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Two-Person Second-Order Games, Part 1: Formulation and Transition Anatomy

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Abstract It is well known that human psychology determines his/her action and behavior. This fact has not been fully incorporated in game theory. This paper intends to incorporate human psychology in formulating games as people play them. In Part 1 of the paper, we formulate a two-person game by the habitual domain theory and the Markov chain theory. Using the habitual domains theory, we present a new model describing the evolution of the states of mind of players over time, the *two-person second-order game*. We introduce the concept of the *focal mind profile* as well as the solution concept of the *win-win mind profile*. In addition, we provide also a method to predict the average number of steps needed for a game to reach a focal or win-win mind profile. Then, in Part 2 of the paper, under some reasonable assumptions, we derive the *possibility theorem* stating that it is always possible to reach a win-win mind profile when suitable conditions are satisfied.

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1 Introduction

In the traditional noncooperative game theory [1, 2], the data of the game, the set of players, the payoff functions and the set of strategies are supposed to be more or less fixed. The players' psychological states are not fully taken into account. These assumptions limit the application of game theory to real game situations. In fact, all games involve human psychology, which changes dynamically over time and situation as explained in habitual domain (HD) theory [3–7]. We introduce this aspect to game theory so that it could be richer and more applicable. In Part 1 of the paper, we focus on formulating games in terms of Markov chains with players' states of mind as the states of the Markov chain. We study the transition of the states of mind and how fast the game can reach a win-win state, in which the game is solved and both players can declare victory.

The paper is organized as follows. Section 2 presents briefly the HD theory and some examples showing the importance of psychological aspects in games. Section 3 is devoted to the formulation of two-person games in terms of the states of mind of players by HD theory and Markov chains theory. In Sect. 4, we provide an anatomy of the transition probability matrix of two-person second-order games. Section 5 concludes the paper.

2 Preliminaries

In this section, we illustrate the importance of psychological aspects in games and present briefly the *Habitual Domain* (HD) theory.

2.1 Importance of Psychological Aspects in Games

In this section, we show through examples the importance of considering the psychological aspects in a game situation.

Example 2.1 (Prisoner Dilemma). Two firms, Firm 1 and Firm 2, are selling the same product in a the same market. Each of the firms has two strategies, sales price SP or regular price RP with RP > SP. This game situation can be assumingly represented by the following bimatrix game known in literature as the prisoner dilemma game [8]

$$\begin{array}{cc} & \begin{array}{cc} \text{RP} & \text{SP} \end{array} \\ \begin{array}{c} \text{RP} \\ \text{SP} \end{array} & \begin{pmatrix} (11, 11) & (1, 15) \\ (15, 1) & (5, 5) \end{pmatrix} \end{array}. \tag{1}$$

The payoffs are given in terms of monetary benefits. Collectively, the strategy profile (RP, RP) is better than (SP, SP), because RP > SP. In the case (SP, RP) or (RP, SP) the firm which adopts the strategy SP enjoys a larger benefit of 15 units for it attracts most of the buyers. Traditional game theory does not propose a globally stable solution to the game (1). Indeed, the profile (SP, SP) is a Nash equilibrium at which the players can be attracted by the profile (RP, RP) where they all get lager payoffs. Note that Nash equilibrium provides only "local" stability against unilateral moves, not collective moves. If the players are at the profile (RP, RP), each of them has a very strong incentive to move to the strategy *SP* for a better payoff individually. Hence the profile (RP, RP) is also not stable. That (RP, SP) and (SP, RP) are not stable is obvious. Thus, no strategy profile can be considered as a stable solution of the game (1) as long as its structure remains the same. In game (1) the psychological aspects that may generate changes in the game structure including changes of the sets of strategies and payoffs are not considered.

Example 2.2 Game of silence [6]. Silence fell over a young couple after a family quarrel. They did not talk to each other for two days. The situation became uneasy for husband and wife. They want to return to normal life without losing face.

The game cannot be represented by a normal form game because the payoff functions and the sets of strategies of players are not determined. Indeed, here the players are in a stressful state and the strategies for solving the game need to be generated.

Example 2.3 Alinsky's Strategy [7]. During the days of the Johnson-Goldwater campaign, commitments made by authorities to the Woodlawn ghetto organization of Chicago were not being met. The organization was powerless. As the organization was already committed to support the democratic administration, the president's campaign did not bring them any help. The organization felt frustrated and did not know how to make the authorities cooperate. The game is about establishing a cooperative atmosphere between the two parties.

This game situation cannot be represented by a normal form game. The payoff functions and the sets of strategies are not determined. The players have to generate new strategies for solving the game. The restructuring of the game is necessary.

Examples 2.1–2.3 show clearly the importance of the psychological aspects in game situations. In fact the states of mind of the players determine the outcome of the game. The games in Examples 2.2–2.3 have no formulation within the framework of traditional noncooperative game theory, because the sets of strategies are not determined and the payoffs or utility functions are not defined or very difficult to estimate. In this paper we will show step by step how the game situations 2.1–2.3 can be solved when they are formulated as second-order games by Habitual Domain theory.

2.2 Habitual Domain Theory

In this section, because of space constraint, we briefly introduce the habitual domain theory. For more details we refer the reader to the books [3, 4, 7] and the paper [5].

The collection of ideas and actions (including ways of perceiving, thinking, responding, acting, and memory) in our brain together with their formation, dynamics, and basis in experience and knowledge, is called our *habitual domain* (HD) [5]. Over time, unless extraordinary or purposeful effort is exerted, our HD will become stabilized within a certain domain. This can be mathematically proved [3, 9]. As a consequence, we observe that each of us has habitual ways of eating, dressing, speaking, etc. Some habitually emphasize economic gains, while others pay attention to social reputation. Some habitually persist in their pursuit of goals, while others change their objectives often. Some are habitually positive and optimistic, while others are negative and pessimistic. Some habitually pay attention to details, others only to generalities.

The concept of individual's HD can be extended to other living entities, such as companies, social organizations, and groups in general. The following are the basic elements of HD.

(i) The *potential domain* PD: the collection of ideas and actions that can potentially be activated to occupy our attention.
(ii) The *actual domain* AD: the set of ideas and actions that are actually activated or occupy our attention.
(iii) The *activation probabilities* AP: the probabilities that ideas or actions in PD also belong to AD.
(iv) The *reachable domain* RD: the set of ideas and actions that can be attained from a given set in an AD.

Thus, the habitual domain can be formally formulated as

$$\mathrm{HD}_t = \{\mathrm{PD}_t, \mathrm{AD}_t, \mathrm{AP}_t, \mathrm{RD}_t\}$$

where t represents time. The theory of HD is based on eight hypotheses H1–H8.

(H1) Circuit Pattern. Thoughts, concepts or ideas are represented by circuit patterns of the brain. A circuit patterns will be reinforced when the corresponding ideas are repeated. Furthermore, the stronger the circuit patterns, the more easily the corresponding thoughts are retrieved in our thinking and decision making process.
(H2) Unlimited Capacity. Practically every normal brain has capacity to encode and store all thoughts, concepts and messages that one intends to.
(H3) Efficient Restructuring. The encoded thoughts, concepts and messages (H1) are organized and stored systematically as data bases for efficient retrieving. Furthermore, according to the dictation of attention they are continuously restructured so that relevant ones can be efficiently retrieved to release charge.
(H4) Analogy and Association. The perception of new events, subjects, or ideas can be learned primarily by analogy and/or association with what is already known. When faced with a new event, subject, or idea, the brain first investigates its features and attributes in order to establish a relationship with what is already known by analogy and/or association. Once the right relationship has been established, the whole of the past knowledge (preexisting memory structure) is automatically brought to bear on the interpretation and understanding of the new event, subject, or idea.
(H5) Goal Setting and State Evaluation. Each one of us has a set of goal functions and for each goal function we have an ideal state or equilibrium to reach and maintain (goal setting). We continuously monitor, consciously or subconsciously,

where we are relative to the ideal state or equilibrium point (state evaluation). Goal setting and state evaluation are dynamic, interactive and are subject to physiological forces, self-suggestion, external information forces, current data bank (memory) and information processing capacity.

(H6) Charge Structure and Attention Allocation. Each event is related to a set of goal functions. When there is an unfavorable deviation of perceived value from the ideal, each goal function will produce various levels of charge. The totality of the charges created by all goal functions is called the *charge structure* and it can change dynamically. At any point in time, our attention will be paid to the event which has the most influence on our charge structure.

(H7) Discharge. To release charges, we tend to select the action which yields the lowest remaining charge (the remaining charge is the resistance to the total discharge) and this is called the *least resistance principle*.

(H8) Information Input. Humans have innate needs to gather external information. Unless attention is paid, external information inputs may not be processed.

Note that hypotheses H1–H4 describe how the brain functions and hypotheses H5–H8 describe how the mind functions. For the details on these hypotheses we refer the reader to [4, 7].

Remark 2.1 It is clear that an analysis of a game situation based on hypotheses H1–H8 will capture more psychological aspects than the traditional game theory does (see [10] and references therein). Most of aspects of hypotheses H1–H8 are not considered in the traditional game theory framework. In fact, most of the models of traditional game theory do not consider the psychological states of players and their changes during the game. For example if the players do not interact between them and with the external world, they do not use hypothesis H8 at all. H2 suggests that players have unlimited capacity to learn if they are willing to. According to H5, players have goal functions and an ideal state for each of them; they continuously monitor where they are relative to the ideal states. According to H6, a charge is a precursor of a mental force to action or inaction, which could lead to drive or stress. In this paper, for simplicity, the charge structure will be called *charge level*. At any point in time, our attention will be paid to the event which has the most important influence on our charge level. The event or decision problem with the most significant charge commands our attention at any given moment. When our attention is allotted to an event, we use the following modes of action to deal with it: *active problem solving* or *avoidance justification*. The former tries to work actively to move the perceived states closer to the ideal states (discharge H7); while the later tries to rationalize the situation so as to lower the ideal states closer to the perceived states. With active problem solving, charge is transferred to drive, while with avoidance justification, charge may be reduced or transferred into stress.

2.3 Analysis of the Examples by Habitual Domain Theory

Example 2.1. (Prisoner dilemma) Here measurable criterion is given for the evaluation of strategy profiles, the payoffs. Hence, the charge levels of both players will be expressed by payoffs: the better the payoff of a player, the lower his charge level.

The strategy profile (RP, RP) may be attractive for the players because they receive the maximum collective payoff $11 + 11 = 22$. However, at this strategy profile, by goal setting and state evaluation H5, the players see that their charge levels are not at their minimum levels. Each of them has a great incentive to move to the strategy SP (discharge H7). Indeed, if one of the players deviates to the strategy *SP* and the other remains in strategy RP, then the deviating player will enjoy a significant increase in his payoff from 11 to 15 (discharge H7). At the profile (RP, SP) or (SP, RP) the charge level of the player who uses strategy RP is very high for he receives only a payoff of 1 (charge structure and attention allocation H6), while the other receives a payoff of 15. At the profile (SP, SP) the charge level of both players is not at the lowest level for they may do better if they both move to the strategy RP (goal setting and state evaluation H5). If there is no interaction between players and their habitual domains are not expanded (information input H8 is not used), there is no win-win profile where no player has any desire to deviate. In each of these possible strategy profiles at least one player is willing to deviate for his charge level is not at the lowest level (discharge H7). Very likely, the players will choose the strategy profile (SP, SP) and remain there. But staying in the strategy profile (SP, SP) may be harmful for both firms in the short or long term. Such situation is known as a *decision trap* in the theory of Habitual Domains. Although the players remain in (SP, SP), their level of charge will be increasing (charge structure and attention allocation H6) because they know that they can do better if they choose the strategy profile (RP, RP) (goal setting and state evaluation H5). Thus, the strategy profile (RP, RP) has some appeal for the players, but it lacks stability.

Example 2.2. After the quarrel there was no communication between wife and husband. Self esteem became the most important goal for both players. Each of them thought that breaking the silence first would hurt the self-esteem and create high charge level because of a large deviation from its ideal value (goal setting and state evaluation hypothesis H5). The absence of communication means that the hypothesis H8 (information input) is not used at all.

Example 2.3. An agreement has been made, the charge level of both players (the organization and the authorities) was low; a strong circuit pattern of cooperation, a state of mind, has formed in the brain of the players (circuit pattern hypothesis H1). After knowing the authorities didn't meet their commitment, the organization saw a large deviation from its ideal goals and felt very frustrated (goal setting and state evaluation H5). This created a high charge level on the organization (charge structure and attention allocation H6). The frustration worsened when the organization didn't find any solution to make authorities change their mind and release their charge (charge structure and attention allocation H6). On the other hand, the charge level of the authorities was low for the organization could not find or make any action that may increase it (goal setting and state evaluation H5).

3 Formulation of a Two-Person Game in Terms of HD Theory

Let us consider a game situation involving two players, I and II. Denote by PD^I, AD^I and PD^{II}, AD^{II} the potential domain (PD) and the actual domain (AD) of Player I and Player II respectively. In terms of HD theory a state of mind of a player can be considered as a collection of ideas or thoughts that could be activated from his potential domain PD to the actual domain. It can also be considered as a part of his reachable domain. Hence we will identify a *state of mind* of a player with its corresponding ideas or thoughts in the potential domain of the player.

In applications, the players could be hostile (noncooperative) or friendly (cooperative) to each other (two states of mind). If needed, the moods or states of mind could be further decomposed into extremely hostile, hostile, neutral, friendly and extremely friendly. The number of states of mind in almost all practical cases is still finite, even if we further decompose them. Therefore, we have the following assumption.

Assumption 3.1 The number of states of mind of each player is finite for the game under consideration.

Let us assume that Player I has s states of mind and Player II has l states of mind regarding the game, denoted respectively by

$$S^{\mathrm{I}} = \{S_1^{\mathrm{I}}, S_2^{\mathrm{I}}, \dots, S_s^{\mathrm{I}}\} \quad \text{and} \quad S^{\mathrm{II}} = \{S_1^{\mathrm{II}}, S_2^{\mathrm{II}}, \dots, S_l^{\mathrm{II}}\}.$$

We refer to the Cartesian product $S = S^{\mathrm{I}} \times S^{\mathrm{II}}$ as the *profile* (states) *space* and its elements as the *mind profiles* or simply the profiles of the game.

Definition 3.1 Player i is in the state of mind S_j^i at time t, if S_j^i is activated from his potential domain PD^i to his actual domain AD^i at time $t, i = \mathrm{I}, \mathrm{II}$. That is, $S_j^i \in \mathrm{AD}^i$ at time $t, i = \mathrm{I}, \mathrm{II}$.

Assumption 3.2 At any time t the actual domain of each player contains only one state of mind and only one state can be activated from the potential domain to the actual domain.

This assumption reflects the fact that, at any moment, the actual domain (attention) of each player is occupied by only one state of mind.

3.1 Formulation of the Game as a Markov Chain

We first formulate the game as a stochastic process. In order to simplify the presentation, in the remaining part of the paper, we will identify the set of states of mind of Player I, S^{I}, with $\{1, 2 \dots, s\}$ and the set of states of mind of Player II, S^{II}, with $\{1, 2, \dots, l\}$. Thus, any profile of the game $(S_i^{\mathrm{I}}, S_j^{\mathrm{II}})$ in $S = S^{\mathrm{I}} \times S^{\mathrm{II}}$ will be denoted by the pair (i, j). Since the activation process from potential domain to the actual domain is probabilistic and the cardinality of $S = S^{\mathrm{I}} \times S^{\mathrm{II}}$ is $s \times l$, we can consider the problem as a stochastic process $\{X_n, n = 0, 1, 2, \dots\}$ with a finite number $s \times l$ of states.

Now let us define the transition probability from any profile to any other profile. Given that a profile (i, j) is activated at the present step, the probability that a profile $(i_1, j_1) \in S$ will be brought to the actual domain in the next step is given by the conditional probability $P_{(i,j),(i_1,j_1)}$.

In practice, the probability $P_{(i,j),(i_1,j_1)}$ can be evaluated by two ways. The first one is by subjective evaluation (by players or/and experts); the second one is by frequency approach i.e. the frequency of activation of the profile (i_1, j_1) if the present profile is (i, j), when the game is repeated a certain number of times.

Assumption 3.3 The profile of the process in the next step depends only on the profile of the process in the present step, i.e.

$$P\big(X_{p+1} = (i, j)/X_p = (i_p, j_p), X_{p-1} = (i_{p-1}, j_{p-1}), \dots, X_0 = (i_0, j_0)\big) \\ = P_{(i_p,j_p)(i,j)}.$$

Assumption 3.3 may be justified as follows. According to HD theory [3–5, 7, 9], unless extraordinary events occur or a special effort is exerted, the activation probability of ideas and actions stabilizes over time. Thus, when there is no occurrence of extraordinary event or a special effort is exerted by players to restructure the game, we may assume that the transition probability of the processes has the Markov property.

Assumption 3.3 implies that the problem can be considered as a Markov chain with a finite number of states. Hence the results of Markov chain theory can be used to study the considered game. The matrix

$$P = (P_{(i_1,j_1)(i,j)}), \quad ((i, j), (i_1, j_1)) \in S^{\mathrm{I}} \times S^{\mathrm{II}},$$

is the *transition probability matrix* of the process. It is a stochastic matrix that describes the behavior of the players with respect to the game situation. According to HD theory, it tends to be stable, but it is subject to changes over time when some new relevant idea or event arrives. Thus, the game can be represented by the following model:

$$\langle \{\mathrm{I}, \mathrm{II}\}, \{\mathrm{HD}^{\mathrm{I}}, \mathrm{HD}^{\mathrm{II}}\}, \{S^{\mathrm{I}}, S^{\mathrm{II}}\}, P\rangle \tag{2}$$

where HD^{I} and $\mathrm{HD}^{\mathrm{II}}$ are the habitual domains of Player I and Player II respectively. If the transition probability matrix P changes at some time t, then we obtain a new game of type (2) with a new transition probability matrix that starts at time t. Note that P is a function of time and HD^{I} and $\mathrm{HD}^{\mathrm{II}}$, and P captures the psychological atmosphere of the game, which is an AD of the game situation.

Definition 3.2 A profile $(i, j) \in S^{\mathrm{I}} \times S^{\mathrm{II}}$ is called a *focal profile* if it is a profile that has some special appeal to the players, and it is desirable to reach it and make it stable.

A focal profile can be chosen based on general principles such as peace, equity, fairness, self-interest, collective interest, stability, harmony, etc. For example, in the prisoner dilemma game (Example 2.1), the profile where both players choose the strategy RP can be considered as a focal profile based on collective interest. Indeed, when the players play (RP, RP) they get the highest total payoff, $11 + 11 = 22$. The collective and individual interests can be satisfied through arrangements [11]. We can qualify the state of mind of both players when they play (RP, RP) by “cooperative”.

Thus the profile where both players are in a "cooperative" state of mind is a focal profile of the game (1). Note that the "cooperative" mind profile is not stable, because both players have a great incentive to shift to "noncooperative" state of mind by using SP once they are there.

Definition 3.3 A profile $(i, j) \in S^{\mathrm{I}} \times S^{\mathrm{II}}$ is called *win-win profile* of the game (2) if

(i) (i, j) is a focal profile.
(ii) (i, j) is an *absorbing* profile or equivalently $P_{(i,j)(i,j)} = 1$.

Assumption 3.4 The game (2) has a focal profile.

This assumption is quite natural, since usually in any game there is some profile that has some appeal to both players. When there is no explicit focal profile, we can consider the profile where both players have a "Cooperative" state of mind as a focal profile. With cooperative spirit, the players can find a third new alternative such that both players can claim victory. Thus, the existence of a focal profile in the game is not a serious problem. The following are challenging problems: (i) under what conditions is it possible to activate a focal profile in a conflict situation? That is, when is it possible to reach it and how to reach it? (ii) How to make it a win-win profile? In other words, how to make it stable once it is reached? This paper is to answer these questions with the help of HD theory and the theory of Markov Chains.

Remark 3.1 Assume that (i, j) is a win-win mind profile. By definition $P_{(i,j)(i,j)} = 1$, hence $P_{(i,j)(i',j')} = 0$, for all $(i', j') \neq (i, j)$. This means that once the players reach the win-win mind profile (i, j), any deviation, unilateral or multilateral, from it is impossible. Thus a win-win mind profile is stable locally and globally. This makes it definitely different and more general than Nash equilibrium. Indeed, Nash equilibrium [2] is immune to unilateral deviations but it is not immune to multilateral deviations. Many researchers have introduced various refinements of Nash equilibrium that are immune against multilateral deviations (coalition deviation) such as the strong equilibrium [12], Coalition Proof Nash equilibrium [13] and Strong Berge equilibrium [14]. However, the major difficulty with these equilibria is that they do not exist in most of the games and the few existence results established are restrictive e.g. [15]. Furthermore, Nash equilibrium is based on strategies and utility function, while the win-win mind profile is based on states of mind and charge level.

Remark 3.2 In order to avoid distraction we give the following short comparison between the traditional games in normal form and the second-order game (2).

(i) In traditional games of normal form, the outcomes of the game are expressed in terms of a utility function. In game (2) the outcome of the game is not in terms of payoff, rather it is in terms of charge level [4, 7]. The objective of each player is to reduce his/her charge level.
(ii) In traditional games of normal form, the sets of strategies are somehow predetermined and fixed. In the game (2) there are no restrictions or limitations on strategies. New strategies can be generated as deeper parts of HD are reached. The players' HDs can generate new strategies as the game evolves.

(iii) In traditional games of normal form, the structure of the game, the sets of strategies and the payoffs are fixed, no reframing is allowed. In second-order games the structure of the game is dynamic and subject to restructuring.
(iv) In traditional games of normal form, the interaction between the players and with the external world is not fully considered. In second-order games, the information input hypothesis H8 plays an important role in reaching a win-win profile.
(v) In traditional games of normal form the players are assumed to be *rational*; in second-order games each player plays the game as he perceives it by his habitual domain and hypotheses H1–H8.

Thus, the difference between the two models is structural and conceptual at the same time. In psychological games [16] the psychological states of players are partially taken into account by incorporating the beliefs of each player about the other players' strategies in the payoff functions. However, the structure of the game is still as in the traditional games. Indeed, the sets of strategies are still more or less fixed too, the outcomes are expressed in terms of utility function, the concept of solution used is Nash equilibrium, not a win-win profile, and no restructuring of the game is considered. Finally, note that the game (2) is beyond stochastic games [17] and [18]. Stochastic games did make a considerable progress in considering a variety of situations (states) in the game; however, the set of strategies is more or less fixed, the solution concept is basically Nash equilibrium, not a win-win profile, and no restructuring of the game is considered. The transition probability matrix is not based on psychological aspect as in game (2).

In [19] the theory of fuzzy games has been introduced as a generalization of the traditional strategic games. In the Butnariu's framework the beliefs of players are expressed by fuzzy sets, payoffs are not involved. It is a considerable advance, but traditional sets of strategies of players are involved in the model and the concept of solution used is Nash equilibrium not a win-win profile. In addition, the restructuring of the game is not considered.

Now let us illustrate by examples how the game (2) can be constructed.

Example 3.1 (Prisoner Dilemma). Let us formulate the prisoner dilemma game of Example 2.1 as a second-order game. First we identify the states of mind of players. Based on previous analysis of this game, each player has at least two states of mind regarding this game: "Cooperation" and "Noncooperation", denoted by C and N respectively. For simplicity of presentation, let us assume that each of the players has only these two states of mind regarding the game. Here by "Cooperation" we mean that a player is in a state of mind such that he is willing to play the strategy RP, and by "Noncooperation" we mean the player is in a state of mind where he is willing to play the strategy SP. Indeed, based on collective interest, the strategy profile (RP, RP) is the best since it yields a maximum total payoff of $11 + 11 = 22$. Thus, based on collective interest, we can consider the profile (C, C) where both players are in cooperative state of mind as a focal profile of the game. From previous discussion we can reasonably assume that in a game like the prisoner dilemma,

$$P_{(\mathrm{C,C})(\mathrm{C,C})} \le P_{(\mathrm{C,C})(\mathrm{N,C})}, \quad \text{and} \quad P_{(\mathrm{C,C})(\mathrm{C,C})} \le P_{(\mathrm{C,C})(\mathrm{C,N})},$$

and $P_{\rm (N,N)(N,N)}$, $P_{(N,C)(N,N)}$, $P_{(C,N)(N,N)}$ should be large; $P_{\rm (C,C)(C,C)}$, $P_{\rm (N,N)(N,C)}$ and $P_{\rm (N,N)(C,N)}$ should be small. As an example, we may assume that the transition probability matrix of the game is

$$P = \begin{array}{c} \\ (\mathrm{C,C}) \\ (\mathrm{C,N}) \\ (\mathrm{N,C}) \\ (\mathrm{N,N}) \end{array} \begin{array}{c} \begin{array}{cccc} (\mathrm{C,C}) & (\mathrm{C,N}) & (\mathrm{N,C}) & (\mathrm{N,N}) \end{array} \\ \begin{pmatrix} 0.1 & 0.4 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.6 \\ 0.15 & 0.1 & 0.1 & 0.65 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{pmatrix} \end{array}$$

Example 3.2 (Game of Silence). There are two players, the husband (Player I) and the wife (Player II). Assumingly, there are also two states of mind for each player: C (cooperation) and N (noncooperation). The state C means to be willing to break the silence and return to harmony and normal life, and N means to continue to be silent and hostile. Obviously, the focal profile is (C, C). After the quarrel there was no cooperation at all since there was no communication. At this phase of the game we assumingly estimate P as

$$P = \begin{array}{c} \\ (\mathrm{C,C}) \\ (\mathrm{C,N}) \\ (\mathrm{N,C}) \\ (\mathrm{N,N}) \end{array} \begin{array}{c} \begin{array}{cccc} (\mathrm{C,C}) & (\mathrm{C,N}) & (\mathrm{N,C}) & (\mathrm{N,N}) \end{array} \\ \begin{pmatrix} 0 & 0 & 0.1 & 0.9 \\ 0 & 0.1 & 0.1 & 0.8 \\ 0 & 0.1 & 0.1 & 0.8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example 3.3 (Alinsky's Strategy). There are two players, the organization (Player I) and the authorities (Player II). Again we can reasonably identify the states of mind of each player as cooperative and noncooperative, denoted by C and N respectively. We can reasonably assume that the (C, C) is a focal profile. In this game we identify two phases. The Phase I, before the organization knew that the commitments are not being met; the Phase II is after this event. We obtain the game

$$\langle \{\mathrm{I, II}\}, \{\mathrm{HD^I}, \mathrm{HD^{II}}\}, \{\{\mathrm{C, N}\}, \{\mathrm{C, N}\}\}, P\rangle,$$

where the matrix P changes the form when the players shift from Phase I to Phase II. At Phase I, since an agreement has been made, the charge level of both players was low; the transition probability matrix of this game can be assumingly estimated as P given below.

$$P = \begin{array}{c} \\ (\mathrm{C,C}) \\ (\mathrm{C,N}) \\ (\mathrm{N,C}) \\ (\mathrm{N,N}) \end{array} \begin{array}{c} \begin{array}{cccc} (\mathrm{C,C}) & (\mathrm{C,N}) & (\mathrm{N,C}) & (\mathrm{N,N}) \end{array} \\ \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.7 & 0.1 & 0.1 & 0.1 \\ 0.8 & 0.1 & 0.1 & 0 \\ 0.7 & 0.1 & 0.1 & 0.1 \end{pmatrix} \end{array}$$

In Phase 2, by (information input H8), the organization knew that commitments are not being met by the authorities. This information completely restructured the game. We can assumingly estimate the transition probability matrix of the game at this phase as

$$P' = \begin{array}{c} \\ (C,C) \\ (C,N) \\ (N,C) \\ (N,N) \end{array} \begin{array}{c} \begin{array}{cccc} (C,C) & (C,N) & (N,C) & (N,N) \end{array} \\ \begin{pmatrix} 0.1 & 0.7 & 0.1 & 0.1 \\ 0 & 0.8 & 0.0 & 0.2 \\ 0.7 & 0.1 & 0.0 & 0.2 \\ 0.1 & 0.1 & 0.0 & 0.8 \end{pmatrix} \end{array}$$

4 Anatomy of the Transition Probability Matrix

As described in the previous section, P is a function of time. Indeed, it is an AD (actual domain) of the game situation. It reflects the psychological climate of the game (players). Given P and a focal profile (i^0, j^0), we want to know with this particular P, can the focal profile (i^0, j^0) be reached, and at what speed. This will be explored in this section with the help of finite Markov chains. In case the focal point cannot be reached or cannot become win-win profile, we want to restructure the game so that P will become P^*. With P^*, the focal point will become a reachable win-win profile, this will be explored in the Part 2 of the paper.

4.1 Classification of States in Finite Markov Chains

In order to facilitate our presentation let us state some basic definitions and results of finite Markov chains theory. The material is taken mainly from [20]. Let $P_{ij}^{(n)}$ be the (i, j)th entry of the power matrix P^n.

Definition 4.1 A state j is said to be *accessible* from a state i if j can be reached from i in a finite number of steps. If two states i and j are accessible to each other, then they are said to *communicate*. Probabilistically, these definitions imply $i \to j$ (j accessible from i), if for some $n \geq 0$, $P_{ij}^{(n)} > 0$; $i \leftrightarrow j$ (i and j communicate), if for some $n \geq 0$, $P_{ij}^{(n)} > 0$ and for some $k \geq 0$, $P_{ji}^{(k)} > 0$.

The communication relation $i \leftrightarrow j$ is an equivalence relation. That is, it is reflexive, symmetric and transitive. The equivalence classes form a partition of the set of states.

Definition 4.2 If all the states of the Markov chain communicate with one another, we say that the chain is *irreducible*.

Definition 4.3 A state i is said to be *recurrent* if and only if, starting from state i, eventual return to this state is certain.

Definition 4.4 A state i is said to be *transient* if and only if, starting from state i, there is a positive probability that the process may not eventually return to this state.

Theorem 4.1 *Suppose that $i \leftrightarrow j$. Then if i is recurrent, then j is also recurrent. If i is transient, then j is also transient.*

This means that recurrence and transience are class properties. If a state of a class is recurrent then all the other states of the class are recurrent.

Remark 4.1 An equivalence class is recurrent if and only if a transition from a state within the class to a state outside the class is not possible. Thus, if the process reaches a recurrent class it will remain there for ever.

Theorem 4.2 *In a finite Markov chain, as the number of steps tends to infinity, the probability that the process is in a transient state tends to zero irrespective of the state at which the process starts.*

In other words, starting at any state, the process will ultimately join a recurrent class and remain there.

Corollary 4.1 *In a finite Markov chain, not all states can be transient.*

Definition 4.2, Theorem 4.1 and Corollary 4.1 imply that an irreducible Markov chain consists of only one class and all the states of this class are recurrent.

Let C be a recurrent class of the Markov chain of the game, according to Remark 4.1 the transition probability matrix P_C within C is a stochastic matrix and if the process starts in C it will never leave it. That is, when the process enters C or starts in C it behaves as a Markov chain having C as the set of states. Such subset will be referred to as Markov *sub chain*.

4.2 Reaching the Focal Profile: Two Particular Cases

Henceforth, without loss of generality, we assume that the profile $(1,1)$ is a focal profile for the game (2), also when we say that a profile will be reached, we mean that it will be reached with probability 1, i.e. with certainty.

Proposition 4.1 [21] *Assume that the Markov chain of the game* (2) *is irreducible. Then, starting at any profile* (i,j), *the focal profile* $(1,1)$ *will be reached.*

Proposition 4.2 *Assume that the Markov chain of the game* (2) *has only one recurrent class R_1, and that R_1 contains the focal profile* $(1,1)$. *Then starting from any profile* (i,j), *the focal profile* $(1,1)$ *will be reached.*

Proof Let P_{R_1} be the transition probability matrix within R_1. If the process starts in the unique recurrent class R_1, it will remain there. Then by applying Proposition 4.1 to the Markov sub chain consisting of R_1 with the transition probability matrix P_{R_1}, we can conclude that the focal profile $(1,1)$ will be reached. If the process starts in a transient profile, then according to Theorem 4.2, certainly the process will join the unique recurrent class R_1. Then again applying Proposition 4.1, we conclude that the process will reach the profile $(1,1)$. □

Remark 4.2 According to Propositions 4.1–4.2, when the Markov chain of the game (2) has only one recurrent class, and the class contains the focal profile (1, 1), the focal profile (1, 1) will be reached. In all other cases it may be impossible to reach the focal profile (1, 1). Let us explain this. Recall that, according to Remark 4.1, once the process joins a recurrent class it will stay there forever. There are two cases where the focal profile (1, 1) may not be reached.

Case 1. When there is only one recurrent class and the focal profile (1, 1) is not in that class, the process may enter that class and remain there, before it visits the focal profile (1, 1). Thus, it would be impossible for the process to reach the focal profile (1, 1).

Case 2. When there are more than one recurrent class, there is at least one class not containing the focal profile (1, 1). The process may join one of the recurrent classes not containing the focal profile (1, 1) and remain there, before visiting the focal profile (1, 1). Thus, it would be impossible for the process to reach the focal profile (1, 1).

4.3 Average Number of Steps for Reaching the Focal Profile (1, 1)

Since not all states can be transient in a finite Markov chain, there must be at least one recurrent class. After rearranging the profiles, the transition probability matrix can be represented by the following *canonical form*:

$$\tilde{P} = \begin{pmatrix} P_1 & 0 & \\ & \ddots & 0 \\ & 0 & P_d \\ & R & & Q \end{pmatrix} \tag{3}$$

where $P_i, i = \overline{1, d}$ represent the transition probability matrices within the corresponding recurrent classes; with r being the total number of recurrent states, R is a $(l \times s - r) \times r$ matrix representing the transition probabilities from transient states to recurrent states and Q is a $(l \times s - r) \times (l \times s - r)$ matrix representing the probability transition within the set of transient states.

Definition 4.5 The matrix $M = (I - Q)^{-1}$ is called the *fundamental matrix* of (3).

Lemma 4.1 [20] *Starting from a transient state i, the expected number of times that the process visits a transient state j before it eventually enters a recurrent class is $M_{i,j}$, which is the (i, j)th element of M, the fundamental matrix. Starting from a transient state i, the expected number of steps the process spends in transient states before entering a recurrent class is $\sum_{k \in T} M_{i,k}$, where T is the set of transient states.*

Lemma 4.2 [20] *Let $P = (P_{ij})_{i=\overline{1,m}, j=\overline{1,m}}$ be the transition probability matrix of a finite irreducible Markov chain. Consider any state i and the matrix*

$$\tilde{P} = \begin{pmatrix} 1 & 0 \\ \tilde{S} & Q \end{pmatrix} \tag{4}$$

where Q is obtained from P by deleting the ith row and ith column corresponding to state i and $\tilde{S}$ is obtained from the ith column of P by deleting its ith entry. Then the expected number of steps starting at state j until reaching state i is given by

$$E(T_i/X_0 = j) = E\left(\sum_{k\neq i} T_{j,k}/X_0 = j\right) = \sum_{k\neq i} M_{j,k},$$

where $M_{j,k}$ is the expected number of visits to k starting at j which is given by the (j,k)th entry of the matrix $M = (I - Q)^{-1}$; T_i is the random variable representing the number of steps needed to reach state i, that is, T_i is the smallest time n such that $X_n = i$; $T_{i,k}$ is the random variable representing the number of visits to k before reaching i (if we start at k, we include this as one visit to k).

Returning to game (2), let C be a recurrent class containing the focal profile $(1, 1)$. Denote by P_C and $\tilde{P}_C$ the transition probability matrix within the recurrent class C and the corresponding transition probability matrix of the form (4) obtained from P_C with respect to the focal profile $(1, 1)$, respectively.

Proposition 4.3 *Assume that the process of the game (2) has only one recurrent class C, and C contains the focal profile $(1, 1)$. Then:*

(i) *Starting from any transient profile (i, j), the expected number of steps needed for the process to reach $(1, 1)$ satisfies the inequality*

$$E(T_{(1,1)}/X_0 = (i,j)) \le \sum_{(i_1,j_1)\in S\setminus C} M_{(i,j),(i_1,j_1)} + \operatorname*{Max}_{(u,v)\in C\setminus\{(1,1)\}} \sum_{(i_1,j_1)\in C\setminus\{(1,1)\}} M'_{(u,v),(i_1,j_1)}, \quad (5)$$

where $T_{(1,1)}$ is the random variable representing the number of steps needed to reach the profile $(1, 1)$, M is the fundamental matrix corresponding to the transition probability matrix of the process in the form (3) with $d = 1$, and M' is the fundamental matrix corresponding to $\tilde{P}_C$ of the form (4) obtained from P_C.

(ii) *Starting from a profile (i, j) within the class C, the expected number of steps needed for the process to reach $(1, 1)$ is*

$$E(T_{(1,1)}/X_0 = (i,j)) = \sum_{(i_1,j_1)\in C\setminus\{(1,1)\}} M'_{(i,j),(i_1,j_1)}. \quad (6)$$

(iii) *In the case, the Markov chain is irreducible, then starting from any profile (i, j), the expected number of steps needed for the process to reach the focal profile $(1, 1)$ is*

$$E(T_{(1,1)}/X_0 = (i,j)) = \sum_{(i_1,j_1)\in S\setminus\{(1,1)\}} M''_{(i,j),(i_1,j_1)} \quad (7)$$

where M'' is the fundamental matrix corresponding to the transition probability matrix of the form (4) with respect to the focal profile $(1, 1)$.

Proof See Appendix A.1. □

Example 4.1 Consider the example of the prisoner dilemma again (Example 2.1). Suppose that the transition probability matrix of the Markov chain of the game is P as given in (8) below. Since we have $P > 0$, the Markov chain of the game is irreducible, then from Proposition 4.1, we know that the focal profile (C, C) will be reached. Let us now compute the expected number of steps needed to reach (C, C) from any other state (i, j), according to Proposition 4.3. First, we have to construct the matrix $\tilde{P}$ of the form (4), then calculate the matrix M,

$$
\begin{aligned}
P &= \begin{matrix} \\ (\mathrm{C,C}) \\ (\mathrm{C,N}) \\ (\mathrm{N,C}) \\ (\mathrm{N,N}) \end{matrix}
\begin{matrix} \begin{matrix} (\mathrm{C,C}) & (\mathrm{C,N}) & (\mathrm{N,C}) & (\mathrm{N,N}) \end{matrix} \\
\begin{pmatrix} 0.35 & 0.19 & 0.45 & 0.01 \\ 0.01 & 0.35 & 0.19 & 0.45 \\ 0.05 & 0.45 & 0.1 & 0.4 \\ 0.01 & 0.4 & 0.19 & 0.4 \end{pmatrix} \end{matrix} \\
\tilde{P} &= \begin{matrix} \\ (\mathrm{C,N}) \\ (\mathrm{N,C}) \\ (\mathrm{N,N}) \end{matrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.01 & 0.35 & 0.19 & 0.45 \\ 0.05 & 0.45 & 0.1 & 0.4 \\ 0.01 & 0.4 & 0.19 & 0.4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tilde{S} & Q \end{pmatrix}, \\
M &= (I - Q)^{-1} = \begin{pmatrix} 0.65 & -0.19 & -0.45 \\ -0.45 & 0.9 & -0.4 \\ -0.4 & -0.19 & 0.6 \end{pmatrix}^{-1} \\
&= \begin{matrix} (\mathrm{C,N}) \\ (\mathrm{N,C}) \\ (\mathrm{N,N}) \end{matrix}
\begin{pmatrix} 23.88 & 10.27 & 24.76 \\ 22.13 & 10.81 & 23.80 \\ 22.93 & 10.27 & 25.71 \end{pmatrix}.
\end{aligned}
\tag{8}
$$

According to Proposition 4.3, the average number of steps required to reach the focal profile (C, C) when starting from the profile (C, N) is $23.88 + 10.27 + 24.76 = 58.91$.

Similarly, the average number of steps needed to reach the focal profile (1, 1) from any other profile can be calculated for the games of Examples 3.2 and 3.3.

4.4 Improving the Speed for Reaching the Focal Profile (1, 1)

In this section we show that when the Markov chain of the game is irreducible, starting from any profile (k, t), the larger the probabilities $P_{(i,j),(1,1)}$, for all $(i, j) \neq (1, 1)$, the smaller the expected number of steps $M_{(k,t)(1,1)}$ needed to reach the focal profile (1, 1). Thus to reduce the expected number of steps needed to reach the focal profile (1, 1), one needs to increase the probabilities $P_{(i,j),(1,1)}$, for all $(i, j) \neq (1, 1)$. We start by illustrating this fact by an example.

Example 4.2 In Example 4.1, we have seen that the average number of steps for reaching the focal profile (C, C) when starting from the profile (C, N) is $23.88 + 10.27 + 24.76 = 58.91$, if the transition probability matrix is P given in (8). Suppose now that the transition probability matrix of the Markov chain of the game is

$$P' = \begin{array}{c} \\ (\mathrm{C,C}) \\ (\mathrm{C,N}) \\ (\mathrm{N,C}) \\ (\mathrm{N,N}) \end{array} \begin{array}{c} \begin{array}{cccc} (\mathrm{C,C}) & (\mathrm{C,N}) & (\mathrm{N,C}) & (\mathrm{N,N}) \end{array} \\ \begin{pmatrix} 0.5 & 0.2 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.4 & 0.25 & 0.1 & 0.25 \\ 0.2 & 0.4 & 0.2 & 0.2 \end{pmatrix} \end{array} \tag{9}$$

Note that compared with the matrix P in (8), we have

$$P_{(i,j)(\mathrm{C,C})} < P'_{(i,j)(\mathrm{C,C})}, \quad \text{for all } (i,j) \neq (\mathrm{C,C}), \tag{10}$$

i.e. the transition probabilities from any profile to the focal profile (C, C) in (9) are larger than those of P given in (8), respectively. Proceeding as in Example 4.1, we get the average number of steps needed to reach the focal profile (C, C), starting from the profile (C, N) as $3.471 + 1.036 + 1.191 = 5.698$, while in the Example 4.1, the corresponding expected number is 58.91. The example shows that the average number of steps needed to reach the focal profile (C, C) from the profile (C, N) is smaller when the probabilities $P_{(i,j),(C,C)}$, for all $(i,j) \neq$ (C, C) are larger. In fact, it is a general property.

Proposition 4.4 *Suppose that the transition probability matrix of the Markov chain of the game is irreducible. The lager the probabilities $P_{(i,j),(1,1)}$, for all $(i,j) \neq (1,1)$, the smaller the number of steps $M_{(k,t)(1,1)}$ needed to reach the focal profile $(1,1)$, starting from any profile (k,t).*

The Proposition 4.4 is a special case of the Proposition A.1 derived in Appendix A.2.

5 Conclusions

In the Part 1 of the present paper, a HD theory based model for two person games has been formulated and presented. In the proposed model, sets of strategies and utility functions are not involved; it focuses on the states of mind of players and their charge levels. Thus, the model could significantly enlarge the scope of applications of games in strategic (normal) form. The assumptions of our model allowed us to use the theory of Markov chains to analyze the game. Further, we have studied the anatomy of the transition probability matrix P. Particularly, when it is possible to reach the focal mind profile $(1,1)$, we provided the average number of steps needed for the process to reach $(1,1)$ and showed how to reduce this average number by changing some relevant entries in P. In Part 2 of the paper, we will focus on restructuring games in operations for reaching a win-win profile to deal with games where, initially, the focal profile $(1,1)$ cannot be reached or it is not a win-win profile. We want to restructure the game, step by step in operations, so that the transition probability matrix P of the game will become P^*. With P^*, the focal point $(1,1)$ will become a reachable win-win profile. As a consequence of this restructuring in operations, we derive the Possibility Theorem that it is always possible to reach a win-win mind profile when suitable conditions are satisfied.

Appendix A

A.1 Proof of Proposition 4.3

Let us prove (5). According to Lemma 4.1, starting from a transient profile (i, j) the expected number of steps required to reach the unique recurrent state C is given by $\sum_{(i_1,j_1)\in S\backslash C} M_{(i,j),(i_1,j_1)}$.

Once the process has reached the unique recurrent class C, it will remain there and its evolution is totally independent from its past evolution before entering the recurrent class C. Let (k, t) be the first profile reached when the process enters the class C. Then, the average number of steps required to reach the focal profile $(1, 1)$ can be calculated based on Lemma 4.2, since the class C is a recurrent and irreducible Markov sub chain. Hence, the expected number of steps needed to reach the focal profile $(1, 1)$, starting from (k, t) is

$$\sum_{(i_1,j_1)\in C\backslash\{(1,1)\}} M'_{(k,t),(i_1,j_1)}.$$

Thus, the expected number of steps needed to reach $(1, 1)$, starting from (i, j), is

$$E(T_{(1,1)}/X_0 = (i, j)) = \sum_{(i_1,j_1)\in S\backslash C} M_{(i,j),(i_1,j_1)} + \sum_{(i_1,j_1)\in C\backslash\{(1,1)\}} M'_{(k,t),(i_1,j_1)}.$$

The inequality (5) follows immediately. Equation (6) is a straightforward consequence of Lemma 4.2 when P, $\tilde{P}$ and M are replaced by P_C, $\tilde{P}_C$ and M'. Let us now prove (7). Since the Markov chain of the game is irreducible, it consists of only one class, which is a recurrent class. Then applying Lemma 4.2, we get (7).

A.2 Reducing the Average Number of Steps for Reaching the Focal Profile (1, 1)

Proposition A.1 *Let Q_1 and Q_2 be two $n \times n$ matrices with entries a_{ij} and b_{ij}, respectively. Assume that*

$$Q_s \geq 0, \quad s = 1, 2, \qquad \sum_{j=1}^{n} a_{ij} \leq 1, \qquad \sum_{j=1}^{n} b_{ij} \leq 1, \quad \textit{for } i = \overline{1, n}$$

and

$$\sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{n} b_{ij}, \quad \textit{for } i = \overline{1, n}.$$

Then

$$M^s = (I - Q_s)^{-1}, \quad s = 1, 2$$

exist and

$$M^s = (I - Q_s)^{-1} \geq 0, \quad s = 1, 2.$$

Furthermore,

$$\sum_{j=1}^{n} c_{ij} \le \sum_{j=1}^{n} d_{ij}, \quad i=\overline{1,n}$$

where c_{ij} *and* d_{ij} *are the entries of* $M^s=(I-Q_s)^{-1}, s=1,2$, *respectively.*

Proof Since we have

$$Q_s \ge 0, \quad s=1,2, \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} \le 1, \quad \sum_{j=1}^{n} b_{ij} \le 1, \quad \text{for } i=\overline{1,n}, \tag{11}$$

then $(I-Q_s)^{-1}, s=1,2$ exist [20] and we have also

$$(I-Q_s)^{-1} = I + Q_s + Q_s^2 + \cdots + Q_s^p + \cdots, \quad s=1,2. \tag{12}$$

By (11) we have $Q_s \ge 0, s=1,2$, then $Q_s^p \ge 0, s=1,2$, for any integer p. Thus, $(I-Q_s)^{-1} \ge 0, s=1,2$. Furthermore, we have

$$\sum_{j=1}^{n} a_{ij} \le \sum_{j=1}^{n} b_{ij}, \quad i=\overline{1,n} \quad \Rightarrow \quad \sum_{j=1}^{n} a_{ij}^p \le \sum_{j=1}^{n} b_{ij}^p, \quad i=\overline{1,n}, \text{ for any integer } p, \tag{13}$$

where a_{ij}^p and b_{ij}^p represent the entries of $Q_s^p, s=1,2$ respectively. We prove this relation by induction. By assumption this is true for $p=1$. Let us assume that it is true for p, then prove that it is true for $p+1$. Thus, we want to prove the following relation:

$$\sum_{j=1}^{n} a_{ij}^p \le \sum_{j=1}^{n} b_{ij}^p, \quad i=\overline{1,n} \quad \Rightarrow \quad \sum_{j=1}^{n} a_{ij}^{p+1} \le \sum_{j=1}^{n} b_{ij}^{p+1}, \quad i=\overline{1,n}. \tag{14}$$

Note that

$$Q_s^{p+1} = Q_s^p \times Q_s, \quad s=1,2, \qquad a_{ij}^{p+1} = \sum_k a_{ik}^p a_{kj},$$
$$b_{ij}^{p+1} = \sum_k b_{ik}^p b_{kj}, \quad i=\overline{1,n}.$$

Thus,

$$\sum_j a_{ij}^{p+1} = \sum_j \sum_k a_{ik}^p a_{kj} \quad \text{and} \quad \sum_j b_{ij}^{p+1} = \sum_j \sum_k b_{ik}^p b_{kj}.$$

Using (13)–(14) and the fact that all the matrices involved have positive entries, we get

$$\sum_j a_{ij}^{p+1} = \sum_j \sum_k a_{ik}^p a_{kj} = \sum_k a_{ik}^p \sum_j a_{kj} \le \sum_k a_{ik}^p \sum_j b_{kj}$$

$$= \sum_j \Big(\sum_k a_{ik}^p\Big) b_{kj} \le \sum_j \Big(\sum_k b_{ik}^p\Big) b_{kj}$$
$$= \sum_j \Big(\sum_k b_{ik}^p b_{kj}\Big) = \sum_j b_{ij}^{p+1}, \quad i = \overline{1,n}.$$

Hence,

$$\sum_{j=1}^n a_{ij}^{p+1} \le \sum_{j=1}^n b_{ij}^{p+1}, \quad i = \overline{1,n}.$$

Using (12), we get

$$\sum_{j=1}^n c_{ij} \le \sum_{j=1}^n d_{ij}, \quad i = \overline{1,n}.$$

□

Relation between Proposition A.1 and Proposition 4.4 Let P and P' be two transition probability matrices of the game (2) such that

$$P'_{(i,j),(1,1)} \le P_{(i,j)(1,1)}, \quad \text{for all } (i,j) \neq (1,1).$$

Assume that the Markov chain is irreducible for P and P'. Let

$$\tilde{P}' = \begin{pmatrix} 1 & 0 \\ \tilde{S}' & Q' \end{pmatrix}, \qquad \tilde{P} = \begin{pmatrix} 1 & 0 \\ \tilde{S} & Q \end{pmatrix}$$

be the matrices of the form (4) corresponding to P' and P respectively. Note that the components of the column vectors $\tilde{S}'$ and $\tilde{S}$ are $P'_{(i,j),(1,1)}$, for all $(i,j) \neq (1,1)$ and $P_{(i,j),(1,1)}$, for all $(i,j) \neq (1,1)$, respectively. Hence, we have

$$P'_{(i,j),(1,1)} \le P_{(i,j)(1,1)}, \quad \text{for all } (i,j) \neq (1,1) \quad \Leftrightarrow \quad \tilde{S}' \le \tilde{S}.$$

Now let (k,t) be a profile such that $(k,t) \neq (1,1)$, then $P'_{(k,t),(1,1)} \le P_{(k,t)(1,1)}$. Since P and P' are stochastic, then for all $(k,t) \neq (1,1)$,

$$P'_{(k,t),(1,1)} \le P_{(k,t)(1,1)} \quad \Leftrightarrow \quad 1 - \sum_{(i,j)\neq(1,1)} P'_{(k,t)(i,j)} \le 1 - \sum_{(i,j)\neq(1,1)} P_{(k,t)(i,j)}$$
$$\Rightarrow \quad \sum_{(i,j)\neq(1,1)} P_{(k,t)(i,j)} \le \sum_{(i,j)\neq(1,1)} P'_{(k,t)(i,j)}.$$

This means that the sum of the entries of the row (k,t) of the matrix Q' is greater than or equal to the sum of the entries of the row (k,t) of the matrix Q. The other conditions of Proposition A.1 can be easily verified. Thus, according to Proposition A.1 the sum of the entries of each row of the fundamental matrix $M = (I - Q)^{-1}$ is less or equal than the sum of the entries of the corresponding row of the fundamental matrix $M' = (I - Q')^{-1}$, i.e.

$$\sum_{(k,t)\neq(1,1)} M_{(i,j)(k,t)} \le \sum_{(k,t)\neq(1,1)} M'_{(i,j)(k,t)}, \quad \text{for all } (i,j) \neq (1,1).$$

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