Contents

1 Introduction

We study the exact theory and its nonlinear approximation of the nonlinear Schrodinger equation (NLS, [1][2][3])

$$iq_t + q_{xx} + 2|q|^2 q = 0. \tag{1.1}$$

The nonlinear approximation of NLS has solutions reside on the Riemann surface of genus N-1 denoted by R_{N-1} :

$$R(E) = \sqrt{\prod_{k=1}^{2N}(E - E_k)} \tag{1.2}$$

where $N \in \mathbb{N}$, $E_k \in \mathbb{C} \setminus \mathbb{R}$ and $E^*_{2k-1} = E_{2k}$.

We first study the theory of R_{N-1} [5][6] from which we can analyze the nonlinear approximations and the theories about the Riemann surfaces will be studied in section 2. Secondly, we study the classical elliptic functions [4] which are close related to the theory of Riemann surfaces and are applied to solve some special solutions of NLS. The theories and properties of the classical elliptic functions will be studied in section 3.

Finally, we use the theories of Riemann surfaces and the classical elliptic functions to solve some special solutions of NLS and analyze the degenerates of the NLS solutions in section 4.

2 The Riemann surfaces

2.1 The construction of Riemann surface

First, take a simple case $f(z) = \sqrt{z}$ for example, $f : \mathbb{C} \to \mathbb{C}$. When $z \in \mathbb{C}$, z can be expressed as $z = |z|e^{i\theta} = |z|e^{i(\theta+2n\pi)}$ where $n \in \mathbb{N}$, then

$$\begin{aligned} f(z) &= \sqrt{z} = |z|^{\frac{1}{2}} e^{i(\frac{\theta+2n\pi}{2})} \\ &= \begin{cases} |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} & \text{if n is even,} \\ -|z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} & \text{if n is odd,} \end{cases} \end{aligned}$$

which implies that f is a two-valued function. Now we want to let $f(z)$ be a single-valued function, so we need to find the corresponding Riemann surface such that f becomes a single-valued function.

Second, by stereographic projection, we know that there is a mapping that projects a sphere onto a plane, then for any $z \in \mathbb{C}$, we can find exactly one corresponding point on the sphere, and we call this sphere extended complex plane, denoted by $\mathbb{C}^*$. So by this projection, we can visualize the point at infinity in the complex plane, and it corresponds to the north pole denoted by N of the extended complex plane.

Note that $f(z) = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}$, when θ increases by 2π, $f(z) = |z|^{\frac{1}{2}} e^{i\frac{\theta+2\pi}{2}} = -|z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ which is just the negative of its original value, again when θ increases by 2π, $f(z) = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ which is the original value. Since $f(z)$ changes its value when θ increases by 2π and $f(z) = \sqrt{z} = \sqrt{z-0}$ where 0 is a branch point, so image there are two sheets lying on the complex plane. By stereographic projection, we can consider the two sheets to be the spheres ($\mathbb{C}^*$) and cut them from the branch point 0 to N which corresponds to cut the complex plan along the negative real axis, that is, from 0 to $-\infty$. Then we know each sheet has a corresponding cut plane and we get two single-valued branches of $f(z)$. Define that

$$f(z) = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}, -\pi \le \theta < \pi, \text{ as } z \text{ is in the sheet-I,}$$

$$f(z) = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}, \pi \leq \theta < 3\pi, \text{as } z \text{ is in the sheet-II.}$$

The cut on each sheet has two edges, label the starting edge with "+" and the terminal edge with "−".

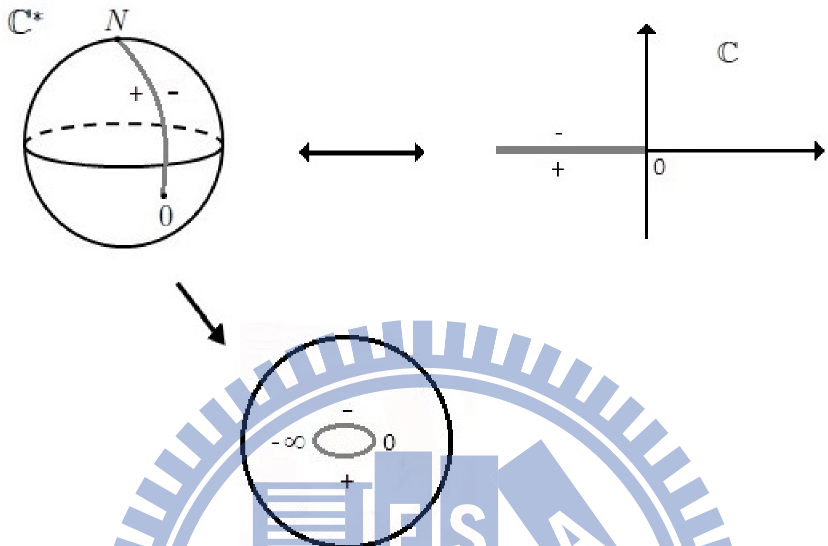

Figure 1.The extended complex plane and cut plane

Third, open two spheres from the cuts and stretch two cuts into circular holes, then rotate two circular holes such that the (+) edge of sheet-I face the (-) edge of sheet-II and the (-) edge of sheet-I face the (+) edge of sheet-II. Then glue two holes together and mold it into a sphere. We call this sphere, Riemann surface of genus 0 for it does not have any holes, denoted by R_0.

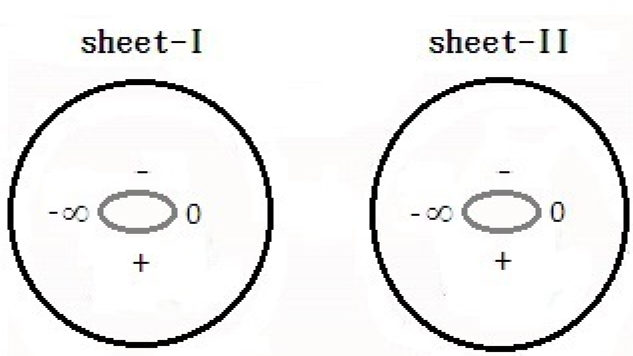

Figure 2.Placing the cuts open

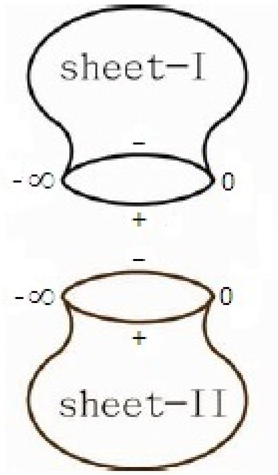

Figure 3.Together with two sheets

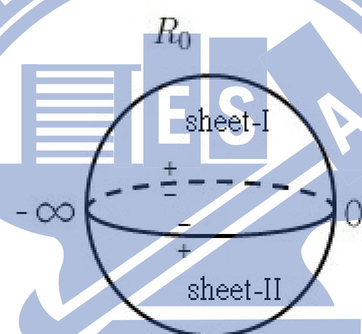

Figure 4.The Riemann surface of genus 0

Since we glue two sheets together and each sheet has a corresponding cut plane, then there must be a corresponding cut plane to Riemann surface. Note that when we cross the cut, we pass from one sheet to another so for convenience, we use the solid line to represent the curve in sheet-I and use the dash line to represent the curve in sheet-II on the cut plane. Moreover, the (+) edge of sheet-I is equivalent to the (-) edge of sheet-II, and the (-) edge of sheet-I is equivalent to the (+) edge of sheet-II in the Riemann surface.

Next, we introduce the relation between the curve in algebraic structure and geometric structure when $f(z) = \sqrt{z}$.

Example 2.1.1. The curve γ_1 is start from a point in the (+) edge of sheet-I to the (-) edge of sheet-I and γ_2 is start from a point in the (+) edge of sheet-II to the (-) edge of sheet-II.

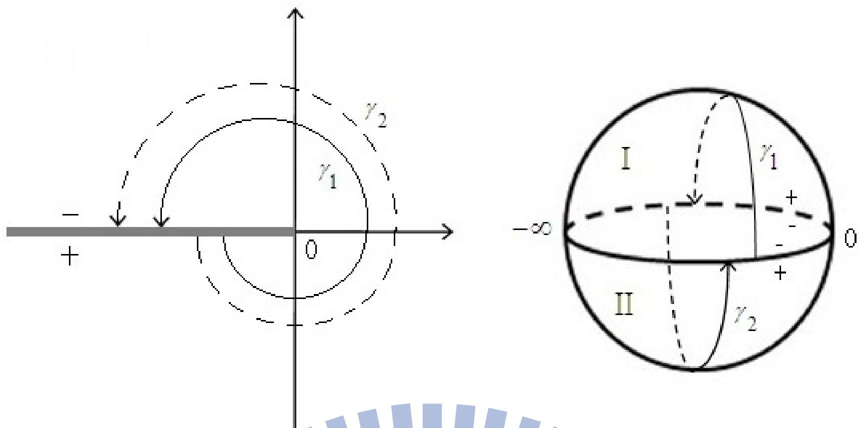

Figure 5.The algebraic structure and geometric structure of γ_1 and γ_2

Last, if a point z_1 is in sheet-I, we can write $z_1 = |z_1|e^{i\theta_1}$ where $\theta_1 \in [-\pi, \pi)$, then the argument of $\sqrt{z_1}$ is $\frac{\theta_1}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2})$, and if a point z_2 is in sheet-II, write $z_2 = |z_2|e^{i\theta_2}$ where $\theta_2 \in [\pi, 3\pi)$, then the argument of $\sqrt{z_2}$ is $\frac{\theta_2}{2} \in [\frac{\pi}{2}, \frac{3\pi}{2})$, therefore $f(z) = \sqrt{z}$ become a single-valued function whenever the point z is in sheet-I or sheet-II.

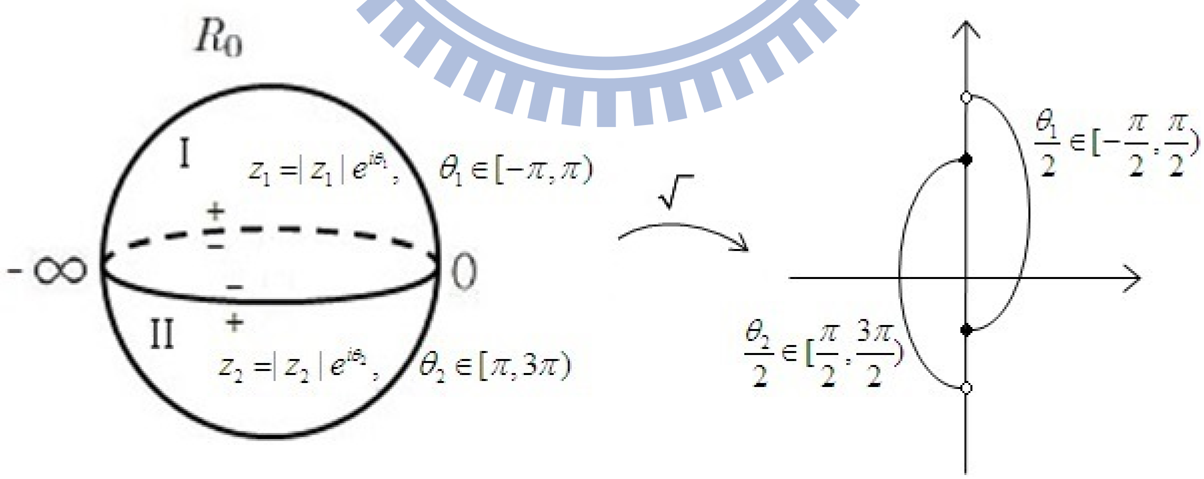

Figure 6.The argument of $f(z)$

Example 2.1.2. Construct the corresponding Riemann surface for $f(z) = \sqrt{\prod_{k=1}^{5} (z - z_k)}$ where

$z_k \in \mathbb{R}$ and $z_1 < z_2 < \cdots < z_5$.

To begin with, cut the plane from z_k to $-\infty$, $k = 1, 2, \cdots 5$.

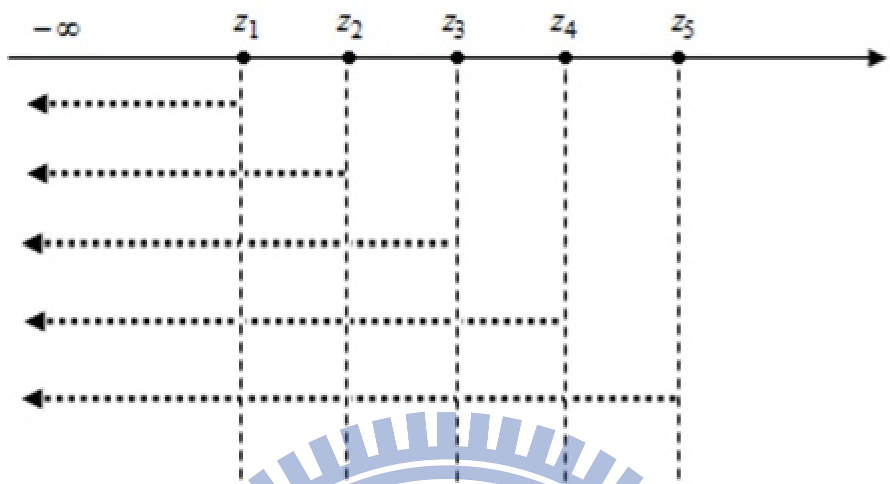

Figure 7.Cutting the plane from z_k to $-\infty$

Since the argument of z increases by 2π when we cross one cut which implies that the argument of $f(z)$ increases by π, then $f(z)$ becomes the negative of its original value. That is, the value of $f(z)$ changes one sign when we cross one cut. Thus the value of $f(z)$ will change sign when crossing the cuts odd times and not change sign when crossing the cuts even times.

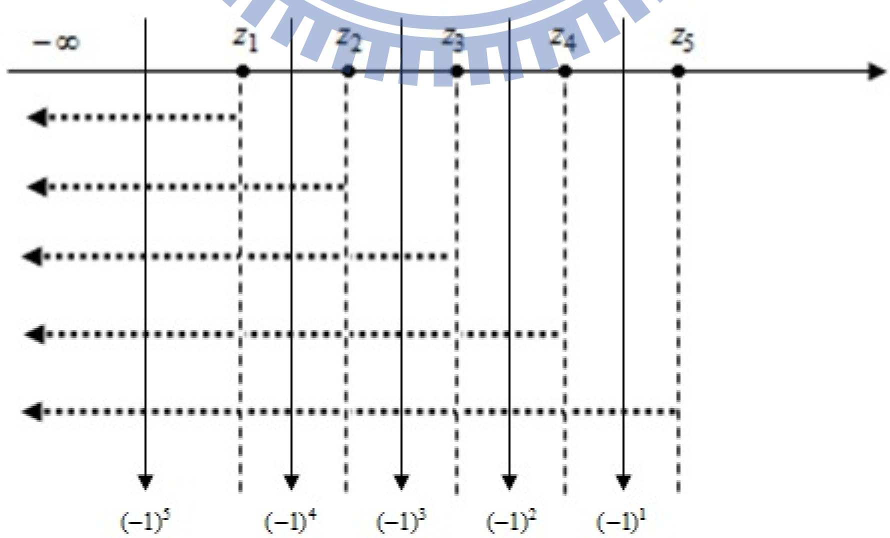

Figure 8.Determining the sign of the value of $f(z)$

Then the cut structure becomes:

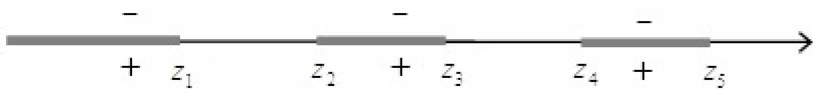

Figure 9.The cut structure

The branch cuts are in $[-\infty, z_1]$, $[z_2, z_3]$, $[z_4, z_5]$.

Furthermore, open the cuts and rotate two sheets such that the (+) edge of sheet-I face the (-) edge of sheet-II and the (-) edge of sheet-I face the (+) edge of sheet-II.

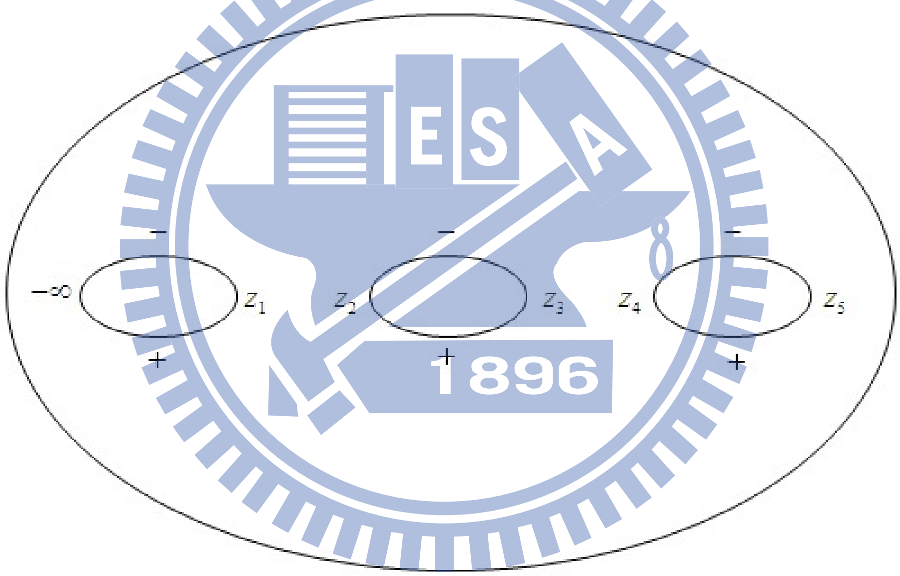

Figure 10.Placing the cuts open

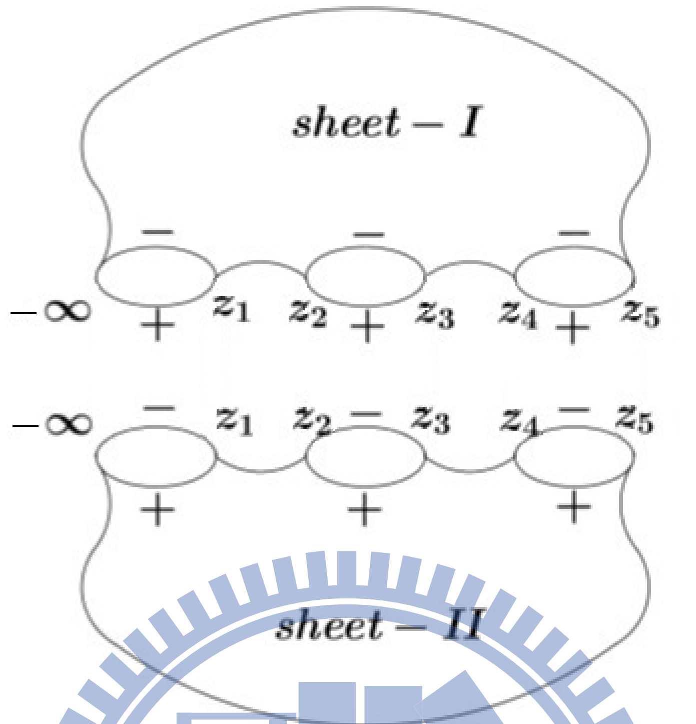

Figure 11.Together with two sheets

Finally, glue two sheets together, then we get the corresponding Riemann surface of $f(z)$ and call it Riemann surface of genus 2.

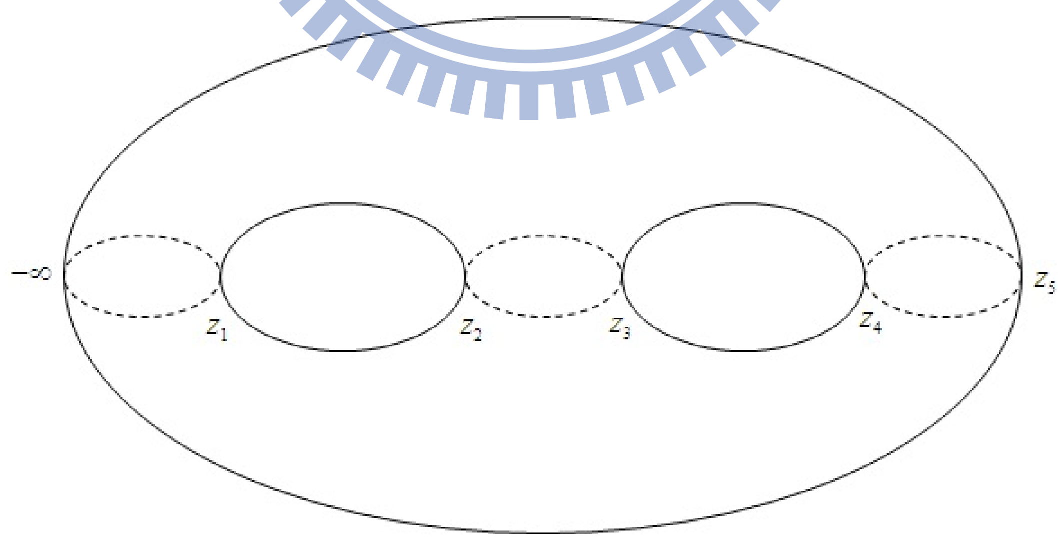

Figure 12.The Riemann surface of genus 2

Example 2.1.3. Construct the corresponding Riemann surface for $f(z) = \sqrt{\prod_{k=1}^{6} (z - z_k)}$ where $z_k \in \mathbb{R}$ and $z_1 < z_2 < \cdots < z_6$.

We do the same process as in Example 1.1.1. Cut the plane from z_k to $-\infty$, $k = 1, 2, \cdots 6$.

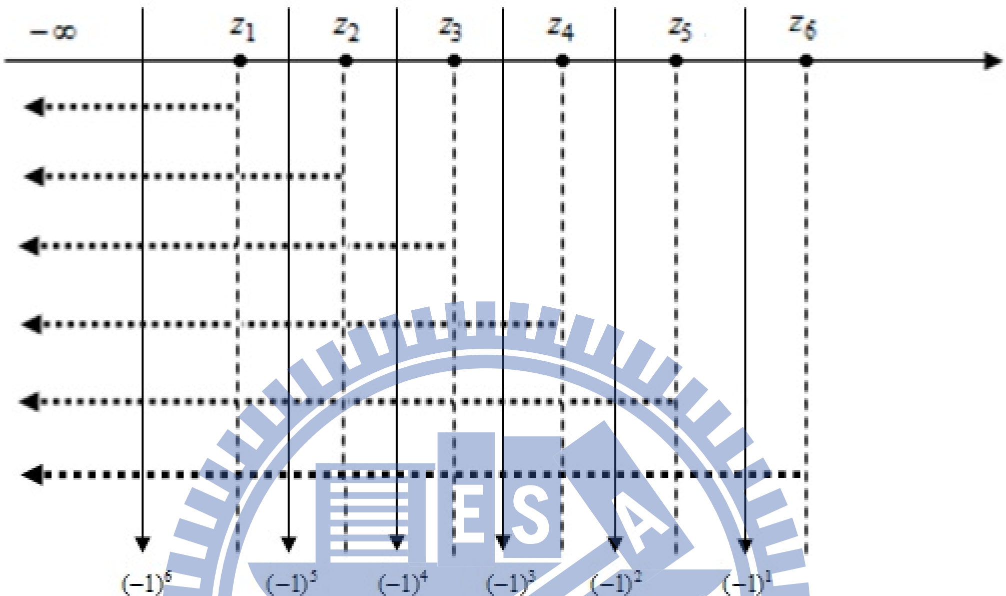

Figure 13.Cutting the plane from z_k to $-\infty$

The cut structure becomes:

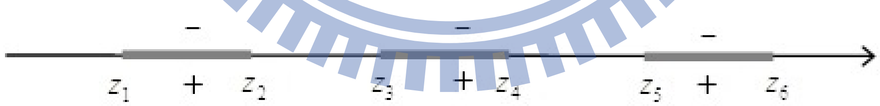

Figure 14.The cut structure

The branch cuts are in $[z_1, z_2]$, $[z_3, z_4]$, $[z_5, z_6]$.

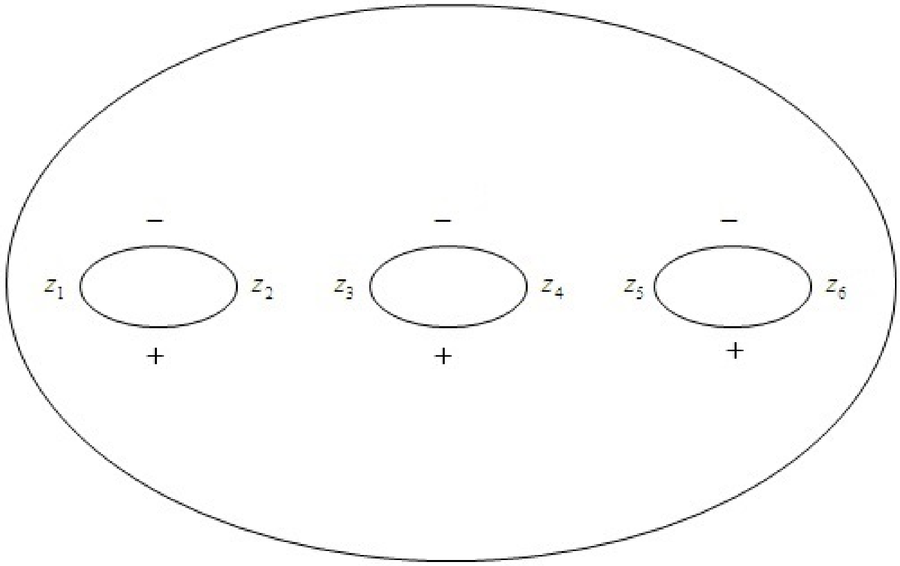

Figure 15.Placing the cuts open

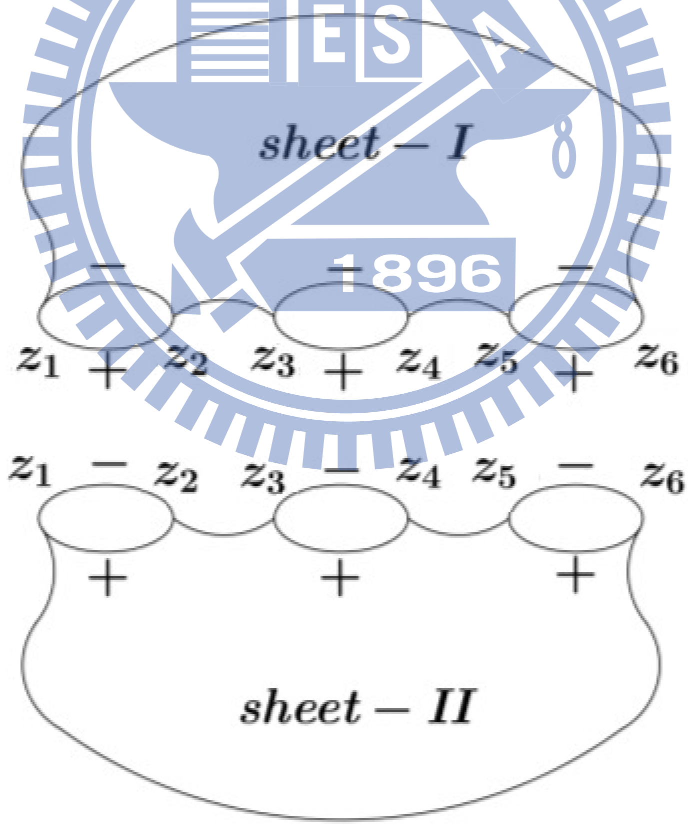

Figure 16.Together with two sheets

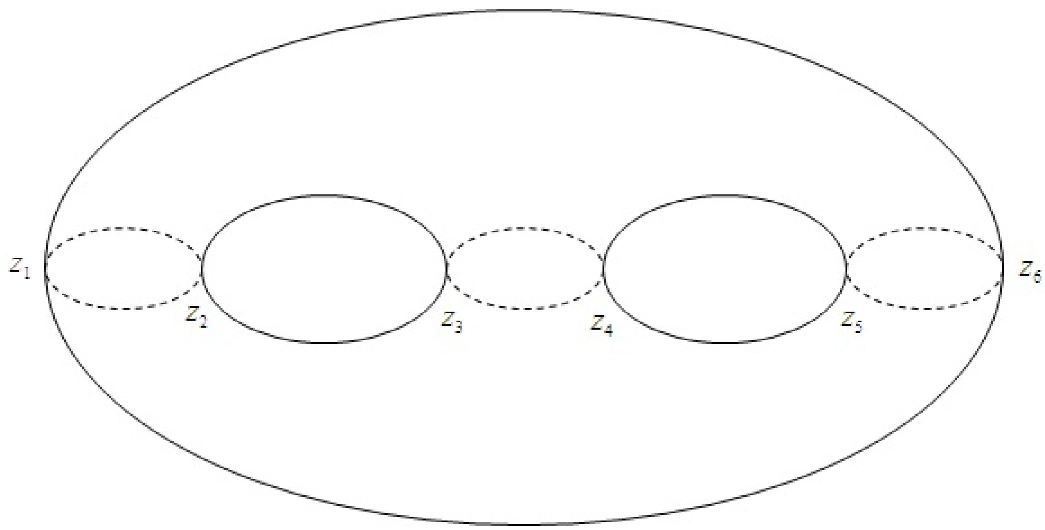

Figure 17.The Riemann surface of genus 2

In the preceding two examples, we know that two functions, one with 5 roots, the other with 6 roots, have different algebraic structures but have the same geometric graph, that is, they all obtain Riemann surface of genus 2.

Now using the same method as before to find the corresponding Riemann surface of $f(z)$ in general case where $f(z) = \sqrt{\prod_{k=1}^{n}(z-z_k)}$, $z_k \in \mathbb{R}$ and $z_1 < z_2 < \cdots < z_n$. Cut the plane from z_k to $-\infty$, $k = 1, 2, \cdots n$.

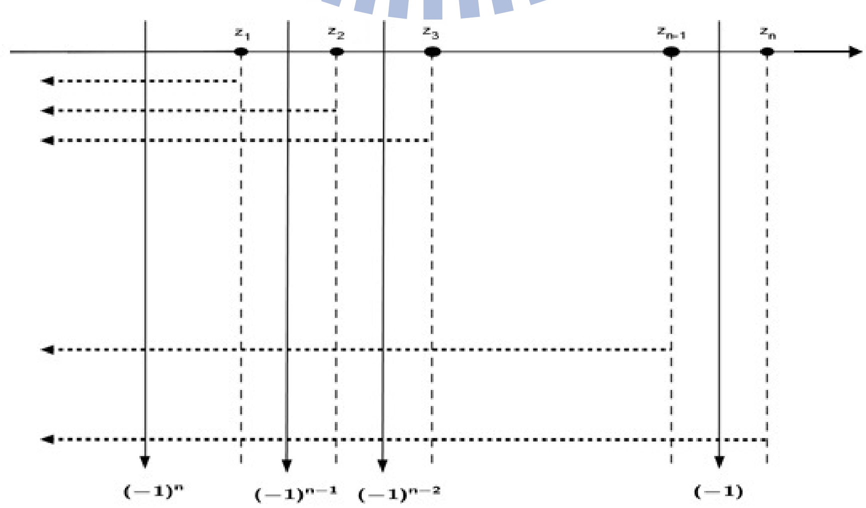

Figure 18.Cutting the plane from z_k to $-\infty$

Case 1. $n = 2N - 1$, $N \in \mathbb{N}$.

The cut structure becomes:

Figure 19.The cut structure

The branch cuts are in $[-\infty, z_1]$, $[z_2, z_3]$, $\cdots$, $[z_{2N-2}, z_{2N-1}]$.

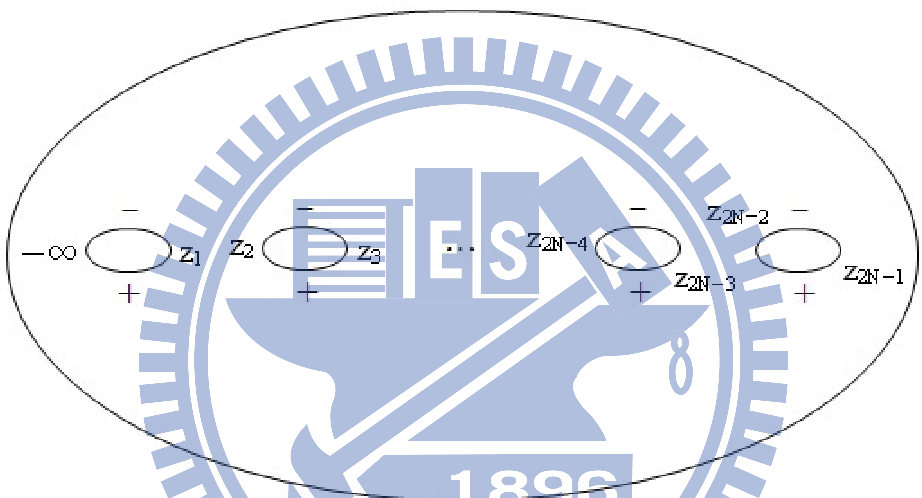

Figure 20.Placing the cuts open

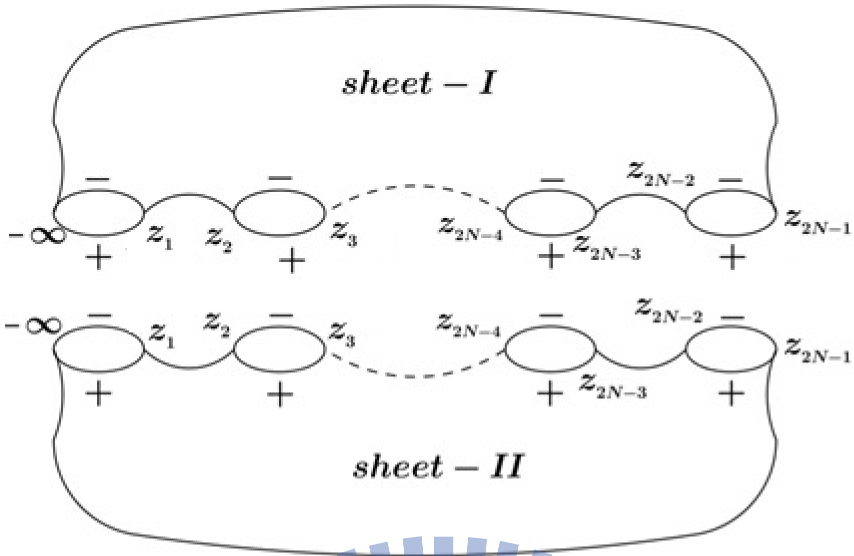

Figure 21.Together with two sheets

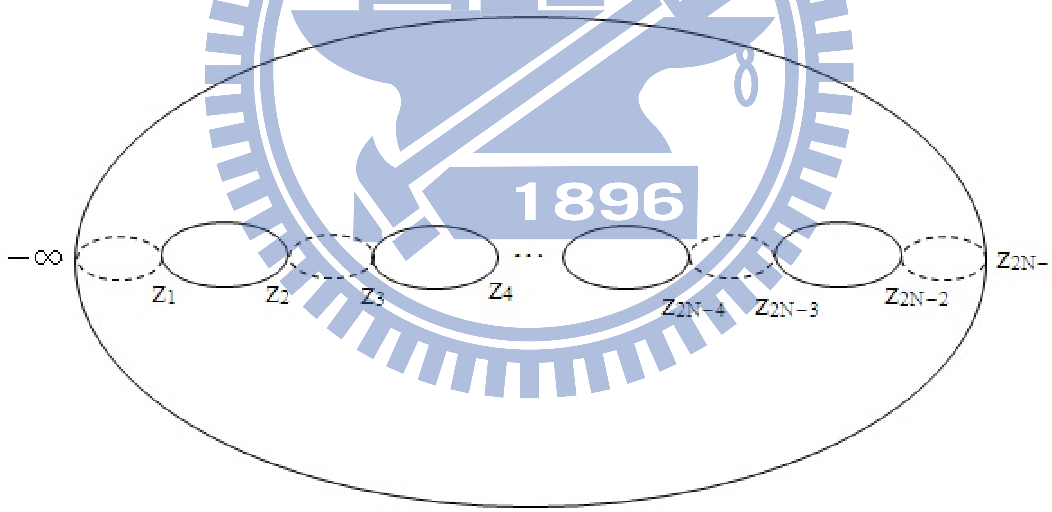

Figure 22.The Riemann surface of genus N-1

Case 2. $n = 2N$, $N \in \mathbb{N}$.

The cut structure becomes:

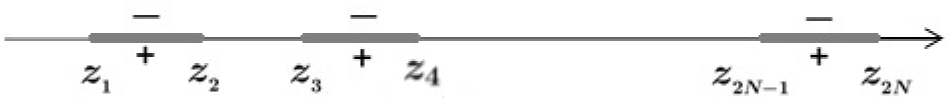

Figure 23.The cut structure

The branch cuts are in $[z_1, z_2]$, $[z_3, z_4]$, $\cdots$, $[z_{2N-1}, z_{2N}]$.

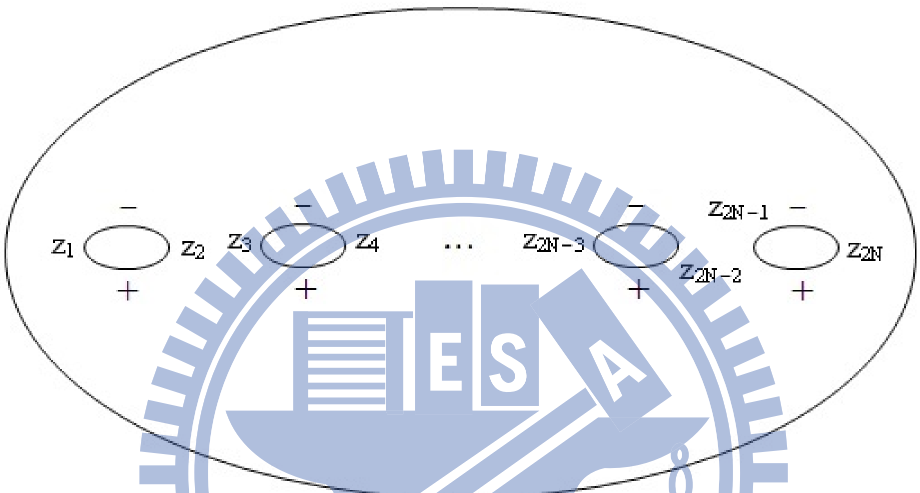

Figure 24.Placing the cuts open

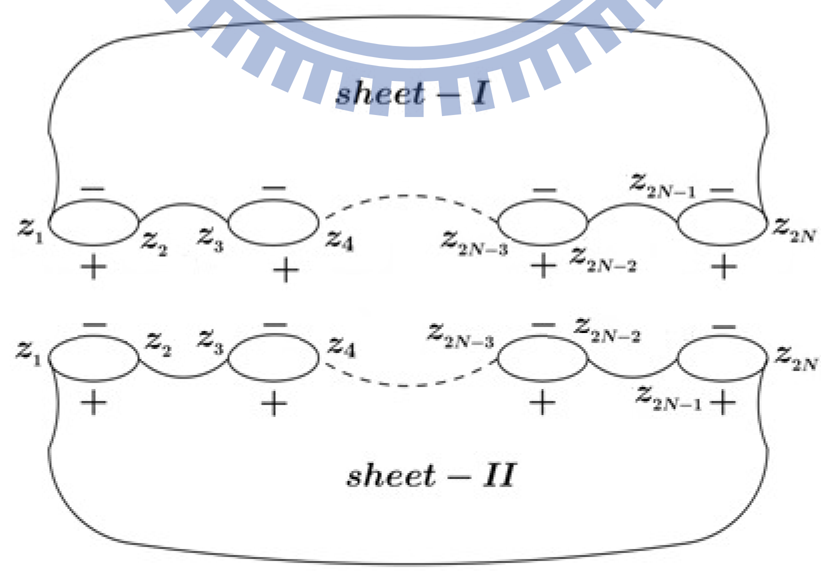

Figure 25.Together with two sheets

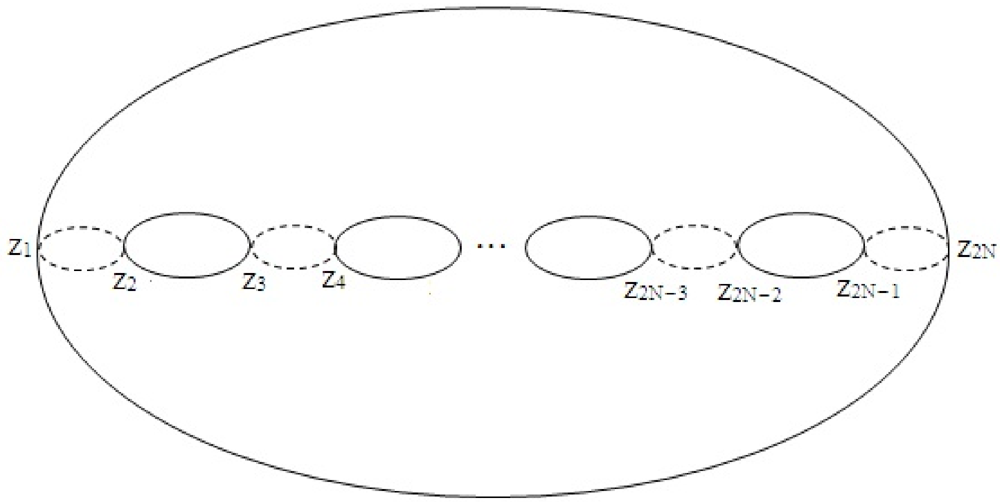

Figure 26.The Riemann surface of genus N-1

In these cases, we all make N branch cuts and get Riemann surface of genus $N-1$ whenever $f(z)$ has $2N-1$ or $2N$ roots.

2.2 The a, b cycles

We introduce the a, b cycles since the simple closed curves on Riemann surface can be written as the linear combination of them. Take two examples to illustrate the a, b cycles of $f(z)$ on the cut plane and the corresponding Riemann surface.

Example 2.2.1. Let $f(z) = \sqrt{\prod_{k=1}^{5}(z-z_k)}$ where $z_1 = -2$, $z_2 = -1$, $z_3 = 0$, $z_4 = 1$, and $z_5 = 2$. From Example 2.1.2, the branch cuts are in $[-\infty, -2]$, $[-1, 0]$, $[1, 2]$. The a, b cycles on the cut plane:

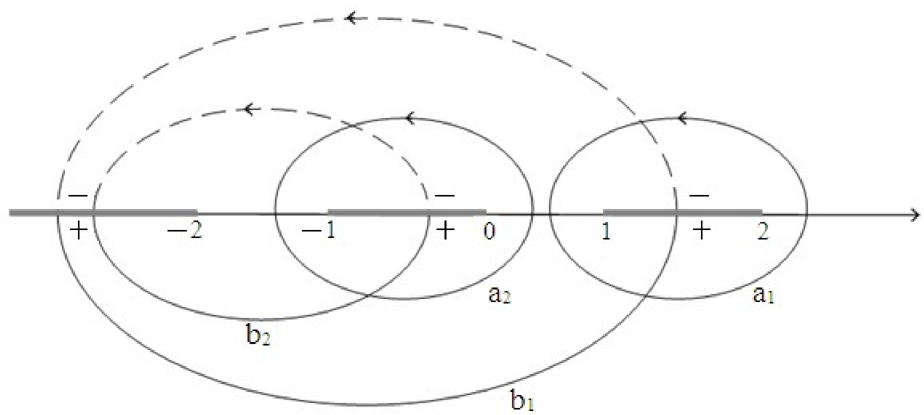

Figure 27.The a, b cycles of $f(z)$

The process of finding the a, b cycles on the corresponding Riemann surface is shown below.

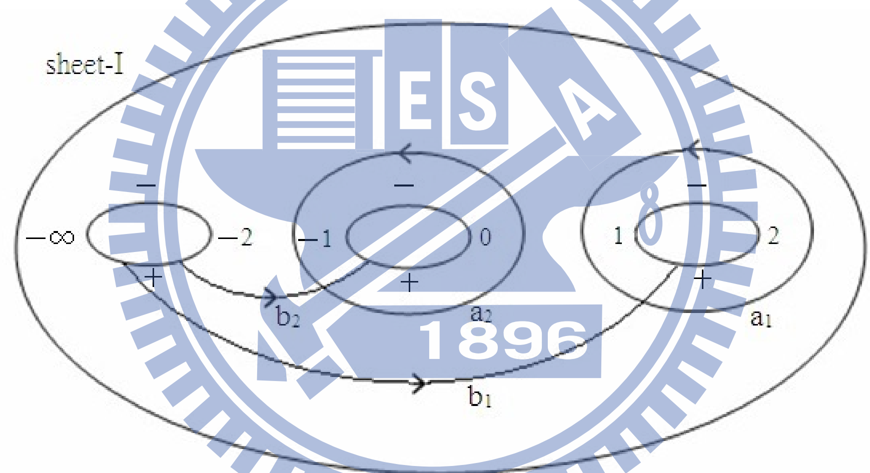

Figure 28.The a, b cycles in sheet-I

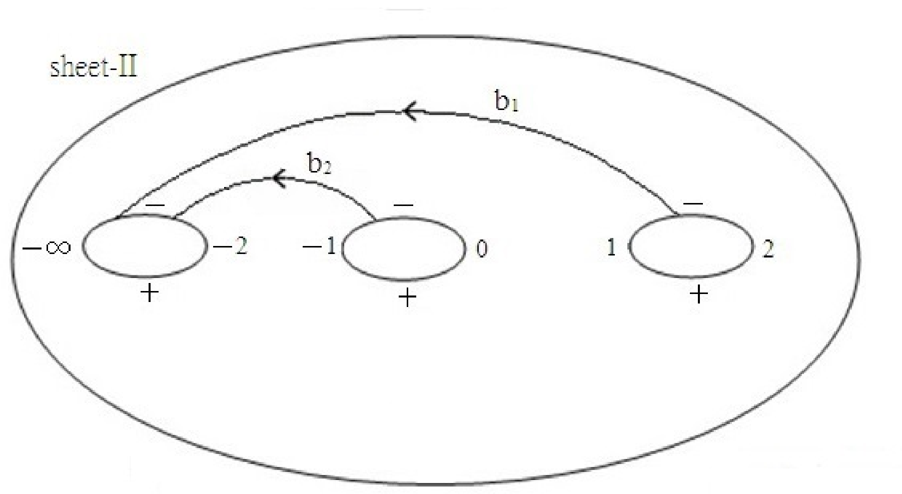

Figure 29.The b cycles in sheet-II

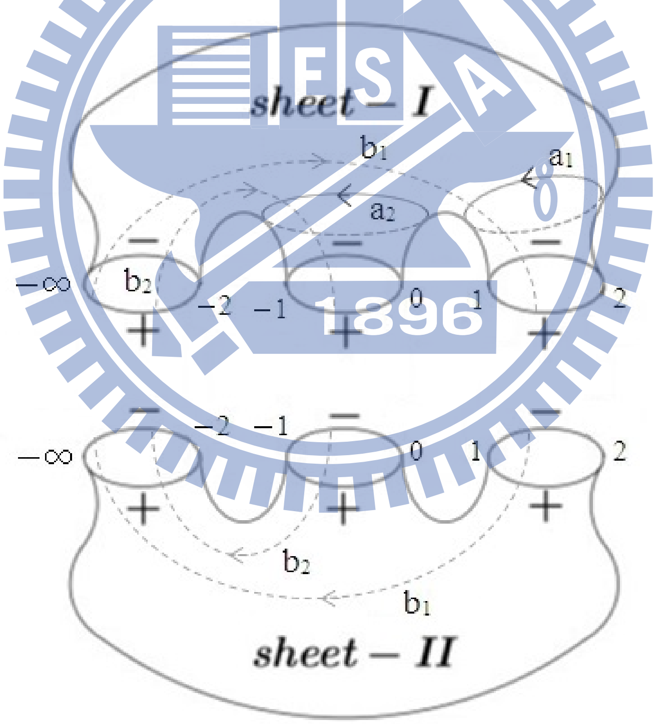

Figure 30.Together with two sheets

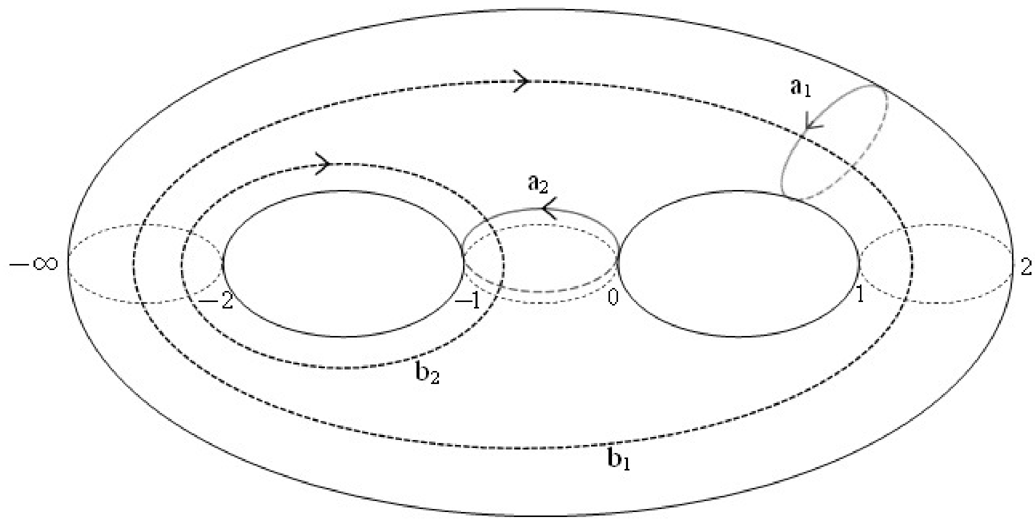

Figure 31.The a, b cycles of $f(z)$ on Riemann surface

Example 2.2.2. Let $f(z) = \sqrt{\prod_{k=1}^{6}(z - z_k)}$ where $z_1 = -2$, $z_2 = -1$, $z_3 = 0$, $z_4 = 1$, $z_5 = 2$, and $z_6 = 3$.

From Example 2.1.3, the branch cuts are in $[-2, -1]$, $[0, 1]$, $[2, 3]$. The a, b cycles on the cut plane:

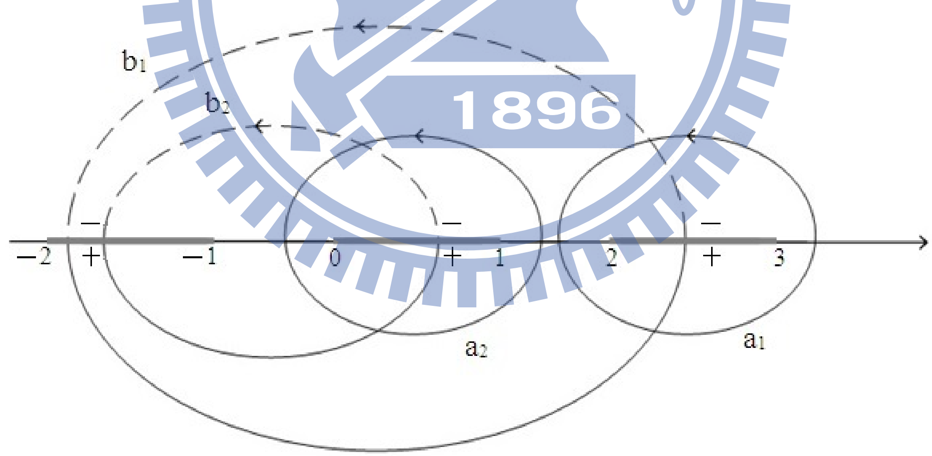

Figure 32.The a, b cycles of $f(z)$

The process of finding the a, b cycles on the corresponding Riemann surface is shown below.

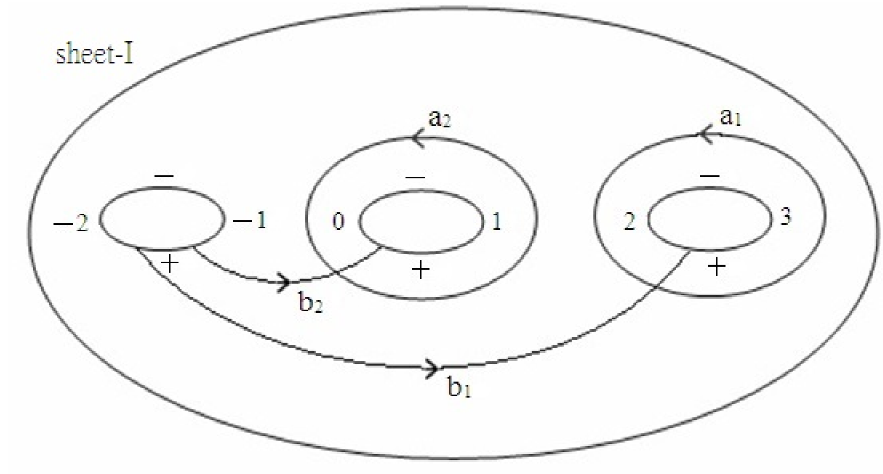

Figure 33.The a, b cycles in sheet-I

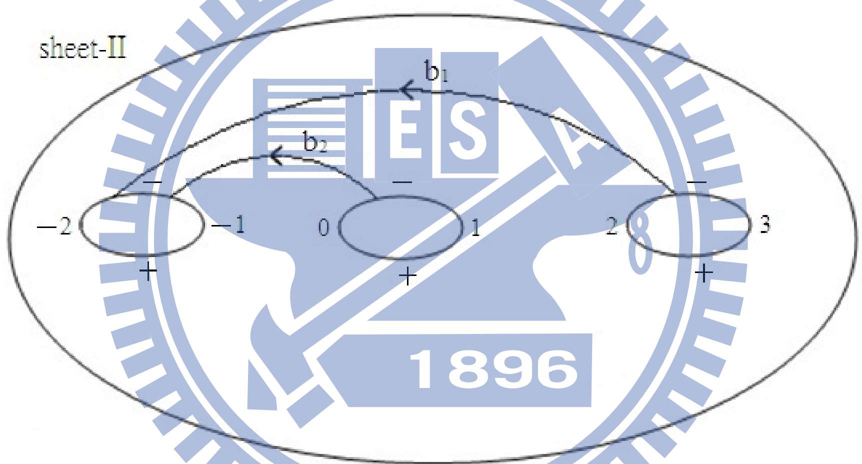

Figure 34.The b cycles in sheet-II

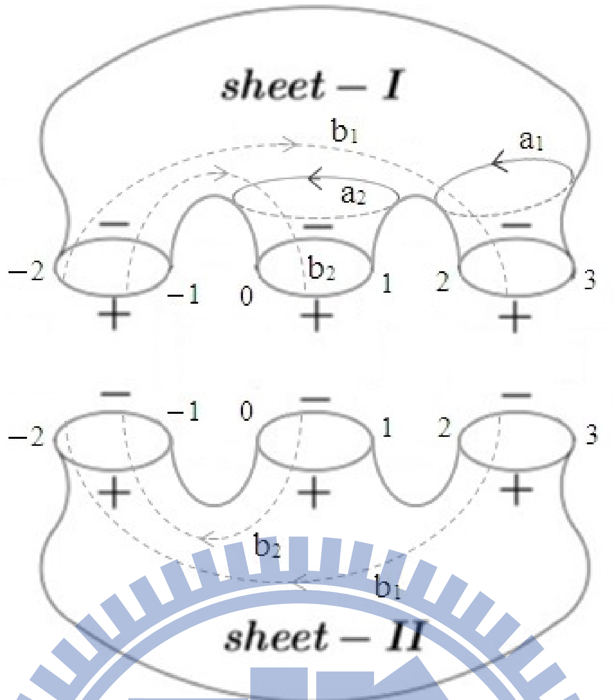

Figure 35.Together with two sheets

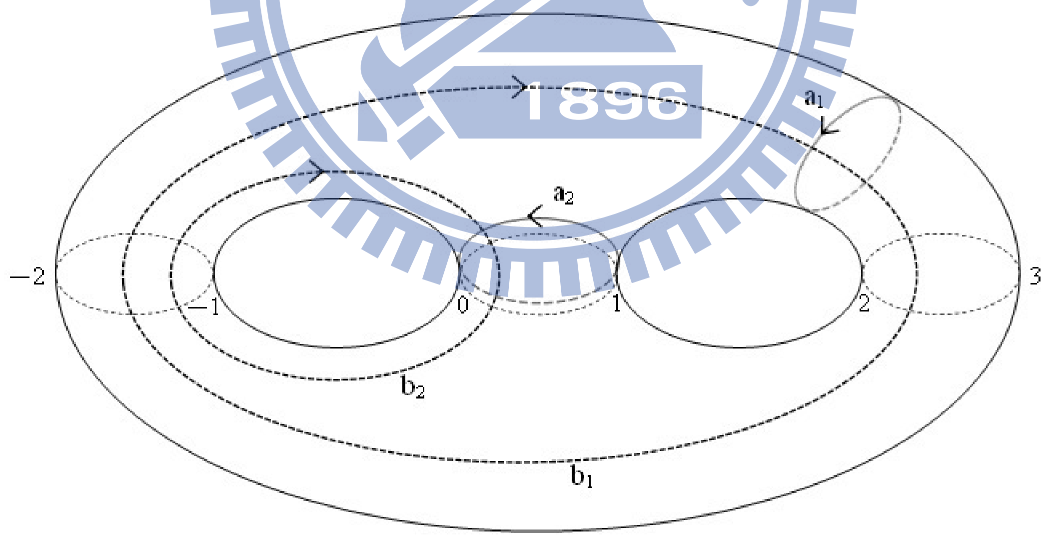

Figure 36.The a, b cycles of $f(z)$ on Riemann surface

After the two examples given above, we can also find the a, b cycles of $f(z)$ on the cut plane and the corresponding Riemann surface in general case where $f(z) = \sqrt{\prod_{k=1}^{n} (z - z_k)}$, $z_k \in \mathbb{R}$ and $z_1 < z_2 < \cdots < z_n$.

Case 1. $n = 2N - 1$, $N \in \mathbb{N}$.

The branch cuts are in $[-\infty, z_1]$, $[z_2, z_3]$, $\cdots$, $[z_{2N-2}, z_{2N-1}]$. The a, b cycles on the cut plane:

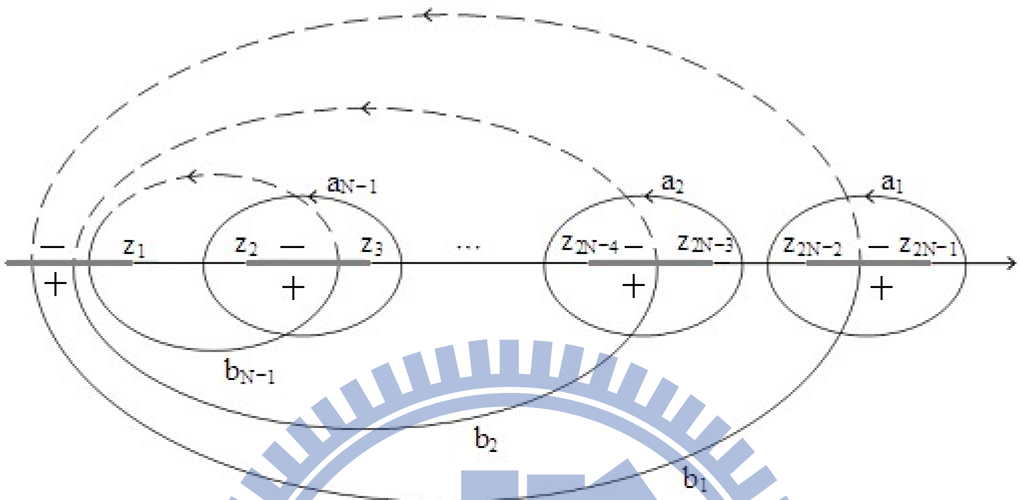

Figure 37.The a, b cycles of $f(z)$

The a, b cycles on the corresponding Riemann surface:

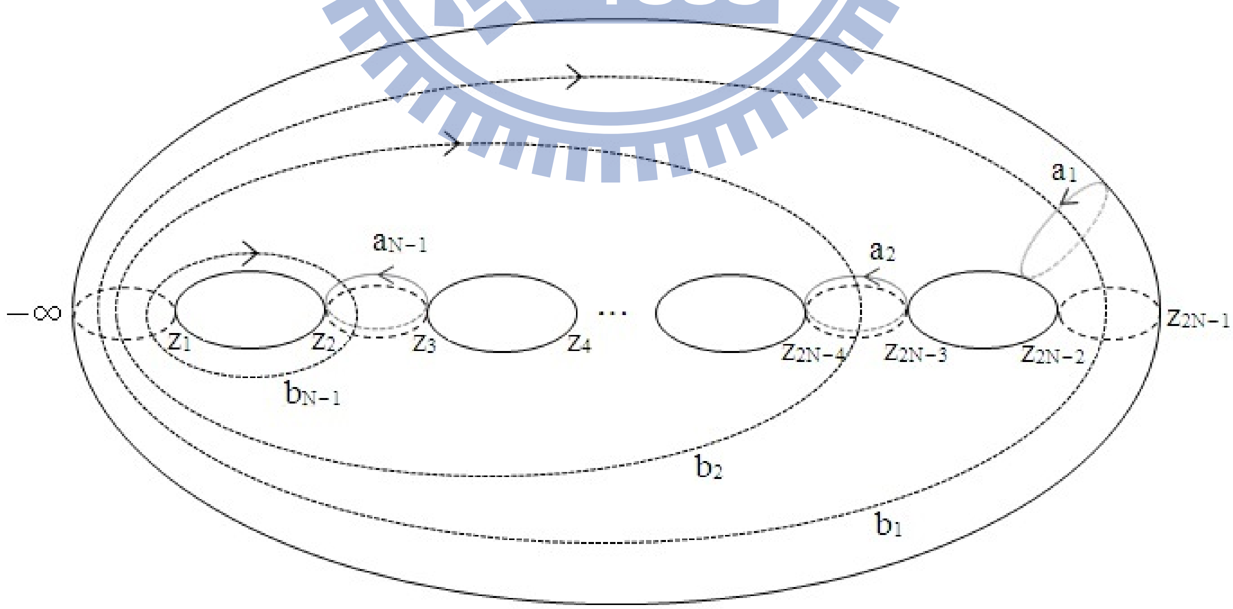

Figure 38.The a, b cycles of $f(z)$ on Riemann surface

Case 2. $n = 2N$, $N \in \mathbb{N}$.

The branch cuts are in $[z_1, z_2]$, $[z_3, z_4]$, $\cdots$, $[z_{2N-1}, z_{2N}]$. The a, b cycles on the cut plane:

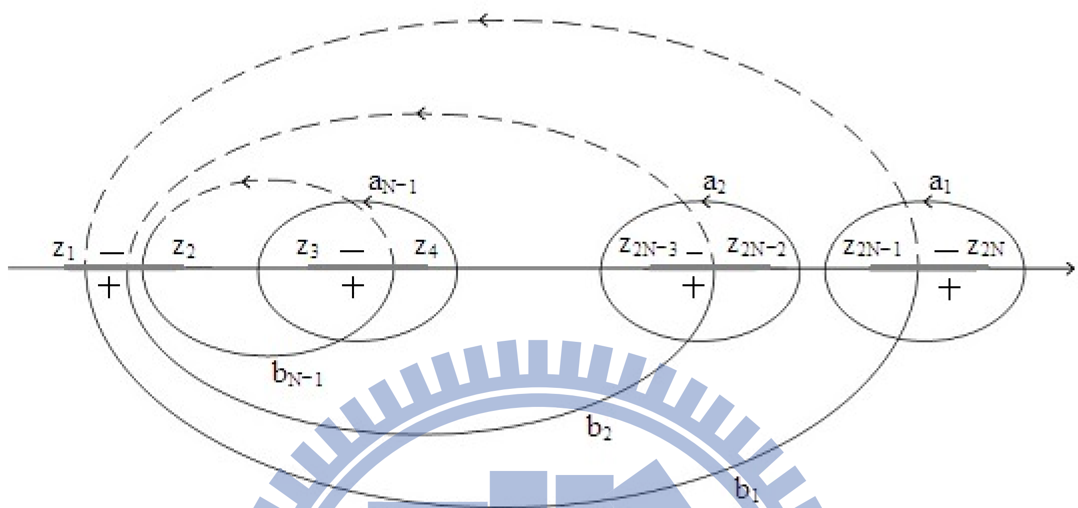

Figure 39. The a, b cycles of $f(z)$

The a, b cycles on the corresponding Riemann surface:

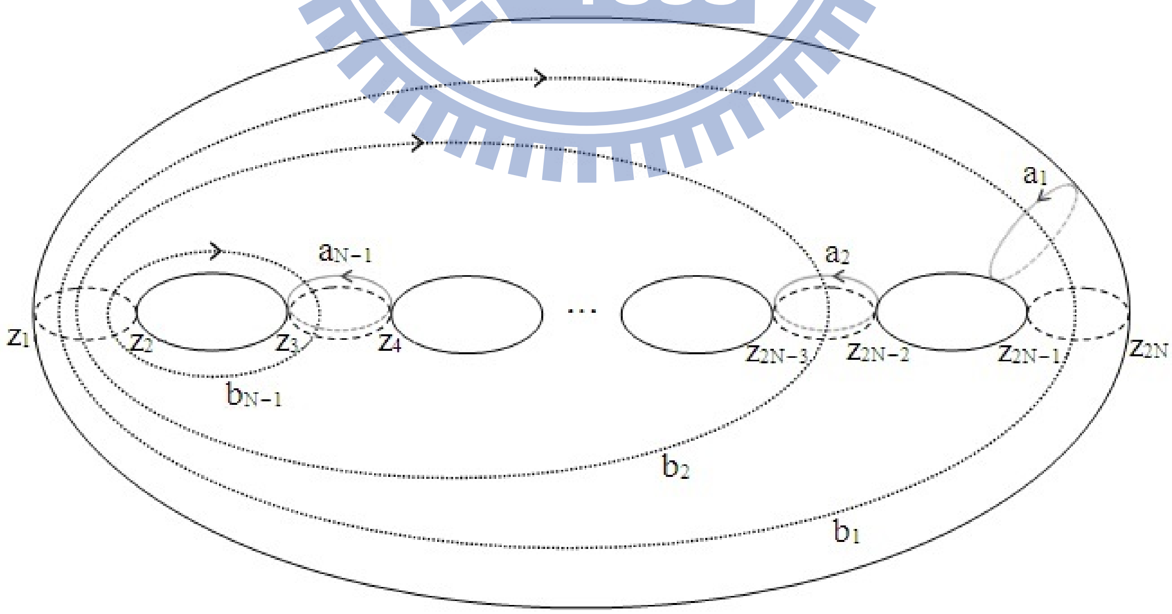

Figure 40. The a, b cycles of $f(z)$ on Riemann surface

Note that each a cycle is non-overlapping and each b cycle is,too. Besides, a cycles and b cycles have the same amount.

2.3 The equivalent paths of a, b cycles

If we want to evaluate the integrals of $f(z)$ over a, b cycles, we can find the equivalent paths of a, b cycles so that our calculation might be much easier than the original. Therefore, find the simplest paths of a, b cycles and then evaluate the integrals.

Let's start off finding the equivalent path of a cycle.

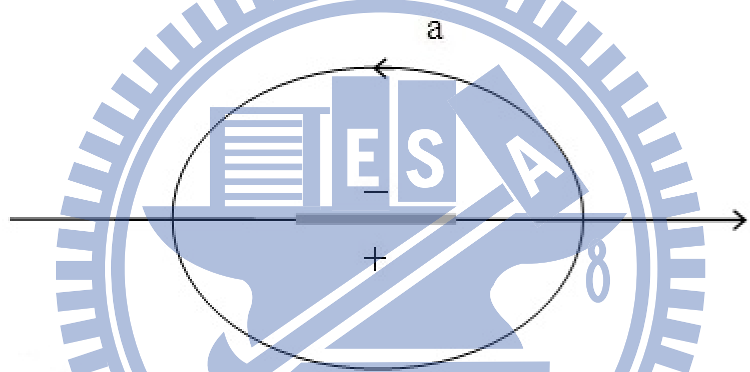

Figure 41.The a cycle

We construct some disjoint contours, L_1, L_2, Γ_1, and Γ_2, that make the a cycle become two closed contours, $K_1 = a^* + L_1 - \Gamma_2 + L_2$ and $K_2 = a^{**} - L_2 - \Gamma_1 - L_1$, which is shown below.

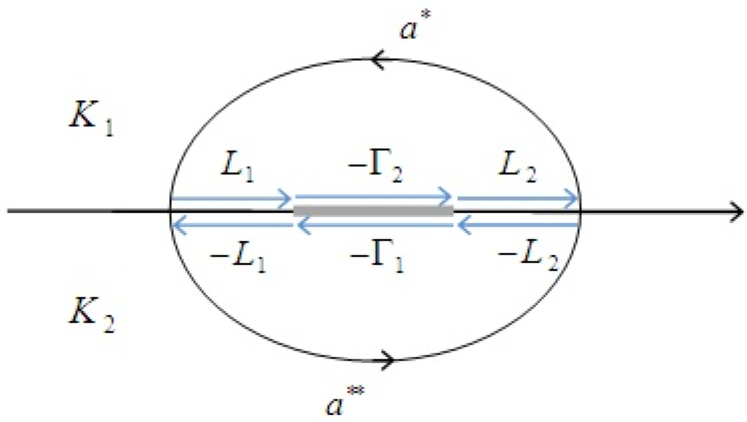

Figure 42.The closed contours K_1 and K_2

By Cauchy theorem, $\int_{K_1} f(z)dz = 0$ and $\int_{K_2} f(z)dz = 0$. Adding contours gives

$$K_1 + K_2 = a^* + L_1 - \Gamma_2 + L_2 + a^{**} - L_2 - \Gamma_1 - L_1 = a^* + a^{**} - \Gamma_1 - \Gamma_2$$

which implies that $a^* + a^{**} = K_1 + K_2 + \Gamma_1 + \Gamma_2$. Then

$$\begin{aligned}\int_a f(z)dz &= \int_{a^*} f(z)dz + \int_{a^{**}} f(z)dz \\ &= \int_{K_1} f(z)dz + \int_{K_2} f(z)dz + \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz \\ &= \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz.\end{aligned}$$

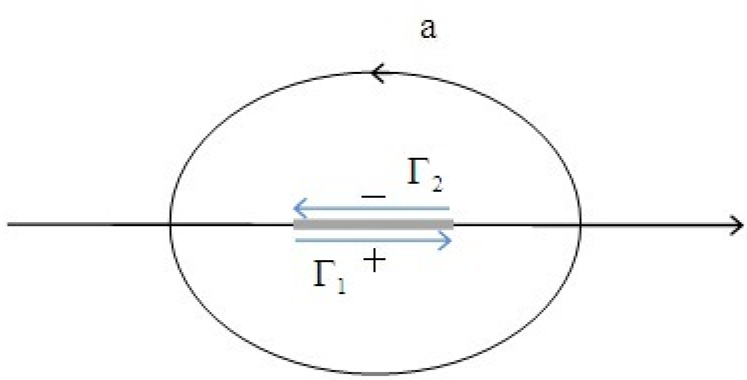

Figure 43.The equivalent paths of a cycle

Next, find the equivalent path of b cycle.

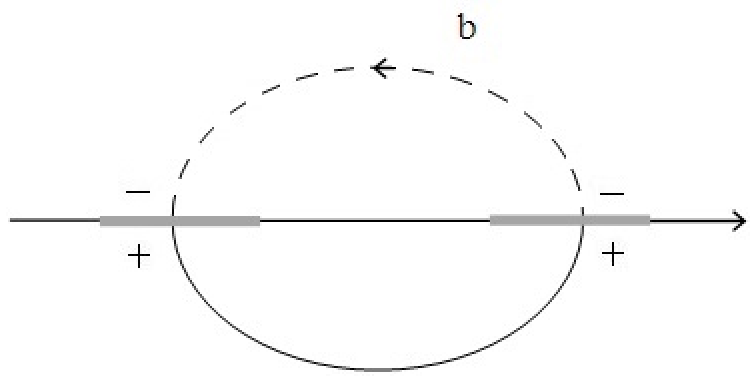

Figure 44.The b cycle

In the similar way, we construct some disjoint contours, L_1, L_2, L_3, L_4, Γ_1, and Γ_2, that make the b cycle become two closed contours, $K_1 = b^* + L_1 - \Gamma_2 + L_2$ and $K_2 = b^{**} - L_3 - \Gamma_1 - L_4$, which is shown below.

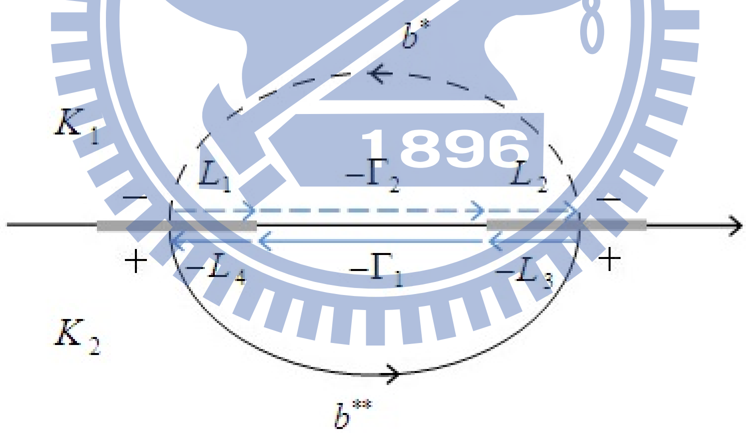

Figure 45.The closed contours K_1 and K_2

By Cauchy theorem, $\int_{K_1} f(z)dz = 0$ and $\int_{K_2} f(z)dz = 0$.Adding contours gives

$$K_1 + K_2 = b^* + L_1 - \Gamma_2 + L_2 + b^{**} - L_3 - \Gamma_1 - L_4 = b^* + b^{**} - \Gamma_1 - \Gamma_2 + L_1 + L_2 - L_3 - L_4$$

which implies that $b^* + b^{**} = K_1 + K_2 + \Gamma_1 + \Gamma_2 - L_1 - L_2 + L_3 + L_4$. Besides, the (-) edge of sheet-II is equal to the (+) edge of sheet-I so the contour $L_1 = L_4$ and $L_2 = L_3$. Then

$$\begin{aligned}
\int_b f(z)dz &= \int_{b^*} f(z)dz + \int_{b^{**}} f(z)dz \\
&= \int_{K_1} f(z)dz + \int_{K_2} f(z)dz + \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz - \int_{L_1} f(z)dz - \int_{L_2} f(z)dz \\
&\quad + \int_{L_3} f(z)dz + \int_{L_4} f(z)dz \\
&= \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz - \int_{L_1} f(z)dz - \int_{L_2} f(z)dz + \int_{L_3} f(z)dz + \int_{L_4} f(z)dz \\
&= \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz - \int_{L_4} f(z)dz - \int_{L_3} f(z)dz + \int_{L_3} f(z)dz + \int_{L_4} f(z)dz \\
&= \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz.
\end{aligned}$$

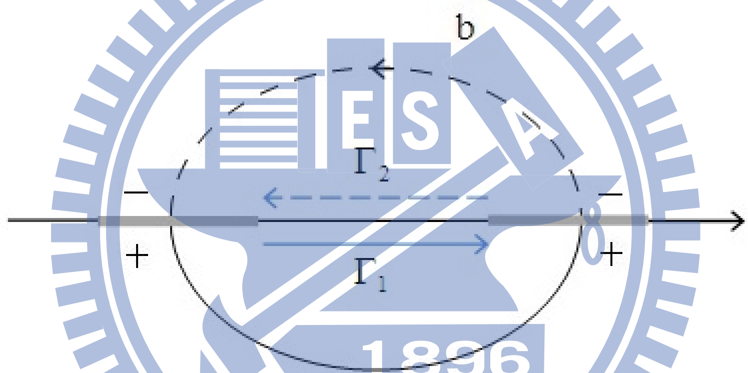

Figure 46.The equivalent paths of b cycle

2.4 The integrals of $\frac{1}{f(z)}$ over a, b cycles with horizontal cut structure

We will use Mathematica to help us evaluate the integrals of $\frac{1}{f(z)}$ over a, b cycles or other paths because most computations are laborious and fairly complex. However, there are some differences between theory and Mathematica so we will discuss the differences between them with different cut structures respectively.

In order to know the value of $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$ about z in sheet-I and in sheet-II, we can write $\prod_{k=1}^{n}(z - z_k)$ in its exponential form: $\prod_{k=1}^{n}(z - z_k) = re^{i\theta}$ where r is the modulus of $\prod_{k=1}^{n}(z - z_k)$

and θ is an argument of $\prod_{k=1}^{n}(z-z_k)$. Let θ_1 denote θ in sheet-I and θ_2 denote θ in sheet-II. Then

$$\begin{aligned} f(z)|_{(II)} &= \sqrt{r}e^{i\frac{\theta_2}{2}} \\ &= \sqrt{r}e^{i\frac{\theta_1+2\pi}{2}} \\ &= \sqrt{r}e^{i\frac{\theta_1}{2}}e^{i\pi} \\ &= -\sqrt{r}e^{i\frac{\theta_1}{2}} \\ &= -f(z)|_{(I)} \end{aligned}$$

since $\theta_2 = \theta_1 + 2\pi$. Here $f(z)|_{(I)}$ and $f(z)|_{(II)}$ denote the value of $f(z)$ with z in sheet-I and sheet-II respectively.

We only need to discuss the difference between the value of $f(z)$ in sheet-I of theory and in Mathematica since $f(z)|_{(II)} = -f(z)|_{(I)}$, the (+) edge of sheet-II is equal to the (-) edge of sheet-I, and the (-) edge of sheet-II is equal to the (+) edge of sheet-I. Note that Mathematica is always considering $\arg(z) \in (-\pi, \pi]$. For convenience, $f(z) \overset{Math.}{=} f(z)$ is used to denote the polynomial $f(z)$ in front of $\overset{Math.}{=}$ is the value in theory and the polynomial $f(z)$ behind is the value in Mathematica.

Define that

$$z = \begin{cases} |z|e^{i\theta}, \theta \in [-\pi, \pi) & \text{if and only if z is in sheet-I;} \\ |z|e^{i\theta}, \theta \in [\pi, 3\pi) & \text{if and only if z is in sheet-II.} \end{cases}$$

The cut on each sheet has two edges, label the starting edge with "+" and the terminal edge with "−".

Use a simple case $f(z) = \sqrt{z}$ to find the difference between theory and Mathematica. Let z be a point in sheet-I, that is, $\arg(z) \in [-\pi, \pi)$. Obviously when $\arg(z) \in (-\pi, \pi)$, the value of $f(z)$ in theory and Mathematica are the same because Mathematica is always considering $\arg(z) \in (-\pi, \pi]$, and the difference between them is when $\arg(z) = -\pi$. Then let $\arg(z) = -\pi$,

$$\text{in theory}: \mathrm{z} = |\mathrm{z}|\mathrm{e}^{-\mathrm{i}\pi} \text{ and } \sqrt{\mathrm{z}} = |\mathrm{z}|^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} = -\mathrm{i}|\mathrm{z}|^{\frac{1}{2}},$$

$$\text{in Mathematica}: -\pi \text{ is regarded as } \pi \text{ so } \mathrm{z} = |\mathrm{z}|\mathrm{e}^{\mathrm{i}\pi} \text{ and } \sqrt{\mathrm{z}} = |\mathrm{z}|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}|\mathrm{z}|^{\frac{1}{2}}.$$

Therefore, $f(z) \overset{Math.}{=} f(z)$ if $\arg(z) \in (-\pi, \pi)$ and $f(z) \overset{Math.}{=} -f(z)$ if $\arg(z) = -\pi$.

As described above, we know $f(z)$ will change sign when z is in sheet-I and $\arg(z) = -\pi$, hence we have the following result.

Lemma 2.4.1. If z_k is the end point for horizontal cut and z is in sheet-I, then

$$\sqrt{z - z_k} \overset{Math.}{=} \begin{cases} \sqrt{z - z_k} & \text{if } \arg(z - z_k) \in (-\pi, \pi), \\ -\sqrt{z - z_k} & \text{if } \arg(z - z_k) = -\pi. \end{cases}$$

Proof. Let z be a point in sheet-I and $z - z_k = |z - z_k|e^{i\theta}$ where $\theta = \arg(z - z_k)$. For the same reason as the above, we can easy obtain $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ when $\theta \in (-\pi, \pi)$ and when $\theta = -\pi$,

$$\text{in theory}: \sqrt{\mathrm{z} - \mathrm{z_k}} = |\mathrm{z} - \mathrm{z_k}|^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} = -\mathrm{i}|\mathrm{z} - \mathrm{z_k}|^{\frac{1}{2}},$$

$$\text{in Mathematica}: \sqrt{\mathrm{z} - \mathrm{z_k}} = |\mathrm{z} - \mathrm{z_k}|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}|\mathrm{z} - \mathrm{z_k}|^{\frac{1}{2}}.$$

Hence, $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ if $\arg(z - z_k) \in (-\pi, \pi)$ and $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$ if $\arg(z - z_k) = -\pi$.

Theorem 2.4.2. Let $f(z) = \sqrt{z - z_k}$ and z_k be the end point for horizontal cut. If z is in sheet-I, then

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in \{\text{the cut with } (+) \text{ edge of sheet-I}\}, \\ f(z) & \text{otherwise.} \end{cases}$$

Proof. Let z be a point in sheet-I. Since $\arg(z - z_k) = -\pi$ when $z \in$ {the cut with (+) edge of sheet-I}, then by Lemma 2.4.1, $f(z) \overset{Math.}{=} -f(z)$. On the other hand, $\arg(z - z_k) \in (-\pi, \pi)$, then $f(z) \overset{Math.}{=} f(z)$ by Lemma 2.4.1.

Theorem 2.4.3. If $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}}$ where z_k and z_{k+1} are two end points for horizontal cut, and z is in sheet-I, then

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in \{\text{the cut with } (+) \text{ edge of sheet-I}\}, \\ f(z) & \text{otherwise.} \end{cases}$$

Proof. Let z be a point in sheet-I.

(1) $z \in$ {the cut with (+) edge of sheet-I}

Since $z - z_k \geq 0$, then $\arg(z - z_k) = 0$ and $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ by Lemma 2.4.1.

Since $z - z_{k+1} < 0$, then $\arg(z - z_{k+1}) = -\pi$, by Lemma 2.4.1, $\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_{k+1}}$.

Thus, $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_k}\sqrt{z - z_{k+1}} = -f(z)$.

(2) $z \in$ {the cut with (-) edge of sheet-I}

Since $z - z_k \geq 0$, then $\arg(z - z_k) = 0$ and $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ by Lemma 2.4.1.

Since $z - z_{k+1} < 0$ and the (-) edge of sheet-I equals the (+) edge of sheet-II, then $\arg(z - z_{k+1}) = \pi$, by Lemma 2.4.1, $\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_{k+1}}$.

Thus, $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_k}\sqrt{z - z_{k+1}} = f(z)$.

(3) $z \in (-\infty, z_k)$

Since $z - z_k < 0$ and $z - z_{k+1} < 0$, then $\arg(z - z_k) = -\pi$ and $\arg(z - z_{k+1}) = -\pi$, by Lemma 2.4.1, $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$ and $\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_{k+1}}$.

Thus, $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_k}\sqrt{z - z_{k+1}} = f(z)$.

(4) Otherwise

Since $\arg(z - z_k) \in (-\pi, \pi)$ and $\arg(z - z_{k+1}) \in (-\pi, \pi)$, then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ and $\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_{k+1}}$ by Lemma 2.4.1.

Thus, $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_k}\sqrt{z - z_{k+1}} = f(z)$.

Example 2.4.4. Compute $\int \frac{1}{f(z)} dz$ over a_1, a_2, b_1, and b_2 cycles where $f(z) = \sqrt{(z+1)z(z-1)(z-2)(z-3)(z-4)}$.

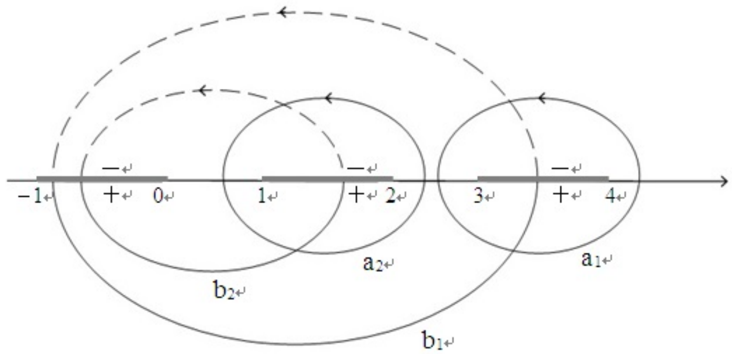

Figure 47.The a, b cycles of $f(z)$

Let $z_1 = -1$, $z_2 = 0$, $z_3 = 1$, $z_4 = 2$, $z_5 = 3$, and $z_6 = 4$.

1. Compute $\int_{a_1} \frac{1}{f(z)} dz$.

   By Cauchy theorem, we can consider that a_1^* is the equivalent path of a_1 and $a_1^* = a_{11}^* \bigcup a_{12}^*$ where a_{11}^* = the path on the horizontal cut from 3 to 4 on the $(+)$ edge of sheet-I and a_{12}^* = the path on the horizontal cut from 4 to 3 on the $(-)$ edge of sheet-I.

   (1) $z \in a_{11}^*$: Let $z = r$, $r : 3 \to 4$ and $dz = dr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.$$

   Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{a_{11}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_3^4 \frac{1}{f(r)} dr$.

   (2) $z \in a_{12}^*$: Let $z = r$, $r : 4 \to 3$ and $dz = dr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

   Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{a_{12}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_4^3 \frac{1}{f(r)} dr$.

Therefore, by (1) and (2),

$$\begin{aligned}\int_{a_1}\frac{1}{f(z)}dz &= \int_{a_1^*}\frac{1}{f(z)}dz\\ &= \int_{a_{11}^*\bigcup a_{12}^*}\frac{1}{f(z)}dz\\ &\overset{Math.}{=} -\int_3^4\frac{1}{f(r)}dr+\int_4^3\frac{1}{f(r)}dr\\ &= 2\int_4^3\frac{1}{f(r)}dr.\end{aligned}$$

2. Compute $\int_{a_2}\frac{1}{f(z)}dz$.

   By Cauchy theorem, we can consider that a_2^* is the equivalent path of a_2 and $a_2^* = a_{21}^*\bigcup a_{22}^*$ where a_{21}^* = the path on the horizontal cut from 1 to 2 on the (+) edge of sheet-I and a_{22}^* = the path on the horizontal cut from 2 to 1 on the (−) edge of sheet-I.

   (1) $z\in a_{21}^*$: Let $z=r$, $r:1\to 2$ and $dz=dr$, then

$$\begin{aligned}\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},\\ \sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4},\\ \sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.\end{aligned}$$

   Thus, $f(z)\overset{Math.}{=}-f(z)$ and $\int_{a_{21}^*}\frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^2\frac{1}{f(r)}dr$.

   (2) $z\in a_{22}^*$: Let $z=r$, $r:2\to 1$ and $dz=dr$, then

$$\begin{aligned}\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},\\ \sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},\\ \sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.\end{aligned}$$

   Thus, $f(z)\overset{Math.}{=}f(z)$ and $\int_{a_{22}^*}\frac{1}{f(z)}dz \overset{Math.}{=} \int_2^1\frac{1}{f(r)}dr$.

Therefore, by (1) and (2),

$$\begin{aligned}\int_{a_2} \frac{1}{f(z)}dz &= \int_{a_2^*} \frac{1}{f(z)}dz \\ &= \int_{a_{21}^* \bigcup a_{22}^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} -\int_1^2 \frac{1}{f(r)}dr + \int_2^1 \frac{1}{f(r)}dr \\ &= 2\int_2^1 \frac{1}{f(r)}dr.\end{aligned}$$

3. Compute $\int_{b_2} \frac{1}{f(z)}dz$.

   By Cauchy theorem, we can consider that b_2^* is the equivalent path of b_2 and $b_2^* = b_{21}^* \bigcup b_{22}^*$ where b_{21}^* = the path on the horizontal line from 0 to 1 on the sheet-I and b_{22}^* = the path on the horizontal line from 1 to 0 on the sheet-II.

   (1) $z \in b_{21}^*$: Let $z = r$, $r : 0 \to 1$ and $dz = dr$, then

$$\begin{aligned}\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\ \sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\ \sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.\end{aligned}$$

   Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{21}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_0^1 \frac{1}{f(r)}dr$.

   (2) $z \in b_{22}^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider b_{22}^{**} = the path on the horizontal line from 1 to 0 on the sheet-I.

   $z \in b_{22}^{**}$: Let $z = r$, $r : 1 \to 0$ and $dz = dr$, then

$$\begin{aligned}\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\ \sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\ \sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.\end{aligned}$$

   Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{22}^*} \frac{1}{f(z)}dz = -\int_{b_{22}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^0 \frac{1}{f(r)}dr$.

Therefore, by (1) and (2),

$$\begin{aligned}\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\ &= \int_{b_{21}^* \bigcup b_{22}^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_0^1 \frac{1}{f(r)}dr - \int_1^0 \frac{1}{f(r)}dr \\ &= 2\int_0^1 \frac{1}{f(r)}dr.\end{aligned}$$

4. Compute $\int_{b_1} f(z)dz$.

By Cauchy theorem, we can consider that b_1^* is the equivalent path of b_1 and $b_1^* = b_2^* \bigcup b_{11}^* \bigcup b_{12}^*$ where b_{11}^* = the path on the horizontal line from 2 to 3 on the sheet-I and b_{12}^* = the path on the horizontal line from 3 to 2 on the sheet-II.

(1) $z \in b_{11}^*$: Let $z = r$, $r : 2 \to 3$ and $dz = dr$, then

$$\begin{aligned}\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\ \sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\ \sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{11}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_2^3 \frac{1}{f(r)}dr$.

(2) $z \in b_{12}^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider b_{12}^{**} = the path on the horizontal line from 3 to 2 on the sheet-I.

$z \in b_{12}^{**}$: Let $z = r$, $r : 3 \to 2$ and $dz = dr$, then

$$\begin{aligned}\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\ \sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\ \sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{12}^*} \frac{1}{f(z)}dz = -\int_{b_{12}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_3^2 \frac{1}{f(r)}dr$.

Therefore, by (1) and (2),

$$\begin{aligned}
\int_{b_1} \frac{1}{f(z)}dz &= \int_{b_1^*} \frac{1}{f(z)}dz \\
&= \int_{b_2^* \bigcup b_{11}^* \bigcup b_{12}^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} 2\int_0^1 \frac{1}{f(r)}dr + \int_2^3 \frac{1}{f(r)}dr - \int_3^2 \frac{1}{f(r)}dr \\
&= 2\int_0^1 \frac{1}{f(r)}dr + 2\int_2^3 \frac{1}{f(r)}dr.
\end{aligned}$$

2.5 The integrals of $\frac{1}{f(z)}$ over a, b cycles with vertical cut structure

Define that

$$z = \begin{cases} |z|e^{i\theta}, \theta \in [-\frac{3\pi}{2}, \frac{\pi}{2}) & \text{if and only if z is in sheet-I;} \\ |z|e^{i\theta}, \theta \in [\frac{\pi}{2}, \frac{5\pi}{2}) & \text{if and only if z is in sheet-II.} \end{cases}$$

The cut on each sheet has two edges, label the starting edge with ”+” and the terminal edge with ”−”.

Lemma 2.5.1. If z_k is the end point for vertical cut and z is in sheet-I, then

$$\sqrt{z-z_k} \overset{Math.}{=} \begin{cases} -\sqrt{z-z_k} & \text{if } \arg(z-z_k) \in [-\frac{3\pi}{2}, -\pi], \\ \sqrt{z-z_k} & \text{if } \arg(z-z_k) \in (-\pi, \frac{\pi}{2}). \end{cases}$$

Proof. Let z be a point in sheet-I and $z-z_k = |z-z_k|e^{i\theta}$ where $\theta = \arg(z-z_k)$. When $\theta \in (-\pi, \frac{\pi}{2})$, the value of $\sqrt{z-z_k}$ in theory and Mathematica all equal $|z-z_k|^{\frac{1}{2}}e^{i\frac{\theta}{2}}$ since Mathematica is always considering $\theta \in (-\pi, \pi]$, then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$. When $\theta \in [-\frac{3\pi}{2}, -\pi]$,

in theory : $\sqrt{\mathrm{z}-\mathrm{z_k}} = |\mathrm{z}-\mathrm{z_k}|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}$,

in Mathematica : $\theta \in [-\dfrac{3\pi}{2}, -\pi]$ is regarded as $\theta + 2\pi \in [\dfrac{\pi}{2}, \pi]$ so $\mathrm{z}-\mathrm{z_k} = |\mathrm{z}-\mathrm{z_k}|\mathrm{e}^{\mathrm{i}(\theta+2\pi)}$

and $\sqrt{\mathrm{z}-\mathrm{z_k}} = |\mathrm{z}-\mathrm{z_k}|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\theta+2\pi}{2}} = -|\mathrm{z}-\mathrm{z_k}|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}$.

Therefore, $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$ if $\arg(z-z_k) \in [-\frac{3\pi}{2}, -\pi]$ and $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$ if $\arg(z-z_k) \in (-\pi, \frac{\pi}{2})$.

Theorem 2.5.2. Let $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}}$ where z_k and z_{k+1} are two end points for vertical cut and the domain be divided into six areas (A), (B), (C), (D), (E), and (F) where

(A)$= \{(\mathrm{x}, \mathrm{y}) : \mathrm{x} < \mathrm{Re}(\mathrm{z_k}),\ \mathrm{y} \geq \mathrm{Im}(\mathrm{z_k})\}$,

(B)$= \{(\mathrm{x}, \mathrm{y}) : \mathrm{x} < \mathrm{Re}(\mathrm{z_k}),\ \mathrm{Im}(\mathrm{z_{k+1}}) \leq \mathrm{y} < \mathrm{Im}(\mathrm{z_k})\}$,

(C)$= \{(\mathrm{x}, \mathrm{y}) : \mathrm{x} < \mathrm{Re}(\mathrm{z_k}),\ \mathrm{y} < \mathrm{Im}(\mathrm{z_{k+1}})\}$,

(D)$= \{(\mathrm{x}, \mathrm{y}) : \mathrm{x} > \mathrm{Re}(\mathrm{z_k}),\ \mathrm{y} \geq \mathrm{Im}(\mathrm{z_k})\}$,

(E)$= \{(\mathrm{x}, \mathrm{y}) : \mathrm{x} > \mathrm{Re}(\mathrm{z_k}),\ \mathrm{Im}(\mathrm{z_{k+1}}) \leq \mathrm{y} < \mathrm{Im}(\mathrm{z_k})\}$,

(F)$= \{(\mathrm{x}, \mathrm{y}) : \mathrm{x} > \mathrm{Re}(\mathrm{z_k}),\ \mathrm{y} < \mathrm{Im}(\mathrm{z_{k+1}})\}$.

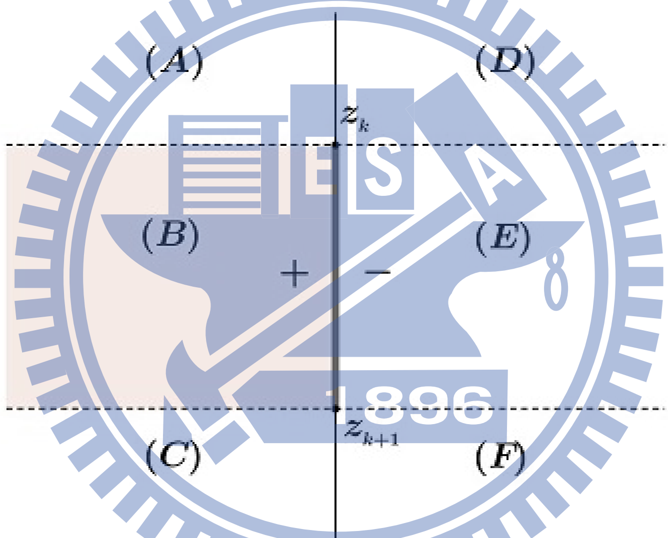

Figure 48.The domain with six areas

If z is in sheet-I, then

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in (B) \bigcup \{\text{the cut with } (+) \text{ edge of sheet-I}\}, \\ f(z) & \text{otherwise.} \end{cases}$$

Proof. Let z be a point in sheet-I.

(1) $z \in$(A)

Since $\arg(z - z_k) \in (-\frac{3\pi}{2}, -\pi]$ and $\arg(z - z_{k+1}) \in (-\frac{3\pi}{2}, -\pi)$, then by Lemma 2.5.1,

$\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_{k+1}}$.

Hence, $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_k}\sqrt{z-z_{k+1}} = f(z)$.

(2) $z \in$(B)

Since $\arg(z-z_k) \in (-\pi, -\frac{\pi}{2})$ and $\arg(z-z_{k+1}) \in (-\frac{3\pi}{2}, -\pi]$, then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_{k+1}}$ by Lemma 2.5.1.

Hence, $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_k}\sqrt{z-z_{k+1}} = -f(z)$.

(3) $z \in$\{the cut with (+) edge of sheet-I\}

Since $\arg(z-z_k) = -\frac{\pi}{2}$ and $\arg(z-z_{k+1}) = -\frac{3\pi}{2}$, then by Lemma 2.5.1,

$\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_{k+1}}$.

Hence, $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_k}\sqrt{z-z_{k+1}} = -f(z)$.

(4) $z \in$\{the cut with (-) edge of sheet-I\}

Since $\arg(z-z_k) = -\frac{\pi}{2}$, then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$ by Lemma 2.5.1.

Since the (-) edge of sheet-I equals the (+) edge of sheet-II, then $\arg(z-z_{k+1}) = \frac{\pi}{2}$, by Lemma 2.5.1, $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$.

Hence, $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_k}\sqrt{z-z_{k+1}} = f(z)$.

(5) $z \in \{(\mathrm{x},\mathrm{y}) : \mathrm{x} = \mathrm{Re}(\mathrm{z_k}),\ \mathrm{y} > \mathrm{Im}(\mathrm{z_k})\}$

Since $\arg(z-z_k) = -\frac{3\pi}{2}$ and $\arg(z-z_{k+1}) = -\frac{3\pi}{2}$, then $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_{k+1}}$ by Lemma 2.5.1.

Hence, $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_k}\sqrt{z-z_{k+1}} = f(z)$.

(6) Otherwise

Since $\arg(z-z_k) \in (-\pi, \frac{\pi}{2})$ and $\arg(z-z_{k+1}) \in (-\pi, \frac{\pi}{2})$, then by Lemma 2.5.1,

$\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_{k+1}}$.

Hence, $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_k}\sqrt{z-z_{k+1}} = f(z)$.

Example 2.5.3. Compute $\int \frac{1}{f(z)}dz$ over a_1, a_2, b_1, and b_2 cycles, where

$f(z) = \sqrt{(z+1-2i)(z+1+2i)(z-1-3i)(z-1+3i)(z-2-i)(z-2+i)}$.

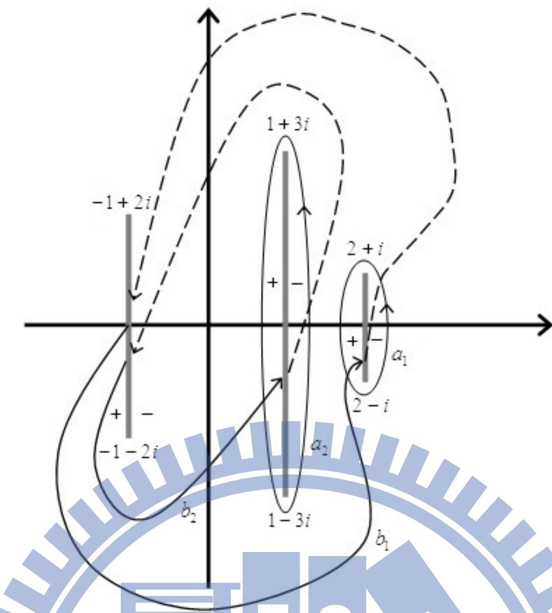

Figure 49.The a, b cycles of $f(z)$

Let $z_1 = -1+2i$, $z_2 = -1-2i$, $z_3 = 1+3i$, $z_4 = 1-3i$, $z_5 = 2+i$, and $z_6 = 2-i$.

1. Compute $\int_{a_1} \frac{1}{f(z)}dz$.

   By Cauchy theorem, we can consider that a_1^* is the equivalent path of a_1 and $a_1^* = a_{11}^* \bigcup a_{12}^*$ where a_{11}^* = the path on the vertical cut from $2+i$ to $2-i$ on the (+) edge of sheet-I and a_{12}^* = the path on the vertical cut from $2-i$ to $2+i$ on the (−) edge of sheet-I.

   (1) $z \in a_{11}^*$: Let $z = 2+ri$, $r : 1 \to -1$ and $dz = idr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.$$

   Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{a_{11}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^{-1} \frac{i}{f(2+ri)}dr$.

(2) $z \in a_{12}^*$: Let $z = 2 + ri$, $r : -1 \to 1$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{a_{12}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{-1}^{1} \frac{i}{f(2+ri)} dr$.

Therefore, by (1) and (2),

$$\begin{aligned}
\int_{a_1} \frac{1}{f(z)} dz &= \int_{a_1^*} \frac{1}{f(z)} dz \\
&= \int_{a_{11}^* \bigcup a_{12}^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} -\int_{1}^{-1} \frac{i}{f(2+ri)} dr + \int_{-1}^{1} \frac{i}{f(2+ri)} dr \\
&= 2\int_{-1}^{1} \frac{i}{f(2+ri)} dr.
\end{aligned}$$

2. Compute $\int_{a_2} \frac{1}{f(z)} dz$.

By Cauchy theorem, we can consider that a_2^* is the equivalent path of a_2 and $a_2^* = a_{21}^* \bigcup a_{22}^* \bigcup a_{23}^* \bigcup a_{24}^* \bigcup a_{25}^* \bigcup a_{26}^*$ where a_{21}^* = the path on the vertical cut from $1+3i$ to $1+i$ on the (+) edge of sheet-I, a_{22}^* = the path on the vertical cut from $1+i$ to $1-i$ on the (+) edge of sheet-I, a_{23}^* = the path on the vertical cut from $1-i$ to $1-3i$ on the (+) edge of sheet-I, a_{24}^* = the path on the vertical cut from $1-3i$ to $1-i$ on the (−) edge of sheet-I, a_{25}^* = the path on the vertical cut from $1-i$ to $1+i$ on the (−) edge of sheet-I, and a_{26}^* = the path on the vertical cut from $1+i$ to $1+3i$ on the (−) edge of sheet-I.

(1) $z \in a_{21}^*$: Let $z = 1 + ri$, $r : 3 \to 1$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{a_{21}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_3^1 \frac{i}{f(1+ri)}dr$.

(2) $z \in a_{22}^*$: Let $z = 1+ri$, $r : 1 \to -1$ and $dz = idr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{a_{22}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^{-1} \frac{i}{f(1+ri)}dr$.

(3) $z \in a_{23}^*$: Let $z = 1+ri$, $r : -1 \to -3$ and $dz = idr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{a_{23}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{-1}^{-3} \frac{i}{f(1+ri)}dr$.

(4) $z \in a_{24}^*$: Let $z = 1+ri$, $r : -3 \to -1$ and $dz = idr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{a_{24}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{-3}^{-1} \frac{i}{f(1+ri)}dr$.

(5) $z \in a_{25}^*$: Let $z = 1+ri$, $r : -1 \to 1$ and $dz = idr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{a_{25}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{-1}^{1} \frac{i}{f(1+ri)}dr$.

(6) $z \in a_{26}^*$: Let $z = 1 + ri$, $r : 1 \to 3$ and $dz = idr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{a_{26}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^3 \frac{i}{f(1+ri)}dr$.

Therefore, by (1), (2), (3), (4), (5), and (6),

$$\begin{aligned}
\int_{a_2} \frac{1}{f(z)}dz &= \int_{a_2^*} \frac{1}{f(z)}dz \\
&= \int_{a_{21}^* \bigcup a_{22}^* \bigcup a_{23}^* \bigcup a_{24}^* \bigcup a_{25}^* \bigcup a_{26}^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} -\int_3^1 \frac{i}{f(1+ri)}dr + \int_1^{-1} \frac{i}{f(1+ri)}dr - \int_{-1}^{-3} \frac{i}{f(1+ri)}dr \\
&\quad + \int_{-3}^{-1} \frac{i}{f(1+ri)}dr - \int_{-1}^{1} \frac{i}{f(1+ri)}dr + \int_1^3 \frac{i}{f(1+ri)}dr \\
&= 2\int_1^3 \frac{i}{f(1+ri)}dr + 2\int_1^{-1} \frac{i}{f(1+ri)}dr + 2\int_{-3}^{-1} \frac{i}{f(1+ri)}dr.
\end{aligned}$$

3. Compute $\int_{b_2} f(z)dz$.

By Cauchy theorem, we can consider that b_2^* is the equivalent path of b_2 and $b_2^* = \bigcup_{j=1}^{8} b_{2j}^*$ where b_{21}^* = the path on the vertical cut from $-1+i$ to $-1-i$ on the (+) edge of sheet-I, b_{22}^* = the path on the vertical cut from $-1-i$ to $-1-2i$ on the (+) edge of sheet-I, b_{23}^* = the path on the vertical cut from $-1-2i$ to $-1-i$ on the (−) edge of sheet-I, b_{24}^* = the path on the vertical cut from $-1-i$ to $-1+i$ on the (−) edge of sheet-I, b_{25}^* = the path on the horizontal line from $-1+i$ to $1+i$ on the sheet-I, b_{26}^* = the path on the horizontal line from $1+i$ to $-1+i$ on the sheet-II, b_{27}^* = the path on the vertical cut from $1+3i$ to $1+i$ on the (+) edge of sheet-II, and b_{28}^* = the path on the vertical cut from $1+i$ to $1+3i$ on the (−) edge of sheet-II.

(1) $z \in b_{21}^*$: Let $z = -1 + ri$, $r : 1 \to -1$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{b_{21}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{1}^{-1} \frac{i}{f(-1+ri)}dr$.

(2) $z \in b_{22}^*$: Let $z = -1 + ri$, $r : -1 \to -2$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{22}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{-1}^{-2} \frac{i}{f(-1+ri)}dr$.

(3) $z \in b_{23}^*$: Let $z = -1 + ri$, $r : -2 \to -1$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{b_{23}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{-2}^{-1} \frac{i}{f(-1+ri)}dr$.

(4) $z \in b_{24}^*$: Let $z = -1 + ri$, $r : -1 \to 1$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{24}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{-1}^{1} \frac{i}{f(-1+ri)}dr$.

(5) $z \in b_{25}^*$: Let $z = r + i$, $r : -1 \to 1$ and $dz = dr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4},$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{b_{25}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{-1}^{1} \frac{1}{f(r+i)}dr$.

(6) $z \in b_{26}^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider b_{26}^{**} = the path on the horizontal line from $1+i$ to $-1+i$ on the sheet-I.

$z \in b_{26}^{**}$: Let $z = r + i$, $r : 1 \to -1$ and $dz = dr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4},$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{b_{26}^*} \frac{1}{f(z)}dz = -\int_{b_{26}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{1}^{-1} \frac{1}{f(r+i)}dr$.

(7) $z \in b_{27}^*$: Since $b_{27}^* \equiv$ the path on the vertical cut from $1+3i$ to $1+i$ on the $(-)$ edge of sheet-I, so let $z = 1 + ri$, $r : 3 \to 1$ and $dz = idr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{27}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{3}^{1} \frac{i}{f(1+ri)}dr$.

(8) $z \in b_{28}^*$: Since $b_{28}^* \equiv$ the path on the vertical cut from $1+i$ to $1+3i$ on the $(+)$ edge of sheet-I, so let $z = 1 + ri$, $r : 1 \to 3$ and $dz = idr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4},$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{b_{27}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^3 \frac{i}{f(1+ri)}dr$.

Therefore, by (1), (2), $\cdots$, (8),

$$\begin{aligned}
\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\
&= \int_{\bigcup_{j=1}^{8} b_{2j}^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} -\int_1^{-1} \frac{i}{f(-1+ri)}dr + \int_{-1}^{-2} \frac{i}{f(-1+ri)}dr - \int_{-2}^{-1} \frac{i}{f(-1+ri)}dr \\
&\quad + \int_{-1}^{1} \frac{i}{f(-1+ri)}dr - \int_{-1}^{1} \frac{1}{f(r+i)}dr + \int_{1}^{-1} \frac{1}{f(r+i)}dr \\
&\quad + \int_{3}^{1} \frac{i}{f(1+ri)}dr - \int_{1}^{3} \frac{i}{f(1+ri)}dr \\
&= 2\int_{-1}^{1} \frac{i}{f(-1+ri)}dr + 2\int_{-1}^{-2} \frac{i}{f(-1+ri)}dr + 2\int_{1}^{-1} \frac{1}{f(r+i)}dr \\
&\quad +2\int_{3}^{1} \frac{i}{f(1+ri)}dr.
\end{aligned}$$

4. Compute $\int_{b_1} f(z)dz$.

By Cauchy theorem, we can consider that b_1^* is the equivalent path of b_1 and $b_1^* = b_2^* \bigcup a_{22}^* \bigcup a_{23}^* \bigcup a_{24}^* \bigcup a_{25}^* \bigcup b_{11}^* \bigcup b_{12}^*$ where b_{11}^* = the path on the horizontal line from $1+i$ to $2+i$ on the sheet-I and b_{12}^* = the path on the horizontal line from $2+i$ to $1+i$ on the sheet-II.

(1) $z \in b_{11}^*$: Let $z = r+i$, $r : 1 \to 2$ and $dz = dr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{11}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^2 \frac{1}{f(r+i)}dr$.

(2) $z \in b_{12}^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider b_{12}^{**} = the path on the horizontal line from $2+i$ to $1+i$ on the sheet-I.

$z \in b_{12}^{**}$: Let $z = r + i$, $r : 2 \to 1$ and $dz = dr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{12}^*} \frac{1}{f(z)}dz = -\int_{b_{12}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_2^1 \frac{1}{f(r+i)}dr$.

Therefore, by (1) and (2),

$$\begin{aligned}
\int_{b_1} \frac{1}{f(z)}dz &= \int_{b_1^*} \frac{1}{f(z)}dz \\
&= \int_{b_2^* \cup a_{22}^* \cup a_{23}^* \cup a_{24}^* \cup a_{25}^* \cup b_{11}^* \cup b_{12}^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} 2\int_{-1}^{1} \frac{i}{f(-1+ri)}dr + 2\int_{-1}^{-2} \frac{i}{f(-1+ri)}dr + 2\int_{1}^{-1} \frac{1}{f(r+i)}dr \\
&\quad +2\int_{3}^{1} \frac{i}{f(1+ri)}dr + 2\int_{1}^{-1} \frac{i}{f(1+ri)}dr + 2\int_{-3}^{-1} \frac{i}{f(1+ri)}dr \\
&\quad +2\int_{1}^{2} \frac{1}{f(r+i)}dr.
\end{aligned}$$

2.6 The integrals of $\frac{1}{f(z)}$ over a, b cycles with slant cut structure

In this section, we will discuss the difference between theory and Mathematica with slant cut. Consider the cut with slope $m = \tan\alpha$ where $0 < \alpha < \pi$ and $\alpha \neq \frac{\pi}{2}$.

Define that

$$z = \begin{cases} |z|e^{i\theta}, \theta \in [\alpha - 2\pi, \alpha) & \text{if and only if z is in sheet-I;} \\ |z|e^{i\theta}, \theta \in [\alpha, \alpha + 2\pi) & \text{if and only if z is in sheet-II.} \end{cases}$$

The cut on each sheet has two edges, label the starting edge with "+" and the terminal edge with "−".

Lemma 2.6.1. If z_k is the end point for slant cut with slope $m = \tan\alpha$, $0 < \alpha < \pi$, $\alpha \neq \frac{\pi}{2}$, and

z is in sheet-I, then

$$\sqrt{z-z_k} \overset{Math.}{=} \begin{cases} -\sqrt{z-z_k} & \text{if } \arg(z-z_k) \in [\alpha-2\pi, -\pi], \\ \sqrt{z-z_k} & \text{if } \arg(z-z_k) \in (-\pi, \alpha). \end{cases}$$

Proof. Let z be a point in sheet-I and $z-z_k = |z-z_k|e^{i\theta}$ where $\theta = \arg(z-z_k)$. When $\theta \in (-\pi, \alpha)$, the value of $\sqrt{z-z_k}$ in theory and Mathematica all equal $|z-z_k|^{\frac{1}{2}}e^{i\frac{\theta}{2}}$ since Mathematica is always considering $\theta \in (-\pi, \pi]$, then $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$. When $\theta \in [\alpha-2\pi, -\pi]$,

$$\text{in theory}: \sqrt{\mathrm{z}-\mathrm{z_k}} = |\mathrm{z}-\mathrm{z_k}|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}},$$

$$\text{in Mathematica}: \theta \in [\alpha-2\pi, -\pi] \text{ is regarded as } \theta+2\pi \in [\alpha, \pi] \text{ so } \mathrm{z}-\mathrm{z_k} = |\mathrm{z}-\mathrm{z_k}|\mathrm{e}^{\mathrm{i}(\theta+2\pi)}$$

$$\text{and } \sqrt{\mathrm{z}-\mathrm{z_k}} = |\mathrm{z}-\mathrm{z_k}|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\theta+2\pi}{2}} = -|\mathrm{z}-\mathrm{z_k}|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}.$$

Therefore, $\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$ if $\arg(z-z_k) \in [\alpha-2\pi, -\pi]$ and $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}$ if $\arg(z-z_k) \in (-\pi, \alpha)$.

Definition 2.6.2. Let a cut with slope $m = \tan\alpha$ where $0 < \alpha < \pi$ and $\alpha \neq \frac{\pi}{2}$, z_k and z_{k+1} are two end points of the cut. A point $(\mathrm{x}, \mathrm{y}) \in \mathrm{L}$ if

$$\mathrm{y}-\mathrm{Im}(\mathrm{z_{k+1}}) > \tan\alpha \cdot (\mathrm{x}-\mathrm{Re}(\mathrm{z_{k+1}})) \text{ as } \tan\alpha > 0,$$

$$\mathrm{y}-\mathrm{Im}(\mathrm{z_{k+1}}) < \tan\alpha \cdot (\mathrm{x}-\mathrm{Re}(\mathrm{z_{k+1}})) \text{ as } \tan\alpha < 0;$$

and $(\mathrm{x}, \mathrm{y}) \in \mathrm{S}$ if

$$\mathrm{y}-\mathrm{Im}(\mathrm{z_{k+1}}) < \tan\alpha \cdot (\mathrm{x}-\mathrm{Re}(\mathrm{z_{k+1}})) \text{ as } \tan\alpha > 0,$$

$$\mathrm{y}-\mathrm{Im}(\mathrm{z_{k+1}}) > \tan\alpha \cdot (\mathrm{x}-\mathrm{Re}(\mathrm{z_{k+1}})) \text{ as } \tan\alpha < 0.$$

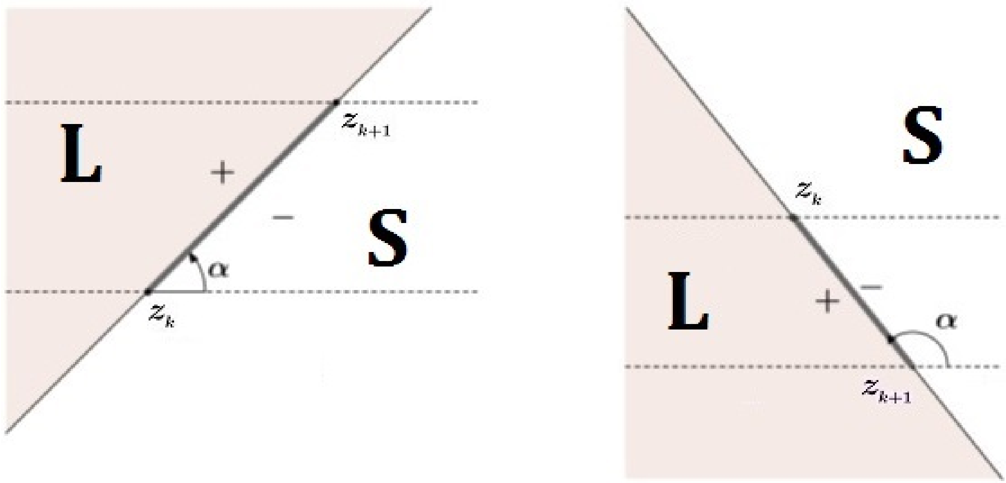

Figure 50.The areas L and S

Theorem 2.6.3. Let $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}}$ where z_k and z_{k+1} are two end points for slant cut and the domain be divided into six areas (A), (B), (C), (D), (E), and (F) where

(A)= $\{(x,y) : (x,y) \in L,\ y \geq \mathrm{Im}(z_k)\}$,

(B)= $\{(x,y) : (x,y) \in L,\ \mathrm{Im}(z_{k+1}) \leq y < \mathrm{Im}(z_k)\}$,

(C)= $\{(x,y) : (x,y) \in L,\ y < \mathrm{Im}(z_{k+1})\}$,

(D)= $\{(x,y) : (x,y) \in S,\ y \geq \mathrm{Im}(z_k)\}$,

(E)= $\{(x,y) : (x,y) \in S,\ \mathrm{Im}(z_{k+1}) \leq y < \mathrm{Im}(z_k)\}$,

(F)= $\{(x,y) : (x,y) \in S,\ y < \mathrm{Im}(z_{k+1})\}$.If z is in sheet-I, then

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in (B) \bigcup \{\text{the cut with } (+) \text{ edge of sheet-I}\}, \\ f(z) & \text{otherwise.} \end{cases}$$

Proof. Let z be a point in sheet-I.

(1) $z \in$(A)

Since $\arg(z-z_k) \in (\alpha - 2\pi, -\pi]$ and $\arg(z-z_{k+1}) \in (\alpha - 2\pi, -\pi)$, then by Lemma 2.6.1,

$\sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}$ and $\sqrt{z-z_{k+1}} \overset{Math.}{=} -\sqrt{z-z_{k+1}}$.

Thus, $f(z) = \sqrt{z-z_k}\sqrt{z-z_{k+1}} \overset{Math.}{=} \sqrt{z-z_k}\sqrt{z-z_{k+1}} = f(z)$.

(2) $z \in$(B)

Since $\arg(z - z_k) \in (-\pi, \alpha - \pi)$ and $\arg(z - z_{k+1}) \in (\alpha - 2\pi, -\pi]$, then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ and $\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_{k+1}}$ by Lemma 2.6.1.

Thus, $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_k}\sqrt{z - z_{k+1}} = -f(z)$.

(3) $z \in$\{the cut with (+) edge of sheet-I\}

Since $\arg(z - z_k) = \alpha - \pi$ and $\arg(z - z_{k+1}) = \alpha - 2\pi$, then by Lemma 2.6.1,

$\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ and $\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_{k+1}}$.

Thus, $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_k}\sqrt{z - z_{k+1}} = -f(z)$.

(4) $z \in$\{the cut with (-) edge of sheet-I\}

Since $\arg(z - z_k) = \alpha - \pi$, then $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ by Lemma 2.6.1.

Since the (-) edge of sheet-I equals the (+) edge of sheet-II, then $\arg(z - z_{k+1}) = \alpha$, by Lemma 2.6.1, $\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_{k+1}}$.

Thus, $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_k}\sqrt{z - z_{k+1}} = f(z)$.

(5) $z \in \{(\mathrm{x}, \mathrm{y}) : \mathrm{y} - \mathrm{Im}(\mathrm{z_k}) = \tan\alpha \cdot (\mathrm{x} - \mathrm{Re}(\mathrm{z_k})),\ \mathrm{y} > \mathrm{Im}(\mathrm{z_k})\}$

Since $\arg(z - z_k) = \alpha - 2\pi$ and $\arg(z - z_{k+1}) = \alpha - 2\pi$, then $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}$ and $\sqrt{z - z_{k+1}} \overset{Math.}{=} -\sqrt{z - z_{k+1}}$ by Lemma 2.6.1.

Thus, $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_k}\sqrt{z - z_{k+1}} = f(z)$.

(6) Otherwise

Since $\arg(z - z_k) \in (-\pi, \alpha)$ and $\arg(z - z_{k+1}) \in (-\pi, \alpha)$, then by Lemma 2.6.1,

$\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}$ and $\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_{k+1}}$.

Thus, $f(z) = \sqrt{z - z_k}\sqrt{z - z_{k+1}} \overset{Math.}{=} \sqrt{z - z_k}\sqrt{z - z_{k+1}} = f(z)$.

Example 2.6.4. Compute $\int \frac{1}{f(z)} dz$ over a_1, a_2, b_1, and b_2 cycles where

$f(z) = \sqrt{(z + \sqrt{3})(z - i)(z - 2i)(z - 1 - (2 + \sqrt{3})i)(z - 1 - i)(z - 2 - 2i)}$.

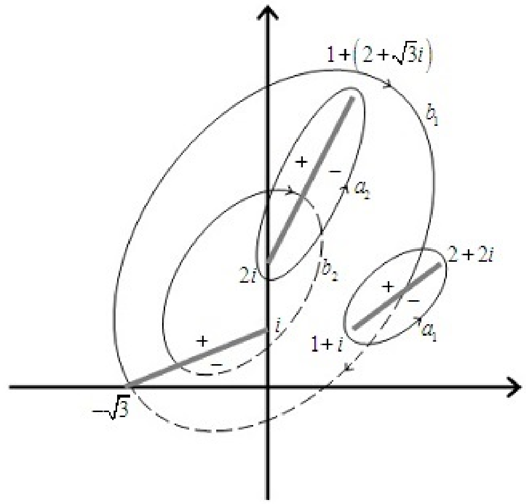

Figure 51.The a, b cycles of $f(z)$

Let $z_1 = -\sqrt{3}$, $z_2 = i$, $z_3 = 2i$, $z_4 = 1 + (2 + \sqrt{3})i$, $z_5 = 1 + i$, and $z_6 = 2 + 2i$.

1. Compute $\int_{a_1} \frac{1}{f(z)} dz$.

   By Cauchy theorem, we can consider that a_1^* is the equivalent path of a_1 and $a_1^* = a_{11}^* \bigcup a_{12}^*$ where a_{11}^* = the path on the slant cut from $2 + 2i$ to $1 + i$ on the $(+)$ edge of sheet-I and a_{12}^* = the path on the slant cut from $1 + i$ to $2 + 2i$ on the $(-)$ edge of sheet-I.

   (1) $z \in a_{11}^*$: Let $z = 2 + 2i + r(-1 - i)$, $r : 0 \to 1$ and $dz = (-1 - i)dr$, then

$$\sqrt{z - z_1}\sqrt{z - z_2} \overset{Math.}{=} \sqrt{z - z_1}\sqrt{z - z_2},$$
$$\sqrt{z - z_3}\sqrt{z - z_4} \overset{Math.}{=} \sqrt{z - z_3}\sqrt{z - z_4},$$
$$\sqrt{z - z_5}\sqrt{z - z_6} \overset{Math.}{=} -\sqrt{z - z_5}\sqrt{z - z_6}.$$

   Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{a_{11}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_0^1 \frac{-1-i}{f(2+2i+r(-1-i))} dr$.

   (2) $z \in a_{12}^*$: Let $z = 2 + 2i + r(-1 - i)$, $r : 1 \to 0$ and $dz = (-1 - i)dr$, then

$$\sqrt{z - z_1}\sqrt{z - z_2} \overset{Math.}{=} \sqrt{z - z_1}\sqrt{z - z_2},$$
$$\sqrt{z - z_3}\sqrt{z - z_4} \overset{Math.}{=} \sqrt{z - z_3}\sqrt{z - z_4},$$
$$\sqrt{z - z_5}\sqrt{z - z_6} \overset{Math.}{=} \sqrt{z - z_5}\sqrt{z - z_6}.$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{a_{12}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^0 \frac{-1-i}{f(2+2i+r(-1-i))}dr$.

Therefore, by (1) and (2),

$$\begin{aligned}
\int_{a_1} \frac{1}{f(z)}dz &= \int_{a_1^*} \frac{1}{f(z)}dz \\
&= \int_{a_{11}^* \bigcup a_{12}^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} -\int_0^1 \frac{-1-i}{f(2+2i+r(-1-i))}dr + \int_1^0 \frac{-1-i}{f(2+2i+r(-1-i))}dr \\
&= 2\int_1^0 \frac{-1-i}{f(2+2i+r(-1-i))}dr.
\end{aligned}$$

2. Compute $\int_{a_2} \frac{1}{f(z)}dz$.

   By Cauchy theorem, we can consider that a_2^* is the equivalent path of a_2 and $a_2^* = a_{21}^* \bigcup a_{22}^*$ where a_{21}^* = the path on the slant cut from $1+(2+\sqrt{3})i$ to $2i$ on the (+) edge of sheet-I and a_{22}^* = the path on the slant cut from $2i$ to $1+(2+\sqrt{3})i$ on the (−) edge of sheet-I.

   (1) $z \in a_{21}^*$: Let $z = 1+(2+\sqrt{3})i + r(-1-\sqrt{3}i)$, $r : 0 \to 1$ and $dz = (-1-\sqrt{3}i)dr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} -\sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

   Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{a_{21}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_0^1 \frac{-1-\sqrt{3}i}{f(1+(2+\sqrt{3})i+r(-1-\sqrt{3}i))}dr$.

   (2) $z \in a_{22}^*$: Let $z = 1+(2+\sqrt{3})i + r(-1-\sqrt{3}i)$, $r : 1 \to 0$ and $dz = (-1-\sqrt{3}i)dr$, then

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$

$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$

$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.$$

   Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{a_{22}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^0 \frac{-1-\sqrt{3}i}{f(1+(2+\sqrt{3})i+r(-1-\sqrt{3}i))}dr$.

Therefore, by (1) and (2),

$$\begin{aligned}
\int_{a_2} \frac{1}{f(z)} dz &= \int_{a_2^*} \frac{1}{f(z)} dz \\
&= \int_{a_{21}^* \bigcup a_{22}^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} -\int_0^1 \frac{-1-\sqrt{3}i}{f(1+(2+\sqrt{3})i + r(-1-\sqrt{3}i))} dr \\
&\quad + \int_1^0 \frac{-1-\sqrt{3}i}{f(1+(2+\sqrt{3})i + r(-1-\sqrt{3}i))} dr \\
&= 2\int_1^0 \frac{-1-\sqrt{3}i}{f(1+(2+\sqrt{3})i + r(-1-\sqrt{3}i))} dr.
\end{aligned}$$

3. Compute $\int_{b_2} \frac{1}{f(z)} dz$.

By Cauchy theorem, we can consider that b_2^* is the equivalent path of b_2 and $b_2^* = b_{21}^* \bigcup b_{22}^*$ where b_{21}^* = the path on the vertical line from i to $2i$ on the sheet-I and b_{22}^* = the path on the vertical line from $2i$ to i on the sheet-II.

(1) $z \in b_{21}^*$: Let $z = ri$, $r : 1 \to 2$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{b_{21}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_1^2 \frac{i}{f(ri)} dr$.

(2) $z \in b_{22}^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider b_{22}^{**} = the path on the vertical line from $2i$ to i on the sheet-I.

$z \in b_{22}^{**}$: Let $z = ri$, $r : 2 \to 1$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{b_{22}^*} \frac{1}{f(z)} dz = -\int_{b_{22}^{**}} \frac{1}{f(z)} dz \overset{Math.}{=} \int_2^1 \frac{i}{f(ri)} dr$.

Therefore, by (1) and (2),

$$\begin{aligned}\int_{b_2}\frac{1}{f(z)}dz &= \int_{b_2^*}\frac{1}{f(z)}dz \\ &= \int_{b_{21}^*\bigcup b_{22}^*}\frac{1}{f(z)}dz \\ &\overset{Math.}{=} -\int_1^2\frac{i}{f(ri)}dr+\int_2^1\frac{i}{f(ri)}dr \\ &= 2\int_2^1\frac{i}{f(ri)}dr.\end{aligned}$$

4. Compute $\int_{b_1} f(z)dz$.

By Cauchy theorem, we can consider that b_1^* is the equivalent path of b_1 and $b_1^* = b_2^*\bigcup b_{11}^*\bigcup b_{12}^*\bigcup b_{13}^*\bigcup b_{14}^*$ where b_{11}^* = the path on the vertical line from $1+(2+\sqrt{3})i$ to $1+2i$ on the sheet-I, b_{12}^* = the path on the vertical line from $1+2i$ to $1+i$ on the sheet-I, b_{13}^* = the path on the vertical line from $1+i$ to $1+2i$ on the sheet-II, and b_{14}^* = the path on the vertical line from $1+2i$ to $1+(2+\sqrt{3})i$ on the sheet-II.

(1) $z\in b_{11}^*$: Let $z=1+ri$, $r:2+\sqrt{3}\to 2$ and $dz=idr$, then

$$\begin{aligned}\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\ \sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\ \sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{11}^*}\frac{1}{f(z)}dz \overset{Math.}{=} \int_{2+\sqrt{3}}^2\frac{i}{f(1+ri)}dr$.

(2) $z\in b_{12}^*$: Let $z=1+ri$, $r:2\to 1$ and $dz=idr$, then

$$\begin{aligned}\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\ \sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\ \sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{b_{12}^*}\frac{1}{f(z)}dz \overset{Math.}{=} -\int_2^1\frac{i}{f(1+ri)}dr$.

(3) $z \in b_{13}^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider b_{13}^{**} = the path on the vertical line from $1+i$ to $1+2i$ on the sheet-I.

$z \in b_{13}^{**}$: Let $z = 1+ri$, $r : 1 \to 2$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},\\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},\\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{b_{13}^*} \frac{1}{f(z)}dz = -\int_{b_{13}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} \int_1^2 \frac{i}{f(1+ri)}dr$.

(4) $z \in b_{14}^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider b_{14}^{**} = the path on the vertical line from $1+2i$ to $1+(2+\sqrt{3})i$ on the sheet-I.

$z \in b_{14}^{**}$: Let $z = 1+ri$, $r : 2 \to 2+\sqrt{3}$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2},\\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4},\\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} \sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{b_{14}^*} \frac{1}{f(z)}dz = -\int_{b_{14}^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_2^{2+\sqrt{3}} \frac{i}{f(1+ri)}dr$.

Therefore, by (1), (2), (3), and (4),

$$\begin{aligned}
\int_{b_1} \frac{1}{f(z)}dz &= \int_{b_1^*} \frac{1}{f(z)}dz\\
&= \int_{b_2^* \bigcup b_{11}^* \bigcup b_{12}^* \bigcup b_{13}^* \bigcup b_{14}^*} \frac{1}{f(z)}dz\\
&\overset{Math.}{=} 2\int_2^1 \frac{i}{f(ri)}dr + \int_{2+\sqrt{3}}^2 \frac{i}{f(1+ri)}dr - \int_2^1 \frac{i}{f(1+ri)}dr\\
&\quad + \int_1^2 \frac{i}{f(1+ri)}dr - \int_2^{2+\sqrt{3}} \frac{i}{f(1+ri)}dr\\
&= 2\int_2^1 \frac{i}{f(ri)}dr + 2\int_{2+\sqrt{3}}^2 \frac{i}{f(1+ri)}dr + 2\int_1^2 \frac{i}{f(1+ri)}dr.
\end{aligned}$$

2.7 The integrals of $\frac{1}{f(z)}$ over other paths

After introducing the integrals of $\frac{1}{f(z)}$ over a, b cycles, we will discuss the integrals of $\frac{1}{f(z)}$ over other paths with three different cut structure. To start with, find the equivalent paths of original paths by Cauchy theorem. Next, use the lemmas and theorems given above to judge where $f(z)$ would change sign. Finally, use Mathematica to evaluate the integrals.

Example 2.7.1. Compute $\int \frac{1}{f(z)}dz$ over γ_1, γ_2, γ_3, and γ_4 paths where $f(z) = \sqrt{\prod_{k=1}^{7}(z-k)}$.

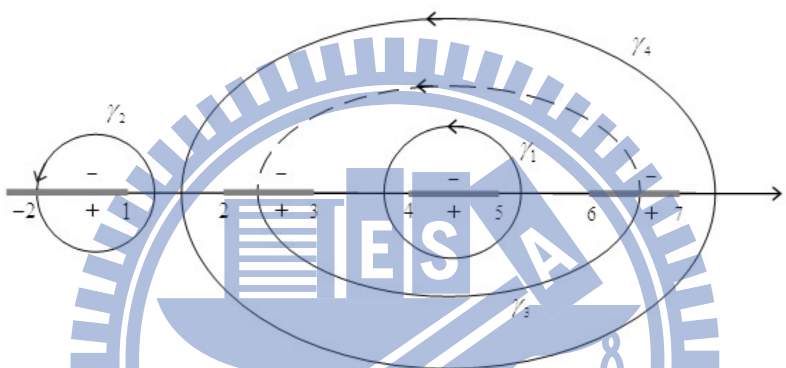

Figure 52.The γ paths of $f(z)$

We evaluate the integrals of different paths respectively.

1. Compute $\int_{\gamma_1} \frac{1}{f(z)}dz$.

   By Cauchy theorem, we can consider that γ_1^* is the equivalent path of γ_1 and $\gamma_1^* = \gamma_{11}^* \bigcup \gamma_{12}^*$ where γ_{11}^* = the path on the horizontal cut from 4 to 5 on the (+) edge of sheet-I and γ_{12}^* = the path on the horizontal cut from 5 to 4 on the $(-)$ edge of sheet-I.

(1) $z \in \gamma_{11}^*$: Let $z = r$, $r : 4 \to 5$ and $dz = dr$, then

$$\begin{aligned}
\sqrt{z-1} &\overset{Math.}{=} \sqrt{z-1},\\
\sqrt{z-2}\sqrt{z-3} &\overset{Math.}{=} \sqrt{z-2}\sqrt{z-3},\\
\sqrt{z-4}\sqrt{z-5} &\overset{Math.}{=} -\sqrt{z-4}\sqrt{z-5},\\
\sqrt{z-6}\sqrt{z-7} &\overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{\gamma_{11}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_4^5 \frac{1}{f(r)}dr$.

(2) $z \in \gamma_{12}^*$: Let $z = r$, $r : 5 \to 4$ and $dz = dr$, then

$$\begin{aligned}
\sqrt{z-1} &\overset{Math.}{=} \sqrt{z-1},\\
\sqrt{z-2}\sqrt{z-3} &\overset{Math.}{=} \sqrt{z-2}\sqrt{z-3},\\
\sqrt{z-4}\sqrt{z-5} &\overset{Math.}{=} \sqrt{z-4}\sqrt{z-5},\\
\sqrt{z-6}\sqrt{z-7} &\overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{\gamma_{12}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_5^4 \frac{1}{f(r)}dr$.

Therefore, by (1) and (2),

$$\begin{aligned}
\int_{\gamma_1} \frac{1}{f(z)}dz &= \int_{\gamma_1^*} \frac{1}{f(z)}dz\\
&= \int_{\gamma_{11}^* \bigcup \gamma_{12}^*} \frac{1}{f(z)}dz\\
&\overset{Math.}{=} -\int_4^5 \frac{1}{f(r)}dr + \int_5^4 \frac{1}{f(r)}dr\\
&= 2\int_5^4 \frac{1}{f(r)}dr.
\end{aligned}$$

2. Compute $\int_{\gamma_2} \frac{1}{f(z)}dz$.

By Cauchy theorem, we can consider that γ_2^* is the equivalent path of γ_2 and $\gamma_2^* = \gamma_{21}^* \bigcup \gamma_{22}^*$ where γ_{21}^* = the path on the horizontal cut from -2 to 1 on the $(+)$ edge of sheet-I and γ_{22}^* = the path on the horizontal cut from 1 to -2 on the $(-)$ edge of sheet-I.

(1) $z \in \gamma_{21}^*$: Let $z = r$, $r : -2 \to 1$ and $dz = dr$, then

$$\begin{aligned}
\sqrt{z-1} &\overset{Math.}{=} -\sqrt{z-1}, \\
\sqrt{z-2}\sqrt{z-3} &\overset{Math.}{=} \sqrt{z-2}\sqrt{z-3}, \\
\sqrt{z-4}\sqrt{z-5} &\overset{Math.}{=} \sqrt{z-4}\sqrt{z-5}, \\
\sqrt{z-6}\sqrt{z-7} &\overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{\gamma_{21}^*} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{-2}^{1} \frac{1}{f(r)} dr$.

(2) $z \in \gamma_{22}^*$: Let $z = r$, $r : 1 \to -2$ and $dz = dr$, then

$$\begin{aligned}
\sqrt{z-1} &\overset{Math.}{=} \sqrt{z-1}, \\
\sqrt{z-2}\sqrt{z-3} &\overset{Math.}{=} \sqrt{z-2}\sqrt{z-3}, \\
\sqrt{z-4}\sqrt{z-5} &\overset{Math.}{=} \sqrt{z-4}\sqrt{z-5}, \\
\sqrt{z-6}\sqrt{z-7} &\overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{\gamma_{22}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{1}^{-2} \frac{1}{f(r)} dr$.

Therefore, by (1) and (2),

$$\begin{aligned}
\int_{\gamma_2} \frac{1}{f(z)} dz &= \int_{\gamma_2^*} \frac{1}{f(z)} dz \\
&= \int_{\gamma_{21}^* \bigcup \gamma_{22}^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} -\int_{-2}^{1} \frac{1}{f(r)} dr + \int_{1}^{-2} \frac{1}{f(r)} dr \\
&= 2\int_{1}^{-2} \frac{1}{f(r)} dr.
\end{aligned}$$

3. Compute $\int_{\gamma_3} \frac{1}{f(z)} dz$.

By Cauchy theorem, we can consider that γ_3^* is the equivalent path of γ_3 and $\gamma_3^* = \gamma_{31}^* \bigcup \gamma_{32}^* \bigcup \gamma_{33}^* \bigcup \gamma_{34}^*$ where γ_{31}^* = the path on the horizontal line from 3 to 4 on the sheet-I, γ_{32}^* = the path on the horizontal line from 4 to 3 on the sheet-II, γ_{33}^* = the path on the horizontal line from 5 to 6 on the sheet-I, and $\gamma_{34}^* =$ the path on the horizontal line from 6 to 5 on the sheet-II.

(1) $z \in \gamma_{31}^*$: Let $z = r$, $r : 3 \to 4$ and $dz = dr$, then

$$\begin{aligned}
\sqrt{z-1} &\overset{Math.}{=} \sqrt{z-1}, \\
\sqrt{z-2}\sqrt{z-3} &\overset{Math.}{=} \sqrt{z-2}\sqrt{z-3}, \\
\sqrt{z-4}\sqrt{z-5} &\overset{Math.}{=} \sqrt{z-4}\sqrt{z-5}, \\
\sqrt{z-6}\sqrt{z-7} &\overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{\gamma_{31}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_3^4 \frac{1}{f(r)} dr$.

(2) $z \in \gamma_{32}^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider $\gamma_{32}^{**} =$ the path on the horizontal line from 4 to 3 on the sheet-I.

$z \in \gamma_{32}^{**}$: Let $z = r$, $r : 4 \to 3$ and $dz = dr$, then

$$\begin{aligned}
\sqrt{z-1} &\overset{Math.}{=} \sqrt{z-1}, \\
\sqrt{z-2}\sqrt{z-3} &\overset{Math.}{=} \sqrt{z-2}\sqrt{z-3}, \\
\sqrt{z-4}\sqrt{z-5} &\overset{Math.}{=} \sqrt{z-4}\sqrt{z-5}, \\
\sqrt{z-6}\sqrt{z-7} &\overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{\gamma_{32}^*} \frac{1}{f(z)} dz = -\int_{\gamma_{32}^{**}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_4^3 \frac{1}{f(r)} dr$.

(3) $z \in \gamma_{33}^*$: Let $z = r$, $r : 5 \to 6$ and $dz = dr$, then

$$\begin{aligned}
\sqrt{z-1} &\overset{Math.}{=} \sqrt{z-1}, \\
\sqrt{z-2}\sqrt{z-3} &\overset{Math.}{=} \sqrt{z-2}\sqrt{z-3}, \\
\sqrt{z-4}\sqrt{z-5} &\overset{Math.}{=} \sqrt{z-4}\sqrt{z-5}, \\
\sqrt{z-6}\sqrt{z-7} &\overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{\gamma_{33}^*} \frac{1}{f(z)} dz \overset{Math.}{=} \int_5^6 \frac{1}{f(r)} dr$.

(4) $z \in \gamma_{34}^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider γ_{34}^{**} = the path on the horizontal line from 6 to 5 on the sheet-I.

$z \in \gamma_{34}^{**}$: Let $z = r$, $r : 6 \to 5$ and $dz = dr$, then

$$\begin{aligned}
\sqrt{z-1} &\overset{Math.}{=} \sqrt{z-1}, \\
\sqrt{z-2}\sqrt{z-3} &\overset{Math.}{=} \sqrt{z-2}\sqrt{z-3}, \\
\sqrt{z-4}\sqrt{z-5} &\overset{Math.}{=} \sqrt{z-4}\sqrt{z-5}, \\
\sqrt{z-6}\sqrt{z-7} &\overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{\gamma_{34}^*} \frac{1}{f(z)} dz = -\int_{\gamma_{34}^{**}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_6^5 \frac{1}{f(r)} dr$.

Therefore, by (1), (2), (3), and (4),

$$\begin{aligned}
\int_{\gamma_3} \frac{1}{f(z)} dz &= \int_{\gamma_3^*} \frac{1}{f(z)} dz \\
&= \int_{\gamma_{31}^* \bigcup \gamma_{32}^* \bigcup \gamma_{33}^* \bigcup \gamma_{34}^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_3^4 \frac{1}{f(r)} dr - \int_4^3 \frac{1}{f(r)} dr + \int_5^6 \frac{1}{f(r)} dr - \int_6^5 \frac{1}{f(r)} dr \\
&= 2\int_3^4 \frac{1}{f(r)} dr + 2\int_5^6 \frac{1}{f(r)} dr.
\end{aligned}$$

4. Compute $\int_{\gamma_4} \frac{1}{f(z)} dz$.

By Cauchy theorem, we can consider that γ_4^* is the equivalent path of γ_4 and $\gamma_4^* = \gamma_1^* \bigcup \gamma_{41}^* \bigcup \gamma_{42}^* \bigcup \gamma_{43}^* \bigcup \gamma_{44}^*$ where γ_{41}^* = the path on the horizontal cut from 2 to 3 on the (+) edge of sheet-I, γ_{42}^* = the path on the horizontal cut from 3 to 2 on the (−) edge of sheet-I, γ_{43}^* = the path on the horizontal cut from 6 to 7 on the (+) edge of sheet-I, and γ_{44}^* = the path on the horizontal cut from 7 to 6 on the (−) edge of sheet-I.

(1) $z \in \gamma_{41}^*$: Let $z = r$, $r : 2 \to 3$ and $dz = dr$, then

$$\sqrt{z-1} \overset{Math.}{=} \sqrt{z-1},$$

$$\sqrt{z-2}\sqrt{z-3} \overset{Math.}{=} -\sqrt{z-2}\sqrt{z-3},$$

$$\sqrt{z-4}\sqrt{z-5} \overset{Math.}{=} \sqrt{z-4}\sqrt{z-5},$$

$$\sqrt{z-6}\sqrt{z-7} \overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{\gamma_{41}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_2^3 \frac{1}{f(r)}dr$.

(2) $z \in \gamma_{42}^*$: Let $z = r$, $r : 3 \to 2$ and $dz = dr$, then

$$\sqrt{z-1} \overset{Math.}{=} \sqrt{z-1},$$

$$\sqrt{z-2}\sqrt{z-3} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-3},$$

$$\sqrt{z-4}\sqrt{z-5} \overset{Math.}{=} \sqrt{z-4}\sqrt{z-5},$$

$$\sqrt{z-6}\sqrt{z-7} \overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{\gamma_{42}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_3^2 \frac{1}{f(r)}dr$.

(3) $z \in \gamma_{43}^*$: Let $z = r$, $r : 6 \to 7$ and $dz = dr$, then

$$\sqrt{z-1} \overset{Math.}{=} \sqrt{z-1},$$

$$\sqrt{z-2}\sqrt{z-3} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-3},$$

$$\sqrt{z-4}\sqrt{z-5} \overset{Math.}{=} \sqrt{z-4}\sqrt{z-5},$$

$$\sqrt{z-6}\sqrt{z-7} \overset{Math.}{=} -\sqrt{z-6}\sqrt{z-7}.$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{\gamma_{43}^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_6^7 \frac{1}{f(r)}dr$.

(4) $z \in \gamma_{44}^*$: Let $z = r$, $r : 7 \to 6$ and $dz = dr$, then

$$\sqrt{z-1} \overset{Math.}{=} \sqrt{z-1},$$

$$\sqrt{z-2}\sqrt{z-3} \overset{Math.}{=} \sqrt{z-2}\sqrt{z-3},$$

$$\sqrt{z-4}\sqrt{z-5} \overset{Math.}{=} \sqrt{z-4}\sqrt{z-5},$$

$$\sqrt{z-6}\sqrt{z-7} \overset{Math.}{=} \sqrt{z-6}\sqrt{z-7}.$$

Thus, $f(z) \overset{Math.}{=} f(z)$ and $\int_{\gamma_{44}^*} \frac{1}{f(z)}dz \overset{Math.}{=} \int_7^6 \frac{1}{f(r)}dr$.

Therefore, by (1), (2), (3), and (4),

$$
\begin{aligned}
\int_{\gamma_4} \frac{1}{f(z)}dz &= \int_{\gamma_4^*} \frac{1}{f(z)}dz \\
&= \int_{\gamma_1^* \bigcup \gamma_{41}^* \bigcup \gamma_{42}^* \bigcup \gamma_{43}^* \bigcup \gamma_{44}^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} 2\int_5^4 \frac{1}{f(r)}dr - \int_2^3 \frac{1}{f(r)}dr + \int_3^2 \frac{1}{f(r)}dr - \int_6^7 \frac{1}{f(r)}dr + \int_7^6 \frac{1}{f(r)}dr \\
&= 2\int_5^4 \frac{1}{f(r)}dr + 2\int_3^2 \frac{1}{f(r)}dr + 2\int_7^6 \frac{1}{f(r)}dr.
\end{aligned}
$$

Example 2.7.2. Compute $\int \frac{1}{f(z)}dz$ over γ path where

$f(z) = \sqrt{(z+\sqrt{3})(z-i)(z-2i)(z-1-(2+\sqrt{3})i)(z-1-i)(z-2-2i)}$.

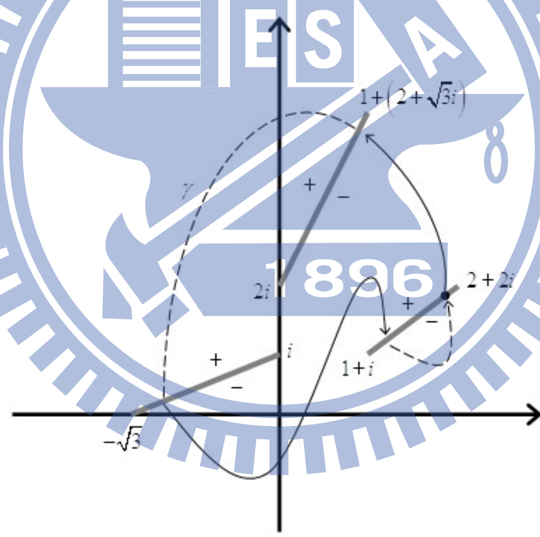

Figure 53.The γ path of $f(z)$

Let $z_1 = -\sqrt{3}$, $z_2 = i$, $z_3 = 2i$, $z_4 = 1+(2+\sqrt{3})i$, $z_5 = 1+i$, and $z_6 = 2+2i$. By Cauchy theorem, we can consider that γ^* is the equivalent path of γ and $\gamma^* = \gamma_1^* \bigcup \gamma_2^*$ where γ_1^* = the path on the vertical line from $2i$ to i on the sheet-II and γ_2^* = the path on the vertical line from i to $2i$ on the sheet-I.

(1) $z \in \gamma_1^*$: We know that $f(z)|_{(I)} = -f(z)|_{(II)}$, so consider γ_1^{**} = the path on the vertical line

from $2i$ to i on the sheet-I.

$z \in \gamma_1^{**}$: Let $z = ri$, $r : 2 \to 1$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{\gamma_1^*} \frac{1}{f(z)}dz = -\int_{\gamma_1^{**}} \frac{1}{f(z)}dz \overset{Math.}{=} \int_2^1 \frac{i}{f(ri)}dr$.

(2) $z \in \gamma_2^*$: Let $z = ri$, $r : 1 \to 2$ and $dz = idr$, then

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{Math.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{Math.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{Math.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}.
\end{aligned}$$

Thus, $f(z) \overset{Math.}{=} -f(z)$ and $\int_{\gamma_2^*} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_1^2 \frac{i}{f(ri)}dr$.

Therefore, by (1) and (2),

$$\begin{aligned}
\int_\gamma \frac{1}{f(z)}dz &= \int_{\gamma^*} \frac{1}{f(z)}dz \\
&= \int_{\gamma_1^* \cup \gamma_2^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_2^1 \frac{i}{f(ri)}dr - \int_1^2 \frac{i}{f(ri)}dr \\
&= 2\int_2^1 \frac{i}{f(ri)}dr.
\end{aligned}$$

3 Elliptic Functions

We discuss the Weierstrassian elliptic functions in section 3.1, the theta functions in section 3.2, and the Jacobian elliptic functions in section 3.3 according to [4]. Now introduce some definitions and properties of elliptic functions.

Let ω_1, ω_2 be any two numbers (real or complex) and $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$. A function $f(z)$ is called a doubly-periodic function of z, with periods $2\omega_1$, $2\omega_2$ if it satisfies $f(z+2\omega_1) = f(z)$, $f(z+2\omega_2) = f(z)$, for all values of z for which $f(z)$ exists.

A doubly-periodic function $f(z;\omega_1,\omega_2)$ is called an elliptic function if it is analytic (except at poles), and has no singularities other than poles in the finite part of the plane. Let $z_0 + 2(m-1)\omega_1 + 2(n-1)\omega_2$, $z_0 + 2(m-1)\omega_1 + 2n\omega_2$, $z_0 + 2m\omega_1 + 2(n-1)\omega_2$, and $z_0 + 2m\omega_1 + 2n\omega_2$ be four vertices for any one of the parallelograms, where $z_0 \in \mathbb{C}$, and m, n $\in \mathbb{Z}$. If there is no point ω inside or on the boundary of these parallelograms (the vertices expected) such that $f(z+\omega) = f(z)$ for all values of z, and none of the poles of f are on the sides of the parallelograms for proper choice of z_0, then such parallelograms are called the cells.

Simple properties of elliptic functions

(I) The number of poles of an elliptic function in any cell is finite.

(II) The number of zeros of an elliptic function in any cell is finite.

(III) The sum of the residues of an elliptic function, $f(z)$, at its poles in any cell is zero.

(IV) Liouville's theorem: An elliptic function, f(z), with no poles in a cell is merely a constant.

3.1 The Weierstrassian elliptic functions

Define the Weierstrassian elliptic function $\wp(z)$ by the equation

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \{ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \} \tag{3.1}$$

where $\sum{}'$ denotes that the term for which $m = n = 0$ has to be omitted from the summation and ω_1, ω_2 satisfy the condition that $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$.

For brevity, we write $\Omega_{m,n}$ in place of $2m\omega_1 + 2n\omega_2$, so that (3.1) becomes

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}'\{\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2}\}. \tag{3.2}$$

Properties of $\wp(z)$

(I) $\wp(z)$ is an even function of z.

(II) $\wp'(z)$ is an odd function of z and it is an elliptic function.

(III) $\wp(z)$ satisfies the nonlinear differential equation

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3, \tag{3.3}$$

where g_2 and g_3 (called the invariants) are given by the equations

$$g_2 = 60\sum_{m,n}{}'\,\Omega_{m,n}^{-4}, \;\; g_3 = 140\sum_{m,n}{}'\,\Omega_{m,n}^{-6}.$$

(IV) The integral representation of $\wp(z)$ is derived from (3.3),

$$z = \int_{\wp(z)}^{\infty} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}}\,dt. \tag{3.4}$$

3.2 The theta functions

Let τ be a (constant) complex number with $\mathrm{Im}(\tau) > 0$; and write $q = e^{\pi i\tau}$, so that $|q| < 1$. Define the theta function $\vartheta(z,q)$ by the series

$$\vartheta(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}, \tag{3.5}$$

qua function of the variable z.

It is evident that

$$\vartheta(z,q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz, \tag{3.6}$$

and that

$$\vartheta(z+\pi,q)=\vartheta(z,q); \tag{3.7}$$

further

$$\begin{aligned}\vartheta(z+\pi\tau,q) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2ni(z+\pi\tau)}\\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} q^{2n} e^{2niz}\\ &= -q^{-1}e^{-2iz}\sum_{n=-\infty}^{\infty}(-1)^{n+1} q^{(n+1)^2} e^{2(n+1)iz}\\ &= -q^{-1}e^{-2iz}\vartheta(z,q). \end{aligned} \tag{3.8}$$

In consequence of (3.7) and (3.8), $\vartheta(z,q)$ is called a quasi doubly-periodic function of z, 1 and $-q^{-1}e^{-2iz}$ are called the multipliers or periodicity factors associated with the periods π and $\pi\tau$ respectively. Moreover, it is obvious that if z_0 be any zero of $\vartheta(z,q)$, then $z_0+m\pi+n\pi\tau$ is also a zero of $\vartheta(z,q)$, for all m, n $\in\mathbb{Z}$.

Definition 3.2.1. Write $\vartheta_4(z,q)$ in place of $\vartheta(z,q)$, define

$$\vartheta_1(z,q)=-ie^{iz+\frac{1}{4}\pi i\tau}\vartheta_4(z+\frac{1}{2}\pi\tau,q), \tag{3.9}$$

$$\vartheta_2(z,q)=\vartheta_1(z+\frac{1}{2}\pi,q), \tag{3.10}$$

$$\vartheta_3(z,q)=\vartheta_4(z+\frac{1}{2}\pi,q). \tag{3.11}$$

From Definition 3.2.1, we can derive

$$\begin{aligned}\vartheta_1(z,q) &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z,\\ \vartheta_2(z,q) &= 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(2n+1)z,\\ \vartheta_3(z,q) &= 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz,\\ \vartheta_4(z,q) &= 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz.\end{aligned}$$

It is obvious that $\vartheta_1(z,q)$ is an odd function of z and that the other Theta-functions are even functions of z. For brevity, let $\vartheta_j(z,q) = \vartheta_j(z)$ for $j = 1,2,3,4$ when the parameter q is not specified. When it is desired to exhibit the dependence of a Theta-function on the parameter τ, it will be written $\vartheta_j(z|\tau)$ for $j = 1,2,3,4$. Also $\vartheta_j(0) = \vartheta_j$ and $\vartheta_j{}'(0) = \vartheta_j{}'$ for $j = 1,2,3,4$.

The four Theta-functions are related in

$$\begin{aligned}
\vartheta_1(z) &= -\vartheta_2(z+\frac{1}{2}\pi) = -iM\vartheta_3(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = -iM\vartheta_4(z+\frac{1}{2}\pi\tau),\\
\vartheta_2(z) &= M\vartheta_3(z+\frac{1}{2}\pi\tau) = M\vartheta_4(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = \vartheta_1(z+\frac{1}{2}\pi),\\
\vartheta_3(z) &= \vartheta_4(z+\frac{1}{2}\pi) = M\vartheta_1(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = M\vartheta_2(z+\frac{1}{2}\pi\tau),\\
\vartheta_4(z) &= -iM\vartheta_1(z+\frac{1}{2}\pi\tau) = iM\vartheta_2(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = \vartheta_3(z+\frac{1}{2}\pi),
\end{aligned} \tag{3.12}$$

where $M = q^{\frac{1}{4}}e^{iz}$.

We can obtain the multipliers of the Theta-functions associated with the periods π and $\pi\tau$ easily by the scheme

	$\vartheta_1(z)$	$\vartheta_2(z)$	$\vartheta_3(z)$	$\vartheta_4(z)$
π	-1	-1	1	1
$\pi\tau$	-N	N	N	-N

Table 3.1

where $N = q^{-1}e^{-2iz}$.

Since one zero of $\vartheta_1(z)$ is obviously $z=0$, it follows that the zeros of $\vartheta_1(z)$, $\vartheta_2(z)$, $\vartheta_3(z)$, $\vartheta_4(z)$, are the points congruent respectively to 0, $\frac{1}{2}\pi$, $\frac{1}{2}\pi+\frac{1}{2}\pi\tau$, $\frac{1}{2}\pi\tau$. The squares of the Theta-functions are related in

$$\begin{aligned}
\vartheta_2^2(z)\vartheta_4^2 &= \vartheta_4^2(z)\vartheta_2^2 - \vartheta_1^2(z)\vartheta_3^2,\\
\vartheta_3^2(z)\vartheta_4^2 &= \vartheta_4^2(z)\vartheta_3^2 - \vartheta_1^2(z)\vartheta_2^2,\\
\vartheta_1^2(z)\vartheta_4^2 &= \vartheta_3^2(z)\vartheta_2^2 - \vartheta_2^2(z)\vartheta_3^2,\\
\vartheta_4^2(z)\vartheta_4^2 &= \vartheta_3^2(z)\vartheta_3^2 - \vartheta_2^2(z)\vartheta_2^2,
\end{aligned} \tag{3.13}$$

and the detailed proof can be found in [4].

The four Theta-functions can be expressed as infinite products

$$\begin{aligned}
\vartheta_4(z) &= G\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2z + q^{4n-2}),\\
\vartheta_3(z) &= G\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos 2z + q^{4n-2}),\\
\vartheta_2(z) &= 2Gq^{\frac{1}{4}}\cos z\prod_{n=1}^{\infty}(1+2q^{2n}\cos 2z + q^{4n}),\\
\vartheta_1(z) &= 2Gq^{\frac{1}{4}}\sin z\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2z + q^{4n}),
\end{aligned} \tag{3.14}$$

where $G = \prod_{n=1}^{\infty}(1-q^{2n})$. It is straightforward that $\vartheta_j(z|\tau)$ satisfies the ordinary differential equation

$$\frac{\partial^2\vartheta_j(z|\tau)}{\partial z^2} = -\frac{4}{\pi i}\frac{\partial\vartheta_j(z|\tau)}{\partial\tau} \tag{3.15}$$

for $j = 1, 2, 3, 4$. By (3.14) and (3.15), we can obtain a relation between Theta-functions of zero argument

$$\vartheta_1{}'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0). \tag{3.16}$$

Remark 3.2.2. The differential equations satisfied by quotients of Theta- functions.

From Table 3.1, we know that the function

$$\frac{\vartheta_1(z)}{\vartheta_4(z)} \text{ and } \frac{\vartheta_2(\mathrm{z})\vartheta_3(\mathrm{z})}{\vartheta_4^2(\mathrm{z})}$$

have periodicity factors -1 and 1 associated with periods π and $\pi\tau$ respectively; and consequently

$$\frac{d}{dz}\{\frac{\vartheta_1(z)}{\vartheta_4(z)}\} = \frac{\vartheta_1{}'(z)\vartheta_4(z) - \vartheta_4{}'(z)\vartheta_1(z)}{\vartheta_4^2(z)}$$

has the same periodicity factors.

If $\phi(z)$ be defined as the quotient

$$\frac{d}{dz}\{\frac{\vartheta_1(z)}{\vartheta_4(z)}\} \div \frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_4^2(z)},$$

that is,

$$\phi(z) = \frac{\vartheta_1{}'(z)\vartheta_4(z) - \vartheta_4{}'(z)\vartheta_1(z)}{\vartheta_4^2(z)} \div \frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_4^2(z)} = \frac{\vartheta_1{}'(z)\vartheta_4(z) - \vartheta_4{}'(z)\vartheta_1(z)}{\vartheta_2(z)\vartheta_3(z)},$$

then $\phi(z)$ is doubly-periodic with periods π and $\pi\tau$; and the only possible poles of $\phi(z)$ are simple poles at points congruent to $\frac{1}{2}\pi$ and $\frac{1}{2}\pi + \frac{1}{2}\pi\tau$.

Now consider $\phi(z + \frac{1}{2}\pi\tau)$; from Definition 3.2.1, we have

$$\vartheta_1(z + \frac{1}{2}\pi\tau) = iq^{-\frac{1}{4}}e^{-iz}\vartheta_4(z),\ \vartheta_4(z + \frac{1}{2}\pi\tau) = iq^{-\frac{1}{4}}e^{-iz}\vartheta_1(z),$$

$$\vartheta_2(z + \frac{1}{2}\pi\tau) = q^{-\frac{1}{4}}e^{-iz}\vartheta_3(z),\ \vartheta_3(z + \frac{1}{2}\pi\tau) = q^{-\frac{1}{4}}e^{-iz}\vartheta_2(z),$$

then

$$\phi(z + \frac{1}{2}\pi\tau) = \frac{-\vartheta_4{}'(z)\vartheta_1(z) + \vartheta_1{}'(z)\vartheta_4(z)}{\vartheta_2(z)\vartheta_3(z)} = \phi(z).$$

Hence $\phi(z)$ is doubly-periodic with periods π and $\frac{1}{2}\pi\tau$; and relative to these periods, the only possible poles of $\phi(z)$ are simple poles at points congruent to $\frac{1}{2}\pi$. By Liouville's theorem, $\phi(z)$ is a constant; and making $z \to 0$, the value of this constant is $\{\vartheta_1{}'\vartheta_4\} \div \{\vartheta_2\vartheta_3\} = \vartheta_4^2$ by (3.16).

Therefore, we have

$$\frac{d}{dz}\{\frac{\vartheta_1(z)}{\vartheta_4(z)}\} = \vartheta_4^2\frac{\vartheta_2(z)}{\vartheta_4(z)} \cdot \frac{\vartheta_3(z)}{\vartheta_4(z)}. \tag{3.17}$$

Let $\xi = \frac{\vartheta_1(z)}{\vartheta_4(z)}$ and use (3.13), then we get

$$(\frac{d\xi}{dz})^2 = (\vartheta_2^2 - \xi^2\vartheta_3^2)(\vartheta_3^2 - \xi^2\vartheta_2^2) \tag{3.18}$$

from (3.17) and this differential equation has the solution $\frac{\vartheta_1(z)}{\vartheta_4(z)}$.

Using the similar argument, we can get

$$\frac{d}{dz}\{\frac{\vartheta_2(z)}{\vartheta_4(z)}\} = -\vartheta_3^2 \frac{\vartheta_1(z)}{\vartheta_4(z)} \frac{\vartheta_3(z)}{\vartheta_4(z)}, \tag{3.19}$$

$$\frac{d}{dz}\{\frac{\vartheta_3(z)}{\vartheta_4(z)}\} = -\vartheta_2^2 \frac{\vartheta_1(z)}{\vartheta_4(z)} \frac{\vartheta_2(z)}{\vartheta_4(z)}. \tag{3.20}$$

3.3 The Jacobian elliptic functions

Let $\frac{\xi\vartheta_3}{\vartheta_2} = y$, $z\vartheta_3^2 = u$; then, if $\kappa^{\frac{1}{2}} = \frac{\vartheta_2}{\vartheta_3}$, the equation (3.18) becomes

$$(\frac{dy}{du})^2 = (1-y^2)(1-\kappa^2 y^2) \tag{3.21}$$

by a slight change of variable. This differential equation (3.21) determines y in terms of u and has the particular solution

$$y = \frac{\xi\vartheta_3}{\vartheta_2} = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(z)}{\vartheta_4(z)} = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(u\vartheta_3^{-2})}{\vartheta_4(u\vartheta_3^{-2})}. \tag{3.22}$$

The differential equation (3.21) may be written as

$$u = \int_0^y (1-t^2)^{-\frac{1}{2}}(1-\kappa^2 t^2)^{-\frac{1}{2}} dt, \tag{3.23}$$

and y can be expressed in terms of u as the quotient of two Theta-functions in the form (3.22). Thus, if

$$u = \int_0^y (1-t^2)^{-\frac{1}{2}}(1-\kappa^2 t^2)^{-\frac{1}{2}} dt,$$

we write

$$y = \mathrm{sn}(u, \kappa)$$

when exhibit y as a function of u and κ or simply

$$y = \mathrm{sn}\, u$$

when it is unnecessary to emphasize κ.

The constant κ is called the modulus; if $\kappa'^{\frac{1}{2}} = \frac{\vartheta_4}{\vartheta_3}$, so that $\kappa^2 + \kappa'^2 = 1$, κ' is called the complementary modulus. The quasi-periods $\pi\vartheta_3^2$, $\pi\tau\vartheta_3^2$ are usually written $2K$, $2iK'$, so that $\mathrm{sn}(u,\kappa)$ has periods $4K$, $2iK'$. The function $\mathrm{sn}\, u$ is known as a Jacobian elliptic function of u, and

$$\mathrm{sn}\, u = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(u\vartheta_3^{-2})}{\vartheta_4(u\vartheta_3^{-2})}.$$

Now write

$$\mathrm{cn}\, u = \frac{\vartheta_4}{\vartheta_2}\frac{\vartheta_2(u\vartheta_3^{-2})}{\vartheta_4(u\vartheta_3^{-2})}, \tag{3.24}$$

$$\mathrm{dn}\, u = \frac{\vartheta_4}{\vartheta_3}\frac{\vartheta_3(u\vartheta_3^{-2})}{\vartheta_4(u\vartheta_3^{-2})}. \tag{3.25}$$

Then, from (3.17), we have

$$\frac{d}{du}\,\mathrm{sn}\, u = \mathrm{cn}\, u\,\mathrm{dn}\, u, \tag{3.26}$$

and from (3.13), we have

$$\mathrm{sn}^2\, u + \mathrm{cn}^2\, u = 1, \tag{3.27}$$

$$\kappa^2\mathrm{sn}^2\, u + \mathrm{dn}^2\, u = 1, \tag{3.28}$$

$$\mathrm{cn}\, 0 = \mathrm{dn}\, 0 = 1. \tag{3.29}$$

Next, let us differentiate the equation (3.27); on using equation (3.26), we get

$$\frac{d}{du}\,\mathrm{cn}\, u = -\mathrm{sn}\, u\,\mathrm{dn}\, u;$$

in like manner, from equations (3.28) and (3.26) we have

$$\frac{d}{du}\,\mathrm{dn}\, u = -\kappa^2\mathrm{sn}\, u\,\mathrm{cn}\, u.$$

Remark 3.3.1. If

$$u = \int_y^1 (1-t^2)^{-\frac{1}{2}}(\kappa'^2 + \kappa^2 t^2)^{-\frac{1}{2}}\,dt,$$

then $y = \mathrm{cn}(u, \kappa)$ or simply $y = \mathrm{cn}\, u$.

If

$$u = \int_y^1 (1-t^2)^{-\frac{1}{2}}(t^2 - \kappa'^2)^{-\frac{1}{2}} dt,$$

then $y = \mathrm{dn}(u, \kappa)$ or simply $y = \mathrm{dn}\, u$.

Glaisher has invented a short and convenience notation to express reciprocals and quotients of the Jacobian elliptic functions :

$$\mathrm{ns}\, u = 1/\mathrm{sn}\, u, \qquad \mathrm{nc}\, u = 1/\mathrm{cn}\, u, \qquad \mathrm{nd}\, u = 1/\mathrm{dn}\, u,$$

$$\mathrm{sc}\, u = \mathrm{sn}\, u/\mathrm{cn}\, u, \quad \mathrm{sd}\, u = \mathrm{sn}\, u/\mathrm{dn}\, u, \quad \mathrm{cd}\, u = \mathrm{cn}\, u/\mathrm{dn}\, u,$$

$$\mathrm{cs}\, u = \mathrm{cn}\, u/\mathrm{sn}\, u, \quad \mathrm{ds}\, u = \mathrm{dn}\, u/\mathrm{sn}\, u, \quad \mathrm{dc}\, u = \mathrm{dn}\, u/\mathrm{cn}\, u.$$

Define

$$K = \int_0^1 (1-t^2)^{-\frac{1}{2}}(1-\kappa^2 t^2)^{-\frac{1}{2}} dt = \int_0^{\frac{1}{2}\pi} (1-\kappa^2 \sin^2\phi)^{-\frac{1}{2}} d\phi, \tag{3.30}$$

$$K' = \int_0^1 (1-t^2)^{-\frac{1}{2}}(1-\kappa'^2 t^2)^{-\frac{1}{2}} dt = \int_0^{\frac{1}{2}\pi} (1-\kappa'^2 \sin^2\phi)^{-\frac{1}{2}} d\phi. \tag{3.31}$$

In (3.31), make the substitution

$$s = (1-\kappa'^2 t^2)^{-\frac{1}{2}},$$

which gives

$$K' = \int_1^{\frac{1}{\kappa}} (s^2-1)^{-\frac{1}{2}}(1-\kappa^2 s^2)^{-\frac{1}{2}} ds. \tag{3.32}$$

Besides, we have some relations

$$K + iK' = \int_0^{\frac{1}{\kappa}} (1-t^2)^{-\frac{1}{2}}(1-\kappa^2 t^2)^{-\frac{1}{2}} dt, \tag{3.33}$$

$$\mathrm{sn}\, K = 1, \quad \mathrm{cn}\, K = 0, \quad \mathrm{dn}\, K = \kappa',$$

$$\mathrm{sn}\,(K+iK') = \frac{1}{\kappa}, \quad \mathrm{cn}\,(K+iK') = -\frac{i\kappa'}{\kappa}, \quad \mathrm{dn}\,(K+iK') = 0.$$

An integral of the form $\int R(\omega, x)dx$, where R denotes a rational function of ω and x, and ω^2 is a quartic or cubic function of x (without repeated factors), is called an elliptic integral.

The following three intgrals were called by Legendre elliptic integrals of the first, second and third kinds, respectively.

$$\text{(i)} \quad \int \{(A_1t^2 + B_1)(A_2t^2 + B_2)\}^{-\frac{1}{2}} dt, \tag{3.34}$$

$$\text{(ii)} \quad \int t^2 \{(A_1t^2 + B_1)(A_2t^2 + B_2)\}^{-\frac{1}{2}} dt, \tag{3.35}$$

$$\text{(iii)} \quad \int (1 + Nt^2)^{-1} \{(A_1t^2 + B_1)(A_2t^2 + B_2)\}^{-\frac{1}{2}} dt. \tag{3.36}$$

The elliptic integral of the first kind presents no difficulty, as it can be integrated at once by a substitution based on the integral formulae of the complementary modulus and Glaisher's notation for quotiens; thus, if A_1, B_1, A_2, B_2 are all positive and $A_2B_1 > A_1B_2$, we write

$$A_1^{\frac{1}{2}} t = B_1^{\frac{1}{2}} \operatorname{cs}(u, \kappa) \qquad [\kappa'^2 = \frac{A_1B_2}{A_2B_1}.]$$

Remark 3.3.2. The degenerates of $\operatorname{dn} u$

By the integral representation of sn$u$,

$$u = \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}$$

and $\kappa^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1$, we get

$$\operatorname{dn} u \text{ degenerates to } \operatorname{sech} u \text{ as } \kappa \to 1,$$

$$\operatorname{dn} u \text{ degenerates to } 1 \text{ as } \kappa \to 0.$$

4 The nonlinear Schrodinger equation

Recall from (1.1), the nonlinear Schrodinger equation is

$$iq_t + q_{xx} + 2|q|^2 q = 0. \tag{4.1}$$

In this paper, we study the special case N=2. Then the NLS solution $q(x,t)$ now resides on the Riemann surface of the elliptic curve

$$R(E) = \sqrt{\prod_{k=1}^{4}(E - E_k)} \tag{4.2}$$

from (1.2) as illustrated in Figure 54, where $E_k \in \mathbb{C} \setminus \mathbb{R}$, $1 \le k \le 4$; $E_2 = E_1^*$ and $E_4 = E_3^*$.

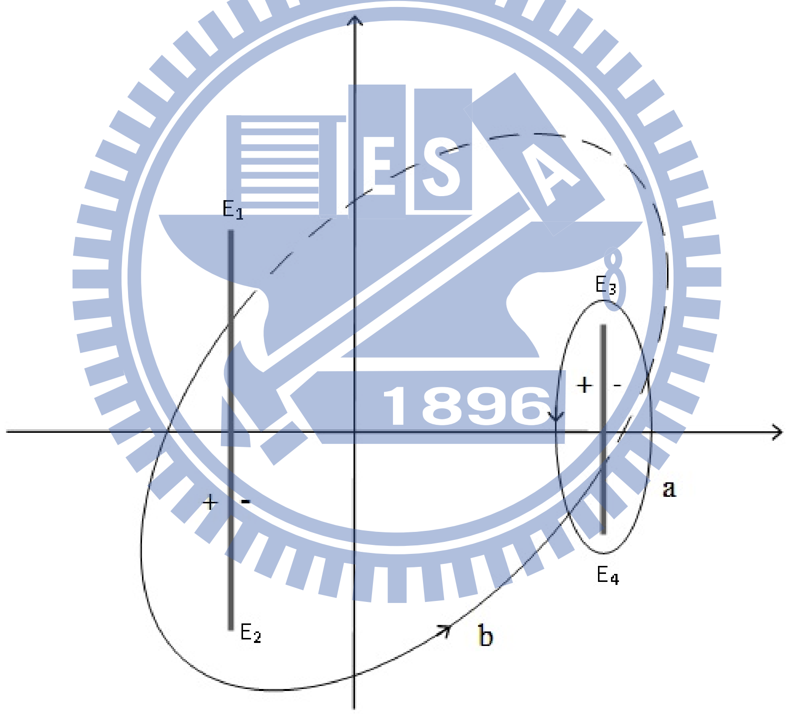

Figure 54.The Riemann surface of $R(E) = \sqrt{\prod_{k=1}^{4}(E - E_k)}$

4.1 The NLS solution $q(x,t)$

Let the constants B and C be defined on this Riemann surface as

$$C = (\int_{a-cycle} \frac{dE}{R(E)})^{-1}, \; B = (\int_{b-cycle} \frac{dE}{R(E)})(\int_{a-cycle} \frac{dE}{R(E)})^{-1}, \tag{4.3}$$

then according to [2][3], $q(x,t)$ may be derived from the following equations,

$$\frac{q_x}{q} = i(2\mu - \sum_{k=1}^{4} E_k), \tag{4.4}$$

$$\frac{q_t}{q} = i[2\mu \sum_{k=1}^{4} E_k + 2\sum_{j>k}^{4} E_j E_k - \frac{3}{2}(\sum_{k=1}^{4} E_k)^2]; \tag{4.5}$$

$$\mu_x = -2iR(\mu), \tag{4.6}$$

$$\mu_t = \mu_x \sum_{k=1}^{4} E_k = -2iR(\mu)\sum_{k=1}^{4} E_k; \tag{4.7}$$

$$\int_{\mu_0}^{\mu(x,t)} \frac{dE}{R(E)} = \int_{\mu_0}^{\mu(x,t)} \frac{dE}{\sqrt{\prod_{k=1}^{4}(E-E_k)}} = -2i(x + \sum_{k=1}^{4} E_k t) + \alpha_0. \tag{4.8}$$

(4.8) shows that $\mu(x,t)$ is the inverse of a Abel integral with the integration constant α_0 determined by the initial condition μ_0.

Rewrite $R(E)$ as

$$R^2(E) = \prod_{k=1}^{4}(E-E_k) = (E^2 - 2b_1E + c_1)(E^2 - 2b_2E + c_2) = S_1(E)S_2(E),$$

where $b_1 = Re(E_1)$, $b_2 = Re(E_3)$, $c_1 = |E_1|^2$, and $c_2 = |E_3|^2$. It is clear that b_1, b_2, c_1, and $c_2 \in \mathbb{R}$. Since $S_1(E) - \lambda S_2(E) = (1-\lambda)E^2 - 2(b_1 - \lambda b_2)E + (c_1 - \lambda c_2)$, then $S_1(E) - \lambda S_2(E)$ is a perfect square if λ is such that

$$(1-\lambda)(c_1 - \lambda c_2) - (b_1 - \lambda b_2)^2 = 0. \tag{4.9}$$

Let $\lambda_1 > \lambda_2$ be the two roots of (4.9), and α_1, α_2 be the two roots of $S_1(E) - \lambda_j S_2(E)$, then for each λ_j,

$$S_1(E) - \lambda_j S_2(E) = (1-\lambda_j)(E-\alpha_j)^2, \;\; j = 1, 2.$$

Therefore,

$$S_j(E) = A_j(E-\alpha_1)^2 + B_j(E-\alpha_2)^2, \;\; j = 1, 2,$$

where $A_1 = \frac{\lambda_2(1-\lambda_1)}{\lambda_2-\lambda_1}$, $B_1 = \frac{\lambda_1(\lambda_2-1)}{\lambda_2-\lambda_1}$, $A_2 = \frac{1-\lambda_1}{\lambda_2-\lambda_1}$, and $B_2 = \frac{\lambda_2-1}{\lambda_2-\lambda_1}$. Note that λ_1, $\lambda_2 > 0$, and $A_2B_1 > A_1B_2$ since

$$\lambda_1 + \lambda_2 = \frac{c_1 + c_2 - 2b_1b_2}{c_2 - b_2^2} > 0$$

and

$$\lambda_1\lambda_2 = \frac{c_1 - b_1^2}{c_2 - b_2^2} > 0.$$

Now let

$$t = \frac{E-\alpha_1}{E-\alpha_2}, \text{ that is, } E = \frac{\alpha_1 - t\alpha_2}{1-t},$$

then

$$dE = \frac{\alpha_1 - \alpha_2}{(1-t)^2}dt.$$

Thus, (4.8) can be written as

$$\begin{aligned}\int \frac{dE}{R(E)} &= \int \frac{dE}{\sqrt{S_1(E)S_2(E)}} = \int \frac{dE}{\sqrt{\prod_{j=1}^{2}[A_j(E-\alpha_1)^2 + B_j(E-\alpha_2)^2]}} \\ &= \frac{1}{\alpha_1 - \alpha_2}\int \frac{dt}{\sqrt{(A_1t^2+B_1)(A_2t^2+B_2)}}, \end{aligned} \tag{4.10}$$

which is the Legendre integral of the first kind since α_j, A_j, B_j $j = 1, 2$, are all real. We see that the Abel integral (4.8) is exactly a Legendre integral of the first kind.

Following (4.4) and (4.5), $q(x,t)$ is found in terms of the Jacobian elleptic function $\mathrm{dn}(u,\kappa)$,

$$q(x,t) = q_0 \cdot \mathrm{dn}(\pi \cdot \vartheta_3^2(0;B) \cdot (\alpha(x,t) + I_0)) \cdot e^{i(Kx-Wt)} \tag{4.11}$$

where

$$\begin{aligned}&\alpha(x,t) = -2iC \cdot (x + \sum_{k=1}^{4} E_k \cdot t),\ q_0 = \pm 2i\pi C \cdot \vartheta_3^2(0;B),\ I_0 \in \mathbb{R},\ K = -\frac{1}{2}\sum_{k=1}^{4} E_k, \text{ and} \\ &W = \frac{1}{4}(\sum_{k=1}^{4} E_k)^2 - 4\pi^2C^2 \cdot (e^{i\pi B} \cdot \vartheta_3^4(\frac{B}{2};B) - 2\vartheta_3^4(0;B)).\end{aligned} \tag{4.12}$$

As shown in [2][3], B, C$\in i\mathbb{R}$, therefore, according to Definition 3.2.1 and (3.12), $\vartheta_j(0;B)$, $\vartheta_3(\frac{1}{2};B)$, $\vartheta_3(\frac{B}{2};B) \in \mathbb{R}$, $j = 1,2,3,4$, which implies that q_0, $\alpha(x,t)$, K, W$\in \mathbb{R}$. It is easy to verify that $q(x,t)$ in (4.11) is an exact NLS two-(real)phase, quasi-periodic solution.

4.2 The degenerates of $q(x,t)$

In this section, we introduce two ways to find the degenerates of $q(x,t)$ in (4.11) when $\{E_k,\ 1 \le k \le 4\}$ collapse. There are several ways for $\{E_k,\ 1 \le k \le 4\}$ to collapse.

Case 1**.**

$$E_1 = re^{i(\epsilon+\frac{\pi}{2})} = ire^{i\epsilon},\ E_3 = re^{i(-\epsilon+\frac{\pi}{2})} = ire^{-i\epsilon},\ \epsilon \ll 1,\ E_2 = E_1^*,\ E_4 = E_3^*,\ \text{and}$$

$$\mu(x,t) = \mu^{(0)}(x,t) + \epsilon \cdot \mu^{(1)}(x,t) + O(\epsilon^2). \tag{4.13}$$

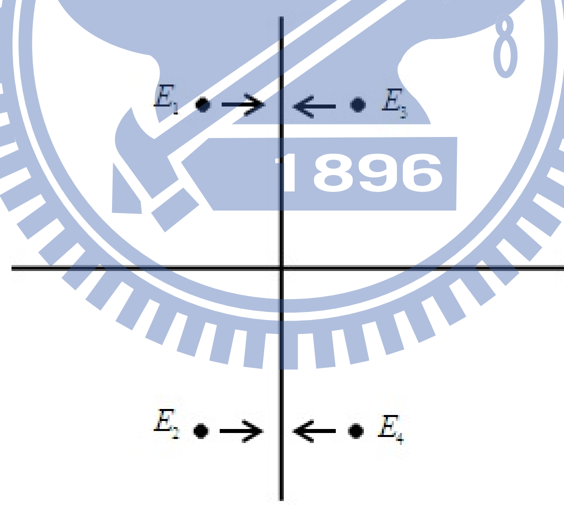

Figure 55.The Case 1

Use Taylor's formula, E_1 and E_3 can be written as

$$E_1 = ir - \epsilon r + O(\epsilon^2),\ E_3 = ir + \epsilon r + O(\epsilon^2),$$

then $Re(E_1) = -\epsilon r + O(\epsilon^2)$ and $Re(E_3) = \epsilon r + O(\epsilon^2)$. Therefore,

$$
\begin{aligned}
R^2(\mu) &= \prod_{k=1}^{4}(\mu - E_k) \\
&= [(\mu - E_1)(\mu - E_2)][(\mu - E_3)(\mu - E_4)] \\
&= [(\mu^{(0)2} + r^2) + \epsilon(2\mu^{(0)}\mu^{(1)} + 2r\mu^{(0)}) + O(\epsilon^2)][(\mu^{(0)2} + r^2) + \epsilon(2\mu^{(0)}\mu^{(1)} - 2r\mu^{(0)}) + O(\epsilon^2)].
\end{aligned}
$$

Take $R(\mu) = -(\mu^{(0)2} + r^2) - \epsilon \cdot 2\mu^{(0)}\mu^{(1)} + O(\epsilon^2)$, then according to (4.6), we have

$$
\mu_x^{(0)} = 2i(\mu^{(0)2} + r^2), \;\; \mu_x^{(1)} = 4i\mu^{(0)}\mu^{(1)}; \tag{4.14}
$$

and according to (4.7) with

$$
\sum_{k=1}^{4} E_k = 2Re(E_1) + 2Re(E_3) = O(\epsilon^2),
$$

we have

$$
\mu_t^{(0)} = 0, \;\; \mu_t^{(1)} = 0. \tag{4.15}
$$

Integrate (4.14) and (4.15) for $\mu^{(0)}$, we get

$$
\mu^{(0)}(x,t) = r \cdot \tan(2irx + rA) = r \cdot \tan(i2r(x+a)) = ir \cdot \tanh(2r(x+a)) \tag{4.16}
$$

where A and a are constants. According to (4.4), we have

$$
\frac{q_x^{(0)}}{q^{(0)}} = 2i\mu^{(0)} = -2r \cdot \tanh(2r(x+a)), \tag{4.17}
$$

and therefore

$$
q^{(0)} = \operatorname{sech}(2r(x+a)) \cdot e^{B(t)}. \tag{4.18}
$$

According to (4.5), we get

$$
\frac{q_t^{(0)}}{q^{(0)}} = 4ir^2. \tag{4.19}
$$

Inserting (4.18) into (4.19) yields

$$
B(t) = 4ir^2 t + c_0 \tag{4.20}
$$

where c_0 is a integration constant, and is to be determined by that (4.18) satisfies NLS (4.1). Finally, we obtain

$$q^{(0)}(x,t) = 2r \cdot \operatorname{sech}(2rx + 2ra) \cdot e^{4ir^2 t}. \tag{4.21}$$

$q^{(0)}(x,t)$ in (4.21) is called a NLS modulated oscillation and such a solution is localized in x, periodic in t.

Case 2.

$$E_1 = re^{i(\pi-\epsilon)} = -re^{-i\epsilon},\ E_3 = re^{i\epsilon},\ \epsilon \ll 1,\ E_2 = E_1^*,\ E_4 = E_3^*,\ \text{and}$$

$$\mu(x,t) = \mu^{(0)}(x,t) + \epsilon \cdot \mu^{(1)}(x,t) + O(\epsilon^2). \tag{4.22}$$

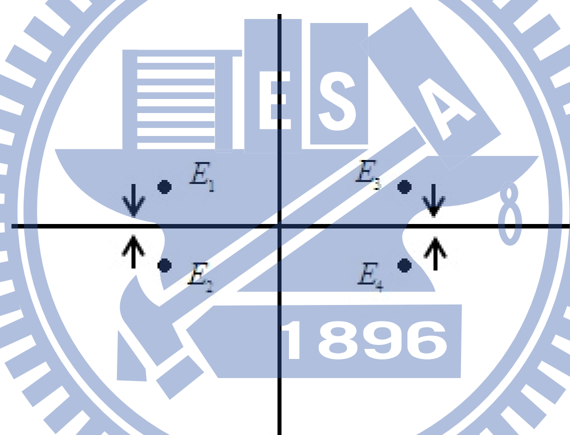

Figure 56.The Case 2

Use the similar way, we can obtain

$$q^{(0)}(x,t) = A \cdot \sec(2rx + B) \cdot e^{-4ir^2 t} \tag{4.23}$$

which is not a NLS solution.

Case 3.

$$E_1 = ir,\ E_3 = s + i\epsilon,\ \epsilon \ll 1,\ E_2 = E_1^*,\ E_4 = E_3^*,\ \text{and}$$

$$\mu(x,t) = \mu^{(0)}(x,t) + \epsilon \cdot \mu^{(1)}(x,t) + O(\epsilon^2),\ \mu^{(0)} \equiv s. \tag{4.24}$$

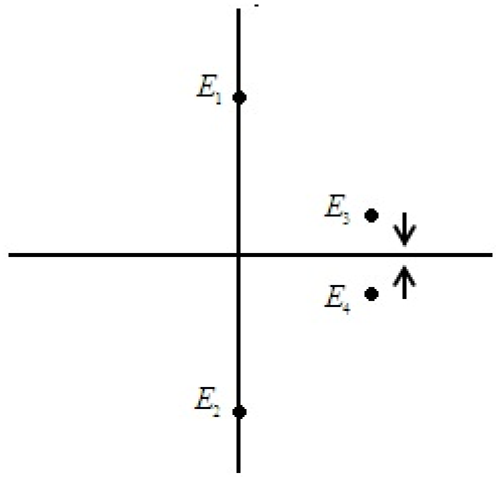

Figure 57.<u>The Case 3</u>

Then

$$\frac{q_x^{(0)}}{q^{(0)}} = 0, \tag{4.25}$$

$$\frac{q_t^{(0)}}{q^{(0)}} = 2ir. \tag{4.26}$$

Integrating (4.25) and (4.26) yield

$$q^{(0)} = re^{i2r^2t} \tag{4.27}$$

which is a NLS plane-wave solution and is x-independent.

Case 4**.**

Same as Case 1 except $\mu^{(0)} \equiv \pm ir$.

In this case, we get

$$q^{(0)} = A \cdot e^{-2rx} e^{i4r^2t} \tag{4.28}$$

which is not a NLS solution.

Case 5**.**

Same as Case 2 except $\mu^{(0)} \equiv -r$.

In this case, we obtain

$$q^{(0)} = A \cdot e^{-2ir(x+2rt)} \tag{4.29}$$

which is not a NLS solution.

Next, we derive the degenerates of $q(x,t)$ in (4.11) by considering the degenerates of $\mathrm{dn}(u;\kappa)$, where $\kappa \to 0$ or $\kappa \to 1$. By (3.25), we can find that one period $2K$ of $\mathrm{dn}(u;\kappa)$,

$$2K \to \pi \text{ as } \kappa \to 0, \text{ and} \tag{4.30}$$

$$2K \to \infty \text{ as } \kappa \to 1. \tag{4.31}$$

Therefore, we expect that the degenerates of $q(x,t)$ for $\kappa \to 0$ will be NLS periodic solution and will be similar to the plane-waves in Case 3, and those for $\kappa \to 1$ will be NLS localized solution (that is, the periods approach to infinitely long) and will be similar to the NLS modulated oscillations in Case 1.

Let $\{E_k,\ 1 \le k \le 4\}$ be denoted as $\{E_k^{(\epsilon)},\ 1 \le k \le 4\}$, $q(x,t)$ in (4.11) be denoted as $q^{(\epsilon)}$, and $\{B,\ C,\ \kappa,\ \vartheta_3(0;B),\ K,\ W\}$ be denoted as $\{B^{(\epsilon)},\ C^{(\epsilon)},\ \kappa^{(\epsilon)},\ \vartheta_3^{(\epsilon)},\ K^{(\epsilon)},\ W^{(\epsilon)}\}$. For the case $\kappa^{(\epsilon)} \to 1$, $\{E_k^{(\epsilon)},\ 1 \le k \le 4\}$ are same as (4.13). Let

$$g^{(\epsilon)} = C^{(\epsilon)} \cdot \vartheta_3^{(\epsilon)2}, \text{ and } r^{(\epsilon)} = -i\pi \cdot g^{(\epsilon)}. \tag{4.32}$$

Let

$$\begin{aligned} &g^{(\epsilon)} \to g^{(0)} \text{ as } \epsilon \to 0 \text{ (that is, } \kappa^{(\epsilon)} \to 1), \\ &r^{(\epsilon)} \to r^{(0)} = -i\pi \cdot g^{(0)} \text{ as } \epsilon \to 0. \end{aligned} \tag{4.33}$$

According to (4.12) with the fact that

$$\sum_{k=1}^{4} E_k^{(\epsilon)} \to 0 \text{ as } \epsilon \to 0, \text{ and } B^{(\epsilon)} \to 0 \text{ as } \epsilon \to 0,$$

we have

$$\begin{aligned} &K^{(\epsilon)} \to 0 \text{ as } \epsilon \to 0, \text{ and} \\ &W^{(\epsilon)} \to -4\pi^2 C^{(0)2}[\vartheta_3^{(0)4} - 2\vartheta_3^{(0)4}] = 4\pi^2 C^{(0)2}\vartheta_3^{(0)4} = 4\pi^2 g^{(0)2} = -4r^2 \text{ as } \epsilon \to 0. \end{aligned}$$

Therefore,

$$q^{(\epsilon)}(x,t) \to q^{(0)}(x,t) = 2r \cdot \mathrm{sech}(2rx + 2ra) \cdot e^{4ir^2t} \text{ as } \kappa^{(\epsilon)} \to 1 \tag{4.34}$$

and $q^{(0)}(x,t)$ is the NLS modulated oscillation which has been obtained in Case 1.

Next, for the case $\kappa^{(\epsilon)} \to 0$, $\{E_k^{(\epsilon)},\ 1 \le k \le 4\}$ are same as (4.24). Now we have

$$\sum_{k=1}^{4} E_k^{(\epsilon)} \to 2s \text{ as } \epsilon \to 0, \text{ and } e^{i\pi B^{(\epsilon)}} \to 0 \text{ as } \epsilon \to 0,$$

which implies that

$$K^{(\epsilon)} \to -s \text{ as } \epsilon \to 0, \text{ and}$$

$$W^{(\epsilon)} \to \frac{1}{4}(2s)^2 - 4\pi^2 C^{(0)2}[-2\vartheta_3^{(0)4}] = s^2 + 8\pi^2 C^{(0)2}\vartheta_3^{(0)4} = s^2 + 8\pi^2 g^{(0)2} = s^2 - 8r^2 \text{ as } \epsilon \to 0.$$

Therefore,

$$q^{(\epsilon)}(x,t) \to q^{(0)}(x,t) = g \cdot e^{i2g^2t} \cdot e^{-i(sx+s^2t)} \text{ as } \kappa^{(\epsilon)} \to 0 \tag{4.35}$$

where $g = 2r$. The degenerates (4.35) satisfy NLS, and are periodic in x for fixed t, and are periodic in t for fixed x. In particular, these degenerates are the NLS plane-wave solutions when $s = 0$, that is, $E_k^{(\epsilon)} \in i\mathbb{R}$ for all k.

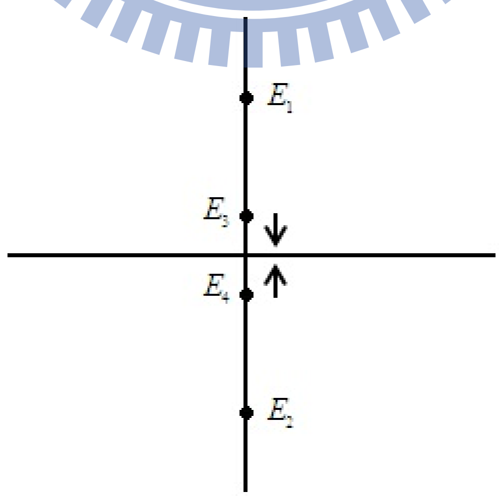

Figure 58.The specified Case 3

5 Conclusion

We know the nonlinear Schrodinger equation $iq_t + q_{xx} + 2|q|^2 q = 0$ (NLS) has solutions $q(x,t)$ reside on the curve

$$R(E) = \sqrt{\prod_{k=1}^{2N}(E - E_k)} \tag{5.1}$$

where $N \in \mathbb{N}$, $E_k \in \mathbb{C} \setminus \mathbb{R}$ and $E_{2k-1}^* = E_{2k}$. But $R(E) = \sqrt{\prod_{k=1}^{2N}(E - E_k)}$ is a two-valued function on complex plane $\mathbb{C}$ for the same z, so we need to modify its domain on a new surface such that it becomes a single-valued and analytic function, and we call this domain Riemann surface.

We first study the theory of Riemann surface. Next, we introduce the a, b cycles since all the simple closed curve on the Riemann surface can be written as the linear combination of them, study finding the simplest equivalent paths of a, b cycles, and then evaluate the integrals over a, b cycles or other pathes with horizontal, vertical, and slant cut structure. In addition, we use Mathematica to help us evaluate the integrals and discuss the differences between theory and Mathematica with different cut structures.

Then we study the classical elliptic functions. To begin with, we introduce some definitions and properties of Weierstrassian elliptic functions. Next, we study the four Theta-functions and some relations between them. Furthermore, we study the Jacobian elliptic functions sn$u$, cn$u$, and dn$u$.

Finally, we use the theories of Riemann surfaces and classical elliptic functions to solve some special solutions of NLS and analyze the degenerates of the NLS solutions $q(x,t)$. There are two degenerates of $q(x,t)$. One is the NLS modulated oscillation $q^0(x,t) = 2r \cdot \mathrm{sech}(2rx + 2ra) \cdot e^{4ir^2 t}$ which is localized in x, and periodic in t. The other is the NLS plane-wave solution $q^0(x,t) = re^{i2r^2 t}$ which is x-independent, and periodic in t. Moreover, the degenerates of $q(x,t)$ for $\kappa \to 0$ is similar to the plane-wave solution, and those for $\kappa \to 1$ is similar to the NLS modulated oscillations.

A The integrals over a, b cycles

We place Mathematica codes of previous examples here.

Example 2.4.4. $f[X\_] = (x+1)^{\frac{1}{2}}\, x\, (x-1)^{\frac{1}{2}}\, (x-2)^{\frac{1}{2}}\, (x-3)^{\frac{1}{2}}\, (x-4)^{\frac{1}{2}}$

1. $\int_{a_1} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_4^3 \frac{1}{f[r]} dr] = 0.474399i$

2. $\int_{a_2} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_2^1 \frac{1}{f[r]} dr] = -3.3715i$

3. $\int_{b_2} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_0^1 \frac{1}{f[r]} dr] = 1.29234$

4. $\int_{b_1} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_0^1 \frac{1}{f[r]} dr + 2\int_2^3 \frac{1}{f[r]} dr] = -3.02069$

Example 2.5.3. $f[X\_] = (x+1-2\mathrm{I})^{\frac{1}{2}}\,(x+1+2\mathrm{I})^{\frac{1}{2}}\,(x-1-3\mathrm{I})^{\frac{1}{2}}\,(x-1+3\mathrm{I})^{\frac{1}{2}}\,(x-2-\mathrm{I})^{\frac{1}{2}}\,(x-2+\mathrm{I})^{\frac{1}{2}}$

1. $\int_{a_1} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_{-1}^1 \frac{\mathrm{I}}{f[2+r\mathrm{I}]} dr] = 0.54204i$

2. $\int_{a_2} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_1^3 \frac{\mathrm{I}}{f[1+r\mathrm{I}]} dr + 2\int_1^{-1} \frac{\mathrm{I}}{f[1+r\mathrm{I}]} dr + 2\int_{-3}^{-1} \frac{\mathrm{I}}{f[1+r\mathrm{I}]} dr] = -0.964009i$

3. $\int_{b_2} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_{-1}^1 \frac{\mathrm{I}}{f[-1+r\mathrm{I}]} dr + 2\int_{-1}^{-2} \frac{\mathrm{I}}{f[-1+r\mathrm{I}]} dr + 2\int_1^{-1} \frac{1}{f[r+\mathrm{I}]} dr + 2\int_3^1 \frac{\mathrm{I}}{f[1+r\mathrm{I}]} dr]$
   $= 0.226455 + 0.692989i$

4. $\int_{b_1} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_{-1}^1 \frac{\mathrm{I}}{f[-1+r\mathrm{I}]} dr + 2\int_{-1}^{-2} \frac{\mathrm{I}}{f[-1+r\mathrm{I}]} dr + 2\int_1^{-1} \frac{1}{f[r+\mathrm{I}]} dr + 2\int_3^1 \frac{\mathrm{I}}{f[1+r\mathrm{I}]} dr$
   $+2\int_1^{-1} \frac{\mathrm{I}}{f[1+r\mathrm{I}]} dr + 2\int_{-3}^{-1} \frac{\mathrm{I}}{f[1+r\mathrm{I}]} dr + 2\int_1^2 \frac{1}{f[r+\mathrm{I}]} dr] = -0.175558 - 0.060036i$

Example 2.6.4. $f[X\_] = (x+\sqrt{3})^{\frac{1}{2}}\,(x-\mathrm{I})^{\frac{1}{2}}\,(x-2\mathrm{I})^{\frac{1}{2}}\,(x-1-2\mathrm{I}-\sqrt{3}\mathrm{I})^{\frac{1}{2}}\,(x-1-\mathrm{I})^{\frac{1}{2}}\,(x-2-2\mathrm{I})^{\frac{1}{2}}$

1. $\int_{a_1} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_1^0 \frac{-1-\mathrm{I}}{f[2+2\mathrm{I}+r(-1-\mathrm{I})]} dr] = -0.717556 + 1.13471i$

2. $\int_{a_2} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_1^0 \frac{-1-\sqrt{3}\mathrm{I}}{f[1+(2+\sqrt{3})\mathrm{I}+r(-1-\sqrt{3}\mathrm{I})]} dr] = -0.502113 - 1.21448i$

3. $\int_{b_2} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_2^1 \frac{\mathrm{I}}{f[r\mathrm{I}]} dr] = 1.13334 - 1.2457i$

4. $\int_{b_1} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_2^1 \frac{\mathrm{I}}{f[r\mathrm{I}]} dr + 2\int_{2+\sqrt{3}}^2 \frac{\mathrm{I}}{f[1+r\mathrm{I}]} dr + 2\int_1^2 \frac{\mathrm{I}}{f[1+r\mathrm{I}]} dr] = -0.466353 - 0.103835i$

B The integrals over other paths

Example 2.7.1. $f[X\_] = (x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}(x-3)^{\frac{1}{2}}(x-4)^{\frac{1}{2}}(x-5)^{\frac{1}{2}}(x-6)^{\frac{1}{2}}(x-7)^{\frac{1}{2}}$

1. $\int_{\gamma_1} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_5^4 \frac{1}{f[r]} dr] = -0.935417i$

2. $\int_{\gamma_2} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_1^{-2} \frac{1}{f[r]} dr] = -0.13304i$

3. $\int_{\gamma_3} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_3^4 \frac{1}{f[r]} dr + 2\int_5^6 \frac{1}{f[r]} dr] = 0.23235$

4. $\int_{\gamma_4} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_5^4 \frac{1}{f[r]} dr + 2\int_3^2 \frac{1}{f[r]} dr + 2\int_7^6 \frac{1}{f[r]} dr] = 0.143249i$

Example 2.7.2. $f[X\_] = (x+\sqrt{3})^{\frac{1}{2}}(x-\mathrm{I})^{\frac{1}{2}}(x-2\mathrm{I})^{\frac{1}{2}}(x-1-2\mathrm{I}-\sqrt{3}\mathrm{I})^{\frac{1}{2}}(x-1-\mathrm{I})^{\frac{1}{2}}(x-2-2\mathrm{I})^{\frac{1}{2}}$

$$\int_{\gamma} \frac{1}{f(z)} dz \overset{Math.}{=} \mathrm{N}[2\int_2^1 \frac{\mathrm{I}}{f[r\mathrm{I}]} dr] = 1.13334 - 1.2457i$$

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