Inverse Gaussian)分布來描述此雜訊。此篇的研究重點是相加性雜訊通道，在有平均以及最大延遲的限制下，基本的通道容量趨勢。

我們深入研究此模型，並且分析出新的通道容量的上界以及下限，而這些界線是逐漸靠近的，也就是說，如果我們允許平均以及最大延遲放寬到無限大，亦或是介質的流體速度趨近無限大，我們可以得到準確的通道容量。

Molecular Communication in a Liquid: Bounds on the Capacity of the Additive Inverse Gaussian Noise Channel with Average and Peak Delay Constraints

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Abstract

In this thesis a very recent and new channel model is investigated that describes communication based on the exchange of chemical molecules in a liquid medium with constant drift. The molecules travel from the transmitter to the receiver at two ends of a one-dimensional axis. A typical application of such communication are nano-devices inside a blood vessel communicating with each other. In this case, we no longer transmit our signal via electromagnetic waves, but we encode our information into the emission time of the molecules. Once a molecule is emitted in the fluid medium, it will be affected by Brownian motion, which causes uncertainty of the molecule's arrival time at the receiver. We characterize this noise with an inverse Gaussian distribution. Here we focus solely on an additive noise channel to describe the fundamental channel capacity behavior with average and peak delay constraints.

This new model is investigated and new analytical upper and lower bounds on the capacity are presented. The bounds are asymptotically tight, i.e., if the average-delay and peak-delay constraints are loosened to infinity, the corresponding asymptotic capacities are derived precisely.

Acknowledgments

First, I want to thank to my advisor, Prof. Stefan M. Moser. Thanks to him, I am able to have such interesting topic as my master thesis. In addition, I have learnt a lot of things from him such as how to analysis problems, some methematical techniques, how to do a good presentation and how to write a good paper and slide. All of these are reallly important to me.

I am really lucky to get in the IT-lab, this lab is like second home to me. In the lab, I can do what I want to do and all members give me full support. Thanks to the IT-lab members, without their help, it is impossible that I can finish everything in time.

Finally, I want to thanks to my family. When I was upset about the thesis or had bad mood, they encouraged me which is the reason that I am able to perservered with thesis.

Lee Ting-Hsuan

Contents

Chapter 1

Introduction

1.1 General Molecular Communication Channel Model

Usually, information carrying signals are transmitted as electromagnetic waves in the air or in wires. Recently, people are more and more interested in communication within nanoscale networks. But when we want to transmit our signal via these tiny devices, we face new problems, for example, the antenna size are restricted or the energy that could be stored is very little. We solve these problems by providing a different type of communication instead. This thesis focuses on a channel which operates in a fluid medium with a constant drift velocity. One application example is blood vessel, which has a blood drift. The nanoscale device could be any medical inspection device that is inserted in our body. The transmitter is a point source with many molecules to be emitted. The receiver waits on the other side for the molecules' arrival. The information is encoded in the emission time of the molecules, X, which takes value in a finite set.

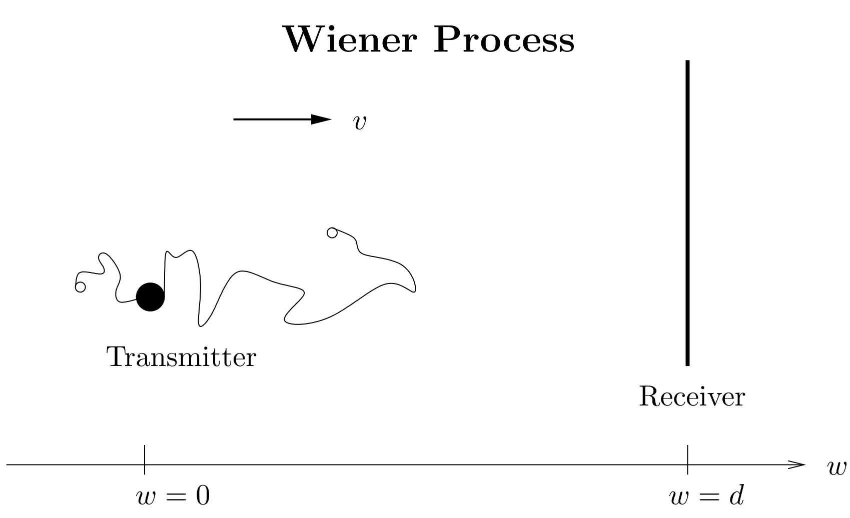


Figure 1.1: Wiener process of molecular communication channel.

Once the nanoscale molecules are emitted into the fluid medium, beside the constant drift, they are affected by Brownian motion, which causes uncertainty of the arrival time at the receiver. We describe this type of channel noise with an inverse Gaussian distribution. The typical situation is shown in Figure 1.1, where w is the position parameter, d is the receiver's position on w axis, and $v > 0$ is the drift velocity. The transmitter is placed at the origin of the w axis. It emits a molecule into a fluid with positive drift velocity v. The information is put on the releasing time. In order to know this information, the receiver ideally subtracts the average traveling time, $\frac{d}{v}$, from the arrival time. Note that once a molecule arrives at the receiver, it is absorbed and never returns to the fluid. Moreover, every molecule is independent of each other.

This molecular communication channel model was proposed by Srinivas, Adve and Eckford [1].

1.2 Mathematical Model

Let $W(x)$ be the position of a molecule at time x that travels via a Brownian motion medium. Let $0 \le x_1 < x_2 < \cdots < x_k$ be a sequence of time indices ordered from small to large. Then, $W(x)$ is a Wiener process if the position increment $R_i = W(x_{i-1}) - W(x_i)$ are independent random variables with

$$R_i \sim \mathcal{N}\big(v(x_i - x_{i-1}), \sigma^2(x_i - x_{i-1})\big) \tag{1.1}$$

where $\sigma^2 = \frac{D}{2}$ with D being the diffusion coefficient, which depends on the temperature and the stickiness of the fluid and the size of the particles. Assuming that the molecule is released at time $x = 0$ at position $W(0) = 0$, the position at time $\tilde{x}$ is $W(\tilde{x}) \sim \mathcal{N}\big(v\tilde{x}, \sigma^2\tilde{x}\big)$. The probability density function (PDF) of W is given by:

$$f_W(w; \tilde{x}) = \frac{1}{\sqrt{2\pi\sigma^2\tilde{x}}} \exp\left(-\frac{(w - v\tilde{x})^2}{2\sigma^2\tilde{x}}\right). \tag{1.2}$$

In our communication system, instead of looking at the position of the molecule at a certain time, we turn our focus on its arriving time at the receiver of a fixed distance d.

We release the molecule at time x from the origin, $W(x) = 0$ and $x \ge 0$. After traveling for a random time N, the molecule then arrives at the receiver for the first time at time Y,

$$Y = x + N. \tag{1.3}$$

Hence, our channel model is characterized by an additive noise in the form of the random propagation time N. This is the only uncertainty we have in the system. When we assume a positive drift velocity $v > 0$, the distribution of the traveling time N is well known to be an *inverse Gaussian* (IG) distribution. As a result, we

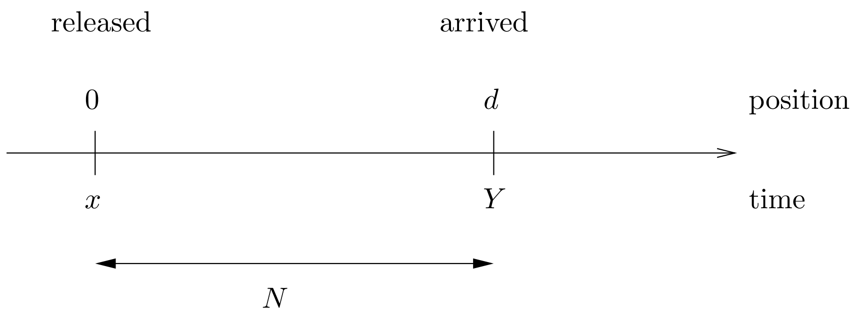


Figure 1.2: The relation between the molecule's time and position.

call this channel the *additive inverse Gaussian noise (AIGN) channel*. Since the PDF of N is

$$f_N(n) = \begin{cases} \sqrt{\frac{\lambda}{2\pi n^3}} \exp\left(-\frac{\lambda(n-\mu)^2}{2\mu^2 n}\right) & n > 0, \\ 0 & n \leq 0, \end{cases} \tag{1.4}$$

we get the conditional probability density of output Y given the channel input $X = x$ as

$$f_{Y|X}(y|x) = \begin{cases} \sqrt{\frac{\lambda}{2\pi(y-x)^3}} \exp\left(-\frac{\lambda(y-x-\mu)^2}{2\mu^2(y-x)}\right) & y > x, \\ 0 & y \leq x. \end{cases} \tag{1.5}$$

There are two important parameters for the inverse Gaussian distribution: the average traveling time

$$\mu = \frac{d}{v} = \frac{\text{distance between transmitter and receiver}}{\text{drift velocity}}, \tag{1.6}$$

and a parameter

$$\lambda = \frac{d^2}{\sigma^2} \tag{1.7}$$

that describes the impact of the noise. Usually we write $N \sim \mathrm{IG}(\mu, \lambda)$. By calculation, we get

$$\mathsf{E}[N] = \mu \quad = \frac{d}{v}, \tag{1.8}$$

$$\mathrm{Var}(N) = \frac{\mu^3}{\lambda} = \frac{d\sigma^2}{v^3}. \tag{1.9}$$

If the drift velocity v increases, the variance decreases, in other words, the distribution is more centered. If the drift velocity is slowed down, we will have a more spread-out noise distribution. Without loss of generality, we normalize the propagation distance to $d = 1$.

For practical reasons, we constrain the transmitter to have a peak delay constraint T and a average delay constraint m at the transmitter, i.e., the input X are subject to the two constraints:

$$\Pr[X > \mathsf{T}] = 0, \tag{1.10}$$

$$\mathsf{E}[X] \leq m. \tag{1.11}$$

We denote the ratio between the allowed average delay and the allowed peak delay by α

$$\alpha \triangleq \frac{m}{\mathsf{T}} \tag{1.12}$$

where $0 < \alpha \leq 1$. Note that for $\alpha = 1$ the average-delay constraint is inactive in the sense that it has no influence on the capacity and is automatically satisfied whenever the peak delay constraint is satisfied. Thus, $\alpha = 1$ corresponds to the case with only a peak-delay constraint. Similarly, $\alpha \ll 1$ corresponds to a dominant average-delay constraint and only a very weak peak-delay constraint.

1.3 Capacity

Since we introduced a new type of channel, the AIGN channel, we are interested in how much information it can carry. In [2], Shannon showed that for memoryless channels with continuous input and output alphabets and an corresponding conditional PDF describing the channel, and under input constraints $\Pr[X > \mathsf{T}] = 0$ and $\mathsf{E}[X] \leq \alpha\mathsf{T}$, the channel capacity is given by

$$\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \triangleq \sup_{f_X(x)\colon \Pr[X>\mathsf{T}]=0,\ \mathsf{E}[X]\leq\alpha\mathsf{T}} I(X;Y) \tag{1.13}$$

where the supremum is taken over all input probability distributions $f(\cdot)$ on X that satisfy (1.10) and (1.11). By $I(X;Y)$ we denote the mutual information between X and Y. For the AIGN channel, we have

$$\begin{aligned}
&\sup_{f_X(x)\colon \Pr[X>\mathsf{T}]=0,\ \mathsf{E}[X]\leq\alpha\mathsf{T}} I(X;Y) \\
&= \sup_{f_X(x)\colon \Pr[X>\mathsf{T}]=0,\ \mathsf{E}[X]\leq\alpha\mathsf{T}} \Big\{h(Y) - h(Y|X)\Big\} && (1.14) \\
&= \sup_{f_X(x)\colon \Pr[X>\mathsf{T}]=0,\ \mathsf{E}[X]\leq\alpha\mathsf{T}} \Big\{h(Y) - h(X+N|X)\Big\} && (1.15) \\
&= \sup_{f_X(x)\colon \Pr[X>\mathsf{T}]=0,\ \mathsf{E}[X]\leq\alpha\mathsf{T}} \Big\{h(Y) - h(N|X)\Big\} && (1.16) \\
&= \sup_{f_X(x)\colon \Pr[X>\mathsf{T}]=0,\ \mathsf{E}[X]\leq\alpha\mathsf{T}} h(Y) - h(N) && (1.17) \\
&= \sup_{f_X(x)\colon \Pr[X>\mathsf{T}]=0,\ \mathsf{E}[X]\leq\alpha\mathsf{T}} h(Y) - h_{\mathrm{IG}(\mu,\lambda)}, && (1.18)
\end{aligned}$$

where (1.17) holds because N and X are independent. The mean constraint (1.11) of the input signal translates to an average constraint for Y:

$$\begin{aligned}\mathsf{E}[Y] &= \mathsf{E}[X+N] && (1.19)\\ &= \mathsf{E}[X]+\mathsf{E}[N] && (1.20)\\ &= \mathsf{E}[X]+\mu && (1.21)\\ &\leq \alpha\mathsf{T}+\mu. && (1.22)\end{aligned}$$

Chapter 2

Mathematical Preliminaries

In this chapter, we will introduce some mathematical properties of the inverse Gaussian random variable and other useful lemmas for future use in this thesis.

2.1 Properties of the Inverse Gaussian Distribution

In [1], the differential entropy of an inverse Gaussian random variable was given in a complicated form that is unyielding for analytical analysis. So we try to modify the original expression and derive a cleaner form for mathematical derivation.

Proposition 2.1 (Differential Entropy of the Inverse Gaussian Distribution)**.**

$$h_{\mathrm{IG}(\mu,\lambda)} = \frac{1}{2}\log\frac{2\pi\mu^3}{\lambda} + \frac{3}{2}\exp\left(\frac{2\lambda}{\mu}\right)\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right) + \frac{1}{2} \tag{2.1}$$

where $\mathrm{Ei}(\cdot)$ *is the exponential integral function defined as*

$$\mathrm{Ei}(-x) \triangleq -\int_x^\infty \frac{e^{-t}}{t}\,\mathrm{d}t = \int_{-\infty}^{-x} \frac{e^t}{t}\,\mathrm{d}t, \quad x > 0. \tag{2.2}$$

In `MATLAB`*, the exponential integral function is implement as* **`expint(x)`**$= -\mathrm{Ei}(-x)$*.*

Proof: see [3]. □

Next, when we want to make an IG random variable add with another IG random variable and end up also in IG distributed, there is a specific way to reach it. Only certain type of IGs will add up to be IG distributed.

Proposition 2.2 (Additivity of the IG distribution)**.** *Let* M *be a linear combination of random variables* M_i*:*

$$M = \sum_{i=0}^{l} c_i M_i, \quad c_i > 0, \tag{2.3}$$

where

$$M_i \sim \mathrm{IG}(\mu_i, \lambda_i), \quad i = 1, \ldots, l. \tag{2.4}$$

Here we assume that M_i are not necessarily independent, but summed up under the constraint that

$$\frac{\lambda_i}{c_i \mu_i^2} = \kappa, \quad \text{for all } i. \tag{2.5}$$

Then

$$M \sim \mathrm{IG}\left(\sum_i c_i \mu_i, \kappa \left(\sum_i c_i \mu_i\right)^2\right) \tag{2.6}$$

Proof: The proof can be found in [4, Sec. 2.4, p. 13]. □

Remark 2.3. *If we simply add two inverse Gaussian random variables, as long as they are in the same fluid, which means they have the same v and σ^2, the result is still inverse Gaussian.*

Consider a Wiener process $X(t)$ beginning with $X(0) = x_0$ with positive drift v and variance σ^2. Choose a and b so that $x_0 < a < b$ and consider the first passage time T_1 from x_0 to a and T_2 from a to b. Then T_1 and T_2 are independent inverse Gaussian variables with parameters

$$\mu_1 = \frac{a - x_0}{v}, \quad \lambda_1 = \frac{(a - x_0)^2}{\sigma^2} \tag{2.7}$$

and

$$\mu_2 = \frac{b - a}{v}, \quad \lambda_2 = \frac{(b - a)^2}{\sigma^2}. \tag{2.8}$$

Now consider $T_3 = T_1 + T_2$, therefore, $c_1 = c_2 = 1$ and

$$\frac{\lambda_i}{\mu_i} = \frac{v^2}{\sigma^2} = \text{constant}, \tag{2.9}$$

T_3 is also an inverse Gaussian variable. That is

$$T_3 \sim \mathrm{IG}\left(\mu_1 + \mu_2, \frac{v^2(\mu_1 + \mu_2)^2}{\sigma^2}\right). \tag{2.10}$$

Since $\mu_1 + \mu_2 = \frac{b - x_0}{v}$,

$$T_3 \sim \mathrm{IG}\left(\frac{b - x_0}{v}, \frac{(b - x_0)^2}{\sigma^2}\right). \tag{2.11}$$

The last observation also follows directly from the realization that T_3 is the first passage time from x_0 to b [4].

Proposition 2.4 (Scaling)**.** *If $N \sim \mathrm{IG}(\mu, \lambda)$, then for any $k > 0$*

$$kN \sim \mathrm{IG}(k\mu, k\lambda). \tag{2.12}$$

Proof: The proof can be found in [4, Sec. 2.4, p. 13]. □

Proposition 2.5. *If N is a random variable distributed as* IG(μ, λ)*. Then*

$$\mathsf{E}[N] = \mu; \tag{2.13}$$
$$\mathsf{E}\left[\frac{1}{N}\right] = \frac{1}{\mu} + \frac{1}{\lambda}; \tag{2.14}$$
$$\mathsf{E}\left[N^2\right] = \mu^2 + \frac{\mu^3}{\lambda}; \tag{2.15}$$
$$\mathsf{E}\left[\frac{1}{N^2}\right] = \frac{1}{\mu^2} + \frac{3}{\lambda^2} + \frac{3}{\mu\lambda}; \tag{2.16}$$
$$\mathrm{Var}\,(N) = \frac{\mu^3}{\lambda}; \tag{2.17}$$
$$\mathrm{Var}\left(\frac{1}{N}\right) = \frac{1}{\mu\lambda} + \frac{2}{\lambda^2}; \tag{2.18}$$
$$\mathsf{E}[N^\nu] = \sqrt{\frac{2\lambda}{\pi}} e^{\frac{\lambda}{\mu}} \mu^{\nu-\frac{1}{2}} K_{\nu-\frac{1}{2}}\left(\frac{\lambda}{\mu}\right), \qquad \nu \in \mathbb{R}. \tag{2.19}$$

where $K_\gamma(\cdot)$ is the order-γ modified Bessel function of the second kind.

Remark 2.6. *From formula [5, (8.486.16)], we get*

$$K_{-\nu}(z) = K_\nu(z) \tag{2.20}$$

we can also write (2.19) *as*

$$\mathsf{E}\left[N^{-\nu}\right] = \sqrt{\frac{2\lambda}{\pi}} e^{\frac{\lambda}{\mu}} \mu^{-\nu-\frac{1}{2}} K_{\nu+\frac{1}{2}}\left(\frac{\lambda}{\mu}\right). \tag{2.21}$$

Proof: The proofs are based on [4, (2.6)], [6, Proposition 2.15], [4, (8.36)] and [5, 3.471 9.]. □

Proposition 2.7. *If $N \sim$* IG(μ, λ)*, then*

$$\mathsf{E}[\log N] = e^{\frac{2\lambda}{\mu}} \,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right) + \log \mu; \tag{2.22}$$
$$\mathsf{E}\left[\frac{N}{\mu} + \frac{\mu}{N}\right] = 2 + \frac{\mu}{\lambda}. \tag{2.23}$$

Proof: A proof is shown in [7]. □

Proposition 2.8. *If N_i are IID $\sim$ IG(μ, λ), then the sample mean from that distribution will be*

$$\frac{1}{n}\sum_{i=1}^{n} N_i = \bar{N} \sim \mathrm{IG}(\mu, n\lambda), \quad \text{for } i = 1, \ldots, n. \tag{2.24}$$

Proof: A proof can be found in [4, Sec. 5.1, p. 56]. □

Lemma 2.9. *Under the three constraints*

$$\mathsf{E}[\log X] = \alpha_1, \tag{2.25}$$
$$\mathsf{E}[X] = \alpha_2, \tag{2.26}$$
$$\mathsf{E}\left[X^{-1}\right] = \alpha_3, \tag{2.27}$$

where α_1, α_2 and α_3 are some fixed values, the maximum entropy distribution is the inverse Gaussian distribution.

Proof: From [8, Chap. 12] we know that if we have the three constraints above, the optimal distribution to maximize the entropy will have the form

$$f(x) = e^{\lambda_0+\lambda_1 \log x+\lambda_2 x+\frac{\lambda_3}{x}} \tag{2.28}$$
$$= x^{\lambda_1} e^{\lambda_0+\lambda_2 x+\frac{\lambda_3}{x}}, \tag{2.29}$$

which is exactly the form of the inverse Gaussian. □

2.2 Related Lemmas and Propositions

In this section, we will show some lemmas and properties that will be used in our proof of bounds.

Proposition 2.10. *Consider a memoryless channel with input alphabet $\mathcal{X} = \mathbb{R}_0^+$ and output alphabet $\mathcal{Y} = \mathbb{R}$, where, conditional on the input $x \in \mathcal{X}$, the distribution on the output Y is denoted by the probability measure $f_{Y|X}(\cdot|x)$. Then, for any distribution $f_Y(\cdot)$ on $\mathcal{Y}$, the channal capacity under a peak-delay constraint T and an average-delay constraint $\alpha\mathsf{T}$ is upper bound by*

$$\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \leq \mathsf{E}_{Q^*}\left[D\big(f_{Y|X}(\cdot|X)||f_Y(\cdot)\big)\right], \tag{2.30}$$

where Q^* is the capacity-achieving distribution satisfying $Q^*(X > \mathsf{T}) = 0$ and $\mathsf{E}_{Q^*}[X] \leq \alpha\mathsf{T}$. Here, $D(\cdot||\cdot)$ denotes relative entropy [8, ch. 2].

Proof: For more details see [9] □

There are two challenges in using (2.30). The first is in finding a clever choice of the law R that will lead to a good upper bound. The second is in upper-bounding the supremum on the right-hand side of (2.30). To handle this challenge we shall resort to some further bounding.

Next, we will list some propositions related to the $\mathcal{Q}$-function.

Definition 2.11. *The $\mathcal{Q}$-function is defined by*

$$\mathcal{Q}(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{t^2/2}\, \mathrm{d}t. \tag{2.31}$$

Note that $\mathcal{Q}(x)$ is the probability that a standard Gaussian random variable will exceed the value x and is therefore monotonically decreasing with an increasing argument.

Proposition 2.12 (Properties for the $\mathcal{Q}$-function)**.** *Bounds for $\mathcal{Q}$-function*

$$\frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}\left(\frac{x}{1+x^2}\right) < \mathcal{Q}(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}, \qquad x > 0; \tag{2.32}$$

$$\mathcal{Q}(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}, \qquad x \geq 0. \tag{2.33}$$

and

$$\mathcal{Q}(x) + \mathcal{Q}(-x) = 1. \tag{2.34}$$

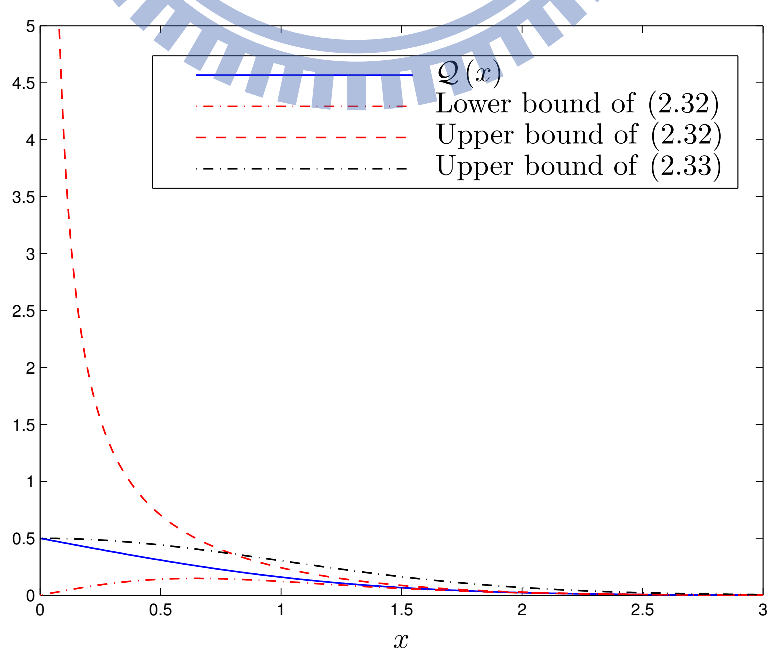
Proof: see [10] □

Remark 2.13. *Let $\Phi(\cdot)$ denote the cumulative distribution function (CDF) of the standard normal distribution:*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{dt}. \tag{2.35}$$

Then

$$\mathcal{Q}(x) = \Phi(-x), \tag{2.36}$$



Proposition 2.14 (Upper and Lower Bound for Exponential Integral Function)**.** *We have*

$$\frac{1}{2}e^{-x}\ln\left(1+\frac{2}{x}\right) < -\operatorname{Ei}(-x) < e^{-x}\ln\left(1+\frac{1}{x}\right), \quad x>0, \tag{2.37}$$

or

$$-e^{-x}\ln\left(1+\frac{1}{x}\right) < \operatorname{Ei}(-x) < -\frac{1}{2}e^{-x}\ln\left(1+\frac{2}{x}\right), \quad x>0. \tag{2.38}$$

Chapter 3

Known Bounds to the Capacity of the AIGN Channel with Only an Average Delay Constraint

We can always bound the capacity with peak and delay constraints by the capacity with only an average delay constraint since adding more constraints will make the capacity smaller. For the case with only an average delay constraint, an upper bound of capacity has been derived in [1]. The entropy maximizing distribution $f^*(y)$ with a mean constraint $\mathsf{E}[Y] \leq m + \mu$ is the exponential distribution with parameter $\frac{1}{m+\mu}$ [8, (12.21)]:

$$f^*(y) = \frac{1}{m+\mu} e^{-\frac{y}{m+\mu}}, \quad y \geq 0. \tag{3.1}$$

The entropy of such a distribution is

$$h^*(Y) = 1 + \ln(m+\mu). \tag{3.2}$$

This can be used to derive a upper bound on the capacity of the AIGN channel:

$$\begin{aligned} \mathsf{C}(m) &\triangleq \sup_{f_X(x)\colon \mathsf{E}[X]\leq m} I(X;Y) && (3.3)\\ &= \sup_{f_X(x)\colon \mathsf{E}[X]\leq m} \left\{ h(Y) - h_{\mathrm{IG}(\mu,\lambda)} \right\} && (3.4)\\ &= \sup_{f_X(x)\colon \mathsf{E}[X]\leq m} h(Y) - h_{\mathrm{IG}(\mu,\lambda)} && (3.5)\\ &= 1 + \ln(m+\mu) - h_{\mathrm{IG}(\mu,\lambda)}. && (3.6) \end{aligned}$$

Hence,

$$\mathsf{C}(m) \leq \frac{1}{2}\log\frac{\lambda(m+\mu)^2}{2\pi\mu^3} - \frac{3}{2}\exp\left(\frac{2\lambda}{\mu}\right)\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right) + \frac{1}{2} \tag{3.7}$$

In [3], the choice of the input $X \sim \text{Exp}\left(\frac{1}{m}\right)$ get the asympototically tight lower bound:

$$\begin{aligned}
\mathsf{C}(m) \geq \log\frac{m}{\lambda} + \frac{\mu}{m} - \frac{\lambda}{\mu} + k\lambda + \frac{3}{2}\log\frac{\lambda}{\mu} + \frac{1}{2}\log\frac{e}{2\pi} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\text{Ei}\left(-\frac{2\lambda}{\mu}\right) \\
- \log\left(1 + \frac{1}{m}e^{\frac{\lambda}{\mu}}\sqrt{\frac{\lambda m}{2+k^2\lambda m}}\mathsf{K}_1\left(\sqrt{\frac{2\lambda}{m} + k^2\lambda^2}\right)\right. \\
\left. + \frac{1}{2m}e^{\frac{\lambda}{\mu}+k\lambda}\sqrt{\frac{\lambda m}{1+k^2\lambda m}}\mathsf{K}_1\left(2\sqrt{\frac{\lambda}{m} + k^2\lambda^2}\right)\right).
\end{aligned} \tag{3.8}$$

where K_1 is type one bessel function.

We can see in Fig. 3.3 that for the capacity with only a average delay constraint is aympototically tight in both m and v.

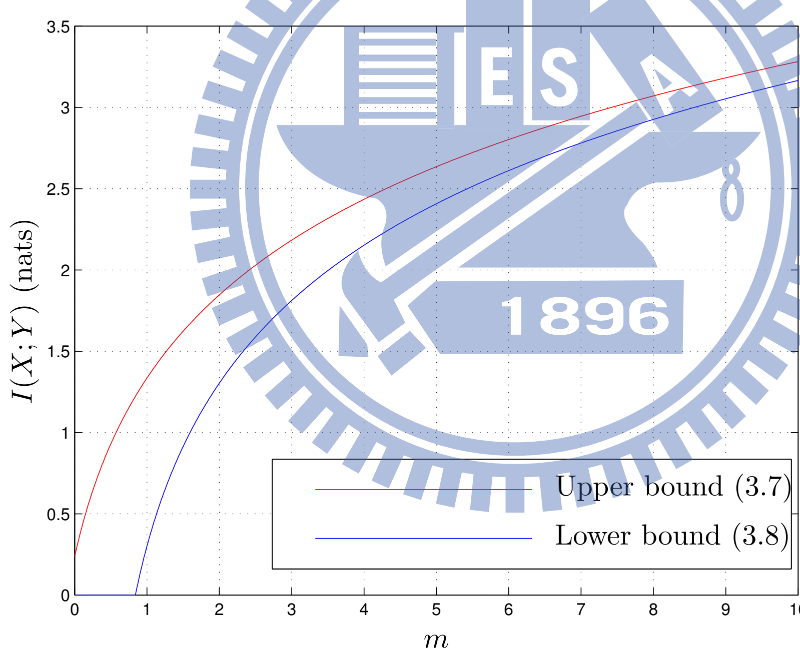


Figure 3.3: Known bounds for the choice: $\mu = 0.5$, $\lambda = 1$ respect to m.

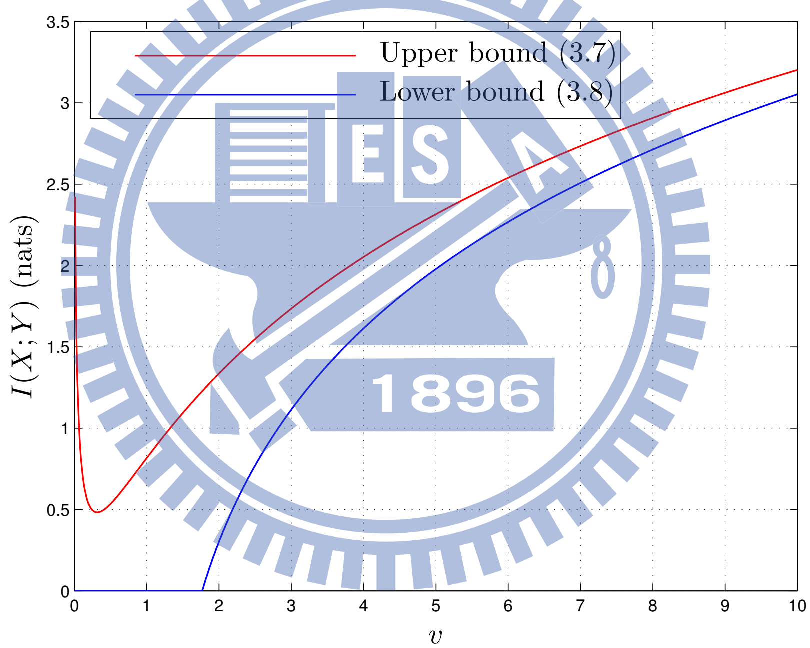


Figure 3.4: Known bounds for the choice: $m = 1$, $\lambda = 1$ respect to v.

Chapter 4

Main Result

To state our results we distinguish between two cases, $0 < \alpha < 0.5$ and $0.5 \leq \alpha < 1$. For the first case, the difference between upper-bound and lower-bound tends to zero as the allowed peak delay tends to infinity, thus, revealing the asymptotic capacity at high power.

Theorem 4.1 (Bounds for $0 < \alpha < 0.5$)**.** *For* $0 < \alpha < 0.5$, $\mathsf{C}(\mathsf{T}, \alpha\mathsf{T})$ *is lower-bounded*

$$\begin{aligned} \mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \geq{}& \log \mathsf{T} - \log \beta^* + \log\left(1 - e^{-\beta^*}\right) - \frac{\lambda}{\mu} + \frac{\beta^*}{\mathsf{T}}(\alpha\mathsf{T} + \mu) + \sqrt{2\lambda\left(\frac{\lambda}{2\mu^2} - \frac{\beta^*}{\mathsf{T}}\right)} \\ & - \frac{1}{2}\log\frac{2\pi e \mu^3}{\lambda} - \frac{3}{2} e^{\frac{2\lambda}{\mu}} \operatorname{Ei}\left(-\frac{2\lambda}{\mu}\right) \end{aligned} \tag{4.1}$$

for

$$\mathsf{T} \geq \frac{2\beta^*\mu^2}{\lambda}. \tag{4.2}$$

And upper-bounded by

$$\begin{aligned} \mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \leq{}& Q^- - Q^+ - 2\left(Q^- - Q^+\right)\log\left(Q^- - Q^+\right) \\ & - 2(1 - Q^- + Q^+)\log(1 - Q^- + Q^+) \\ & + \max\left\{0, (Q^- - Q^+)\log\left(\mu(Q^- + Q^+) + \alpha\mathsf{T}(Q^- - Q^+)\right)\right\} + \alpha\beta^* \\ & - \log\alpha\beta^* + \log(\alpha\mathsf{T} + \mu) + \log\left(1 - e^{-\frac{\alpha\beta^*(T+\delta)}{\alpha\mathsf{T} + \mu - \alpha\mathsf{T}(Q^- - Q^+) - \mu(Q^- + Q^+)}}\right) \\ & - \frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\operatorname{Ei}\left(-\frac{2\lambda}{\mu}\right) \end{aligned} \tag{4.3}$$

for

$$\mathsf{T} > \frac{1}{\alpha}\left(\frac{\alpha\beta^* e^{-\alpha\beta^*}}{1 - e^{-\beta^*\min\left(1, \frac{\alpha}{\mu}\delta_0\right)}} - \mu\right) \tag{4.4}$$

where δ_0 is the minimum value of δ, and

$$Q^- \triangleq \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{\delta}{\mu}}-\sqrt{\frac{\mu}{\delta}}\right)\right) \tag{4.5}$$

$$Q^+ \triangleq e^{2\frac{\lambda}{\mu}}\mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{\delta}{\mu}}+\sqrt{\frac{\mu}{\delta}}\right)\right) \tag{4.6}$$

Here, $\delta > \mu$ is free parameter, and β^* is the unique solution to

$$\alpha = \frac{1}{\beta^*} - \frac{e^{-\beta^*}}{1-e^{-\beta^*}} \tag{4.7}$$

A suboptimal but useful choice of the free parameter in (4.3) *is*

$$\delta = \log(\mathsf{T}+1) + 10\mu. \tag{4.8}$$

Fig. 4.5 and Fig. 4.6 show the bounds of Theorem 4.1 for $\alpha = 0.3$ as a function of T and v, respectly.

Corollary 4.2 (Asymptotics for the case of $0 < \alpha < 0.5$)**.** *If $0 < \alpha < 0.5$, then*

$$\lim_{\mathsf{T}\uparrow\infty}\left\{\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) - \log\mathsf{T}\right\} = \alpha\beta^* - \log\beta^* + \log\left(1-e^{-\beta^*}\right) - \frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right) \tag{4.9}$$

and

$$\lim_{v\uparrow\infty}\left\{\mathsf{C}(v) - \frac{3}{2}\log v\right\} = \log\mathsf{T} - \log\beta^* + \log\left(1-e^{-\beta^*}\right) + \alpha\beta^* + \frac{1}{2}\log\frac{\lambda}{2\pi e}. \tag{4.10}$$

Theorem 4.3 (Bounds for $0.5 \le \alpha \le 1$)**.** *If $0.5 \le \alpha \le 1$, then $\mathsf{C}(\mathsf{T},\alpha\mathsf{T})$ is lower-bounded by*

$$\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) \ge \log\mathsf{T} - \frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right) \tag{4.11}$$

and upper-bounded by

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) \le\ & Q^- - Q^+ - 2\left(Q^- - Q^+\right)\log\left(Q^- - Q^+\right) \\ & - (1-Q^- + Q^+)\log(1-Q^- + Q^+) \\ & + (Q^- - Q^+)\max\left\{0, \log\left(\mu(Q^- + Q^+) + \alpha\mathsf{T}(Q^- - Q^+)\right)\right\} + \log(\mathsf{T}+\delta) \\ & - \frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right).\end{aligned} \tag{4.12}$$

for

$$\mathsf{T} > 1 - \min\delta \tag{4.13}$$

where Q^- and Q^+ are given in the (4.5) *and* (4.6)*, respectively.*
Here, $\delta > \mu$ is free parameter, and β^* is the unique solution to (4.7)*. A suboptimal but useful choice of the free parameter in* (4.12)

$$\delta = \log(\mathsf{T}+1) + 10\mu \tag{4.14}$$

Corollary 4.4 (Asymptotics for the case of $0.5 \leq \alpha \leq 1$)**.** *If* $0.5 \leq \alpha \leq 1$*, then*

$$\lim_{\mathsf{T}\uparrow\infty} \left\{ \mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) - \log \mathsf{T} \right\} = -\frac{1}{2} \log \frac{2\pi e \mu^3}{\lambda} - \frac{3}{2} e^{\frac{2\lambda}{\mu}} \operatorname{Ei}\left(-\frac{2\lambda}{\mu}\right) \tag{4.15}$$

and

$$\lim_{v\uparrow\infty} \left\{ \mathsf{C}(v) - \frac{3}{2} \log v \right\} = \log \mathsf{T} + \frac{1}{2} \log \frac{\lambda}{2\pi e} \tag{4.16}$$

Fig. 4.7 and Fig. 4.8 show the bounds of Theorem (4.3) for $\alpha = 0.7$. In addition, Fig. 4.11 and Fig. 4.12 show how the bounds in (4.3) and (4.12) performance with different α. Fig. 4.9 and Fig. 4.10 show that the difference of the choice of different δ, the blue line is the numerical optimal δ. Furthermore, the Fig. 4.13 shows the continuity of the capacity.

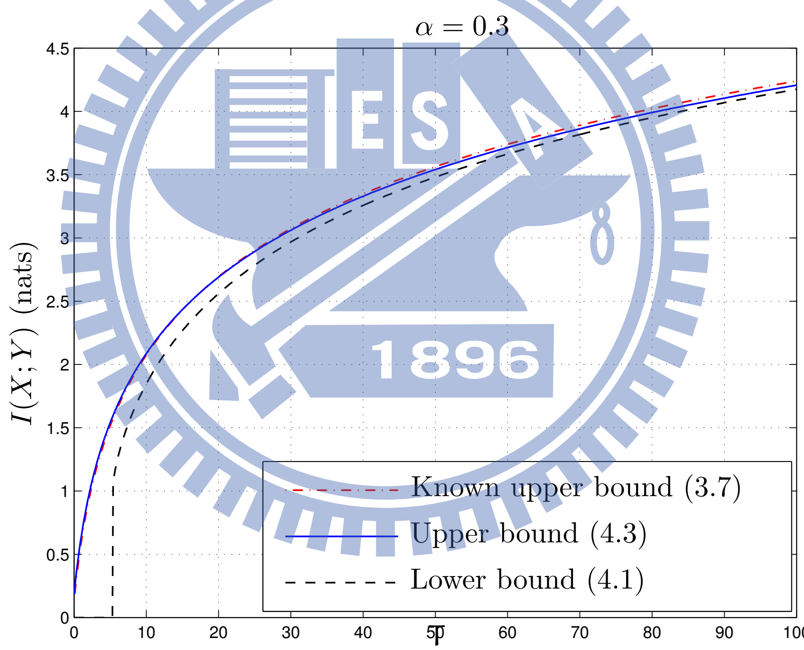


Figure 4.5: Comparison between the upper and lower bound in (4.3) and (4.1) and known upper bound (3.7) for the choice: $\mu = 0.5$, $\lambda = 0.25$, and $\alpha = 0.3$.

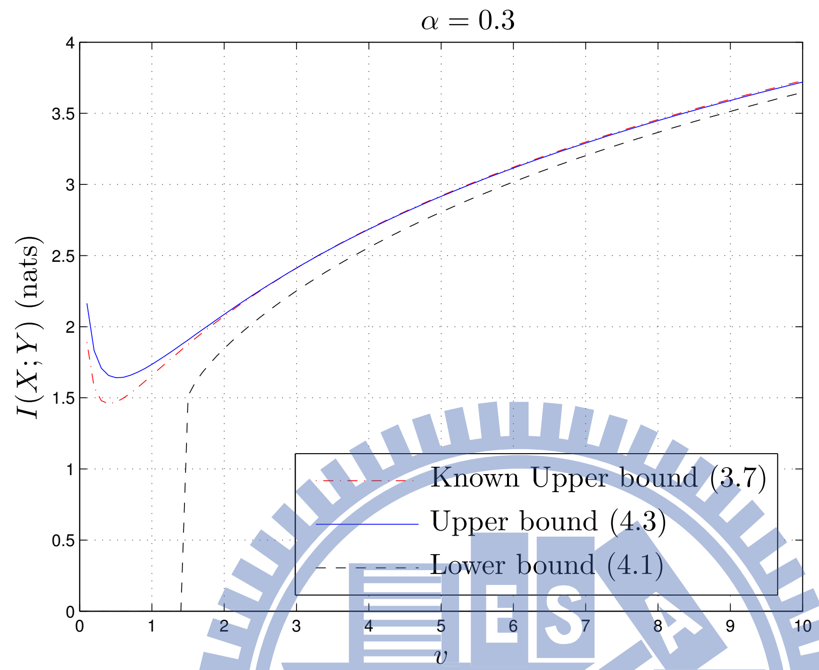


Figure 4.6: Upper bound in (4.3) and (4.1) compare to the (3.7) for the choice: $\mathsf{T} = 10$ $\lambda = 0.25$, $\delta = \log(\mathsf{T}+1) + 10\mu$ and $\alpha = 0.3$.

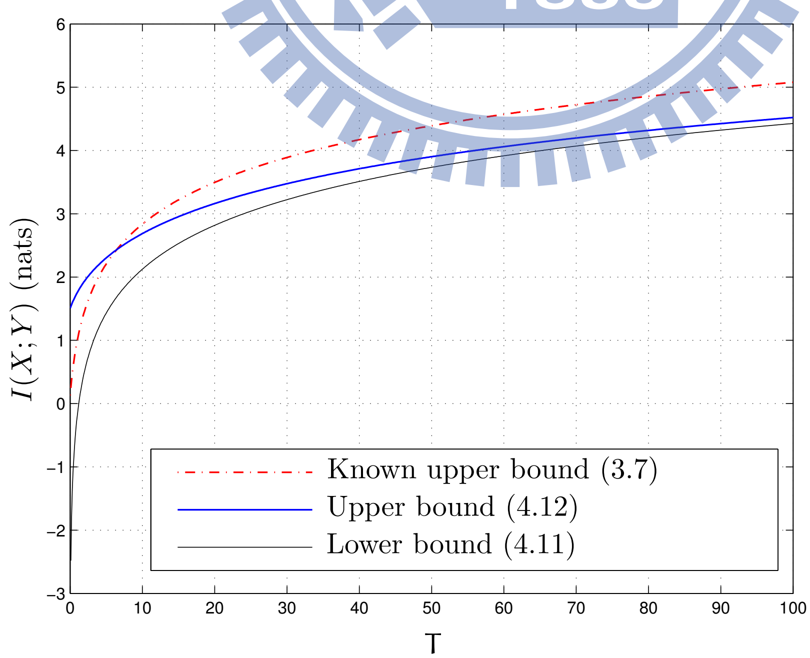


Figure 4.7: Bounds in (4.12), (4.11) and (3.7) for the choice: $\mu = 0.5$, $\lambda = 0.25$, and $\alpha = 0.7$.

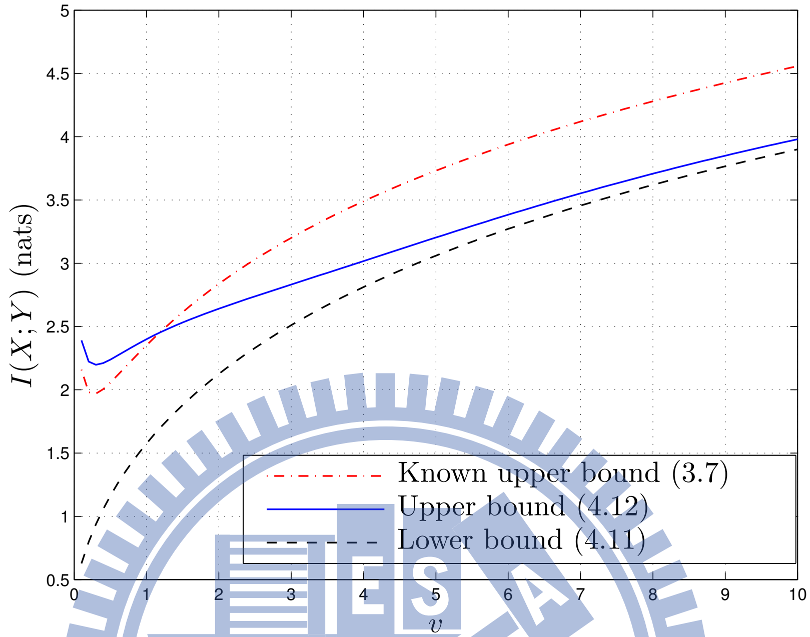


Figure 4.8: Bounds in (4.12), (4.11) and (3.7) for the choice: $\mathsf{T} = 10$, $\lambda = 0.25$, $\delta = \frac{2}{v+1}$ and $\alpha = 0.7$.

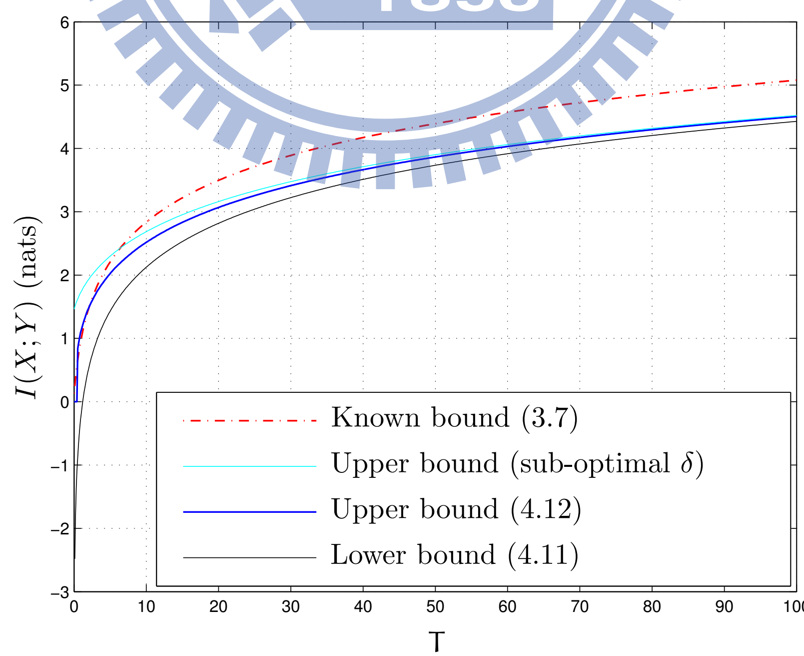


Figure 4.9: Bounds in (4.12), (4.11) with and (3.7) optimal and sub-optimal $\delta = \log(\mathsf{T}+1) + 10\mu$ for the choice: $\mu = 0.5$, $\lambda = 0.25$ and $\alpha = 0.7$.

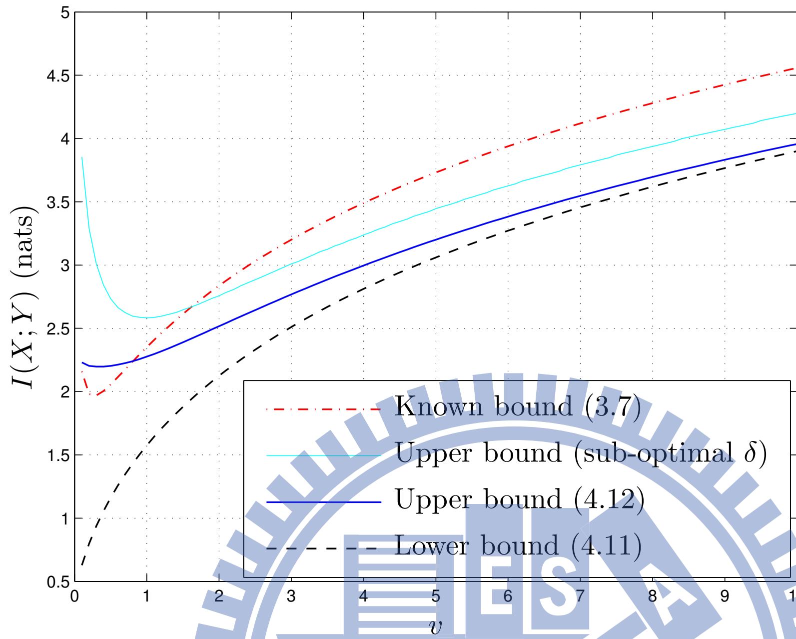


Figure 4.10: Bounds in (4.12), (4.11) with and (3.7) optimal and sub-optimal $\delta = \frac{2}{v+1}$ for the choice: $\mathsf{T} = 10$, $\lambda = 0.25$ and $\alpha = 0.7$.

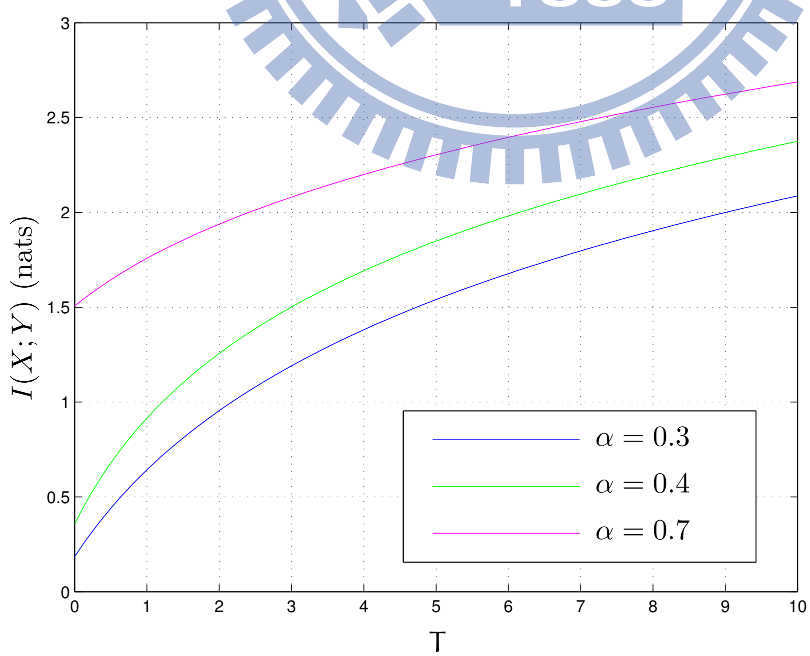


Figure 4.11: Upper bounds, (4.3) and (4.12), with different α for the choice: $\mu = 0.5$ and $\lambda = 0.25$.

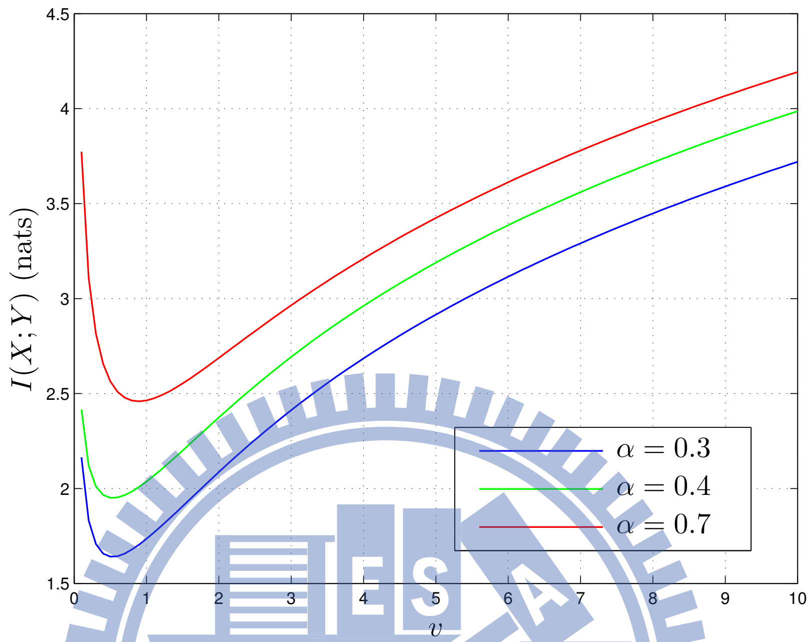


Figure 4.12: Upper bounds, (4.3) and (4.12), with different α for the choice: $\mathsf{T} = 10$ and $\lambda = 0.25$.

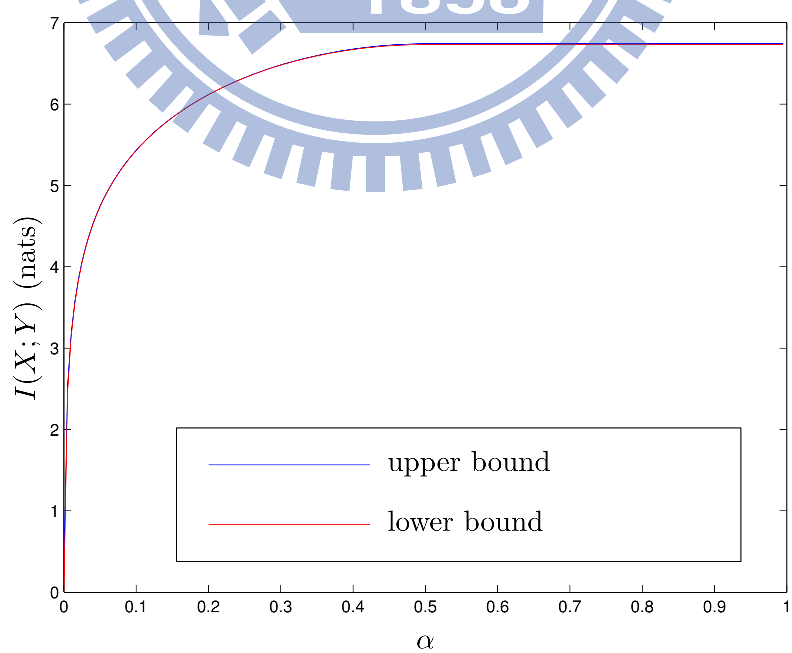


Figure 4.13: Upper bounds and lower bounds with different α for the choice: $\mathsf{T} = 1000$, $\mu = 0.5$, and $\lambda = 0.25$.

Chapter 5

Derivations

5.1 Proof of Upper-Bound of Capacity

5.1.1 For $0 < \alpha < 0.5$

The derivation of the upper bounds (4.3) is based on Proposition (2.10) with the following choice of output distribution:

$$f_Y(y) = \begin{cases} \frac{\beta(1-p)}{\mathsf{T}\left(1-e^{-\beta\left(1+\frac{\delta}{\mathsf{T}}\right)}\right)} e^{\frac{-\beta y}{\mathsf{T}}} & \text{if } 0 \le y \le \mathsf{T}+\delta, \\ \nu p e^{-\nu(y-\mathsf{T}-\delta)} & \text{if } y > \mathsf{T}+\delta, \end{cases} \tag{5.1}$$

where β, p, ν and δ are free parameters with following constraints:

$$\beta > 0, \tag{5.2}$$

$$\delta \ge 0, \tag{5.3}$$

$$\nu > 0, \tag{5.4}$$

$$0 \le p \le 1. \tag{5.5}$$

The capacity is upper bounded as follow:

$$\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \le \mathsf{E}_{Q^*}\left[D\big(f_{Y|X}(\cdot|X)||f_Y(\cdot)\big)\right] \tag{5.6}$$

$$\le -h(N) - \mathsf{E}_{Q^*}\left[\int_0^{\mathsf{T}+\delta} f_{Y|X}(y|x)\log\left(\frac{\beta(1-p)}{\mathsf{T}\left(1-e^{-\beta\left(1+\frac{\delta}{\mathsf{T}}\right)}\right)}e^{-\frac{\beta}{\mathsf{T}}y}\right)\mathrm{d}y\right]$$

$$- \mathsf{E}_{Q^*}\left[\int_{\mathsf{T}+\delta}^{\infty} f_{Y|X}(y|x)\log\left(p\nu e^{-\nu(y-\mathsf{T}-\delta)}\right)\mathrm{d}y\right]. \tag{5.7}$$

Since

$$f_{Y|X}(y|x) = \sqrt{\frac{\lambda}{2\pi(y-x)^3}}\exp\left(-\frac{\lambda(y-x-\mu)^2}{2\mu^2(y-x)}\right) I\{y \ge x\}, \tag{5.8}$$

plugging $f_{Y|X}$ into (5.7) then we can get

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) \leq & -h(N) - \log p - \log\nu - \nu(\mathsf{T}(1-\alpha)+\delta+\mu) \\ & + E_{Q^*}[c_1(X)] \\ & \quad \cdot \left(\log\frac{1-e^{-\beta(1+\frac{\delta}{\mathsf{T}})}}{\beta} + \log\mathsf{T} + \log\frac{p\nu}{1-p} + \nu(\mathsf{T}+\delta)\right) \\ & + \left(\frac{\beta}{\mathsf{T}}-\nu\right)\mathsf{E}_{Q^*}[Xc_1(X)+c_2(X)], \end{aligned} \tag{5.9}$$

where

$$c_1(x) = \int_0^{T+\delta-x}\sqrt{\frac{\lambda}{2\pi t^3}}e^{\frac{-\lambda(t-\mu)^2}{2\mu^2 t}}\,\mathrm{d}t, \tag{5.10}$$

$$c_2(x) = \int_0^{T+\delta-x}\sqrt{\frac{\lambda}{2\pi t}}e^{\frac{-\lambda(t-\mu)^2}{2\mu^2 t}}\,\mathrm{d}t. \tag{5.11}$$

To further bound $c_1(x)$ and $c_2(x)$, we let $w=\frac{t}{u}$, $\tau(x)=\frac{\mu}{\mathsf{T}+\delta-x}$ and $\rho=\frac{\lambda}{\mu}$

$$c_1(x) = \int_0^{T+\delta-x}\sqrt{\frac{\lambda}{2\pi t^3}}e^{\frac{-\lambda(t-\mu)^2}{2\mu^2 t}}\,\mathrm{d}t \tag{5.12}$$

$$= \int_0^{\frac{1}{\tau(x)}}\sqrt{\frac{\rho}{2\pi w^3}}e^{-\rho\frac{(w-1)^2}{2w}}\,\mathrm{d}w \tag{5.13}$$

$$\begin{aligned} &= 1-\mathcal{Q}\left(\sqrt{\rho}\left(\sqrt{\frac{1}{\tau(x)}}-\sqrt{\tau(x)}\right)\right) \\ &\quad + e^{2\rho}\mathcal{Q}\left(\sqrt{\rho}\left(\sqrt{\frac{1}{\tau(x)}}+\sqrt{\tau(x)}\right)\right) \end{aligned} \tag{5.14}$$

$$c_2(x) = \int_0^{T+\delta-x}\sqrt{\frac{\lambda}{2\pi t}}e^{\frac{-\lambda(t-\mu)^2}{2\mu^2 t}}\,\mathrm{d}t \tag{5.15}$$

$$= \int_0^{\frac{1}{\tau(x)}}\mu w\sqrt{\frac{\rho}{2\pi w^3}}e^{-\rho\frac{(w-1)^2}{2w}}\,\mathrm{d}w \tag{5.16}$$

$$\begin{aligned} &= \mu\left[1-\mathcal{Q}\left(\sqrt{\rho}\left(\sqrt{\frac{1}{\tau(x)}}-\sqrt{\tau(x)}\right)\right)\right. \\ &\quad \left. -\mu e^{2\rho}\mathcal{Q}\left(\sqrt{\rho}\left(\sqrt{\frac{1}{\tau(x)}}+\sqrt{\tau(x)}\right)\right)\right]. \end{aligned} \tag{5.17}$$

Since $0\leq x\leq\mathsf{T}$, so $\tau(x)$ must satisfy

$$0\leq\tau(x)\leq\frac{\mu}{\delta}. \tag{5.18}$$

Noted that because of (5.18), $c_1(x)$ and $c_2(x)$ are both monotonically decreasing in $\tau(x)$; furthermore, $c_1(x)$ and $c_2(x)$ goes to zero as $\tau(x)$ goes to zero. Hence, we can the bound of $c_1(x)$ and $c_2(x)$ by using (5.18)

$$0 \leq 1 - \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{\delta}{\mu}} - \sqrt{\frac{\mu}{\delta}}\right)\right) + e^{2\frac{\lambda}{\mu}}\mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{\delta}{\mu}} + \sqrt{\frac{\mu}{\delta}}\right)\right) \\ \leq c_1(x) \leq 1 \tag{5.19}$$

$$0 \leq \mu\left[1 - \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{\delta}{\mu}} - \sqrt{\frac{\mu}{\delta}}\right)\right)\right] - \mu e^{2\frac{\lambda}{\mu}}\mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{\delta}{\mu}} + \sqrt{\frac{\mu}{\delta}}\right)\right) \\ \leq c_2(x) \leq \mu. \tag{5.20}$$

To simplify the notations, we use the shorthand given in (4.5) and (4.6):

$$0 \leq 1 - Q^- + Q^+ \leq c_1(x) \leq 1 \tag{5.21}$$

$$0 \leq \mu(1 - Q^- - Q^+) \leq c_2(x) \leq \mu \tag{5.22}$$

Next, we choose for the free parameters p and ν:

$$p \triangleq 1 - \mathsf{E}_{Q^*}[c_1(X)] \tag{5.23}$$

$$\nu \triangleq \frac{1 - \mathsf{E}_{Q^*}[c_1(X)]}{\mu + \alpha\mathsf{T} - \mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)]} \tag{5.24}$$

From (5.21), we know that

$$0 \leq 1 - \mathsf{E}_{Q^*}[c_1(X)] \leq 1 \tag{5.25}$$

and

$$\alpha\mathsf{T} + \mu - \mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)] \geq 0, \tag{5.26}$$

so that $0 \leq p \leq 1$ and $\nu \geq 0$ as required. Plugging p and ν into the (5.9) then we get the following bound:

$$\begin{aligned} \mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \leq &-h(N) + (1 - \mathsf{E}_{Q^*}[c_1(X)])\left(1 - \frac{(\mathsf{T} + \delta)(1 - \mathsf{E}_{Q^*}[c_1(X)])}{\mu + \alpha\mathsf{T} - \mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)]}\right) \\ &- 2\,(1 - \mathsf{E}_{Q^*}[c_1(X)])\log\left(1 - \mathsf{E}_{Q^*}[c_1(X)]\right) \\ &+ (1 - \mathsf{E}_{Q^*}[c_1(X)])\log\left(\mu + \alpha\mathsf{T} - \mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)]\right) \\ &+ \mathsf{E}_{Q^*}[c_1(X)]\left(\log\mathsf{T} - \log\beta + \log\left(1 - e^{-\beta\left(1 + \frac{\delta}{\mathsf{T}}\right)}\right)\right) \\ &+ \frac{\beta}{\mathsf{T}}\mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)] - \mathsf{E}_{Q^*}[c_1(X)]\log\mathsf{E}_{Q^*}[c_1(X)] \end{aligned} \tag{5.27}$$

then we choose β as:

$$\beta = \frac{\mathsf{T}\mathsf{E}_{Q^*}[c_1(X)]}{\mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)]}\alpha\beta^* \tag{5.28}$$

where β^* is non-zero solution to $\alpha = \frac{1}{\beta^*} - \frac{e^{-\beta^*}}{1 - e^{-\beta^*}}$ for $0 < \alpha < 0.5$. Furthermore,

$$-\log\beta = -\log(\alpha\mathsf{T}) - \log\beta^* - \log\mathsf{E}_{Q^*}[c_1(X)] + \log\left(\mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)]\right) \tag{5.29}$$

$$\leq -\log(\alpha\mathsf{T}) - \log\beta^* - \log\mathsf{E}_{Q^*}[c_1(X)] + \log\left(\alpha\mathsf{T} + \mu\right), \tag{5.30}$$

and

$$\log\left(1-e^{-\beta\left(1+\frac{\delta}{\mathsf{T}}\right)}\right)=\log\left(1-e^{-(T+\delta)\frac{\mathsf{E}_{Q^*}[c_1(X)]}{\mathsf{E}_{Q^*}[Xc_1(X)+c_2(X)]}\alpha\beta^*}\right) \tag{5.31}$$

$$\leq\log\left(1-e^{-\frac{\alpha\beta^*(\mathsf{T}+\delta)}{\alpha\mathsf{T}+\mu-\alpha\mathsf{T}(Q^--Q^+)-\mu(Q^-+Q^+)}}\right) \tag{5.32}$$

Hence, for $0<\alpha<0.5$

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T})\leq&-h(N)+(1-\mathsf{E}_{Q^*}[c_1(X)])\left(1-\frac{(\mathsf{T}+\delta)(1-\mathsf{E}_{Q^*}[c_1(X)])}{\alpha\mathsf{T}(Q^--Q^+)+\mu(Q^-+Q^+)}\right)\\&-2\left(1-\mathsf{E}_{Q^*}[c_1(X)]\right)\log\left(1-\mathsf{E}_{Q^*}[c_1(X)]\right)\\&+(1-\mathsf{E}_{Q^*}[c_1(X)])\log\left(\mu(Q^-+Q^+)+\alpha\mathsf{T}(Q^--Q^+)\right)\\&+\mathsf{E}_{Q^*}[c_1(X)]\Bigg(\alpha\beta^*-\log\alpha-\log\beta^*+\log(\alpha\mathsf{T}+\mu)\\&\quad+\log\left(1-e^{-\frac{\alpha\beta^*(\mathsf{T}+\delta)}{\alpha\mathsf{T}+\mu-\alpha\mathsf{T}(Q^--Q^+)-\mu(Q^-+Q^+)}}\right)\Bigg)\\&-2\mathsf{E}_{Q^*}[c_1(X)]\log\mathsf{E}_{Q^*}[c_1(X)]\end{aligned} \tag{5.33}$$

It can be shown that for $t>\frac{1}{\sqrt{e}+1}$

$$z(t)=1-t-2(1-t)\log(1-t)-2t\log t \tag{5.34}$$

is monotonically decreasing. In our proof, we choose $t=\mathsf{E}_{Q^*}[c_1(X)]$, as long as $1-Q^-+Q^+\geq\frac{1}{\sqrt{e}+1}$, $\mathsf{E}_{Q^*}[c_1(X)]$ always larger than $\frac{1}{\sqrt{e}+1}$. In addition, for $\delta\geq\mu$ this requirement always be satisfied.
In addition,

$$\frac{(\mathsf{T}+\delta)(1-\mathsf{E}_{Q^*}[c_1(X)])}{\alpha\mathsf{T}(Q^--Q^+)+\mu(Q^-+Q^+)}>0. \tag{5.35}$$

Hence, we can get the upper bound for $0<\alpha<0.5$:

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T})\leq&-h(N)+Q^--Q^+-2\left(Q^--Q^+\right)\log\left(Q^--Q^+\right)\\&-2(1-Q^-+Q^+)\log(1-Q^-+Q^+)\\&+(Q^--Q^+)\max\left\{0,\log\left(\mu(Q^-+Q^+)+\alpha\mathsf{T}(Q^--Q^+)\right)\right\}\\&+\mathsf{E}_{Q^*}[c_1(X)]\Bigg(\alpha\beta^*-\log\alpha\beta^*+\log(\alpha\mathsf{T}+\mu)\\&\quad+\log\left(1-e^{-\frac{\alpha\beta^*(T+\delta)}{\alpha\mathsf{T}+\mu-\alpha\mathsf{T}(Q^--Q^+)-\mu(Q^-+Q^+)}}\right)\Bigg)\end{aligned} \tag{5.36}$$

To further bound the capacity, we have to let

$$\mathsf{T}>\frac{1}{\alpha}\left(\frac{\alpha\beta^*e^{-\alpha\beta^*}}{1-e^{-\beta^*\min\left(1,\frac{\alpha}{\mu}\delta_0\right)}}-\mu\right), \tag{5.37}$$

where δ_0 is the minimun value of δ, then it can be shown that

$$\alpha\beta^* - \log\alpha\beta^* + \log(\alpha\mathsf{T}+\mu) + \log\left(1 - e^{-\frac{\alpha\beta^*(T+\delta)}{\alpha\mathsf{T}+\mu-\alpha\mathsf{T}(Q^- - Q^+)-\mu(Q^-+Q^+)}}\right) > 0. \quad (5.38)$$

Hence, in (5.36), the $\mathsf{E}_{Q^*}[c_1(X)]$ need to be upper-bounded, finally we get the upper bound of capacity when $0 < \alpha < 0.5$:

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) \le\ & Q^- - Q^+ - 2\left(Q^- - Q^+\right)\log\left(Q^- - Q^+\right)\\ & - 2(1 - Q^- + Q^+)\log(1 - Q^- + Q^+)\\ & + (Q^- - Q^+)\max\left\{0, \log\left(\mu(Q^- + Q^+) + \alpha\mathsf{T}(Q^- - Q^+)\right)\right\} + \alpha\beta^*\\ & - \log\alpha\beta^* + \log(\alpha\mathsf{T}+\mu) + \log\left(1 - e^{-\frac{\alpha\beta^*(T+\delta)}{\alpha\mathsf{T}+\mu-\alpha\mathsf{T}(Q^- - Q^+)-\mu(Q^-+Q^+)}}\right)\\ & - \frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right). \end{aligned} \quad (5.39)$$

5.1.2 For $0.5 \le \alpha \le 1$

For proving (4.12), we use different output distribution:

$$f_Y(y) = \begin{cases} \frac{1-p}{\mathsf{T}+\delta} & \text{if } 0 \le y \le \mathsf{T}+\delta,\\ \nu p e^{-\nu(y-\mathsf{T}-\delta)} & \text{if } y > \mathsf{T}+\delta, \end{cases} \quad (5.40)$$

where p, ν and δ are free parameters with following constraints:

$$\delta \ge 0, \quad (5.41)$$
$$\nu > 0, \quad (5.42)$$
$$0 \le p \le 1. \quad (5.43)$$

From similar derivation in section 5.1.1, choosing

$$p \triangleq 1 - \mathsf{E}_{Q^*}[c_1(X)], \quad (5.44)$$
$$\nu \triangleq \frac{1 - \mathsf{E}_{Q^*}[c_1(X)]}{\mu + \alpha\mathsf{T} - \mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)]}. \quad (5.45)$$

We get:

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) \le\ & -h(N) + (1 - \mathsf{E}_{Q^*}[c_1(X)])\left(1 - \frac{(\mathsf{T}+\delta)(1 - \mathsf{E}_{Q^*}[c_1(X)])}{\mu + \alpha\mathsf{T} - \mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)]}\right)\\ & - 2\left(1 - \mathsf{E}_{Q^*}[c_1(X)]\right)\log\left(1 - \mathsf{E}_{Q^*}[c_1(X)]\right)\\ & + (1 - \mathsf{E}_{Q^*}[c_1(X)])\log\left(\mu + \alpha\mathsf{T} - \mathsf{E}_{Q^*}[Xc_1(X) + c_2(X)]\right)\\ & - \mathsf{E}_{Q^*}[c_1(X)]\log\mathsf{E}_{Q^*}[c_1(X)] + \mathsf{E}_{Q^*}[c_1(X)]\log(\mathsf{T}+\delta) \end{aligned} \quad (5.46)$$

It can be shown that for $t > \frac{1}{4}(\sqrt{5}-1)^2$

$$z(t) = 1 - t - 2(1-t)\log(1-t) - t\log t \quad (5.47)$$

is monotonically decreasing. Same as before, we choose $t = \mathsf{E}_{Q^*}[c_1(X)]$, as long as $1 - Q^- + Q^+ \geq \frac{1}{4}(\sqrt{5}-1)^2$, the $\mathsf{E}_{Q^*}[c_1(X)]$ always larger than $\frac{1}{4}(\sqrt{5}-1)^2$. In addition, for $\delta \geq \mu$ this requirement always be satisfied.
Hence, we get the upper bound for $0.5 \leq \alpha \leq 1$:

$$\begin{aligned}\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \leq\ & Q^- - Q^+ - 2\left(Q^- - Q^+\right)\log\left(Q^- - Q^+\right) \\ & - (1 - Q^- + Q^+)\log(1 - Q^- + Q^+) \\ & + (Q^- - Q^+)\max\left\{0, \log\left(\mu(Q^- + Q^+) + \alpha\mathsf{T}(Q^- - Q^+)\right)\right\} + \log(\mathsf{T} + \delta) \\ & - \frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right).\end{aligned} \tag{5.48}$$

where

$$\mathsf{T} \geq 1 - \min\delta \tag{5.49}$$

$$1 - Q^- + Q^+ \geq \frac{1}{4}(\sqrt{5}-1)^2 \tag{5.50}$$

5.2 Proof of Lower-Bound of Capacity

5.2.1 For $0 < \alpha < 0.5$

One can always find a lower bound on capacity by dropping the maximization and choosing an arbitrary input distribution $f_X(x)$ in (1.13). To get a tight bound, this choice of $f_X(x)$ should yield a mutual information that is reasonably close to capacity. From [3], we know the lower-bound is tight when the input is an exponential distribution; however, since we have a peak constraint, we only can choose the cut exponential as our input distribution, $f_X(x)$, and $f_N(n)$ denotes the channel distribution,

$$f_X(x) = \frac{\beta}{\mathsf{T}(1 - e^{-\beta})}e^{\frac{-\beta}{\mathsf{T}}x}I\{0 \leq x \leq \mathsf{T}\}, \tag{5.51}$$

$$f_N(n) = \sqrt{\frac{\lambda}{2\pi n^3}}\exp\left(\frac{-\lambda(n-\mu)^2}{2\mu^2 n}\right)I\{n \geq 0\} \tag{5.52}$$

where $\beta > 0$ is free parameter. Therefore, the channel output $f_Y(y)$ is

$$\begin{aligned}f_Y(y) &= (f_X * f_N)(y) &(5.53)\\ &= \int_{-\infty}^{\infty} f_N(x)f_X(y - x)\mathrm{d}x &(5.54)\\ &= \int_{-\infty}^{\infty}\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left(\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right)I\{x \geq 0\} \\ &\qquad \cdot\frac{\beta}{\mathsf{T}(1 - e^{-\beta})}e^{\frac{-\beta}{\mathsf{T}}(y-x)}I\{0 \leq y - x \leq \mathsf{T}\}\,\mathrm{d}x. &(5.55)\end{aligned}$$

To calculate the convolution, we separate it to two cases, $y - \mathsf{T} \leq 0$ and $y - \mathsf{T} > 0$. and follow a similar calculation as given in [11].
For the first case, $y \leq \mathsf{T}$, we have:

$$f_Y(y) = \sqrt{\frac{\lambda}{2\pi}} \frac{\beta}{\mathsf{T}(1-e^{-\beta})} e^{\frac{\lambda}{\mu} - \frac{\beta}{\mathsf{T}} y} \underbrace{\int_0^y x^{-\frac{3}{2}} e^{-x\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right) - \frac{\lambda}{2x}} \, \mathrm{d}x}_{I_1(y)} \tag{5.56}$$

Let $a^2 = \frac{\lambda}{2}$ and $b^2 = \frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}$. We assume $b^2 \geq 0$, so

$$\mathsf{T} \geq \frac{2\beta\mu^2}{\lambda}. \tag{5.57}$$

Then $I_1(y)$ is

$$\begin{aligned} I_1(y) &= \int_0^y x^{-\frac{3}{2}} e^{-x\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right) - \frac{\lambda}{2x}} \, \mathrm{d}x & (5.58) \\ &= \int_0^y x^{-\frac{3}{2}} e^{-\frac{a^2}{x} - xb^2} \mathrm{d}x. & (5.59) \end{aligned}$$

Let $t = x^{-\frac{1}{2}}$,

$$I_1(y) = 2 \int_{\frac{1}{\sqrt{y}}}^{\infty} \exp\left(-a^2 t^2 - \frac{b^2}{t^2}\right) \mathrm{d}t \tag{5.60}$$

from the Abramowitz and Stegun, 7.4.33 [12]:

$$\begin{aligned} I_1(y) = \frac{\sqrt{\pi}}{2a} \Bigg\{ & e^{2ab} \left[1 - \operatorname{erf}\left(b\sqrt{y} + \frac{a}{\sqrt{y}} \right) \right] \\ & + e^{-2ab} \left[1 - \operatorname{erf}\left(-b\sqrt{y} + \frac{a}{\sqrt{y}} \right) \right] \Bigg\} & (5.61) \\ = \sqrt{\frac{2\pi}{\lambda}} \Bigg\{ & e^{\sqrt{2\lambda\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right)}} \mathcal{Q}\left(\sqrt{2y\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right)} + \sqrt{\frac{\lambda}{y}} \right) \\ & + e^{-\sqrt{2\lambda\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right)}} \mathcal{Q}\left(-\sqrt{2y\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right)} + \sqrt{\frac{\lambda}{y}} \right) \Bigg\}. & (5.62) \end{aligned}$$

Similary, for the case that $y - \mathsf{T} > 0$

$$f_Y(y) = \sqrt{\frac{\lambda}{2\pi}} \frac{\beta}{\mathsf{T}(1-e^{-\beta})} e^{\frac{\lambda}{\mu} - \frac{\beta}{\mathsf{T}} y} \underbrace{\int_{y-\mathsf{T}}^y x^{-\frac{3}{2}} e^{-x\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right) - \frac{\lambda}{2x}} \, \mathrm{d}x}_{I_2(y)} \tag{5.63}$$

where

$$I_2(y) = \int_{y-\mathsf{T}}^{y} x^{-\frac{3}{2}} e^{-x\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)-\frac{\lambda}{2x}} \mathrm{d}x \tag{5.64}$$

$$= \int_{y-\mathsf{T}}^{y} x^{-\frac{3}{2}} e^{-\frac{a^2}{x}-xb^2} \mathrm{d}x \tag{5.65}$$

$$= 2\int_{\frac{1}{\sqrt{y}}}^{\sqrt{\frac{1}{y-\mathsf{T}}}} \exp\left(-a^2t^2 - \frac{b^2}{t^2}\right)\mathrm{d}t \tag{5.66}$$

$$= 2\left(\int_{\frac{1}{\sqrt{y}}}^{\infty} \exp\left(-a^2t^2 - \frac{b^2}{t^2}\right)\mathrm{d}t - \int_{\frac{1}{\sqrt{y-\mathsf{T}}}}^{\infty} \exp\left(-a^2t^2 - \frac{b^2}{t^2}\right)\mathrm{d}t\right) \tag{5.67}$$

$$\begin{aligned} &= \frac{\sqrt{\pi}}{2a}\left\{e^{2ab}\left[1-\operatorname{erf}\left(b\sqrt{y}+\frac{a}{\sqrt{y}}\right)\right] + e^{-2ab}\left[1-\operatorname{erf}\left(-b\sqrt{y}+\frac{a}{\sqrt{y}}\right)\right]\right\} \\ &\quad + \frac{\sqrt{\pi}}{2a}\left\{e^{2ab}\left[1-\operatorname{erf}\left(b\sqrt{y-\mathsf{T}}+\frac{a}{\sqrt{y-\mathsf{T}}}\right)\right]\right. \\ &\qquad\qquad \left.+ e^{-2ab}\left[1-\operatorname{erf}\left(-b\sqrt{y-\mathsf{T}}+\frac{a}{\sqrt{y-\mathsf{T}}}\right)\right]\right\} \end{aligned} \tag{5.68}$$

$$\begin{aligned} &= \sqrt{\frac{2\pi}{\lambda}}\left\{e^{\sqrt{2\lambda\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)}}\mathcal{Q}\left(\sqrt{2y\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)}+\sqrt{\frac{\lambda}{y}}\right)\right. \\ &\qquad\qquad \left. + e^{-\sqrt{2\lambda\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)}}\mathcal{Q}\left(-\sqrt{2y\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)}+\sqrt{\frac{\lambda}{y}}\right)\right\} \\ &\quad - \sqrt{\frac{2\pi}{\lambda}}\left\{e^{\sqrt{2\lambda\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)}}\mathcal{Q}\left(\sqrt{2(y-\mathsf{T})\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)}+\sqrt{\frac{\lambda}{y-\mathsf{T}}}\right)\right. \\ &\qquad\qquad \left. + e^{-\sqrt{2\lambda\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)}}\mathcal{Q}\left(-\sqrt{2(y-\mathsf{T})\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)}+\sqrt{\frac{\lambda}{y-\mathsf{T}}}\right)\right\} \end{aligned} \tag{5.69}$$

$$= I_1(y) - \tilde{I}_2(y) \tag{5.70}$$

where $\tilde{I}_2(y) > 0$ for all y.
Hence, we have output distribution $f_Y(y)$:

$$f_Y(y) = \begin{cases} \sqrt{\frac{\lambda}{2\pi}}\frac{\beta}{\mathsf{T}(1-e^{-\beta})}e^{\frac{\lambda}{\mu}-\frac{\beta}{\mathsf{T}}y}I_1(y) & y \le \mathsf{T} \\ \sqrt{\frac{\lambda}{2\pi}}\frac{\beta}{\mathsf{T}(1-e^{-\beta})}e^{\frac{\lambda}{\mu}-\frac{\beta}{\mathsf{T}}y}I_2(y) & y > \mathsf{T}. \end{cases} \tag{5.71}$$

This now yields the following lower bound on capacity:

$$\mathsf{C} \triangleq \max_{f_X(x)} I(X;Y) \tag{5.72}$$

$$\ge I(X;Y)\big|_{X\sim f(x)} \tag{5.73}$$

$$= \left(h(Y) - h(Y|X)\right)\big|_{X\sim f(x)} \tag{5.74}$$

$$= h(Y)\big|_{X\sim f(x)} - h(N) \tag{5.75}$$

$$= \mathsf{E}_Y[-\log f_Y(Y)]\,\big|_{X\sim f(x)} - h(N) \tag{5.76}$$

$$
\begin{aligned}
&= \Pr[Y > \mathsf{T}]\mathsf{E}[-\log f_Y(Y)|Y > \mathsf{T}] + \Pr[Y \le \mathsf{T}]\mathsf{E}[-\log f_Y(Y)|Y \le \mathsf{T}] \\
&\quad - h(N) && (5.77) \\
&= -h(N) + \big(\Pr[Y > \mathsf{T}] + \Pr[Y \le \mathsf{T}]\big) \\
&\qquad \cdot \left(-\log\sqrt{\frac{\lambda}{2\pi}} + \log\mathsf{T} - \log\beta + \log\left(1 - e^{-\beta}\right) - \frac{\lambda}{\mu} \right) \\
&\quad + \Pr[Y \le \mathsf{T}] \left(\frac{\beta}{\mathsf{T}}\mathsf{E}[Y|Y \le \mathsf{T}] - \mathsf{E}[\log I_1(Y)|Y \le \mathsf{T}] \right) \\
&\quad + \Pr[Y > \mathsf{T}] \left(\frac{\beta}{\mathsf{T}}\mathsf{E}[Y|Y > \mathsf{T}] - \mathsf{E}\left[\log\left(I_1(Y) - \tilde{I}_2(Y)\right)\middle| Y > \mathsf{T}\right] \right) && (5.78) \\
&\ge -h(N) + -\log\sqrt{\frac{\lambda}{2\pi}} + \log\mathsf{T} - \log\beta + \log\left(1 - e^{-\beta}\right) - \frac{\lambda}{\mu} \\
&\quad + \frac{\beta}{\mathsf{T}}\mathsf{E}[Y] - \mathsf{E}[\log I_1(Y)] && (5.79) \\
&\ge -\log\sqrt{\frac{\lambda}{2\pi}} + \log\mathsf{T} - \log\beta + \log\left(1 - e^{-\beta}\right) - \frac{\lambda}{\mu} + \frac{\beta}{\mathsf{T}}(\alpha\mathsf{T} + \mu) \\
&\quad - \mathsf{E}[\log I_1(Y)] - h(N) && (5.80)
\end{aligned}
$$

We further bound the log term,

$$
\begin{aligned}
I_1(y) &= \sqrt{\frac{2\pi}{\lambda}} \left\{ e^{\sqrt{2\lambda\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right)}} \mathcal{Q}\left(\sqrt{2y\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right)} + \sqrt{\frac{\lambda}{y}} \right) \right. \\
&\qquad \left. + e^{-\sqrt{2\lambda\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right)}} \left[1 - \mathcal{Q}\left(\sqrt{2y\left(\frac{\lambda}{2\mu^2} - \frac{\beta}{\mathsf{T}}\right)} - \sqrt{\frac{\lambda}{y}} \right) \right] \right\} && (5.81) \\
&= \sqrt{\frac{2\pi}{\lambda}} e^{-\sqrt{\frac{\lambda^2}{\mu^2} - \frac{2\lambda\beta}{\mathsf{T}}}} \left\{ 1 - \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}} \left(\sqrt{\frac{y}{\mu}\left(1 - \frac{2\mu^2\beta}{\lambda\mathsf{T}}\right)} - \sqrt{\frac{\mu}{y}} \right) \right) \right. \\
&\qquad \left. + e^{\frac{2\lambda}{\mu}\sqrt{1 - \frac{2\mu^2\beta}{\lambda\mathsf{T}}}} \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}} \left(\sqrt{\frac{y}{\mu}\left(1 - \frac{2\mu^2\beta}{\lambda\mathsf{T}}\right)} + \sqrt{\frac{\mu}{y}} \right) \right) \right\}, && (5.82)
\end{aligned}
$$

because the function

$$
\begin{aligned}
y \mapsto & 1 - \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}} \left(\sqrt{\frac{y}{\mu}\left(1 - \frac{2\mu^2\beta}{\lambda\mathsf{T}}\right)} - \sqrt{\frac{\mu}{y}} \right) \right) \\
&\qquad + e^{\frac{2\lambda}{\mu}\sqrt{1 - \frac{2\mu^2\beta}{\lambda\mathsf{T}}}} \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}} \left(\sqrt{\frac{y}{\mu}\left(1 - \frac{2\mu^2\beta}{\lambda\mathsf{T}}\right)} + \sqrt{\frac{\mu}{y}} \right) \right) && (5.83)
\end{aligned}
$$

is monotonically increasing, so we upper-bounded it by y goes infinity, then (5.83) is bouned by 1, i.e.

$$
I_1(y) \le \sqrt{\frac{2\pi}{\lambda}} e^{-\sqrt{\frac{\lambda^2}{\mu^2} - \frac{2\lambda\beta}{\mathsf{T}}}} \qquad (5.84)
$$

which does not depend on y anymore, so we can get rid of the expectation directly. Finally, we get the lower bound of capacity:

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) &\geq \log\mathsf{T} - \log\beta + \log\left(1-e^{-\beta}\right) - \frac{\lambda}{\mu} + \frac{\beta}{\mathsf{T}}(\alpha\mathsf{T}+\mu) + \sqrt{2\lambda\left(\frac{\lambda}{2\mu^2}-\frac{\beta}{\mathsf{T}}\right)} \\ &\quad - \frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right) \qquad (5.85)\\ &= \log\mathsf{T} - \log\beta^* + \log\left(1-e^{-\beta^*}\right) - \frac{\lambda}{\mu} + \frac{\beta^*}{\mathsf{T}}(\alpha\mathsf{T}+\mu) + \sqrt{2\lambda\left(\frac{\lambda}{2\mu^2}-\frac{\beta^*}{\mathsf{T}}\right)} \\ &\quad - \frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right) \qquad (5.86)\end{aligned}$$

Here, we choose β as solution of β^* which satisfied $\alpha = \frac{1}{\beta^*} - \frac{e^{\beta^*}}{1-e^{\beta^*}}$ for $0 < \alpha < 0.5$.

5.2.2 For $0.5 \leq \alpha \leq 1$

For $0.5 \leq \alpha \leq 1$, we choose uniform distribution as our input distribution:

$$f_X(x) = \frac{1}{\mathsf{T}} \cdot I\{0 \leq x \leq \mathsf{T}\} \qquad (5.87)$$

since $\mathsf{E}[X] = 0.5$, the average delay constraint is always satisfied.
Hence,

$$\begin{aligned}f_Y(y) &= (f_X * f_N)(y) &(5.88)\\ &= \int_{-\infty}^{\infty} f_N(x) f_X(y-x)\,\mathrm{d}x &(5.89)\\ &= \frac{1}{\mathsf{T}}\int_{-\infty}^{\infty}\sqrt{\frac{\lambda}{2\pi x^3}}e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}} I\{x\geq 0\} I\{0 \leq y-x \leq \mathsf{T}\}\,\mathrm{d}x &(5.90)\end{aligned}$$

Similary, we separate it to two cases, $y-\mathsf{T} \leq 0$ and $y-\mathsf{T} > 0$.
For the first case, $y-\mathsf{T} \leq 0$:

$$f_Y(y) = \frac{1}{\mathsf{T}}\int_0^y \sqrt{\frac{\lambda}{2\pi x^3}}e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}}\,\mathrm{d}x \qquad (5.91)$$

Note that the integration is like the $c_1(x)$ we use in the upper bound, the only different thing is the range of the integration. So, we have:

$$\begin{aligned}f_Y(y) &= \frac{1}{\mathsf{T}}\int_0^{\frac{y}{\mu}} \sqrt{\frac{\lambda}{2\pi\mu w^3}}e^{-\frac{\lambda(w-1)^2}{2\mu w}}\,\mathrm{d}w &(5.92)\\ &= \frac{1}{\mathsf{T}}\Bigg[1 - \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{y}{\mu}}-\sqrt{\frac{\mu}{y}}\right)\right) \\ &\qquad + e^{2\frac{\lambda}{\mu}}\mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{y}{\mu}}+\sqrt{\frac{\mu}{y}}\right)\right)\Bigg] &(5.93)\\ &\triangleq I_3(y) &(5.94)\end{aligned}$$

For the second case, $y - \mathsf{T} > 0$:

$$
\begin{aligned}
f_Y(y) &= \frac{1}{\mathsf{T}} \int_{y-\mathsf{T}}^{y} \sqrt{\frac{\lambda}{2\pi x^3}} e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}} \mathrm{d}x && (5.95)\\
&= \frac{1}{\mathsf{T}} \left[\int_0^{y} \sqrt{\frac{\lambda}{2\pi x^3}} e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}} \mathrm{d}x - \int_0^{y-\mathsf{T}} \sqrt{\frac{\lambda}{2\pi x^3}} e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}} \mathrm{d}x \right] && (5.96)\\
&= I_3(y) - \tilde{I}_3(y) && (5.97)
\end{aligned}
$$

where $\tilde{I}_3 > 0$ for all y. Hence, we have output distribution $f_Y(y)$:

$$
f_Y(y) = \begin{cases} I_3(y) & y \le T \\ I_3(y) - \tilde{I}_3(y) & y > T \end{cases} \tag{5.98}
$$

This now yields the following lower bound on capacity:

$$
\begin{aligned}
\mathsf{C} &\triangleq \max_{f_X(x)} I(X;Y) && (5.99)\\
&\ge I(X;Y)\big|_{X\sim f(x)} && (5.100)\\
&= (h(Y) - h(Y|X))\big|_{X\sim f(x)} && (5.101)\\
&= h(Y)\big|_{X\sim f(x)} - h(N) && (5.102)\\
&= -\mathsf{E}_Y[\log f_Y(Y)]\,\big|_{X\sim f(x)} - h(N) && (5.103)\\
&= \Pr[Y > \mathsf{T}]\mathsf{E}[-\log f_Y(Y)|Y > \mathsf{T}] + \Pr[Y \le \mathsf{T}]\mathsf{E}[-\log f_Y(Y)|Y \le \mathsf{T}]\\
&\quad - h(N) && (5.104)\\
&= -h(N) - \Pr[Y \le \mathsf{T}]\mathsf{E}[\log I_3(Y)|Y \le \mathsf{T}]\\
&\quad - \Pr[Y > \mathsf{T}]\mathsf{E}\Big[\log\big(I_3(Y) - \tilde{I}_3(Y)\big)\Big| Y > \mathsf{T}\Big] && (5.105)\\
&\ge -h(N) - \mathsf{E}[\log I_3(Y)] && (5.106)
\end{aligned}
$$

We futher bound $I_3(y)$:

$$
\begin{aligned}
I_3(y) &= \frac{1}{\mathsf{T}} \left[1 - \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}} \left(\sqrt{\frac{y}{\mu}} - \sqrt{\frac{\mu}{y}} \right) \right) + e^{2\frac{\lambda}{\mu}} \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}} \left(\sqrt{\frac{y}{\mu}} + \sqrt{\frac{\mu}{y}} \right) \right) \right] && (5.107)\\
&\le \frac{1}{\mathsf{T}}. && (5.108)
\end{aligned}
$$

Equation (5.108) is because (5.107) is monotonically decreasing in y.
Hence, we get the lower bound for $0.5 \le \alpha \le 1$:

$$
\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \ge \log \mathsf{T} - \frac{1}{2} \log \frac{2\pi e \mu^3}{\lambda} - \frac{3}{2} e^{\frac{2\lambda}{\mu}} \operatorname{Ei}\left(-\frac{2\lambda}{\mu} \right). \tag{5.109}
$$

5.3 Asymptotic Capacity of AIGN Channel

In this section, we try to figure out how the capacity behaves when the drift velocity v or the peak-delay constraint T tend to infinity.

5.3.1 When T Large

From chapter 4, we have for $0 < \alpha < 0.5$:

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) \geq \log\mathsf{T} - \log\beta^* + \log\left(1-e^{-\beta^*}\right) - \frac{\lambda}{\mu} + \frac{\beta^*}{\mathsf{T}}(\alpha\mathsf{T}+\mu) + \sqrt{2\lambda\left(\frac{\lambda}{2\mu^2}-\frac{\beta^*}{\mathsf{T}}\right)} \\ -\frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right)\end{aligned} \tag{5.110}$$

when T goes to infinity, we get

$$\begin{aligned}\mathsf{C}(\mathsf{T}) \geq \log\mathsf{T} + \alpha\beta^* - \log\beta^* + \log\left(1-e^{-\beta^*}\right) \\ -\frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right) + o(1)\end{aligned} \tag{5.111}$$

and the capacity upper-bounded by

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) \leq Q^- - Q^+ - 2\left(Q^- - Q^+\right)\log\left(Q^- - Q^+\right) \\ - 2(1-Q^- + Q^+)\log(1-Q^- + Q^+) \\ + \left((Q^- - Q^+)\log\left(\mu(Q^- + Q^+) + \alpha\mathsf{T}(Q^- - Q^+)\right)\right)^+ + \alpha\beta^* \\ - \log\alpha\beta^* + \log(\alpha\mathsf{T}+\mu) + \log\left(1 - e^{-\frac{\alpha\beta^*(T+\delta)}{\alpha\mathsf{T}+\mu-\alpha\mathsf{T}(Q^- - Q^+)-\mu(Q^- + Q^+)}}\right) \\ -\frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right)\end{aligned} \tag{5.112}$$

where

$$Q^- \triangleq \mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{\delta}{\mu}} - \sqrt{\frac{\mu}{\delta}}\right)\right) \tag{5.113}$$

$$Q^+ \triangleq e^{2\frac{\lambda}{\mu}}\mathcal{Q}\left(\sqrt{\frac{\lambda}{\mu}}\left(\sqrt{\frac{\delta}{\mu}} + \sqrt{\frac{\mu}{\delta}}\right)\right) \tag{5.114}$$

As T goes to infinity, the capacity become:

$$\begin{aligned}\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) \leq \log\mathsf{T} + \alpha\beta^* - \log\beta^* + \log\left(1-e^{-\beta^*}\right) + o(1) \\ -\frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right)\end{aligned} \tag{5.115}$$

Here, (5.115) is because when T goes to infinity, Q^- and Q^+ will go to zero. From (5.111) and (5.115). Note that the upper bound and lower bound coincide, which gives us

$$\begin{aligned}\lim_{\mathsf{T}\uparrow\infty}\left\{\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) - \log\mathsf{T}\right\} = \alpha\beta^* - \log\beta^* + \log\left(1-e^{-\beta^*}\right) \\ -\frac{1}{2}\log\frac{2\pi e\mu^3}{\lambda} - \frac{3}{2}e^{\frac{2\lambda}{\mu}}\,\mathrm{Ei}\left(-\frac{2\lambda}{\mu}\right)\end{aligned} \tag{5.116}$$

For $0.5 \leq \alpha \leq 1$:

$$\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \geq \log \mathsf{T} - \frac{1}{2} \log \frac{2\pi e \mu^3}{\lambda} - \frac{3}{2} e^{\frac{2\lambda}{\mu}} \operatorname{Ei}\left(-\frac{2\lambda}{\mu}\right) \tag{5.117}$$

when T goes to infinity, we get the same equation:

$$\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \geq \log \mathsf{T} - \frac{1}{2} \log \frac{2\pi e \mu^3}{\lambda} - \frac{3}{2} e^{\frac{2\lambda}{\mu}} \operatorname{Ei}\left(-\frac{2\lambda}{\mu}\right) \tag{5.118}$$

and the upper bound, from (4.12):

$$\begin{aligned}
\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \leq\ & Q^- - Q^+ - 2\left(Q^- - Q^+\right) \log\left(Q^- - Q^+\right) \\
& - \left(1 - Q^- + Q^+\right) \log(1 - Q^- + Q^+) \\
& + \left(Q^- - Q^+\right) \max\left\{0, \log\left(\mu(Q^- + Q^+) + \alpha\mathsf{T}(Q^- - Q^+)\right)\right\} + \log(\mathsf{T} + \delta) \\
& - \frac{1}{2} \log \frac{2\pi e \mu^3}{\lambda} - \frac{3}{2} e^{\frac{2\lambda}{\mu}} \operatorname{Ei}\left(-\frac{2\lambda}{\mu}\right).
\end{aligned} \tag{5.119}$$

where Q^- and Q^+ are same as (5.113) and (5.114).
We choose $\delta = \log \mathsf{T}$, then as T goes to infinity, Q^- and Q^+ will go to zero. Therefore, we get asymptotically upper bound:

$$\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) \leq \log \mathsf{T} - \frac{1}{2} \log \frac{2\pi e \mu^3}{\lambda} - \frac{3}{2} e^{\frac{2\lambda}{\mu}} \operatorname{Ei}\left(-\frac{2\lambda}{\mu}\right) + o(1) \tag{5.120}$$

We can see (5.118) and (5.120) are coincide as T goes to infinity. Hence we get the asympototic capacity when T goes to infinity for $0.5 \leq \alpha \leq 1$:

$$\lim_{\mathsf{T}\uparrow\infty} \left\{\mathsf{C}(\mathsf{T}, \alpha\mathsf{T}) - \log \mathsf{T}\right\} = -\frac{1}{2} \log \frac{2\pi e \mu^3}{\lambda} - \frac{3}{2} e^{\frac{2\lambda}{\mu}} \operatorname{Ei}\left(-\frac{2\lambda}{\mu}\right) \tag{5.121}$$

5.3.2 v Large

We rewrite the bound (4.3) by using $v = \frac{1}{\mu}$, for $0 < \alpha < 0.5$:

$$\begin{aligned}
\mathsf{C}(v) \geq\ & \log \mathsf{T} - \log \beta^* + \log\left(1 - e^{-\beta^*}\right) - \lambda v + \frac{\beta^*}{\mathsf{T}}\left(\alpha\mathsf{T} + \frac{1}{v}\right) + \sqrt{2\lambda\left(\frac{\lambda v^2}{2} - \frac{\beta^*}{\mathsf{T}}\right)} \\
& - \frac{1}{2} \log \frac{2\pi e}{\lambda v^3} - \frac{3}{2} e^{2\lambda v} \operatorname{Ei}(-2\lambda v) && (5.122) \\
\geq\ & \log \mathsf{T} - \log \beta^* + \log\left(1 - e^{-\beta^*}\right) - v\lambda + \frac{\beta^*}{\mathsf{T}}\left(\alpha\mathsf{T} + \frac{1}{v}\right) + \sqrt{2\lambda\left(\frac{v^2\lambda}{2} - \frac{\beta^*}{\mathsf{T}}\right)} \\
& + \frac{3}{2} \log v + \frac{1}{2} \log \frac{\lambda}{2\pi e} + \frac{3}{4} \log\left(1 + \frac{1}{2v\lambda}\right) && (5.123)
\end{aligned}$$

where (5.123) is simply plugging in the lower bound of $\operatorname{Ei}(\cdot)$ from Proposition 2.14. Its asymptotic lower bound is:

$$\mathsf{C}(v) \geq \log \mathsf{T} - \log \beta^* + \log\left(1 - e^{-\beta^*}\right) + \alpha\beta^* + \frac{3}{2} \log v + \frac{1}{2} \log \frac{\lambda}{2\pi e} + o(1). \tag{5.124}$$

Similary, we rewrite (4.12):

$$\begin{aligned}\mathsf{C}(v) \le\ & Q^- - Q^+ - 2\left(Q^- - Q^+\right)\log\left(Q^- - Q^+\right) \\ & - 2(1 - Q^- + Q^+)\log(1 - Q^- + Q^+) \\ & + (Q^- - Q^+)\max\left\{0, \log\left(\frac{1}{v}(Q^- + Q^+) + \alpha\mathsf{T}(Q^- - Q^+)\right)\right\} + \alpha\beta^* \\ & - \log\alpha\beta^* + \log(\alpha\mathsf{T} + \frac{1}{v}) + \log\left(1 - e^{-\frac{\alpha\beta^*(T+\delta)}{\alpha\mathsf{T} + \frac{1}{v} - \alpha\mathsf{T}(Q^- - Q^+) - \frac{1}{v}(Q^- + Q^+)}}\right) \\ & - \frac{1}{2}\log\frac{2\pi e}{\lambda v^3} - \frac{3}{2}e^{2\lambda v}\,\mathrm{Ei}(-2\lambda v) \end{aligned} \tag{5.125}$$

where

$$Q^- \triangleq \mathcal{Q}\left(\sqrt{\lambda v}\left(\sqrt{\delta v} - \sqrt{\frac{1}{v\delta}}\right)\right) \tag{5.126}$$

$$Q^+ \triangleq e^{2\lambda v}\mathcal{Q}\left(\sqrt{v\lambda}\left(\sqrt{v\delta} + \sqrt{\frac{1}{v\delta}}\right)\right) \tag{5.127}$$

We choose $\delta = \frac{1}{\sqrt{v}}$, then when v goes to infinity, it is obvious that Q^- will go to zero. For Q^+:

$$e^{2\lambda v}\mathcal{Q}\left(\sqrt{v\lambda}\left(\sqrt{v\delta} + \sqrt{\frac{1}{v\delta}}\right)\right) \le \frac{1}{2}e^{-\frac{v\lambda}{2}\left(\sqrt{v\delta} + \sqrt{\frac{1}{v\delta}}\right)^2} \tag{5.128}$$

which will also goes to zero as v goes to infinity. Therefore, we get asymptotically upper bound:

$$\mathsf{C}(v) \le \log\mathsf{T} - \log\beta^* + \log\left(1 - e^{-\beta^*}\right) + \alpha\beta^* + \frac{3}{2}\log v + \frac{1}{2}\log\frac{\lambda}{2\pi e} + o(1). \tag{5.129}$$

We can see (5.124) and (5.129) are coincide as v goes to infinity. Hence we get the asympototic capacity for v goes to infinity:

$$\lim_{v\uparrow\infty}\left\{\mathsf{C}(v) - \frac{3}{2}\log v\right\} = \log\mathsf{T} - \log\beta^* + \log\left(1 - e^{-\beta^*}\right) + \alpha\beta^* + \frac{1}{2}\log\frac{\lambda}{2\pi e} \tag{5.130}$$

For $0.5 \le \alpha \le 1$:

$$\begin{aligned}\mathsf{C}(v) &\ge \log\mathsf{T} - \frac{1}{2}\log\frac{2\pi e}{\lambda v^3} - \frac{3}{2}e^{2v\lambda}\,\mathrm{Ei}(-2\lambda v) && (5.131)\\ &\ge \log\mathsf{T} + \frac{3}{2}\log v + \frac{1}{2}\log\frac{\lambda}{2\pi e} + \frac{3}{4}\log\left(1 + \frac{1}{2v\lambda}\right) && (5.132)\end{aligned}$$

where (5.132) is simply plugging in the lower bound of $\mathrm{Ei}(\cdot)$ from Proposition 2.14. Its asymptotic lower bound is:

$$\mathsf{C}(v) \ge \log\mathsf{T} + \frac{3}{2}\log v + \frac{1}{2}\log\frac{\lambda}{2\pi e} + o(1) \tag{5.133}$$

and the upper bound, from (4.12):

$$\begin{aligned}
\mathsf{C}(\mathsf{T},\alpha\mathsf{T}) \leq\ & Q^- - Q^+ - 2\left(Q^- - Q^+\right)\log\left(Q^- - Q^+\right) \\
& - \left(1 - Q^- + Q^+\right)\log(1 - Q^- + Q^+) \\
& + \left(Q^- - Q^+\right)\max\left\{0, \log\left(\frac{1}{v}(Q^- + Q^+) + \alpha\mathsf{T}(Q^- - Q^+)\right)\right\} + \log(\mathsf{T}+\delta) \\
& - \frac{1}{2}\log\frac{2\pi e}{\lambda v^3} - \frac{3}{2}e^{2\lambda v}\,\mathrm{Ei}(-2\lambda v)\,. \qquad (5.134)
\end{aligned}$$

where Q^- and Q^+ are defined by (5.126) and (5.127).
Similary, we choose $\delta = \frac{1}{\sqrt{v}}$, as v goes to infinity, Q^- and Q^+ will go to zero. Therefore, we get asymptotically upper-bound for $0.5 \leq \alpha \leq 1$:

$$\mathsf{C}(v) \leq \log\mathsf{T} + \frac{3}{2}\log v + \frac{1}{2}\log\frac{\lambda}{2\pi e} + o(1) \qquad (5.135)$$

We can see (5.133) and (5.135) are coincide as v goes to infinity. Hence we get the asympototic capacity when v goes to infinity for $0.5 \leq \alpha \leq 1$:

$$\lim_{v\uparrow\infty}\left\{\mathsf{C}(v) - \frac{3}{2}\log v\right\} = \log\mathsf{T} + \frac{1}{2}\log\frac{\lambda}{2\pi e} \qquad (5.136)$$

Chapter 6

Discussion and Conclusion

This thesis provide new upper bound and lower bound on the capacity of the AIGN channel in the situation of both a peak delay constraint and average delay constraint.

We have derived the upper bound and lower bound of capacity, and they are very tight when either T or v goes to infinity. We also have found that as α increased, our upper bound become the difference between known upper bound is increased. For the first point, it is because for fixed peak delay constraint, the average delay constraint become weaker and weaker as α growing;however, as α increased, the peak constraint become stronger, and that's the main reason for the second phenomenon. Moreover, when the fluid velocity, v, is extremely small, from the Fig. 4.6 and Fig. 4.8, the noise caused by Brownian motion actually helps the transmission.

With the help of [11], we were able to compute the exact output distribution of an exponential input. This lower bound (4.1) was much tighter than the known bound with respect to both v and T. It turned out that together with the known upper bound, this lower bound allowed us to derive the asymptotic capacity at high T and high v.

For future research, we propose the following problems related to the additive inverse Gaussian noise channel:

- Derivation of the channel capacity when T and v is small.
- Proof for $\alpha \geq 0.5$, the capacity dose not depend on the α.

List of Figures

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