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Patterns of high energy massive string scatterings in the Regge regime

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ABSTRACT: We calculate high energy massive string scattering amplitudes of open bosonic string in the Regge regime (RR). We found that the number of high energy amplitudes for each fixed mass level in the RR is much more numerous than that of Gross regime (GR) calculated previously. Moreover, we discover that the leading order amplitudes in the RR can be expressed in terms of the Kummer function of the second kind. In particular, based on a summation algorithm for Stirling number identities developed recently, we discover that the ratios calculated previously among scattering amplitudes in the GR can be extracted from this Kummer function in the RR. We conjecture and give evidences that the existence of these GR ratios in the RR persists to subleading orders in the Regge expansion of all string scattering amplitudes. Finally, we demonstrate the universal power-law behavior for all massive string scattering amplitudes in the RR.

KEYWORDS: Bosonic Strings, Discrete and Finite Symmetries

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1 Introduction

There are two fundamental regimes of high energy string scattering amplitudes. These are the fixed angle regime or Gross regime (GR), and the fixed momentum transfer regime or Regge regime (RR). These two regimes represent two different high energy perturbation expansions of the scattering amplitudes, and contain complementary information of the theory. The UV behavior of high energy string scatterings in the GR is well known to be very soft exponential fall-off, while that of RR is hard power-law. The high energy string scattering amplitudes in the GR [1–3] were recently intensively reinvestigated for massive string states at arbitrary mass levels [4–12]. See also the developments in [13–15]. An infinite number of linear relations, or stringy symmetries, among string scattering amplitudes of different string states were obtained. Moreover, these linear relations can be solved for each fixed mass level, and ratios $T^{(N,2m,q)}/T^{(N,0,0)}$ among the amplitudes can be obtained. An important new ingredient of these calculations is the decoupling of zero-norm states (ZNS) [16–18] in the old covariant first quantized (OCFQ) string spectrum. It is interesting to note that the calculation in [1–3] is valid only for four-tachyon amplitude, but not for all other amplitudes of excited string states. This was pointed out and the calculation was corrected by two independent groups [6, 14] with two different approaches. Since there does not exist any algebraic structure (or group structure) of this high energy 26D spacetime symmetry, mathematically the meaning of these infinite number of ratios remains mysterious.

Another fundamental regime of high energy string scattering amplitudes is the RR [19–24]. See also [25–27]. Since the decoupling of ZNS applies to all kinematic regimes, one

expects some implication of this decoupling in the RR. Moreover, it is conceivable that there exists some link between the patterns of the high energy scattering amplitudes of GR and RR. With this in mind, in this paper, we give a detail calculation of high energy string scattering amplitudes in the RR. We will find that the number of high energy scattering amplitudes for each fixed mass level in the RR is much more numerous than that of GR calculated previously. On the other hand, it seems that both the saddle-point method and the method of decoupling of high energy ZNS adopted in the calculation of GR do not apply to the case of RR. However the calculation is still manageable, and the general formula for the high energy scattering amplitudes for each fixed mass level in the RR can be written down explicitly.

In contrast to the case of scatterings in the GR, we will see that there is no linear relation among scatterings in the RR. Moreover, we discover that the leading order amplitudes at each fixed mass level in the RR can be expressed in terms of the Kummer function of the second kind. More surprisingly, for those leading order high energy amplitudes $A^{(N,2m,q)}$ in the RR with the same type of $(N, 2m, q)$ as those of GR, we can extract from them the ratios $T^{(N,2m,q)}/T^{(N,0,0)}$ in the GR by using this Kummer function. Mathematically, the proof of this result turns out to be highly nontrivial and is based on a summation algorithm for Stirling number identity derived by Mkauers in 2007 [28]. It is very interesting to see that the identity in eq. (4.3) suggested by string theory calculation can be rigorously proved by a totally different but sophisticated mathematical method. The derivation of these physical ratios from Kummer function through Stirling number identities seems to suggest another interpretation of these infinite number of ratios mathematically. We then proceed to calculate Regge string scattering amplitudes to subleading orders. We conjecture and give evidences that these ratios persist to all orders in the Regge expansion of high energy string scattering amplitudes for the even mass level with $(N-1) = \frac{M^2}{2} = \text{even}$. For the odd mass levels with $(N-1) = \frac{M^2}{2} = \text{odd}$, the existence of the GR ratios shows up only in the first $[N/2]+1$ terms in the Regge expansion of the amplitudes. At last, as an application of our results, we show that the well known $s^{\alpha(t)}$ power-law behavior of the four tachyon string scattering amplitude in the RR can be extended to all high energy massive string scattering amplitudes.

This paper is organized as following. In section II, after a brief review of high energy string scatterings in the GR, we first calculate all leading high energy scattering amplitudes for the mass level $M^2 = 4$ in the RR. We compare the two sets of amplitudes and discover a link between the two. The calculation is then generalized to general mass level in the RR in section III. We show that the leading order amplitudes can be expressed in terms of the Kummer function of the second kind. In section IV, based on a summation algorithm for Stirling number identity, we show that the ratios among scattering amplitudes in the GR can be extracted from Kummer function derived in section III. In section V, we give evidences that the existence of these ratios in the RR persists to subleading orders in the Regge expansion of all high energy string scattering amplitudes. In section VI, we demonstrate the universal power-law behavior for all massive string scattering amplitudes in the RR. Finally, an appendix is devoted to the kinematics used in the text.

2 Regge scattering for $M_2^2 = 4$

We begin with a brief review of high energy string scatterings in the GR. That is in the kinematic regime

$$s, -t \rightarrow \infty, t/s \approx -\sin^2 \frac{\theta}{2} = \text{fixed (but } \theta \neq 0) \quad (2.1)$$

where s, t and u are the Mandelstam variables and θ is the CM scattering angle. It was shown [7, 8] that for the 26D open bosonic string the only states that will survive the high-energy limit at mass level $M_2^2 = 2(N - 1)$ are of the form

$$|N, 2m, q\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q |0\rangle, \quad (2.2)$$

where the polarizations of the 2nd particle with momentum k_2 on the scattering plane were defined to be $e^P = \frac{1}{M_2}(E_2, k_2, 0) = \frac{k_2}{M_2}$ as the momentum polarization, $e^L = \frac{1}{M_2}(k_2, E_2, 0)$ the longitudinal polarization and $e^T = (0, 0, 1)$ the transverse polarization. Note that e^P approaches to e^L in the GR, and the scattering plane is defined by the spatial components of e^L and e^T . Polarizations perpendicular to the scattering plane are ignored because they are kinematically suppressed for four point scatterings in the high-energy limit. One can use the saddle-point method to calculate the high energy scattering amplitudes. For simplicity, we choose k_1, k_3 and k_4 to be tachyons and the final result of the ratios of high energy, fixed angle string scattering amplitude are [7, 8]

$$\frac{T^{(N,2m,q)}}{T^{(N,0,0)}} = \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!! \quad (2.3)$$

The ratios in eq. (2.3) can also be obtained by using the decoupling of two types of ZNS in the spectrum

$$\text{Type I : } L_{-1} |x\rangle, \quad \text{where } L_1 |x\rangle = L_2 |x\rangle = 0, \quad L_0 |x\rangle = 0; \quad (2.4)$$

$$\text{Type II : } (L_{-2} + \frac{3}{2}L_{-1}^2) |\tilde{x}\rangle, \quad \text{where } L_1 |\tilde{x}\rangle = L_2 |\tilde{x}\rangle = 0, \quad (L_0 + 1) |\tilde{x}\rangle = 0. \quad (2.5)$$

As examples, for $M_2^2 = 4, 6$, we get [4, 5]

$$T_{TTT} : T_{LLT} : T_{(LT)} : T_{[LT]} = 8 : 1 : -1 : -1, \quad (2.6)$$

$$\begin{aligned} T_{TTTT} : T_{TTLL} : T_{LLLL} : T_{TTL} : T_{LLL} : \tilde{T}_{LT,T} : \tilde{T}_{LP,P} : T_{LL} : \tilde{T}_{LL} & \dots \\ 16 : \frac{4}{3} : \frac{1}{3} : -\frac{4\sqrt{6}}{9} : -\frac{\sqrt{6}}{9} : -\frac{2\sqrt{6}}{3} : 0 : \frac{2}{3} : 0 & \dots \end{aligned} \quad (2.7)$$

We now turn to the discussion on high energy string scatterings in the RR. That is in the kinematic regime

$$s \rightarrow \infty, \sqrt{-t} = \text{fixed (but } \sqrt{-t} \neq \infty). \quad (2.8)$$

As in the case of GR, we only need to consider the polarizations on the scattering plane, which is defined in appendix A. Appendix A also includes the kinematic set up and some formulas we need in our calculation. However, instead of using (E, θ) as the two independent kinematic variables in the GR, we choose to use (s, t) in the RR. One of the reason

has been, in the RR, $t \sim E\theta$ is fixed, and it is more convenient to use (s, t) rather than (E, θ) . In the RR, to the lowest order, equations (A.13) to (A.18) reduce to

$$e^P \cdot k_1 = -\frac{1}{M_2} \left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right) \simeq -\frac{s}{2M_2}, \quad (2.9a)$$

$$e^L \cdot k_1 = -\frac{p}{M_2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) \simeq -\frac{s}{2M_2}, \quad (2.9b)$$

$$e^T \cdot k_1 = 0 \quad (2.9c)$$

and

$$e^P \cdot k_3 = \frac{1}{M_2} \left(\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \theta \right) \simeq -\frac{\tilde{t}}{2M_2} \equiv -\frac{t - M_2^2 - M_3^2}{2M_2}, \quad (2.10a)$$

$$e^L \cdot k_3 = \frac{1}{M_2} \left(p \sqrt{q^2 + M_3^2} - q \sqrt{p^2 + M_2^2} \cos \theta \right) \simeq -\frac{\tilde{t}'}{2M_2} \equiv -\frac{t + M_2^2 - M_3^2}{2M_2}, \quad (2.10b)$$

$$e^T \cdot k_3 = -q \sin \phi \simeq -\sqrt{-t}. \quad (2.10c)$$

Note that e^P does not approach to e^L in the RR. This is very different from the case of GR. In the following discussion, we will calculate the amplitudes for the longitudinal polarization e^L . For the e^P amplitudes, the results can be trivially modified. There is another important difference between the high energy scattering amplitudes in the RR and in the GR. We will find that the number of high energy scattering amplitudes for each fixed mass level in the RR is much more numerous than that of GR calculated previously. On the other hand, it seems that both the saddle-point method and the method of decoupling of high energy ZNS adopted in the calculation of GR do not apply to the case of RR. In this section, we will explicitly calculate the string scattering amplitudes on the scattering plane (e^L, e^T) for the mass level $M_2^2 = 4$. In the mass level $M_2^2 = 4$ ($M_1^2 = M_3^2 = M_4^2 = -2$), it turns out that there are eight high energy amplitudes in the RR

$$\begin{aligned} & \alpha_{-1}^T \alpha_{-1}^T \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^T \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^L |0\rangle, \\ & \alpha_{-1}^T \alpha_{-2}^T |0\rangle, \alpha_{-1}^T \alpha_{-2}^L |0\rangle, \alpha_{-1}^L \alpha_{-2}^T |0\rangle, \alpha_{-1}^L \alpha_{-2}^L |0\rangle. \end{aligned} \quad (2.11)$$

The $s - t$ channel of these amplitudes can be calculated to be

$$\begin{aligned} A^{TTT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right)^3 \\ &\simeq -i (\sqrt{-t})^3 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left(-\frac{1}{8}s^3 + \frac{1}{2}s \right), \end{aligned} \quad (2.12)$$

$$\begin{aligned} A^{LTT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right)^2 \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right) \\ &\simeq -i (\sqrt{-t})^2 \left(-\frac{1}{2M_2} \right) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left[\frac{3}{4}s^3 - \frac{t}{4}s^2 - \left(\frac{t}{2} + 3 \right) s \right], \end{aligned} \quad (2.13)$$

$$\begin{aligned}
A^{LLT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right)^2 \\
&\simeq -i(\sqrt{-t}) \left(-\frac{1}{2M_2} \right)^2 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \\
&\quad \cdot \left[\left(\frac{1}{4}t - \frac{9}{2} \right) s^3 + \left(\frac{1}{4}t^2 + \frac{7}{2}t \right) s^2 + \frac{(t+6)^2}{2} s \right], \tag{2.14}
\end{aligned}$$

$$\begin{aligned}
A^{LLL} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right)^3 \\
&\simeq -i \left(-\frac{1}{2M_2} \right)^3 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \\
&\quad \cdot \left[-\left(\frac{11}{2}t - 27 \right) s^3 - 6(t^2 + 6t) s^2 - \frac{(t+6)^3}{2} s \right], \tag{2.15}
\end{aligned}$$

$$\begin{aligned}
A^{TTT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left[\frac{e^T \cdot k_1}{x^2} + \frac{e^T \cdot k_3}{(1-x)^2} \right] \\
&\simeq -i(\sqrt{-t})^2 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \left(-\frac{1}{8}s^3 + \frac{1}{2}s \right), \tag{2.16}
\end{aligned}$$

$$\begin{aligned}
A^{TTL} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left[\frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1-x)^2} \right] \\
&\simeq i(\sqrt{-t}) \left(-\frac{1}{2M_2} \right) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \\
&\quad \cdot \left[-\left(\frac{1}{8}t + \frac{3}{4} \right) s^3 - \frac{1}{8}(t^2 - 2t) s^2 - \left(\frac{1}{4}t^2 - t - 3 \right) s \right], \tag{2.17}
\end{aligned}$$

$$\begin{aligned}
A^{LTL} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right) \left[\frac{e^T \cdot k_1}{x^2} + \frac{e^T \cdot k_3}{(1-x)^2} \right] \\
&\simeq i(\sqrt{-t}) \left(-\frac{1}{2M_2} \right) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left[\frac{3}{4}s^3 - \frac{t}{4}s^2 - \left(\frac{t}{2} + 3 \right) s \right], \tag{2.18}
\end{aligned}$$

and

$$\begin{aligned}
A^{LLL} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right) \left[\frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1-x)^2} \right] \\
&\simeq i \left(-\frac{1}{2M_2} \right)^2 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \\
&\quad \cdot \left[\left(\frac{3}{4}t + \frac{9}{2} \right) s^3 + (t^2 - 4t) s^2 + \left(\frac{1}{4}t^3 + \frac{1}{2}t^2 - 9t - 18 \right) s \right]. \tag{2.19}
\end{aligned}$$

From the above calculation, one can easily see that all the amplitudes are in the same leading order ($\sim s^3$) in the RR, while in the GR only A^{TTT} , A^{LLT} and A^{TTL} are in the leading order ($\sim t^{3/2}s^3$ or $t^{5/2}s^2$), all other amplitudes are in the subleading orders. On

the other hand, one notes that, for example, the term $\sim \sqrt{-t}t^2s^2$ in A^{LLT} and A^{TL} are in the leading order in the GR, but are in the subleading order in the RR. On the contrary, the terms $\sqrt{-t}s^3$ in A^{LLT} and A^{TL} are in the subleading order in the GR, but are in the leading order in the RR. These observations suggest that the high energy string scattering amplitudes in the GR and RR contain information complementary to each other.

One can now see that the number of high energy scattering amplitudes in the RR is much more numerous than that of GR. One important observation for high energy amplitudes in the RR is for those amplitudes with the same structure as those of the GR in eq. (2.2). For these amplitudes, the relative ratios of the coefficients of the highest power of t in the leading order amplitudes in the RR can be calculated to be

$$A^{TTT} = -i(\sqrt{-t}) \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left(\frac{1}{8}ts^3\right) \sim \frac{1}{8}, \tag{2.20}$$

$$A^{LLT} = -i(\sqrt{-t}) \left(-\frac{1}{2M_2}\right)^2 \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \left(\frac{1}{4}ts^3\right) \sim \frac{1}{64}, \tag{2.21}$$

$$A^{TL} = i(\sqrt{-t}) \left(-\frac{1}{2M_2}\right) \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left(-\frac{1}{8}ts^3\right) \sim -\frac{1}{32}, \tag{2.22}$$

which reproduces the ratios in the GR in eq. (2.6). Note that the symmetrized and anti-symmetrized amplitudes are defined as

$$T^{(TL)} = \frac{1}{2}(T^{TL} + T^{LT}), \tag{2.23}$$

$$T^{[TL]} = \frac{1}{2}(T^{TL} - T^{LT}); \tag{2.24}$$

and similarly for the amplitudes $A^{(TL)}$ and $A^{[TL]}$ in the RR. Note that $T^{LT} \sim (\alpha_{-1}^L)(\alpha_{-2}^T)|0\rangle$ in the GR is of subleading order in energy, while A^{LT} in the RR is of leading order in energy. However, the contribution of the amplitude A^{LT} to $A^{(TL)}$ and $A^{[TL]}$ in the RR will not affect the ratios calculated above. As we will see in section IV, this interesting result can be generalized to all mass levels in the string spectrum.

3 General mass levels

In this section, we calculate high energy string scattering amplitudes in the RR for the arbitrary mass levels. Instead of states in eq. (2.2) for the GR, one can easily argue that the most general string states one needs to consider at each fixed mass level $N = \sum_{n,m} nk_n + mq_m$ for the RR are

$$|k_n, q_m\rangle = \prod_{n>0} (\alpha_{-n}^T)^{k_n} \prod_{m>0} (\alpha_{-m}^L)^{q_m} |0\rangle. \tag{3.1}$$

It seems that both the saddle-point method and the method of decoupling of high energy ZNS adopted in the calculation of GR do not apply to the case of RR. However the

calculation is still manageable, and the general formula for the high energy scattering amplitudes in the RR can be written down explicitly. In fact, by the simple kinematics $e^T \cdot k_1 = 0$, and the energy power counting of the string amplitudes, we end up with the following rules to simplify the calculation for the leading order amplitudes in the RR:

$$\alpha_{-n}^T : 1 \text{ term (contraction of } ik_3 \cdot X \text{ with } \varepsilon_T \cdot \partial^n X), \quad (3.2)$$

$$\alpha_{-n}^L : \begin{cases} n > 1, & 1 \text{ term} \\ n = 1 & 2 \text{ terms (contraction of } ik_1 \cdot X \text{ and } ik_3 \cdot X \text{ with } \varepsilon_L \cdot \partial^n X). \end{cases} \quad (3.3)$$

The $s - t$ channel scattering amplitudes of this state with three other tachyonic states can be calculated to be

$$\begin{aligned} A^{(k_n, q_m)} &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{ie^L \cdot k_1}{-x} + \frac{ie^L \cdot k_3}{1-x} \right]^{q_1} \\ &\quad \cdot \prod_{n=1}^{q_1} \left[\frac{ie^T \cdot k_3 (n-1)!}{(1-x)^n} \right]^{k_n} \prod_{m=2}^{q_1} \left[\frac{ie^L \cdot k_3 (m-1)!}{(1-x)^m} \right]^{q_m} \\ &= \left(\frac{-i\tilde{t}'}{2M_2} \right)^{q_1} \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{s}{-\tilde{t}} \right)^j \int_0^1 dx x^{k_1 \cdot k_2 - j} (1-x)^{k_2 \cdot k_3 + j - \sum_{n,m} (nk_n + mq_m)} \\ &\quad \cdot \prod_{n=1}^{q_1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2}^{q_1} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2} \right) \right]^{q_m} \\ &= \left(\frac{-i\tilde{t}'}{2M_2} \right)^{q_1} \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{s}{-\tilde{t}} \right)^j B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 + j - N + 1) \\ &\quad \cdot \prod_{n=1}^{q_1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2}^{q_1} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2} \right) \right]^{q_m}. \end{aligned} \quad (3.4)$$

The Beta function above can be approximated in the large s , but fixed t limit as follows

$$\begin{aligned} &B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 + j - N + 1) \\ &= B\left(-1 - \frac{s}{2} + N - j, -1 - \frac{t}{2} + j\right) \\ &= \frac{\Gamma(-1 - \frac{s}{2} + N - j)\Gamma(-1 - \frac{t}{2} + j)}{\Gamma(\frac{u}{2} + 2)} \\ &\approx B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \left(-1 - \frac{s}{2}\right)^{N-j} \left(\frac{u}{2} + 2\right)^{-N} \left(-1 - \frac{t}{2}\right)_j \\ &\approx B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{-j} \left(-1 - \frac{t}{2}\right)_j. \end{aligned} \quad (3.5)$$

where

$$(a)_j = a(a+1)(a+2)\dots(a+j-1) \quad (3.6)$$

is the Pochhammer symbol. The leading order amplitude in the RR can then be written as

$$A^{(k_n, q_m)} = \left(\frac{-i\tilde{t}'}{2M_2}\right)^{q_1} B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2}{\tilde{t}'}\right)^j \left(-1 - \frac{t}{2}\right)_j \cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2}\right)\right]^{q_m}, \quad (3.7)$$

which is UV power-law behaved as expected. The summation in eq. (3.7) can be represented by the Kummer function of the second kind U as follows,

$$\sum_{j=0}^p \binom{p}{j} \left(\frac{2}{\tilde{t}'}\right)^j \left(-1 - \frac{t}{2}\right)_j = 2^p (\tilde{t}')^{-p} U\left(-p, \frac{t}{2} + 2 - p, \frac{\tilde{t}'}{2}\right) \dots \quad (3.8)$$

Finally, the amplitudes can be written as

$$A^{(k_n, q_m)} = \left(-\frac{i}{M_2}\right)^{q_1} U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}'}{2}\right) B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2}\right)\right]^{q_m}. \quad (3.9)$$

In the above, U is the Kummer function of the second kind and is defined to be

$$U(a, c, x) = \frac{\pi}{\sin \pi c} \left[\frac{M(a, c, x)}{(a-c)!(c-1)!} - \frac{x^{1-c} M(a+1-c, 2-c, x)}{(a-1)!(1-c)!} \right] \quad (c \neq 2, 3, 4 \dots) \quad (3.10)$$

where $M(a, c, x) = \sum_{j=0}^{\infty} \frac{(a)_j x^j}{(c)_j j!}$ is the Kummer function of the first kind. U and M are the two solutions of the Kummer Equation

$$xy''(x) + (c-x)y'(x) - ay(x) = 0. \quad (3.11)$$

It is crucial to note that $c = \frac{t}{2} + 2 - q_1$, and is not a constant as in the usual case, so U in eq. (3.9) is not a solution of the Kummer equation. This will make our analysis in the next section more complicated as we will see soon. On the contrary, since $a = -q_1$ an integer, the Kummer function in eq. (3.8) terminated to be a finite sum. This will simplify the manipulation of Kummer function used in this paper.

4 Reproducing the GR ratios in the RR

In section II, we have learned that the relative coefficients of the highest power t terms in the leading order amplitudes in the RR can reproduce the ratios of the amplitudes in the GR for the mass level $M_2^2 = 4$. Now we are going to generalize the calculation to the string states of the arbitrary mass levels. The leading order amplitudes of string states in the RR, which share the same structure as eq. (2.2) in the GR can be written as

$$A^{(N, 2m, q)} = B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \sqrt{-t}^{N-2m-2q} \left(\frac{1}{2M_2}\right)^{2m+q} 2^{2m} (\tilde{t}')^q U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'}{2}\right). \quad (4.1)$$

It is important to note that there is no linear relation among high energy string scattering amplitudes of different string states for each fixed mass level in the RR as can be seen from eq. (4.1). This is very different from the result in the GR. In other words, the ratios $A^{(N,2m,q)}/A^{(N,0,0)}$ are t -dependent functions. As was done in section II for the mass level $M_2^2 = 4$, we can extract the coefficients of the highest power of t in $A^{(N,2m,q)}/A^{(N,0,0)}$. We can use the identity of the Kummer function in eq. (3.8) to calculate

$$\frac{A^{(N,2m,q)}}{A^{(n,0,0)}} = (-1)^q \left(\frac{1}{2M_2}\right)^{2m+q} (-t)^m \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2}\right)_j \frac{(-2/t)^j}{j!} + O\left\{\left(\frac{1}{t}\right)^{m+1}\right\}. \quad (4.2)$$

where we have replaced \tilde{t} by t as t is large. If the leading order coefficients in eq. (4.2) extracted from the high energy string scattering amplitudes in the RR are to be identified with the ratios calculated previously among high energy string scattering amplitudes in the GR in eq. (2.3), we need the following identity

$$\begin{aligned} & \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2}\right)_j \frac{(-2/t)^j}{j!} \\ &= 0(-t)^0 + 0(-t)^{-1} + \dots + 0(-t)^{-m+1} + \frac{(2m)!}{m!} (-t)^{-m} + O\left\{\left(\frac{1}{t}\right)^{m+1}\right\} \dots \end{aligned} \quad (4.3)$$

The coefficient of the term $O\left\{(1/t)^{m+1}\right\}$ in eq. (4.3) is irrelevant for our discussion. The proof of eq. (4.3) suggested by string theory calculation turns out to be nontrivial mathematically. Presumably, the difficulty of the rigorous proof of eq. (4.3) is associated with the unusual non-constant c in the argument of Kummer function in eq. (4.1) as mentioned above. We first rewrite the summation on the left hand side of eq. (4.3) as

$$\begin{aligned} & \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2}\right) \left(-\frac{t}{2}\right)_{j-1} \left(-\frac{2}{t}\right)^j \frac{1}{j!} \\ &= \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2}\right) \sum_{k=0}^{j-1} (-1)^{j-1-k} s(j-1, k) \left(-\frac{t}{2}\right)^k \left(-\frac{2}{t}\right)^j \frac{1}{j!} \\ &\stackrel{\mathcal{O}(t^{-m})}{\sim} \sum_{j=m}^{2m} (-2m)_j s(j-1, j-m) \frac{2^m}{j!} + \sum_{j=m+1}^{2m} (-2m)_j s(j-1, j-m-1) \frac{2^m}{j!} \\ &= 2^m \sum_{j=0}^m (-1)^{j+m} \binom{2m}{j+m} s(j+m-1, j) \\ &\quad + 2^m \sum_{j=1}^m (-1)^{j+m} \binom{2m}{j+m} s(j+m-1, j-1) \end{aligned} \quad (4.4)$$

where we have used the signed Stirling number of the first kind $s(n, k)$ to expand the Pochhammer symbol. The definition of $s(n, k)$ is

$$(x)_n = \sum_{k=0}^n (-1)^{n-k} s(n, k) x^k. \quad (4.5)$$

Thus the leading order nontrivial identity of eq. (4.3) can be written as ($m \geq 0$)

$$f(m) \equiv \sum_{j=0}^m (-1)^j \binom{2m}{j+m} [s(j+m-1, j-1) + s(j+m-1, j)] = (2m-1)!! \quad (4.6)$$

where we have used the convention that

$$s(m-1, -1) \begin{cases} = 0 & , \text{ for } m \geq 1 \\ = 1 & , \text{ for } m = 0 \end{cases} , \quad s(-1, 0) = 0. \quad (4.7)$$

With the help of the algorithm developed by Mkauers in 2007 [28], this identity can be proved. The point is that we can find a recurrence relation of $f(m)$ by his algorithm. However, to utilize the algorithm, we need to introduce an auxiliary variable u and define

$$\begin{aligned} f(u, m) &\equiv \sum_{j=0}^{m+u} (-1)^j \binom{2m+u}{j+m} [s(j+m-1, j-1) + s(j+m-1, j)] \\ &\equiv f_1(u, m) + f_2(u, m) \end{aligned} \quad (4.8)$$

where f_1 and f_2 are the two summations, each with one Stirling number, and $f(0, m) = f(m)$. By the algorithm, we can prove that both f_1, f_2 satisfy the following recurrence relation [28]

$$-(1+2m+u)f(u, m) + (2m+u)f(u+1, m) + f(u, m+1) = 0, \quad (4.9)$$

hence, so is f . eq. (4.9) is the most nontrivial step to prove eq. (4.6). Now, note that

$$f(u, 0) = \sum_{j=0}^u (-1)^j \binom{u}{j} = \begin{cases} 1 & , u = 0 \\ 0 & , u > 0 \end{cases} \quad \dots \quad (4.10)$$

Using the recurrence relation eq. (4.9) and substituting $(u, m) = (1, 0), (2, 0) \dots$, one can prove that

$$f(u, 1) = 0, \quad \forall u > 0. \quad (4.11)$$

Similarly, by substituting $(u, m) = (1, 1), (2, 1), (3, 1) \dots$, one can get $f(u, 2) = 0, \forall u > 0$. In general, we have

$$f(u, m) = 0, \quad \forall u > 0. \quad (4.12)$$

Finally we substitute $u = 0$ in the eq. (4.9) to obtain

$$-(1+2m)f(0, m) + 2mf(1, m) + f(0, m+1) = 0, \quad (4.13)$$

which implies

$$f(m+1) = (2m+1)f(m). \quad (4.14)$$

Eq. (4.6) is thus proved by mathematical induction.

The vanishing of the coefficients of $(-t)^0, (-t)^{-1}, \dots, (-t)^{-m+1}$ terms on the l.h.s. of eq. (4.3) means, for $1 \leq i \leq m$,

$$g(m, i) \equiv \sum_{j=0}^{m+i} (-1)^{j-i} \binom{2m}{j+m-i} [s(j+m-1-i, j) + s(j+m-1-i, j-1)] = 0. \tag{4.15}$$

To prove this identity, we need the recurrence relation [28]

$$-2(1+m)^2(1+2m)g(m, i) + (2+7m+4m^2)g(m+1, i) - 2m(1+m)(1+2m)g(m+1, i+1) - mg(m+2, i) = 0. \tag{4.16}$$

Putting $i=0, 1, 2, \dots$, and using the fact we have just proved, i.e. $g(m+1, 0) = (2m+1)g(m, 0)$, one can show that

$$g(m, i) = 0 \quad \text{for } 1 \leq i \leq m. \tag{4.17}$$

Eq. (4.3) is finally proved. We thus have shown that for those leading order high energy amplitudes $A^{(N, 2m, q)}$ in the RR with the same type of $(N, 2m, q)$ as those of GR, we can extract from them the ratios $T^{(N, 2m, q)}/T^{(N, 0, 0)}$ in the GR by using the Kummer function. Mathematically, the proof of this result turns out to be highly nontrivial and is based on a summation algorithm for Stirling number identity derived by Mkauers [28]. It is very interesting to see that the identity in eq. (4.3) suggested by string scattering amplitude calculation can be rigorously proved by a totally different but sophisticated mathematical method. In the next section, we discuss the generalization to subleading order amplitudes in the RR.

5 Subleading orders

In this section, we calculate the next few subleading order amplitudes in the RR for the mass level $M_2^2 = 4, 6$. We will see that the ratios in eqs. (2.6) and (2.7) persist to subleading order amplitudes in the RR. For the even mass levels with $(N-1) = \frac{M_2^2}{2} = \text{even}$, we conjecture and give evidences that the existence of these ratios in the RR persists to all orders in the Regge expansion of all high energy string scattering amplitudes. For the odd mass levels with $(N-1) = \frac{M_2^2}{2} = \text{odd}$, the existence of these ratios will show up only in the first $[N/2]+1$ terms in the Regge expansion of the amplitudes.

We will extend the kinematic relations in the RR to the subleading orders. We first express all kinematic variables in terms of s and t , and then expand all relevant quantities in s :

$$E_1 = \frac{s - (m_2^2 + 2)}{2\sqrt{2}}, \tag{5.1}$$

$$E_2 = \frac{s + (m_2^2 + 2)}{2\sqrt{2}}, \tag{5.2}$$

$$|\mathbf{k}_2| = \sqrt{E_1^2 + 2}, \quad |\mathbf{K}_3| = \sqrt{\frac{s}{4} + 2}; \tag{5.3}$$

$$e_P \cdot k_1 = -\frac{1}{2m_2}s + \left(-\frac{1}{m_2} + \frac{m_2}{2}\right), \quad (\text{exact}) \quad (5.4)$$

$$e_L \cdot k_1 = -\frac{1}{2m_2}s + \left(-\frac{1}{m_2} + \frac{m_2}{2}\right) - 2m_2s^{-1} - 2m_2(m_2^2 - 2)s^{-2} \\ - 2m_2(m_2^4 - 6m_2^2 + 4)s^{-3} - 2m_2(m_2^6 - 12m_2^4 + 24m_2^2 - 8)s^{-4} + O(s^{-5}), \quad (5.5)$$

$$e_T \cdot k_1 = 0. \quad (5.6)$$

A key step is to express the scattering angle θ in terms of s and t . This can be achieved by solving

$$t = -\left(-\left(E_2 - \frac{\sqrt{s}}{2}\right)^2 + (|\mathbf{k}_2| - |\mathbf{k}_3| \cos \theta)^2 + |\mathbf{k}_3|^2 \sin^2 \theta\right) \quad (5.7)$$

to obtain

$$\theta = \arccos\left(\frac{s + 2t - m_2^2 + 6}{\sqrt{s + 8}\sqrt{\frac{(s+2)^2 - 2(s-2)m_2^2 + m_2^4}{s}}}\right). \quad (\text{exact}) \quad (5.8)$$

One can then calculate the following expansions

$$e_P \cdot k_3 = \frac{1}{m_2}\left(E_2 \frac{\sqrt{s}}{2} - |\mathbf{k}_2||\mathbf{k}_3| \cos \theta\right) = -\frac{t + 2 - m_2^2}{2m_2}, \quad (5.9)$$

$$e_L \cdot k_3 = \frac{1}{m_2}\left(k_2 \frac{\sqrt{2}}{2} - E_2 k_3 \cos \theta\right) \\ = -\frac{t + 2 + m_2^2}{2m_2} - m_2 t s^{-1} - m_2[-4(t + 1) + m_2^2(t - 2)]s^{-2} \\ - m_2[4(4 + 3t) - 12tm_2^2 + (t - 4)m_2^4]s^{-3} \\ - m_2[-16(3 + 2t) + 24(2 + 3t)m_2^2 \\ - 24(-1 + t)m_2^4 + (-6 + t)m_2^6]s^{-4} + O(s^{-5}), \quad (5.10)$$

$$e_T \cdot k_3 = -|\mathbf{k}_3| \sin \theta \\ = -\sqrt{-t} - \frac{1}{2}\sqrt{-t}(2 + t + m_2^2)s^{-1} \\ - \frac{1}{8\sqrt{-t}}[32 + 52t + 20t^2 + t^3 + (32 + 20t - 6t^2)m_2^2 + (8 - 3t)m_2^4]s^{-2} \\ + \frac{1}{16\sqrt{-t}}[320 + 456t + 188t^2 + 22t^3 + t^4 - (-224 + 36t + 132t^2 + 5t^3)m_2^2 \\ + (-16 - 122t + 15t^2)m_2^4 + (-24 + 5t)m_2^6]s^{-3} \\ + \frac{1}{128(-t)^{3/2}}[1024 + 12032t + 16080t^2 + 7520t^3 + 1432t^4 + 136t^5 + 5t^6 \\ - 4(-512 - 896t + 2232t^2 + 1844t^3 + 170t^4 + 7t^5)m_2^2 \\ + 2(768 - 2240t - 2372t^2 + 1172t^3 + 35t^4)m_2^4 \\ - 4(-128 + 288t - 450t^2 + 35t^3)m_2^6 + (64 + 240t - 35t^2)m_2^8]s^{-4} + O(s^{-5}). \quad (5.11)$$

We are now ready to calculate the expansions of the four amplitudes A_{TTT} , A_{LLT} , $A_{[LT]}$,

$A_{(LT)}$ for the mass level $M_2^2 = 4$ to subleading orders in s in the RR. These are

$$A^{TTT} \sim \frac{1}{8}\sqrt{-t}s^3 + \frac{3}{16}\sqrt{-t}t(t+6)s^2 + \frac{3t^3 + 84t^2 - 68t - 864}{64}\sqrt{-t}s + O(1), \quad (5.12)$$

$$A^{LLT} \sim \frac{1}{64}\sqrt{-t}(t-6)s^3 + \frac{3}{128}\sqrt{-t}(t^2 - 20t - 12)s^2 + \frac{3t^3 - 342t^2 - 92t + 5016 + 1728(-t)^{-1/2}}{512}\sqrt{-t}s + O(1), \quad (5.13)$$

$$A^{[LT]} \sim -\frac{1}{64}\sqrt{-t}(t+2)s^3 - \frac{3}{128}\sqrt{-t}(t+2)^2s^2 + O(s) - \frac{(3t-8)(t+6)^2[1-2(-t)^{-1/2}]}{512}\sqrt{-t}s + O(1), \quad (5.14)$$

$$A^{(LT)} \sim -\frac{1}{64}\sqrt{-t}(t+10)s^3 - \frac{1}{128}\sqrt{-t}(3t^2 + 52t + 60)s^2 + O(s) - \frac{3[t^3 + 30t^2 + 76t - 1080 - 960(-t)^{-1/2}]}{512}\sqrt{-t}s + O(1). \quad (5.15)$$

One can now easily see that the ratios of the coefficients of the highest power of t in the leading order coefficient functions $\frac{1}{8} : \frac{1}{64} : -\frac{1}{64} : -\frac{1}{64}$ agree with the ratios in the GR $8 : 1 : -1 : -1$ calculated in eq. (2.6) as expected. Moreover, one further observation is that these ratios remain the same for the coefficients of the highest power of t in the subleading orders (s^2) $\frac{3}{16} : \frac{3}{128} : -\frac{3}{128} : -\frac{3}{128}$ and (s) $\frac{3}{64} : \frac{3}{512} : -\frac{3}{512} : -\frac{3}{512}$. We conjecture that these ratios persist to all energy orders in the Regge expansion of the amplitudes. This is consistent with the results of GR by taking both $s, -t \rightarrow \infty$. For the mass level $M_2^2 = 6$ [5], the amplitudes can be calculated to be

$$\begin{aligned} A^{TTTT} &\sim \left(\frac{s^2}{4} - s\right) \left(\frac{s^2}{4} - 1\right) (e^T \cdot k_3)^4 \\ &= \frac{t^2}{16}s^4 + \frac{t^2(t+6)}{8}s^3 + \frac{t(t^3 + 24t^2 - 4t - 256)}{16}s^2 \\ &\quad + \frac{t(3t^3 - 2t^2 - 396t - 768)}{4}s - \left(\frac{t^4}{4} + 166t^3 + 960t^2 - 64t - 1024\right)s^0 \\ &\quad + (-83t^4 - 1536t^3 + 384t^2 + 21248t + 12288)s^{-1} + O(s^{-2}), \end{aligned} \quad (5.16)$$

$$\begin{aligned} A^{TTLL} &\sim \left(\frac{s^2}{4} - s\right) \left(\frac{s^2}{4} - 1\right) (e^T \cdot k_3)^2 (e^L \cdot k_3)^2 \\ &\quad + \frac{3st}{2} \left(\frac{s}{2} + 1\right) \left(\frac{t}{2} + 1\right) (e^L \cdot k_1)^2 (e^T \cdot k_3)^2 \times \frac{1}{6} \\ &\quad - s \left(\frac{s^2}{4} - 1\right) (t+2) (e^L \cdot k_1) (e^L \cdot k_3) (e^T \cdot k_3)^2 \times \frac{1}{2} \\ &= \frac{t(t-16)}{192}s^4 + \frac{t(t^2 - 41t - 32)}{96}s^3 + \frac{t^4 - 132t^3 - 328t^2 + 1984t + 2048}{192}s^2 \\ &\quad + \left(-\frac{11t^4}{32} - \frac{11t^3}{4} + \frac{163t^2}{3} + 184t + \frac{128}{3}\right)s^1 \\ &\quad + \left(-\frac{11}{8}t^4 + 88t^3 + 744t^2 + 304t - 1408\right)s^0 \\ &\quad + 4(11t^4 + 280t^3 + 204t^2 - 4448t - 4480)s^{-1} + O(s^{-2}), \end{aligned} \quad (5.17)$$

$$\begin{aligned}
A^{LLLL} &\sim \left(\frac{s^2}{4} - s\right) \left(\frac{s^2}{4} - 1\right) (e^L \cdot k_3)^4 - t \left(\frac{t^2}{4} - 1\right) (s+2)(e^L \cdot k_1)^3 (e^L \cdot k_3) \\
&\quad + \frac{3st}{2} \left(\frac{s}{2} + 1\right) \left(\frac{t}{2} + 1\right) (e^L \cdot k_1)^2 (e^L \cdot k_3)^2 \\
&\quad - s \left(\frac{s^2}{4} - 1\right) (t+2)(e^L \cdot k_1)(e^L \cdot k_3)^3 \\
&\quad + \left(\frac{t^2}{4} - t\right) \left(\frac{t^2}{4} - 1\right) (e^L \cdot k_1)^4 \\
&= \frac{t(t-52)}{768} s^4 + \frac{t(t^2-140t+256)}{384} s^3 + \frac{t^4-456t^3+2816t^2-512t-16384}{768} s^2 \\
&\quad \left(-\frac{19t^4}{64} + 6t^3 - \frac{17t^2}{3} - 176t - \frac{256}{3}\right) s^1 \\
&\quad + (3t^4 - 10t^3 - 528t^2 - 672t + 1792) s^0 + O(s^{-1}), \tag{5.18}
\end{aligned}$$

$$\begin{aligned}
A^{TTL} &\sim -\left(\frac{s^2}{4} - s\right) \left(\frac{s^2}{4} - 1\right) (e^T \cdot k_3)^2 (e^L \cdot k_3) \\
&\quad - \frac{st}{4} \left(\frac{s}{2} + 1\right) \left(\frac{t}{2} + 1\right) (e^T \cdot k_3)^2 (e^L \cdot k_1) \times \frac{1}{3} \\
&\quad + s \left(\frac{s^2}{4} - 1\right) \left(\frac{t}{2} + 1\right) (e^T \cdot k_3)^2 (e^L \cdot k_1) \times \frac{1}{3} \\
&= -\frac{(t+20)t}{96\sqrt{6}} s^4 - \frac{t(t^2+31t+40)}{48\sqrt{6}} s^3 - \frac{t^4+38t^3+224t^2-1520t-2560}{96\sqrt{6}} s^2 \\
&\quad + \frac{-3t^4-72t^3+2248t^2+12000t+5120}{48\sqrt{6}} s^1 \\
&\quad + \frac{67t^3+1194t^2+1344t-3712}{2\sqrt{6}} s^0 + O(s^{-1}), \tag{5.19}
\end{aligned}$$

$$\begin{aligned}
A^{LLL} &\sim -\left(\frac{s^2}{4} - s\right) \left(\frac{s^2}{4} - 1\right) (e^L \cdot k_3)^3 + t \left(\frac{t^2}{4} - 1\right) \left(\frac{s}{2} + 1\right) (e^L \cdot k_1)^2 (e^L \cdot k_3) \\
&\quad - \frac{st}{4} \left(\frac{s}{2} + 1\right) \left(\frac{t}{2} + 1\right) [(e^L \cdot k_1)^2 (e^L \cdot k_3) + (e^L \cdot k_3)^2 (e^L \cdot k_1)] \\
&\quad + s \left(\frac{s^2}{4} - 1\right) \left(\frac{t}{2} + 1\right) (e^L \cdot k_3)^2 (e^L \cdot k_1) - \left(\frac{t^2}{4} - t\right) \left(\frac{t^2}{4} - 1\right) (e^L \cdot k_1)^3 \\
&= -\frac{t^2-8t-128}{384\sqrt{6}} s^4 - \frac{t^3-52t^2-412t+256}{192\sqrt{6}} s^3 \\
&\quad - \frac{t^4-236t^3-1272t^2+4832t+15872}{384\sqrt{6}} s^2 \\
&\quad + \frac{35t^4+50t^3-3008t^2-23728t-14848}{96\sqrt{6}} s^1 \\
&\quad - \frac{47t^4+1432t^3+24796t^2+40640t-101376}{48\sqrt{6}} s^0 + O(s^{-1}), \tag{5.20}
\end{aligned}$$

$$\begin{aligned}
\tilde{A}^{LT,T} &\sim -\left(\frac{s^2}{4} - s\right) \left(\frac{s^2}{4} - 1\right) (e^T \cdot k_3)^2 (e^L \cdot k_3) \times 0 \\
&\quad - \frac{st}{4} \left(\frac{s}{2} + 1\right) \left(\frac{t}{2} + 1\right) (e^L \cdot k_1) (e^T \cdot k_3)^2 \times \frac{1}{2} \\
&\quad + s \left(\frac{s^2}{4} - 1\right) \left(\frac{t}{2} + 1\right) (e^T \cdot k_3)^2 (e^L \cdot k_1) \times \left(-\frac{1}{4}\right) \\
&= -\frac{t(t+2)}{64\sqrt{6}} s^4 - \frac{t(t+2)^2}{32\sqrt{6}} s^3 - \frac{t^4 + 12t^3 + 8t^2 - 152t - 256}{64\sqrt{6}} s^2 \\
&\quad + \frac{-3t^4 + 196t^2 + 624t + 512}{32\sqrt{6}} s^1 + \sqrt{\frac{3}{8}} (5t^3 + 30t^2 + 24t - 32) s^0 + O(s^{-1}), \quad (5.21)
\end{aligned}$$

$$\begin{aligned}
A^{LL} &\sim \left(\frac{s^2}{4} - s\right) \left(\frac{s^2}{4} - 1\right) (e^L \cdot k_3)^2 + \frac{st}{2} \left(\frac{s}{2} + 1\right) \left(\frac{t}{2} + 1\right) (e^L \cdot k_1) (e^L \cdot k_3) \\
&\quad + \left(\frac{t^2}{4} - t\right) \left(\frac{t^2}{4} - 1\right) (e^L \cdot k_1)^2 \\
&= \frac{(t+8)^2}{384} s^4 + \frac{(t^3 + 20t^2 + 80t - 128)}{192} s^3 + \frac{t^4 + 16t^3 + 96t^2 - 880t - 3328}{384} s^2 \\
&\quad + \frac{-t^4 + 8t^3 - 110t^2 - 1648t - 1408}{48} s^1 \\
&\quad + \frac{t^4 - 4t^3 - 202t^2 - 704t + 1728}{6} s^0 + O(s^{-1}). \quad (5.22)
\end{aligned}$$

In the above calculations, as in the case of $M_2^2 = 4$, we have ignored a common overall factor which will be discussed in section VI. Note that the ratios of the coefficients in the leading order t for the energy orders s^4, s^3, s^2 reproduced the GR ratios in eq. (2.7). However, the subleading terms for orders s^1, s^0 contain no GR ratios. Mathematically, this is because the highest power of t in the coefficient functions of s^1 is 4 rather than 5, and those of s^0 is 4 rather than 6. This is because the power of t in the kinematic relation eq. (5.11) can be as high as one wants if one goes to subleading orders, while that of eq. (5.10) is not. The $\sin \theta$ factor in eq. (5.11) contributes terms of higher order powers of t , while $\cos \theta$ factor in eq. (5.10) does not. This can be seen from the kinematic relation in eq. (5.8). In general, one can easily show that the $\sin \theta$ factor will contribute only for the even mass levels with $(N - 1) = \frac{M_2^2}{2} = \text{even}$. We thus conjecture that the existence of the GR ratios in the RR persists to all orders in the Regge expansion of all string amplitudes for the even mass level. For the odd mass levels with $(N - 1) = \frac{M_2^2}{2} = \text{odd}$, the existence of the GR ratios will show up only in the first $[N/2] + 1$ terms in the Regge expansion of the amplitudes. An interesting question is whether this phenomena persists for the case of superstring where GSO projection needs to be imposed.

6 Universal power law behavior

In the discussion of section V, we ignored an overall common factor $\frac{\Gamma(-1-s/2)\Gamma(-1-t/2)}{\Gamma(u/2+2)}$ of the amplitudes for mass levels $M_2^2 = 4, 6$. We paid attention only to the ratios among scattering amplitudes of different string states. In this section, we calculate the high energy behavior

of string scattering amplitudes for string states at arbitrary mass levels in the RR. The power law behavior $\sim s^{\alpha(t)}$ of the four-tachyon amplitude in the RR is well known in the literature. Here we want to generalize this result to string states at arbitrary mass levels. We can use the saddle point method to calculate the leading term of gamma functions in the RR

$$\frac{\Gamma(-1 - s/2)\Gamma(-1 - t/2)}{\Gamma(u/2 + 2)} = \frac{\Gamma(-1 - s/2)\Gamma(-1 - t/2)}{\Gamma(-s/2 - t/2 + N - 2)} \sim s^{t/2 - N + 1} \quad (\text{in the RR}). \quad (6.1)$$

Thus, the overall s -dependence in the amplitudes is of the form

$$A^{(kn, qm)} \sim s^{\alpha(t)} \quad (\text{in the RR}) \quad (6.2)$$

where

$$\alpha(t) = \alpha(0) + \alpha' t, \quad \alpha(0) = 1 \text{ and } \alpha' = 1/2. \quad (6.3)$$

This generalizes the high energy behavior of the four-tachyon amplitude in the RR to string states at arbitrary mass levels. The new result here is that the behavior is universal and is mass level independent. In fact, as a simple application, one can also derive eq. (6.2) directly from eq. (3.9) by using

$$B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \sim s^{\alpha(t)}. \quad (\text{in the RR}) \quad (6.4)$$

We conclude that the well known $\sim s^{\alpha(t)}$ power-law behavior of the four tachyon string scattering amplitude in the RR can be extended to high energy string scattering amplitudes of arbitrary string states.

7 Conclusion

In this paper, we calculate high energy massive string scattering amplitudes of 26D open bosonic string in the Regge regime (RR). It turns out that both the saddle-point method and the method of decoupling of high energy ZNS adopted in the calculation of GR [4–12] do not apply to the case of RR. However, the general formula for the high energy scattering amplitudes for each fixed mass level in the RR can still be written down explicitly. We have found that the number of high energy amplitudes for each fixed mass level in the RR is much more numerous than that of Gross regime (GR) calculated previously [4–12].

On the other hand, there is no linear relation among scatterings in the RR in contrast to the case of scatterings in the GR. Moreover, we discover that the leading order amplitudes in the RR can be expressed in terms of the Kummer function of the second kind. In particular, based on a summation algorithm for Stirling number identity in the combinatoric number theory, we discover that the ratios calculated previously among scattering amplitudes in the GR can be extracted from this Kummer function in the RR. We conjecture and give evidences that the existence of the GR ratios in the RR persists to all orders in the Regge expansion of all string amplitudes for the even mass level with $(N - 1) = \frac{M_2^2}{2} = \text{even}$. For the odd mass levels with $(N - 1) = \frac{M_2^2}{2} = \text{odd}$, the existence of the GR ratios shows up only in the first $[N/2] + 1$ terms in the Regge expansion of the amplitudes. An interesting question

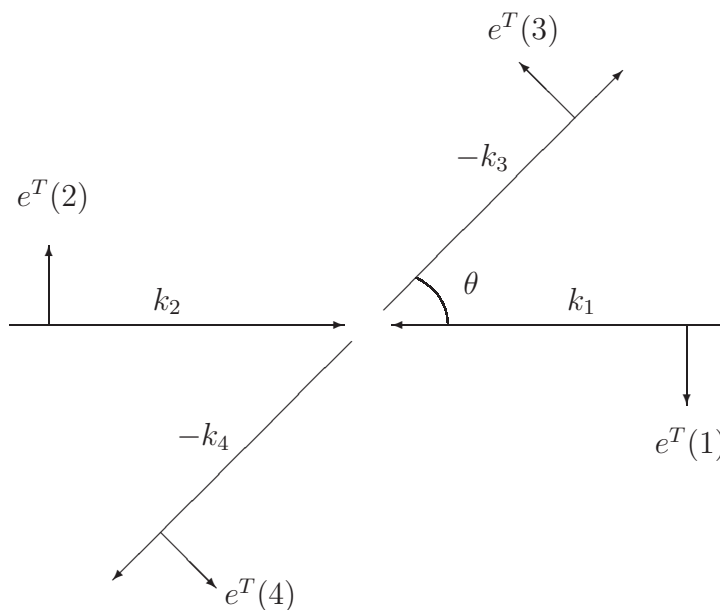


Figure 1. Kinematic variables in the center of mass frame.

is whether this phenomena persists for the case of superstring where GSO projection needs to be imposed. Finally, we demonstrate the universal power-law behavior for all massive string scattering amplitudes in the RR. This result generalizes the well known result for the case of high energy four-point tachyon amplitudes.

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A Kinematic variables and notations

In this appendix, we list the expressions of the kinematic variables we used in the evaluation of 4-point functions in this paper. For convenience, we take the center of momentum frame and choose the momenta of particles 1 and 2 to be along the X^1 -direction. The high energy scattering plane is defined to be on the $X^1 - X^2$ plane.

The momenta of the four particles are

$$k_1 = \left(+\sqrt{p^2 + M_1^2}, -p, 0 \right), \quad (\text{A.1})$$

$$k_2 = \left(+\sqrt{p^2 + M_2^2}, +p, 0 \right), \quad (\text{A.2})$$

$$k_3 = \left(-\sqrt{q^2 + M_3^2}, -q \cos \phi, -q \sin \theta \right), \quad (\text{A.3})$$

$$k_4 = \left(-\sqrt{q^2 + M_4^2}, +q \cos \phi, +q \sin \theta \right) \quad (\text{A.4})$$

where $p \equiv |\vec{p}|$, $q \equiv |\vec{q}|$ and $k_i^2 = -M_i^2$. In the calculation of the string scattering amplitudes, we use the following formulas

$$-k_1 \cdot k_2 = \sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2 = \frac{1}{2} (s - M_1^2 - M_2^2), \quad (\text{A.5})$$

$$-k_2 \cdot k_3 = -\sqrt{p^2 + M_2^2} \cdot \sqrt{q^2 + M_3^2} + pq \cos \theta = \frac{1}{2} (t - M_2^2 - M_3^2), \quad (\text{A.6})$$

$$-k_1 \cdot k_3 = -\sqrt{p^2 + M_1^2} \cdot \sqrt{q^2 + M_3^2} - pq \cos \theta = \frac{1}{2} (u - M_1^2 - M_3^2) \quad (\text{A.7})$$

where the Mandelstam variables are defined as usual with

$$s + t + u = \sum_i M_i^2 = 2(N - 4). \quad (\text{A.8})$$

The center of mass energy E is defined as

$$E = \frac{1}{2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) = \frac{1}{2} \left(\sqrt{q^2 + M_3^2} + \sqrt{q^2 + M_4^2} \right). \quad (\text{A.9})$$

We define the polarizations of the string state on the scattering plane as

$$e^P = \frac{1}{M_2} \left(\sqrt{p^2 + M_2^2}, p, 0 \right), \quad (\text{A.10})$$

$$e^L = \frac{1}{M_2} \left(p, \sqrt{p^2 + M_2^2}, 0 \right), \quad (\text{A.11})$$

$$e^T = (0, 0, 1). \quad (\text{A.12})$$

The projections of the momenta on the scattering plane can be calculated as (here we only list the ones we need for our calculations)

$$e^P \cdot k_1 = -\frac{1}{M_2} \left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right), \quad (\text{A.13})$$

$$e^L \cdot k_1 = -\frac{p}{M_2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right), \quad (\text{A.14})$$

$$e^T \cdot k_1 = 0 \quad (\text{A.15})$$

and

$$e^P \cdot k_3 = \frac{1}{M_2} \left(\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \theta \right), \quad (\text{A.16})$$

$$e^L \cdot k_3 = \frac{1}{M_2} \left(p \sqrt{q^2 + M_3^2} - q \sqrt{p^2 + M_2^2} \cos \theta \right), \quad (\text{A.17})$$

$$e^T \cdot k_3 = -q \sin \theta = -\sqrt{-t}. \quad (\text{A.18})$$

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