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Faithful 1-edge fault tolerant graphs [1]

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Abstract

A graph G^* is 1-edge fault tolerant with respect to a graph G, denoted by 1-EFT(G), if any graph obtained by removing an edge from G^* contains G. A 1-EFT(G) graph is said to be optimal if it contains the minimum number of edges among all 1-EFT(G) graphs. Let G_i^* be 1-EFT(G_i) for $i = 1,2$. It can be easily verified that the cartesian product graph $G_1^* \times G_2^*$ is 1-edge fault tolerant with respect to the cartesian product graph $G_1 \times G_2$. However, $G_1^* \times G_2^*$ may contain too many edges; hence it may not be optimal for many cases. In this paper, we introduce the concept of faithful graph with respect to a given graph, which is proved to be 1-edge fault tolerant. Based on this concept, we present a construction method of the 1-EFT graph for the cartesian product of several graphs. Applying this construction scheme, we can obtain optimal 1-edge fault tolerant graphs with respect to n-dimensional tori $C(m_1, m_2, \ldots, m_n)$, where $m_i \geqslant 4$ are even integers for all $1 \leqslant i \leqslant n$. © 1997 Elsevier Science B.V.

Keywords: Cartesian product; Kronecker product; Edge fault tolerance; Meshes; Tori

1. Introduction and notations

Computer and communication networks are usually represented by graphs where nodes represent processors and edges represent links between processors. Among them, meshes, tori and hypercubes are widely used graph models for networks [3], which can be expressed as cartesian products of graphs.

In this paper, a *graph* means an undirected graph in which multiple edges are allowed. In order to formally define cartesian product, we first introduce the definition of isomorphism. Two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ are said to be *isomorphic*, denoted by $G = H$, if there is a bijection ϕ mapping V_1 to V_2 such that $(u, v) \in E_1$ if and only if $(\phi(u), \phi(v)) \in E_2$. The bijection ϕ is called an *isomorphism* from G to H; ϕ is also called an *automorphism* if $V(H) = V(G)$. Two vertices u and v in G are said to be *similar* if there is an automorphism mapping u to v. G is said to be *node-symmetric* if all vertices are similar to one another. The *cartesian product* of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph with the vertex set $V_1 \times V_2$ such that (u_1, v_1) is joined to (u_2, v_2) k times if and only if either $u_1 = u_2$ and v_1 is joined to v_2 k times in G_2 or $v_1 = v_2$ and u_1 is joined to u_2 k times in G_1. An *n-dimensional mesh* (abbreviated as *mesh*) $M(m_1, m_2, \ldots, m_n)$ is defined as the cartesian product of n paths P_i of length m_i, de-

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noted by $P_{m_1} \times P_{m_2} \times \cdots \times P_{m_n}$, and an *n-dimensional torus* (abbreviated as *torus*) $C(m_1, m_2, \ldots, m_n)$ as the cartesian product of n cycles C_i of length m_i, denoted by $C_{m_1} \times C_{m_2} \times \cdots \times C_{m_n}$ [2]. In particular, an *n-dimensional hypercube* (abbreviated as *hypercube*) Q_n is a mesh $M(2, 2, \ldots, 2)$.

Motivated by the study of computer and communication networks that can tolerate failure of their components, Harary and Hayes [2] formulated the concept of edge fault tolerance in graphs. Let G be a graph with p nodes. A p-node graph G' is said to be *k-edge fault tolerant*, or *k-EFT*, with respect to (abbreviated as w.r.t.) G, if every graph obtained by removing any k edges from G' contains G. For brevity, we refer to G^* as a k-EFT(G) graph or simply k -EFT(G). A k-EFT(G) graph G^* is called *optimal* if it contains the least number of edges among all k-EFT(G) graphs. We use $eft_k(G)$ to denote the difference between the number of edges in an optimal k-EFT(G) graph and that in G.

Families of k-EFT graphs w.r.t. some graphs have been studied in literature [1,2,5,6]. It is observed that there is no general approach to the construction of edge fault tolerant graphs. However, we note that meshes, tori and hypercubes can be expressed as cartesian products of several primal graphs. In this paper, we aim at providing a scheme for constructing 1-edge fault tolerant graphs w.r.t. some graph products. Once we can find certain 1-EFT graphs w.r.t. these primal graphs having some desired properties, this scheme enables us to construct a 1-EFT graph w.r.t. the graph product. In particular, we apply this scheme to construct a 1-EFT$(C(m_1, m_2, \ldots, m_n))$ and show it is optimal, where $m_1, m_2, \ldots, m_n$ are positive even integers with each $m_i \geqslant 4$.

In Section 2, some graph products and graph operations are introduced. In Section 3, we define the concept of *faithful graphs*. Faithful graphs are shown to be 1-EFT w.r.t. an underlying graph and are called faithful 1-EFT graphs. Based on the concept of faithful graphs, we can show that the graph obtained from a graph operation introduced in Section 2 is 1-EFT w.r.t. a cartesian product graph. This enables us to construct 1-EFT graphs. In Section 4, we apply this construction to obtain optimal 1-EFT graphs with respect to some graphs, for example, $C(m_1, m_2, \ldots, m_n)$ where $m_i \geqslant 4$ is even for all $1 \leqslant i \leqslant n$. Concluding remarks are made in Section 4.

2. Graph products and operations

Besides cartesian product, the Kronecker product is another useful graph product. The *Kronecker product* of G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph with the vertex set $V_1 \times V_2$ such that (u_1, v_1) is joined to (u_2, v_2) k times if and only if u_1 is joined to u_2 m times in G_1 and v_1 is joined to v_2 n times in G_2 with $k = mn$. The Kronecker product was first introduced by Weichsel [7].

Since multiple edges are allowed in graphs studied in this paper, all set operations are defined on multisets; e.g., $\{a, b\} \uplus \{a\} = \{a, a, b\}$ and $\{a, a, b\} - \{a, c\} = \{a, b\}$, where "$\uplus$" denotes the sum operation of two multisets. Let $G = (V, E)$ be a graph where V $(= V(G))$ is the vertex set of G and E $(= E(G))$ is the edge set of G. $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degrees, respectively, of vertices in G. Let E' be a subset of E. We use $G - E'$ to denote the spanning subgraph of G with the edge set $E - E'$. For convenience, $G - e$ denotes $G - \{e\}$. We use ${}_kG$ to denote the graph obtained by duplicating k times of each edge in G; in particular, G denotes ${}_1G$. We call (G^*, G) a *graph pair* if G^* is a spanning supergraph of G. Moreover, (G^*, G) is a *1-EFT pair* if G^* is a 1-EFT(G). Throughout this paper, let (G_i^*, G_i) be a graph pair for all i with $G_i^* = (V_i, E_i^*)$ and $G_i = (V_i, E_i)$.

We use $(G_1^*, G_1) \oplus (G_2^*, G_2)$ to denote the graph with $V_1 \times V_2$ as its vertex set and $E(G_1 \times G_2) \uplus E((G_1^* - E_1) \circ (G_2^* - E_2))$ as its edge set. Obviously, $G_1 \times G_2$ is a spanning subgraph of $(G_1^*, G_1) \oplus (G_2^*, G_2)$. Then, we define an operator $\otimes$ on two graph pairs (G_1^*, G_1) and (G_2^*, G_2), denoted by $(G_1^*, G_1) \otimes (G_2^*, G_2)$, as the graph pair $((G_1^*, G_1) \oplus (G_2^*, G_2), G_1 \times G_2)$. For example, let $G_1 = C_6$, $G_2 = C_4$, G_1^* be the graph in Fig. 1(a), and G_2^* be the graph in Fig. 1(b). In Figs. 1(a) and (b), dashed lines represent edges $G_1^* - E_1$ and $G_2^* - E_2$. We illustrate $G_1 \times G_2$, $(G_1^* - E_1) \circ (G_2^* - E_2)$, and $(G_1^*, G_1) \oplus (G_2^*, G_2)$ in Figs. 1(c), (d), and (e), respectively. Since $\times$ and $\circ$ are commutative and associative, the following theorem can easily be obtained.

Theorem 1.

$$(G_1^*, G_1) \otimes (G_2^*, G_2) = (G_2^*, G_2) \otimes (G_1^*, G_1),$$

and

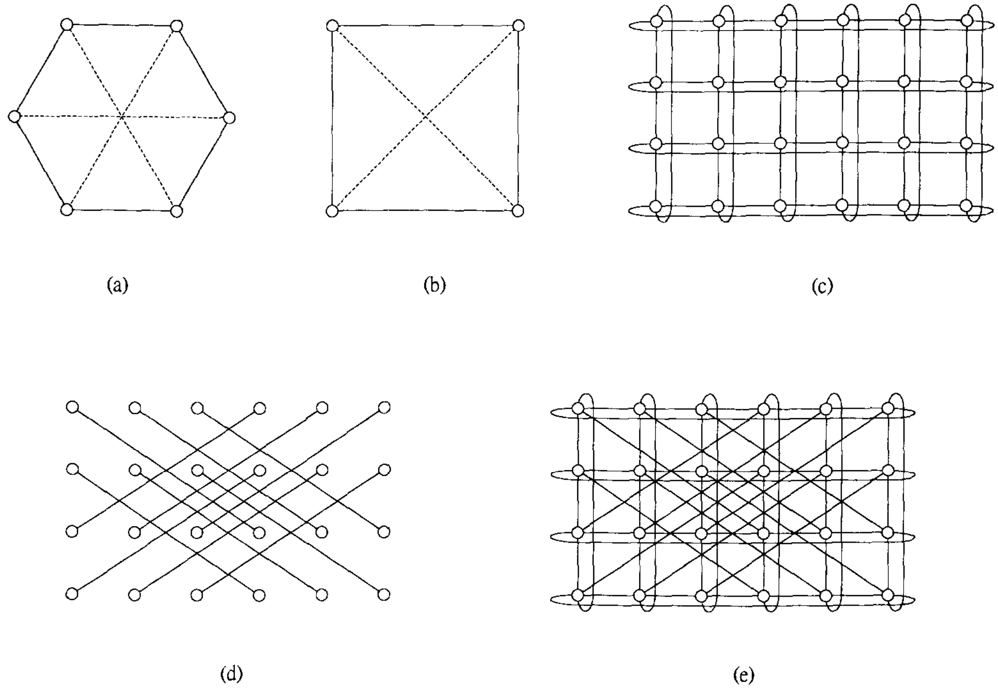

Fig. 1. (a) graph G_1^*. (b) Graph G_2^*. (c) Graph $G_1 \times G_2$. (d) Graph $(G_1^* - E_1) \circ (G_2^* - E_2)$. (e) Graph $(G_1^*, G_1) \oplus (G_2^*, G_2)$.

$$((G_1^*, G_1) \otimes (G_2^*, G_2)) \otimes (G_3^*, G_3)$$
$$= (G_1^*, G_1) \otimes ((G_2^*, G_2) \otimes (G_3^*, G_3)).$$

We can recursively define $(G_1^*, G_1) \otimes (G_2^*, G_2) \otimes \cdots \otimes (G_n^*, G_n)$ as $((G_1^*, G_1) \otimes (G_2^*, G_2) \otimes \cdots \otimes (G_{n-1}^*, G_{n-1})) \otimes (G_n^*, G_n)$. We define $(G_1^*, G_1) \oplus (G_2^*, G_2) \oplus \cdots \oplus (G_n^*, G_n)$ as $((G_1^*, G_1) \otimes (G_2^*, G_2) \otimes \cdots \otimes (G_{n-1}^*, G_{n-1})) \oplus (G_n^*, G_n)$. We have the following corollary.

Corollary 2. *For any permutation π on the set $\{1, 2, \ldots, n\}$, we have*

(i) $$(G_1^*, G_1) \otimes (G_2^*, G_2) \otimes \cdots \otimes (G_n^*, G_n) = (G_{\pi(1)}^*, G_{\pi(1)}) \otimes (G_{\pi(2)}^*, G_{\pi(2)}) \otimes \cdots \otimes (G_{\pi(n)}^*, G_{\pi(n)});$$

(ii) $$(G_1^*, G_1) \oplus (G_2^*, G_2) \oplus \cdots \oplus (G_n^*, G_n) = (G_{\pi(1)}^*, G_{\pi(1)}) \oplus (G_{\pi(2)}^*, G_{\pi(2)}) \oplus \cdots \oplus (G_{\pi(n)}^*, G_{\pi(n)}).$$

Let G_i^* be a 1-EFT(G_i) graph for $i = 1, 2$. It is easy to verify that $G_1^* \times G_2^*$ is a 1-EFT($G_1 \times G_2$) graph. However, $G_1^* \times G_2^*$ may contain much more edges than that of optimal 1-EFT($G_1 \times G_2$). For example, let $G_1 = C_6$ and $G_2 = C_4$. The graphs G_1^* shown in Fig. 1(a) and G_2^* shown in Fig. 1(b) are 1-EFT(G_1) and 1-EFT(G_2), respectively. Hence, the graph $G_1^* \times G_2^*$ is 1-EFT($G_1 \times G_2$). It can be verified that the graph $(G_1^*, G_1) \oplus (G_2^*, G_2)$ in Fig. 1(e) is also 1-EFT($G_1 \times G_2$). Since that the number of edges in $(G_1^*, G_1) \oplus (G_2^*, G_2)$ is less than that in $G_1^* \times G_2^*$, $G_1^* \times G_2^*$ is not an optimal 1-EFT($G_1 \times G_2$) graph.

For $1 \leqslant i \leqslant n$, we define the ith *projection* of $V_1 \times V_2 \times \cdots \times V_n$ as the function $p_i : V_1 \times V_2 \times \cdots \times V_n \to V_i$ given by $p_i((x_1, x_2, \ldots, x_n)) = x_i$ where $x_j \in V_j$ for $1 \leqslant j \leqslant n$.

3. Faithful edge fault tolerant graphs

Let K_2 be the complete graph on two vertices z_1 and z_2. We refer to C_2 as ${}_2K_2$. Obviously, C_2 is 1-EFT(K_2). Let $G = (V(G), E(G))$ be a graph with $V(G) =$

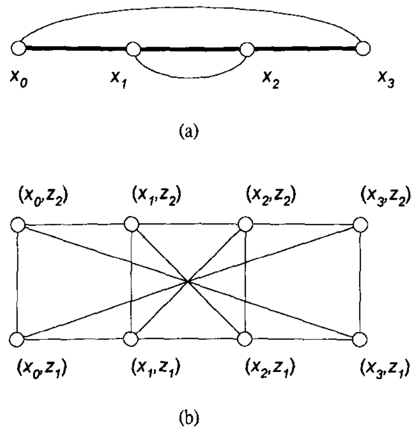

Fig. 2. (a) Graph P_4^*. (b) Graph $(P_4^*, P_4) \oplus (C_2, K_2)$.

$\{x_0, x_1, \ldots, x_{n-1}\}$, and $G^* = (V, E^*)$ be a spanning supergraph of G. Then $G \times K_2$ is a spanning subgraph of $(G^*, G) \oplus (C_2, K_2)$. Any edge in $G \times K_2$ is of the form either $((x_i, z_1), (x_i, z_2))$ for some $x_i \in V(G)$ or $((x_i, z_k), (x_j, z_k))$ for some $(x_i, x_j) \in E(G)$ and $k = 1, 2$. Let X be a set of edges given by

$$X = \{((x_i, z_1), (x_i, z_2)) \mid \forall x_i \in V(G)\}.$$

For an edge $e = (x_i, x_j)$ in G, let $Y_e = \{((x_i, z_1), (x_j, z_1)), ((x_i, z_2), (x_j, z_2))\}$. The graph G^* is said to be *faithful* or a *faithful graph* w.r.t. G, denoted by $\mathrm{FG}(G)$, if it satisfies the following two conditions:

(1) There exists a function $\sigma : V(G) \to V(G)$ such that the function $h : V(G \times K_2) \to V(G \times K_2)$ given by $h((x_i, z_1)) = (x_i, z_1)$ and $h((x_i, z_2)) = (\sigma(x_i), z_2)$ induces an isomorphism from $G \times K_2$ into a subgraph of $((G^*, G) \oplus (C_2, K_2)) - X$.

(2) For any edge $e = (x_i, x_j)$ in G, there exists an isomorphism f_e from $G \times K_2$ into a subgraph of $((G^*, G) \oplus (C_2, K_2)) - Y_e$ such that $p_1(f_e((x_i, z_1))) = p_1(f_e((x_i, z_2)))$ for every $x_i \in V(G)$, where p_1 is the 1st projection of the specified vertex.

Remark. If function σ satisfies condition (1), then σ is an automorphism on G. We call such σ an *inversion* of G^*.

Let P_n be a path of n vertices, or simply called an n-path, with $V(P_n) = \{x_0, x_1, \ldots, x_{n-1}\}$ and $E(P_n) = \{(x_i, x_{i+1}) \mid 0 \leqslant i < n - 1\}$. Consider a spanning supergraph P_n^* of P_n given by $E(P_n^*) = E(P_n) \cup \{(x_i, x_{n-i-1}) \mid 0 \leqslant i < \lceil n/2 \rceil\}$. We illustrate P_4^* and $(P_4^*, P_4) \oplus (C_2, K_2)$ in Fig. 2. We define the function $\sigma : V(P_n) \to V(P_n)$ as $\sigma(x_i) = x_{n-i-1}$ for every i. It can be easily verified that σ satisfies condition (1). We use $(P_5^*, P_5) \oplus (C_2, K_2)$ for illustration. Figs. 3(a) and (b) show P_5^* and $(P_5^*, P_5) \oplus (C_2, K_2)$. In Fig. 3(c), we illustrate $(P_5^*, P_5) \oplus (C_2, K_2)$, where the vertices are labelled according to the function h, and the the graph isomorphic to $P_5 \times K_2$ is shown by dark lines. Let $e = (x_i, x_{i+1})$ be an edge of P_n. Let the mapping f_e be defined as $f_e((x_j, z_k)) = (x_{n-i+j-1}, z_k)$ for $0 \leqslant j \leqslant i$ and $f_e((x_j, z_k)) = (x_{j-i-1}, z_{3-k})$ for $i < j < n$. It is observed that the function f_e is an isomorphism from $P_n \times K_2$ into a subgraph of $((P_n^*, P_n) \oplus (C_2, K_2)) - Y_e$ such that $p_1(f_e((x_i, z_1))) = p_1(f_e((x_i, z_2)))$ for every $x_i \in V(P_n)$. Thus the function f_e satisfies condition (2). In Fig. 3(d), we illustrate the case of $n = 5$ and $e = (x_1, x_2)$ where the vertices are labelled according to the function f_e, and the dark lines represent the graph isomorphic to $P_5 \times K_2$. Hence P_n^* is $\mathrm{FG}(P_n)$, which is stated in the following lemma.

Lemma 3. *P_n^* is* $\mathrm{FG}(P_n)$.

Consider the example illustrated in Fig. 3(d). Identifying $f_e((x_i, z_1))$ and $f_e((x_i, z_2))$ specified in condition (2), we can obtain a graph that can tolerate fault on edge (x_1, x_2). It is not surprising that the faithful graph P_n^* is 1-EFT(P_n). To generalize this result, let G be an arbitrary graph, and G^* be $\mathrm{FG}(G)$. Obviously, $(G^*, G) \oplus (C_2, K_2)$ is 1-EFT$(G \times K_2)$. The number of edges in $((G^*, G) \oplus (C_2, K_2)) - X$ is at most $|E(G \times K_2)| + 2(|E(G^*)| - |E(G)|) - |V(G)|$. Since the function h is an isomorphism from $G \times K_2$ into a subgraph of $((G^*, G) \oplus (C_2, K_2)) - X$, it follows that $2(|E(G^*)| - |E(G)|) - |V(G)| \geqslant 0$, i.e., $|E(G^*)| - |E(G)| \geqslant \lceil |V(G)|/2 \rceil$. Let f_e be a function satisfying condition (2). The function $g_e : V(G) \to V(G)$ given by $g_e(x_i) = p_1(f_e((x_i, z_1)))$ is called an *e-rotation* of G^*. Obviously, g_e induces an isomorphism from G into $G^* - e$. Thus we have the following lemma.

Lemma 4. *Any faithful graph G^* w.r.t. G is 1-*EFT(G). *Moreover, $|E(G^*)| - |E(G)| \geqslant \lceil |V(G)|/2 \rceil$ for any faithful graph G^* w.r.t. G.*

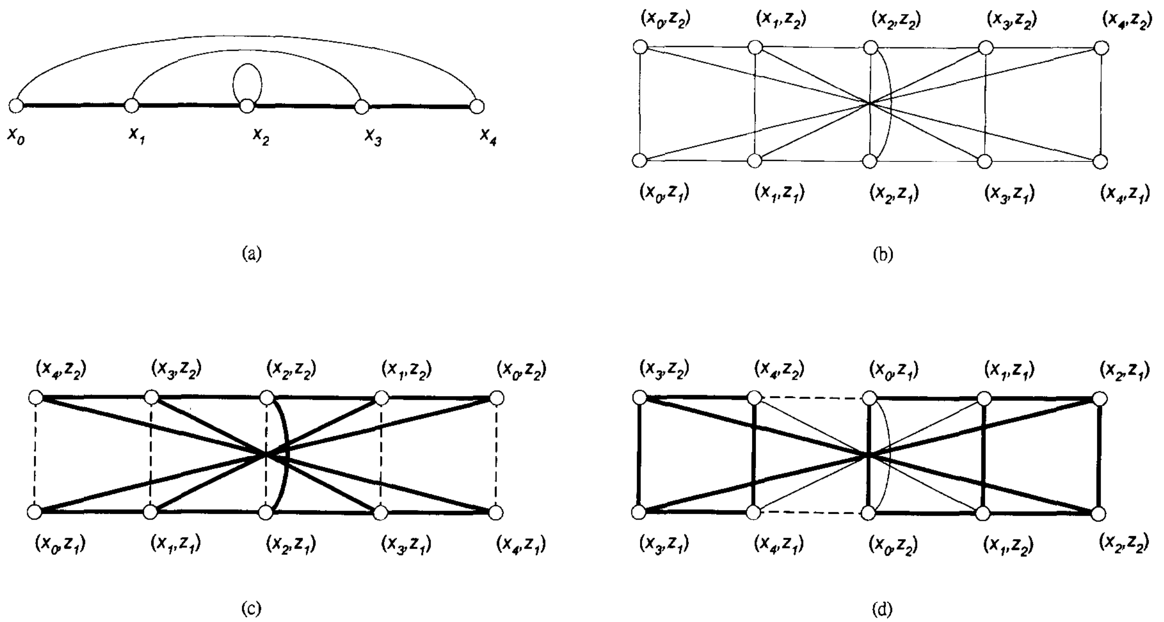

Fig. 3. (a) Graph P_5^*. (b) Graph $(P_5^*, P_5) \oplus (C_2, K_2)$. (c) The function h satisfies condition (1). (d) The function f_e satisfies condition (2) where $e = (x_1, x_2)$.

Since any faithful graph G^* w.r.t. G is also 1-EFT(G), we call G^* a *faithful* 1-EFT(G).

Let G' be the spanning supergraph of G with $E(G') = E(_2G) \uplus \{(x_i, x_i) \mid x_i \in V(G)\}$. Let σ be the identity function defined on $V(G)$, and f_e be the identity function defined on $V(G \times K_2)$ for every $e \in E(G)$. Given these functions, it can be easily verified that G' is a faithful graph w.r.t. G. We have the following lemma.

Lemma 5. *Any graph has a faithful supergraph.*

The question whether any 1-EFT(G) is FG(G) naturally arises. Consider G to be an n-path P_n. The n-cycle C_n is a spanning supergraph of P_n with $E(C_n) = E(P_n) \cup \{(x_0, x_{n-1})\}$. Harary and Hayes [2] pointed out that C_n is an optimal 1-EFT(P_n) graph. It follows from Lemma 4 that C_n is not FG(P_n) if $n \geqslant 3$. Therefore C_n is FG(P_n) if and only if $n = 2$. Thus any 1-EFT(G) graph is not necessarily FG(G).

Harary and Hayes [2] presented an optimal 1-EFT(C_n) graph $\hat{C}_n$ as follows:

$$E(\hat{C}_n) = \begin{cases} E(C_n) \cup H_n \cup \{(x_0, x_{n/2})\} \\ \quad \text{when } n \text{ is even}, \\ E(C_n) \cup H_n \\ \cup \{(x_0, x_{(n-1)/2}), (x_0, x_{(n+1)/2})\} \\ \quad \text{when } n \text{ is odd}, \end{cases}$$

where $H_n = \{(x_i, x_{n-i}) \mid 1 \leqslant i < \lfloor n/2 \rfloor\}$. (See Fig. 4.) However, $\hat{C}_n$ is not FG(C_n) if $n \geqslant 5$. For ease of exposition, we here only prove that $\hat{C}_n$ is not FG(C_n) when n is even, and the cases that n is odd can be similarly proved.

Let n be a positive even integer. Suppose $\hat{C}_n$ is FG(C_n). There exists a function $\sigma : V(C_n) \to V(C_n)$ which is an automorphism on $V(C_n)$ such that condition (1) is satisfied. Since the function h specified in condition (1) induces an isomorphism from $C_n \times K_2$ into a subgraph of $((\hat{C}_n, C_n) \oplus (C_2, K_2)) - X$, it follows that $\sigma(x_0) = x_{n/2}$, $\sigma(x_{n/2}) = x_0$, and $\sigma(x_i) = x_{n-i}$ for $i \neq 0, n/2$. But $(x_{n/2}, x_{n-1}) = (\sigma(x_0), \sigma(x_1)) \notin E(C_n)$, while $(x_0, x_1) \in E(C_n)$. (In Fig. 5, we illustrate the case $n = 6$ where the vertices are labelled according to h.) Thus there is no iso-

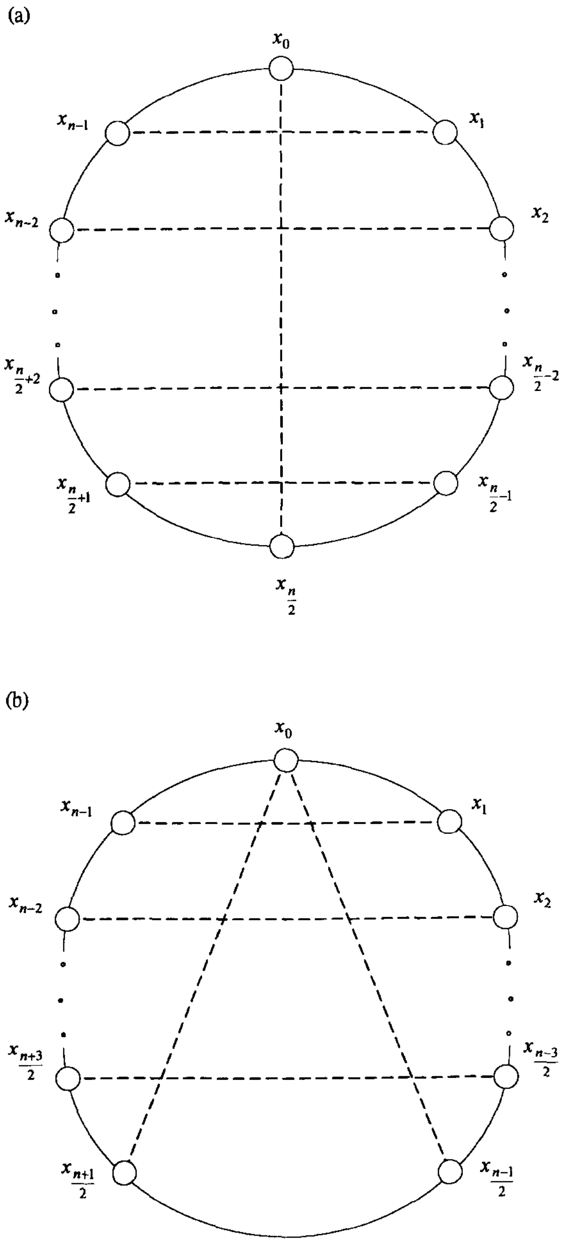

Fig. 4. (a) $\hat{C}_n$ for even n. (b) $\hat{C}_n$ for odd n.

morphic image of $C_n \times K_2$ in $((\hat{C}_n, C_n) \oplus (C_2, K_2)) - X$, which leads to a contradiction. Hence $\hat{C}_n$ is not $\mathrm{FG}(C_n)$.

Let n be a positive even integer. We construct a supergraph C_n^* of C_n as follows: $E(C_n^*) = E(C_n) \cup \{(x_i, x_{i+n/2}) \mid 0 \leqslant i < n/2\}$.

Lemma 6. *Let n be a positive even integer. C_n^* is $\mathrm{FG}(C_n)$ if and only if $n \geqslant 4$.*

Proof. Note that $C_2 = {}_2K_2$ and $C_2^* = {}_3K_2$. It is observed that there are two parallel edges in $C_2 \times K_2$ but there is no parallel edge in $((C_2^*, C_2) \oplus (C_2, K_2)) - Y_e$ for any $e \in E(C_2)$. Hence, there is no f_e satisfies the condition (2). Thus, C_2^* is not $\mathrm{FG}(C_2)$.

Now, we discuss the case $n \geqslant 4$. We define a function $\sigma : V(C_n) \to V(C_n)$ by assigning $\sigma(x_i) = x_{((i+n/2) \bmod n)}$ for every i. It can be observed that σ satisfies condition (1). (See Fig. 6(a) for the case $n = 4$; the vertices are labelled according to the function h.) Choose an arbitrary edge e from C_n, say $e = (x_{n/2-1}, x_{n/2})$. Consider a mapping $f_e : C_n \times K_2 \to ((C_n^*, C_n) \oplus (C_2, K_2))$ given by $f_e((x_j, z_k)) = (x_j, z_k)$ if $0 \leqslant j < n/2$, and $f_e((x_j, z_k)) = (x_{3n/2-j-1}, z_{3-k})$ otherwise. It follows that f_e satisfies condition (2). (See Fig. 6(b) for the case $n = 4$ with $e = (x_1, x_2)$; the vertices are labelled according to f_e.) Since $(C_n^*, C_n) \oplus (C_2, K_2)$ is node-symmetric, we can always find $f_{e'}$ for every edge $e' \in E(C_n)$ which satisfies condition (2). Hence C_n^* is $\mathrm{FG}(C_n)$. $\square$

Theorem 7. *Let G_i^* be $\mathrm{FG}(G_i)$ for $i = 1, 2$. The graph $(G_1^*, G_1) \oplus (G_2^*, G_2)$ is $\mathrm{FG}(G_1 \times G_2)$. In other words, let $W = \{(G^*, G) \mid G^*$ is $\mathrm{FG}(G)\}$. Then W is closed under the operation $\otimes$.*

Proof. Let $V_1 = \{x_0, x_1, \ldots, x_{m-1}\}$ and $V_2 = \{y_0, y_1, \ldots, y_{n-1}\}$. Since G_i^* is $\mathrm{FG}(G_i)$, G_i^* has an inversion σ_i for $i = 1, 2$. We define a function $\sigma : V(G_1 \times G_2) \to V(G_1 \times G_2)$ by assigning $\sigma((x_r, y_s)) = (\sigma_1(x_r), \sigma_2(y_s))$. Obviously, σ is a one-to-one mapping on $V(G_1 \times G_2)$. Since σ_i is an automorphism on G_i for $i = 1, 2$, it follows that $(x_i, x_j) \in E(G_1)$ implies $(\sigma_1(x_i), \sigma_1(x_j)) \in E(G_1)$ and that $(y_k, y_l) \in E(G_2)$ implies $(\sigma_2(y_k), \sigma_2(y_l)) \in E(G_2)$. If $((x_i, y_k), (x_i, y_l)) \in E(G_1 \times G_2)$, it follows that $(\sigma((x_i, y_k)), \sigma((x_i, y_l))) \in E(G_1 \times G_2)$. Similarly, we have $(\sigma((x_i, y_k)), \sigma((x_j, y_k))) \in E(G_1 \times G_2)$ if $((x_i, y_k), (x_j, y_k)) \in E(G_1 \times G_2)$. Thus σ is an automorphism on $G_1 \times G_2$ and satisfies condition (1).

Let e be an edge of $G_1 \times G_2$. We assume without loss of generality that $e = ((x_i, y_j), (x_k, y_j))$ where

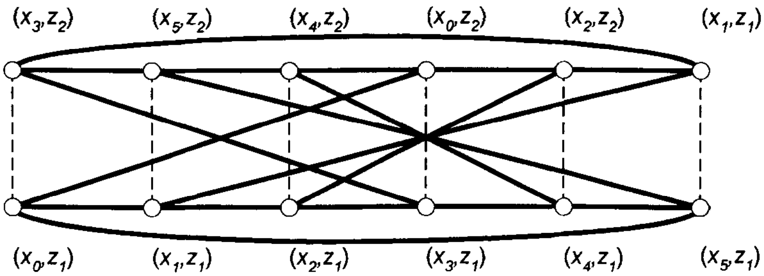

Fig. 5. C_6 is not a faithful graph of C_6.

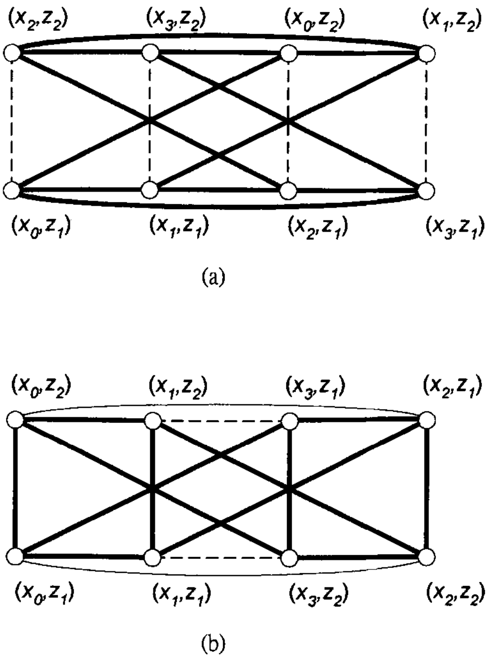

Fig. 6. The graph $(C_4^*, C_4) \oplus (C_2, K_2)$. (a) The function h satisfies condition (1). (b) The function f_e satisfies condition (2) where $e = (x_1, x_2)$.

$e' = (x_i, x_k)$ is an edge of G_1. Let $f_{e'}$ be a function from $G_1 \times K_2$ into a subgraph of $((G_1^*, G_1) \oplus (C_2, K_2)) - Y_{e'}$ such that $p_1(f_{e'}((x_i, z_1))) = p_1(f_{e'}((x_i, z_2)))$ for every $x_i \in V_1$. Define a function $f_e : V_1 \times V_2 \times V(K_2) \to V_1 \times V_2 \times V(K_2)$ as follows: $f_e((x_r, y_s, z_t)) = (x_u, y_v, z_w)$ where $(x_u, z_w) = f_{e'}(x_r, z_t)$, and $y_v = y_s$ if $z_w = z_t$, and $y_v = \sigma_2(y_s)$ otherwise.

For every $y_s \in V_2$, the image of f_e for $V_1 \times \{y_s\} \times V(K_2)$ is either

$$\{(x_u, y_v, z_w) \mid (x_u, z_w) = f_{e'}(x_r, z_t), y_v = y_s \in V_2, \text{ and } x_r \in V_1, z_t \in V(K_2), z_w = z_t\},$$

or

$$\{(x_u, y_v, z_w) \mid (x_u, z_w) = f_{e'}(x_r, z_t), y_v = \sigma_2(y_s), \text{ and } x_r \in V_1, z_t \in V(K_2), z_w \neq z_t\}.$$

In both cases, f_e induces an isomorphism from $V_1 \times V(K_2)$ into its image because the function $f_{e'}$ satisfies condition (2).

For every $(x_r, z_t) \in V_1 \times V(K_2)$ the image of f_e for $\{x_r\} \times V_2 \times \{z_t\}$ is either

$$\{(x_u, y_v, z_w) \mid (x_u, z_w) = f_{e'}(x_r, z_t), y_v = y_s \text{ and } y_s \in V_2, z_w = z_t\},$$

or

$$\{(x_u, y_v, z_w) \mid (x_u, z_w) = f_{e'}(x_r, z_t), y_v = \sigma_2(y_s), \text{ and } y_s \in V_2, z_w \neq z_t\}.$$

In both cases, f_e induces an isomorphism from V_2 into its image because the function σ_2 is an automorphism on V_2.

From the above discussion, we know that the function f_e induces an isomorphism from $V_1 \times V_2 \times V(K_2)$ into a subgraph of $(G_1^*, G_1) \oplus (G_2^*, G_2) \oplus (C_2, K_2) - \{((x_r, y_s, z_1), (x_r, y_s, z_2)) \mid x_r \in V_1, y_s \in V_2\}$. Furthermore, $p_{1,2}(f_e((x_r, y_s, z_1))) = p_{1,2}(f_e((x_r, y_s, z_2)))$ where $p_{1,2}(x, y, z) = (x, y)$. Hence f_e satisfies condition (2). Thus $(G_1^*, G_1) \oplus (G_2^*, G_2)$ is FG$(G_1 \times G_2)$. $\square$

Corollary 8. *Let G_i^* be a faithful graph of G_i for $i = 1, 2$. The graph $(G_1^*, G_1) \oplus (G_2^*, G_2)$ is 1-EFT$(G_1 \times G_2)$. Furthermore, $eft_1(G_1 \times G_2) \leqslant 2(|E(G_1^*)| - |E(G_1)|)(|E(G_2^*)| - |E(G_2)|)$.*

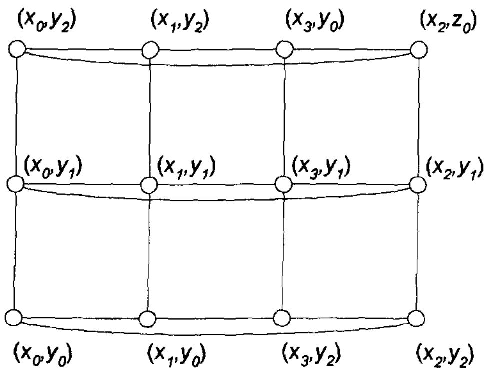

Fig. 7. A reconfiguration of $C_4 \times P_3$ in $((C_4^*, C_4) \oplus (P_3^*, P_3)) - ((x_1, y_1), (x_2, y_1))$.

Proof. It follows from Theorem 7 and Lemma 4 that $(G_1^*, G_1) \oplus (G_2^*, G_2)$ is 1-EFT$(G_1 \times G_2)$. Furthermore, $|E((G_1^*, G_1) \oplus (G_2^*, G_2))| - |E(G_1 \times G_2)| \leqslant 2(|E(G_1^*) - E(G_1)|)(|E(G_2^*)| - |E(G_2)|)$. Therefore the corollary follows. □

Corollary 9. *Let G_i^* be* FG(G_i) *for $i = 1, 2$, and $e' = (x_j, x_k)$ be any edge of G_1. Let $f : V(G_1 \times G_2) \to V(G_1 \times G_2)$ be a function given by $f((x_r, y_s)) = (x_u, y_v)$ where $x_u = g_{e'}(x_r)$ (i.e., e'-rotation of G_1^*), and $y_v = y_s$ if $p_2(f_{e'}((x_r, z_t))) = z_t$, and $y_v = \sigma_2(y_s)$ otherwise. Then f induces an isomorphism from $G_1 \times G_2$ into a subgraph of $(G_1^*, G_1) \oplus (G_2^*, G_2) - e$ for any edge $e = ((x_i, y_j), (x_k, y_j))$ with $y_j \in V(G_2)$.*

Corollary 9 can be found in the proof of Theorem 7 for the satisfaction of condition (2). Using Corollary 9 we can construct reconfigurations for any 1-edge fault. We illustrate in Fig. 7 an isomorphism from $C_4 \times P_3$ into a subgraph of $(C_4^*, C_4) \oplus (P_3^*, P_3) - ((x_1, y_1), (x_2, y_1))$, i.e., a reconfiguration for 1-edge fault on $((x_1, y_1), (x_2, y_1))$.

4. Discussion

The cartesian product is one of useful graph products. Many popular interconnection networks are built as cartesian product graphs [3]. As shown by Sabidussi [4], all graphs have a unique prime factorization with respect to the cartesian product. Applying Theorem 7, we can easily construct a 1-EFT graph with respect to some graph G if we know the faithful extensions of all of its prime factors. For example, let $m_1, m_2, \ldots, m_n$ be positive even integers with each $m_i \geqslant 4$. We use $C^*(m_1, m_2, \ldots, m_n)$ to denote the graph $(C_{m_1}^*, C_{m_1}) \oplus (C_{m_2}^*, C_{m_2}) \oplus \cdots \oplus (C_{m_n}^*, C_{m_n})$. It follows from Lemma 6 and Theorem 7 that $C^*(m_1, m_2, \ldots, m_n)$ is a faithful graph, and thus 1-EFT$(C(m_1, m_2, \ldots, m_n))$. It is observed that $\delta(H) \geqslant 1 + \delta(G)$ for any 1-EFT(G) graph H. Since $\delta(C(m_1, m_2, \ldots, m_n)) = 2^n$ and $\Delta(C^*(m_1, m_2, \ldots, m_n)) = 2^n + 1$, $C^*(m_1, m_2, \ldots, m_n)$ is an optimal 1-EFT graph w.r.t. $C(m_1, m_2, \ldots, m_n)$, which is concluded in the following lemma.

Lemma 10. *Let $m_i \geqslant 4$ be a positive even integer for all i. $C^*(m_1, m_2, \ldots, m_n)$ is an optimal* 1-EFT *graph w.r.t. $C(m_1, m_2, \ldots, m_n)$. Furthermore, $eft_1(C(m_1, m_2, \ldots, m_n)) = \frac{1}{2}\prod_{i=1}^{n} m_i$.*

However, the problem of deciding $eft_1(C(m_1, m_2, \ldots, m_n))$ with some odd m_i seems very difficult. We have the following conjecture.

Conjecture 11.

$$eft_1(C(m_1, m_2, \ldots, m_n)) > 2^{n-1} \prod_{i=1}^{n} \lceil m_i/2 \rceil$$

if each m_i is an odd integer with $m_i \geqslant 5$ and $n \geqslant 2$.

On the other hand, since the graph $\hat{C}_n$ is not FG(C_n) for $n \geqslant 4$, there is no straightforward method for us to use this family $\hat{C}_n$ for constructing 1-EFT graphs w.r.t. tori.

A hypercube Q_n can be treated as a mesh $M(2, 2, \ldots, 2)$. Let $C_2^{n^*}$ denote the graph $(C_2, K_2) \oplus (C_2, K_2) \oplus \cdots \oplus (C_2, K_2)$ (n times). It follows from Theorem 7 that $C_2^{n^*}$ is FG(Q_n) and thus 1-EFT(Q_n). It was also proved in [2] that $C_2^{n^*}$ is an optimal 1-EFT(Q_n) graph.

Lemma 5 states that any graph has a faithful supergraph. We can apply Theorem 7 to obtain a faithful graph for the cartesian product of several graphs. Note that our construction method enables us to find a 1-EFT(G) though not necessarily optimal when G is the cartesian product of several graphs. Take the mesh $M(m_1, m_2, \ldots, m_n)$ as an example. It follows from Lemma 3 and Theorem 7 that the graph $(P_{m_1}^*, P_{m_1}) \oplus (P_{m_2}^*, P_{m_2}) \oplus \cdots \oplus (P_{m_n}^*, P_{m_n})$

is 1-EFT$(M(m_1,m_2,\ldots,m_n))$. The difference of the number of edges between these two graphs is $2^{n-1}\prod_{i=1}^{n}\lceil m_i/2\rceil$. However, it was proved in [1] that $eft_1(M(m_1,m_2,\ldots,m_n)) \leqslant \frac{1}{2}(\prod_{i=1}^{n} m_i - \prod_{i=1}^{n}(m_i - 2))$. Thus, our construction is not optimal for meshes. On the other hand, some graphs obtained from the operator $\oplus$ on graph pairs are optimal 1-EFT graphs. Consider the example of $P_3 \times K_2$. It is known that C_3 is an optimal 1-EFT(P_3). But C_3 is not FG(P_3). Though $(C_3,P_3)\oplus(C_2,K_2)$ is not FG$(P_3\times K_2)$, Chou and Hsu [1] proved that $(C_3,P_3)\oplus(C_2,K_2)$ is an optimal 1-EFT$(P_3\times K_2)$.

Though (optimal) 1-EFT(G) graphs may not be necessarily FG(G), the concept of faithful graphs incorporating with the operator $\oplus$ provides a construction scheme for 1-EFT(G). In other words, applying Theorem 7 we can construct a FG(G), which is always 1-EFT(G), especially when G is the cartesian product of several graphs. Furthermore, we also note that cartesian product and Kronecker product are widely studied in graph theory. To our knowledge, no connection between these two products are known. Our result provides a possible connection between these two products.

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